P1

Diverse Sub Prob $\in NP$

Give a plan, we can check whether it is correct by linear scan (Using count array/hashing) in polynomial time ($O(nm)$ or $O(nm^2)$)

Independent Set $\in$ Diverse Sub Prob

We define that in an independent set problem (For instance I), nodes are n.

In this problem (Instance I'), we set $n' = m' = n$。

And for i th row, we set $A[i, j] = 1$ if and only if $i = j$ or (i, j) has an edge.

$I \to I'$

Let $S(i)$ be the set of j that $A[i, j] = 1$

For any solution that has k nodes, we can choose corresponding rows, and by definition, we can know that there $S(i)$ has no intersection (Because no two nodes has an edge).

$I' \to I$

For any solution that has k rows， we can choose nodes if we choose corresponding rows. By definition, it's an independent set of k

Since Independent Set $\in$ NP-Complete, So diverse Subset Problem is also in NP-Complete

P2

For a 3-coloring instance I, we build a 4-color instance I' as follows: let Graph in I be G, Graph G' in I' based on G (copy every node and edges) and then create a new dummy node T' and add edge between every node and T'。

$I \to I'$

For every previous node, we copy the color pattern in I to I', and we use the 4th color (The color that not used in I) to color T', thus for every edge, it's two sides has a different color (For previous edge it's obvious, for edge between T' and previous nodes, since T' has unique new color, so it must meet condition)

$I' \to I$

For every previous node, we copy the color pattern in I' to I. Since T' links to every other node, so previous nodes must not use the color that T' uses, so they only can be colored in three colors, which is the same as the 3-color problem.

P3

We assume $|X| = |Y| = |Z| = n$, we need to find a algorithm that $\frac{|ALG|}{|OPT|} = \frac{|M_{ALG}|}{|M_{OPT}|} \geq \frac{1}{3}$

Design algorithm: we choose any $v \in T$, and remove all other element T which has common point with v until we can not pick anything (In other word, we pick any match until there no any match available anymore)

Prove: Let set $m = |M_{OPT}|, k = |M_{ALG}|,\ i$ th 3DM contain exactly one node in X, Y, Z。

For every element in M_{ALG} , it has three node, so it at most cover 3 of m matching in OPT (Cover means at least have one element in common). We should notice that $3k$ element must cover all m matching (if not, we can choose another available which is contradiction). So $3k \geq m$ 。 $3|M_{ALG}| \geq |M_{OPT}|$ which is equivlent to $\frac{|M_{ALG}|}{|M_{OPT}|} \geq \frac{1}{3}$.

Q.E.D.

P4

(1)

Bin Packing Prob $\in NP$

Give a plan, we can check whether it is correct by linear scan in polynomial time ($O(nk)$ or $O(n)$ (Using count array of k))

Partition $\in$ Bin Packing Prob

We define that in a partition problem (For instance I), there n item, i-th is w_i and the total weight is $W = \sum w_i$.

In this problem (Instance I'), we set $n' = n, K = 2,$ and the $s_i = \frac{w_i}{W/2}$ and run the algorithm.

$I \rightarrow I'$

If we have a plan in partition, which means we find several item (We called them left side) which $\sum w_i = W/2,$ so we consider there corresponding item in I', there sum $\sum s_i = 1$。 The sum of s_i in rest part is also 1。

$I' \rightarrow I$

If we have a plan in Bin Packing problem, we can also choose the first bin to correspond to first half of the partition problem. Same as previous, it's $\sum w_i = W/2$。

(2)

$\alpha = 2$

Let $S = \sum s_i$, then $|OPT| \geq S$ (At least use $\lceil S \rceil$ bin)。

Design algorithm: we pick any available item into the bin now until no one can be filled in this bin, and then we create a new bin. Also, we test the first bin and last bin, if sum of them ≤ 1, we put all last item into first one, and until the sum of first bin and last bin is greater than 1

Prove: let $m = |ALG|$, and the sum of s_j in i-th bin is v_i。 Then we must have $v_i \leq 1$ and for any $1 \leq i < m, 1 < v_i + v_{i+1} \leq 2$ (If ≤ 1, then our algorithm can put them together), also $v_1 + v_m > 1$

Then $2S = 2\sum v_i = v_1 + v_m + \sum_{i=1}^{m-1}(v_i + v_{i+1}) > m = |ALG|$

So $2|OPT| \geq 2S \geq |ALG|$, we have $\frac{|ALG|}{|OPT|} \leq 2$