Functional programming, Lecture No. 0
A bit of theoretical flashbacks

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Introduction

General words on Haskell

- The language is named after Haskell Curry, an American logician
- The first implementation: 1990
- The current language standard: Haskell2010
- Default compiler: Glasgow Haskell Compiler (GHC)
- Haskell is a strongly-typed, polymorphic, and purely functional programming language

GHC

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- Very and very roughly and approximately, the compiling pipeline is arranged as follows:

$$\text{parsing} \Rightarrow \text{compile-time (type-checking)} \Rightarrow \text{runtime}$$

A bit of history

Lambda calculus and type theory. Historical notes

- At the end of the 1920-s, Alonzo Church provided an alternative approach to the foundations of mathematics. Here, the notion of a function is the primitive one
- Lambda calculus is a formal system that describes arbitrary abstract functions

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- Lambda calculus is a formal system that describes arbitrary abstract functions
- Moreover, Church used lambda calculus to show that Peano arithmetic is undecidable.

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- The first system of typed lambda calculus is a hybrid from lambda calculus and type theory developed by Bertrand Russell and Alfred North Whitehead (1910-s).

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- Polymorphic type inference (Roger Hindley, Robin Milner and Luis Damas (1970-1980-s))
- ML: the very first language with a polymorphic inferred type system (Robin Milner, 1973)
- The language Haskell appeared at the beginning of 1990-s. Haskell desinged by Simon Peyton Jones, Philip Wadler, and others

General conceptual aspects

The notion of a function

Let us recall the ordinary notion of a function, the set-theoretical one.

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A function is a triple $\langle A, B, f \rangle$, where f is a functional relation.
$f(x) = y \Leftrightarrow \langle x, y \rangle \in f$.

The notion of a function

- In such a set-theoretical approach, we identify a function and its graph
- In lambda calculus, the notion of a function is the primitive one
- Such an understanding of a function provides us a Turing-complete model of computation

Lambda calculus in a nutshell

The formal definition

Definition
Let $V = \{x, y, z, \dots\}$ *be the set of variables. The set of preterms is generated by the following grammar*

$$M ::= x \mid (\lambda x.M) \mid (MM)$$

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Definition
The set of lambda terms is defined as the set of preterms modulo $\equiv_\alpha$.

$\Lambda = \Lambda_{pre} / \equiv_\alpha = \{[M]_N \mid M \equiv_\alpha N\}$

The reduction relation

Definition
An operational semantics is defined as the following rewriting rules:

Reduction rules

$$(\lambda x.M)N \to_\beta M[x := N]$$

$$\frac{M_1 \to_\beta M_2}{M_1 N \to_\beta M_2 N} \qquad \frac{M_1 \to_\beta M_2}{N M_1 \to_\beta N M_2}$$

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Definition

- *A term that has the form* $(\lambda x.M)N$ *is called* β*-redex*
- *A term is in normal form if it has no redexes in its subterms*

Reduction strategies

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It is clear, that SN implies WN, not vice versa. In other words, there exists a term that has an infinite reduction path, but it has a finite reduction path at the same time.

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\begin{aligned}
&(\lambda xy.x)(\lambda z.z)((\lambda x.xx)(\lambda x.xx)) \to_\beta \\
&(\lambda y.\lambda z.z)((\lambda x.xx)(\lambda x.xx)) \to_\beta \\
&(\lambda z.z)[y := (\lambda x.xx)(\lambda x.xx)] \to_\beta \\
&\lambda z.z
\end{aligned}
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$(\lambda y.\lambda z.z)((\lambda x.xx)(\lambda x.xx)) \to_\beta$	$(\lambda xy.x)(\lambda z.z)(xx)(x := [\lambda x.xx]) \to_\beta$
$(\lambda z.z)[y := (\lambda x.xx)(\lambda x.xx)] \to_\beta$	$(\lambda xy.x)(\lambda z.z)((\lambda x.xx)(\lambda x.xx)) \to_\beta$
$\lambda z.z$	$\ldots$

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Theorem
Let M be a term such that M has a normal form M', then M might be reduced to M' via normal order

Lambda calculus in a nutshell. Call-by-value and call-by-name

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- The Haskell reduction has a call-by-name strategy. Informally, such a stragety is called lazy. Laziness denotes that Haskell doesn't compute a value if it's not needed at the moment
- Call-by-name reduction reduces reducible terms to the bitter end, but it's not always optimal, unfortunately
- In Haskell, the reduction is arranged as call-by-name evaluation up to so called weak head normal form

Pure functions and side-effects

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- It means that, such a function has the same behaviour at every point. This principle is also called referential transparency
- A side-effect function is a function that may yield different value passing the same arguments. Mathematically, such a function is not function at all.
- Haskell functions are (mostly) pure ones, but Haskell isn't confluent as a version of lambda calculus

A couple of words on types

Motivation

- A type is a syntax construction that should be assigned to terms and values according to the list of rules
- Types define a sort of partial specification
- Type checking allows one to catch an enormous class of errors

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- Types define a sort of partial specification
- Type checking allows one to catch an enormous class of errors
- The standard definition by Benjamin Peirce:

> *A type system is a tractable syntactic method for proving the absence of certain program behaviours by classifying phrases according to the kinds of values they compute*

General persective

Type theory has two perspectives that often intersect with each other.

- Type theory as the branch of proof theory and constructive mathematics
- Type theory as the branch of computer science and programming language theory

A landscape of typing from a bird's eye view

We may classify the possible ways of typing as follows:

- Strong and weak typing:
  - Strong typing: Java, Haskell, Ocaml, Rust, etc
  - Weak typing: JavaScript, e.g.
- Static and dynamic typing:
  - C, C++, Java, Haskell, etc
  - JavaScript, Ruby, PHP, etc
- Implicit and explicit typing:
  - JavaScript, Ruby, PHP, etc
  - C++, Java, etc
- Inferred typing:
  - Haskell, Standard ML, Ocaml, Idris, etc

A short reminder on simply typed lambda calculus

The typing rules are:

Axiom

$$\Gamma, x : \sigma \vdash x : \sigma$$

Lambda abstraction

$$\frac{\Gamma, x : \sigma \vdash M : \psi}{\Gamma \vdash (\lambda x.M) : \sigma \to \psi}$$

Application

$$\frac{\Gamma \vdash M : \sigma \to \psi \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \psi}$$

Higher order functions

- Higher-order functions are widely used in ordinary mathematics, such as differential operator, that has type $\mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$

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- In typed lambda calculus, a function of type $\varphi \to \psi$ is called higher-order, if $\varphi = \theta \to \delta$, for example:

$$\lambda fgx.g(fx) : (\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$$

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- A function is a first-class object

Type systems and their metatheoretical properties

- Progress and preservation = safety
- Weak normalisation
- The type uniqueness
- The inversion property
- Weakening and permutation
- Canonicity
- Type preservation under substitution
- etc...

The type system classification

As you know, we have the following ways of dependency between terms and types:

- A term depends on type (polymorphism in system F)
- A type depends on type (so-called type operators in $\lambda_{\bar{\omega}}$)
- A type depends on terms (dependent types in the basic DT system called P and its extensions)

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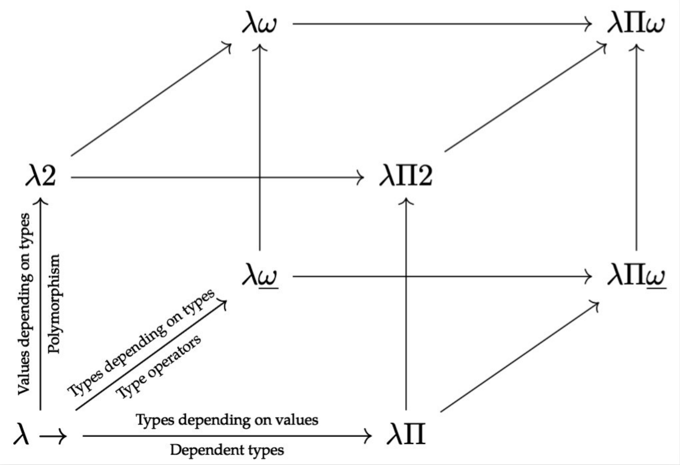
All possible combinations of these dependencies might be illustrated via Barendregt's lambda cube, the lattice of type theories

Lambda cube



λω
λΠω
λ2
λΠ2
λω
λΠω
λ →
λΠ
Values depending on types
Polymorphism
Types depending on types
Type operators
Types depending on values
Dependent types

Polymorphism

Motivation

- A polymorphism is a quite powerful tool in making a code more general and abstract
- Such an abstraction allows one to avoid a boilerplate. Let us take a look at the following functions:

```haskell
twiceInt :: (Int -> Int) -> Int -> Int
twiceInt f v = f (f v)

twiceBool :: (Bool -> Bool) -> Bool -> Bool
twiceBool f v = f (f v)
```

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```

- It is clear that all these functions do the same work.
- We don't want to reproduce the same pattern each time

System F

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- The initial motivation was a characterisation of computable functions that are provably recursive in second-order arithmetic

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- The initial motivation was a characterisation of computable functions that are provably recursive in second-order arithmetic
- From an engineering perspective, System F is an axiomatic representation of parametric polymorpism
- The problem of type inference in the Curry-style typed System F is undecidable

System F. Typing rules

The Curry style typing rules:

Generalisation rule

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \forall \alpha . \sigma} \; \alpha \notin \mathrm{rng}(\Gamma)$$

Type instantiation

$$\frac{\Gamma \vdash M : \forall \alpha . \sigma}{\Gamma \vdash M : \sigma[\alpha := \psi]}$$

System F. Typing rules

The Church style typing rules:

Type abstraction

$$\frac{\Gamma, \alpha \vdash M : \sigma}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.\sigma}$$

Explicit type instantiation

$$\frac{\Gamma \vdash M : \forall\alpha.\sigma}{\Gamma \vdash M[\psi] : \sigma[\alpha := \psi]}$$

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- One may enable the System F polymorphism in Haskell with the extension called `RankNTypes`
- The language extension `TypeApplications` provides an explicit type instantiation. In Haskell notation, `f @ a`

Polymorphism. Hindley-Milner type system

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The inference rules are the following restricted Curry-style system F rules:

Type scheme introduction

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \forall \vec{\alpha}.\sigma} \ \vec{\alpha} \cap \mathrm{rng}(\Gamma) = \emptyset$$

Type instantiation

$$\frac{\Gamma \vdash M : \forall \alpha.\sigma}{\Gamma \vdash M : \sigma[\alpha := \psi]}$$

Type instantiation

$$\frac{\Gamma, x : \sigma \vdash M : \psi}{\Gamma, \vdash \lambda x.M : (\sigma \to \psi)} \ \sigma \text{ is quantifier-free}$$

Polymorphism. Hindley-Milner type system

This version of polymorphism is a "default" polymorphism in the basic Haskell. In fact, the following code:

```
const :: a -> b -> a
const x _ = x
```

denotes something like this

```
{-# LANGUAGE ExplicitForAll #-}
const :: forall a b. a -> b -> a
const x _ = x
```

A couple of words on kinds

Extending polymorpism with kinds

- We have already resigned to the fact that all objects are classified with types
- How can we classify types themselves?
- Types have kinds in the same sense as terms have types
- In particular, such a type classification provide way to writing type operators

Extending polymorpism with kinds. Some examples

$$\mathsf{Id} = \Lambda\sigma.\sigma$$

$$\mathsf{Id} = \Lambda\sigma : *.\sigma$$

Extending polymorpism with kinds. Some examples

$$\mathsf{Id} = \Lambda\sigma.\sigma$$

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$$\mathsf{Pair} = \Lambda\sigma : *.\Lambda\psi : *.\forall\vartheta.(\sigma \to \psi \to \vartheta) \to \vartheta$$

$$\mathsf{Pair} : * \to * \to *$$

Extending polymorpism with kinds. Some examples

$$\mathsf{Id} = \Lambda\sigma.\sigma$$

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$$\mathsf{Pair} = \Lambda\sigma : *.\Lambda\psi : *.\forall\vartheta.(\sigma \to \psi \to \vartheta) \to \vartheta$$

$$\mathsf{Pair} : * \to * \to *$$

$$\mathsf{Nat} : *$$

$$\mathsf{String} : *$$

$$\mathsf{PairNS} = \mathsf{Pair}\ [\mathsf{Nat}][\mathsf{String}]$$

System F_ω

The basic characteristics of the system F_ω are:

- Kinds: $\kappa, \mu ::= * \mid (\kappa \to \mu)$
- One needs to extend the system with kinding rules and kinding judgements that have the form $\Gamma \vdash \psi : \kappa$, where ψ is a type and κ is a kind

Let us take a look at some kinding rules:

Kinding abstraction

$$\frac{\Gamma, \varphi : \kappa_1 \vdash \psi : \kappa_2}{\Gamma \vdash \Lambda\varphi : \kappa_1.\psi : \kappa_1 \to \kappa_2}$$

Kinding generalisation

$$\frac{\Gamma, \varphi : \kappa \vdash \psi : *}{\Gamma \vdash \forall\varphi : \kappa.\psi : *}$$

The modified System F rules

The quantifier rules have the modified form:

Type abstraction with kinding

$$\frac{\Gamma, \alpha : \kappa \vdash M : \sigma}{\Gamma \vdash \Lambda\alpha : \kappa.M : \forall\alpha.\sigma}$$

Type instantiation with kinding

$$\frac{\Gamma \vdash M : \forall\alpha : \kappa_1.\sigma \qquad \Gamma \vdash \psi : \kappa_1}{\Gamma \vdash M[\psi] : \sigma[\alpha := \psi]}$$

System F_ω in Haskell

One may enable the extension called `TypeFamilies` to define type-level functions

```haskell
{-# LANGUAGE TypeFamilies #-}

type family G a where
  G Int = Bool
  G a   = Char

type family AnotherG a :: *
type instance AnotherG Int = Bool
type instance AnotherG String = Char
```

The relevant Haskell type system

System FC

- The system F_ω enriched with algebraic data types was the underlying Haskell type system till the mid-2000-s
- At the moment, F_ω is extended to the system FC. Let me drop the full definition of this system

System FC

- The system F_ω enriched with algebraic data types was the underlying Haskell type system till the mid-2000-s
- At the moment, F_ω is extended to the system FC. Let me drop the full definition of this system
- The features of *FC* are:
    1. Coercions and equality constraints
    2. Generalised algebraic data types
    3. et cetera

System FC

The example of a generalised algebraic data type is the following one:

```haskell
{-# LANGUAGE GADTs #-}

  data Term a where
    Lit    :: Int -> Term Int
    Succ   :: Term Int -> Term Int
    IsZero :: Term Int -> Term Bool
    If     :: Term Bool -> Term a -> Term a -> Term a
    Pair   :: Term a -> Term b -> Term (a,b)
```

System FC

```haskell
eval :: Term a -> a
eval (Lit i)       = i
eval (Succ t)      = 1 + eval t
eval (IsZero t)    = eval t == 0
eval (If b e1 e2) = if eval b then eval e1 else eval e2
eval (Pair e1 e2) = (eval e1, eval e2)
```

System FC

Formally, the type above is defined as follows:

```
{-# LANGUAGE ExistentialQuantification #-}
{-# LANGUAGE TypeFamilies #-}

data Term a =
  a ~ Int => Lit a |
  a ~ Int => Succ a |
 (a ~ Bool) => IsZero (Term Int) |
  If Bool (Term a) (Term a) |
  forall b c. (a ~ (b, c)) => Pair (Term b) (Term c)
```

Summary

Finally

- We discussed the general aspects of functional programming itself and its story
- We briefly and brutally reintroduced you to lambda calculus and the variety of type systems
- We overview the main directions of type theory influence on Haskell
- We said a couple of words on the underlying Haskell type system called System *FC*

Thank you for your kind attention!