Reverse Engineering Inductance RLC properties using Resonance

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1 Abstract

In our experiment, we worked to reverse engineer the basics of an RLC circuit and the characteristics of the components in the circuit. The Resistor, Inductor, and Capacitor make up the components of the circuit, providing an impedance to the circuit that provides an implicit change in energy. Because the theoretical impedance of the Inductor and Capacitor work in the complex plane, utilization of the resonance of the circuit allows us to analyze if the inductance properties are true. Through our results, we were successful in this proof by an acceptable average deviation of 52.28 Hz overall and an average percentage for deviation of 16%. These deviations stem from the systematic errors from the inductor, such as deviations in coil lengths, the loop created by the circuit, and more. Furthermore, there exists an implicit random error from data collection of the component values of the RLC Circuit. In analysis of resonance in both the frequency versus voltage and frequency versus current explicitly display the accuracy in resonant theory given the resultant graphs. We see the 16% deviation in frequency that is for all but one data set an overestimate, where this one data set causes a large average percentage deviation. Lastly, the efforts made toward calculating the number of coils seemed to be partially fruitful, but not verifiable. Furthermore, the expected number of coils can not be accurately determined given a non-uniform core.

2 Introduction

When we study electromagnetism, particularly the topics of voltage and current of a circuit, the foundations of our knowledge lie upon the implied resistance from our voltage and current relationship. Specifically, we derive that the The voltage drop across a whole entire circuit over the current through the circuit is an impedance, known also as a resistance in the real plane and reactance in the complex. In DC, this is a constant value irrespective of time, while in AC this relationship is based upon time.

$$\text{DC Solution} : V = I * Z_T$$
$$\text{AC Solution} : V(t) = I(t) * Z_T$$

$$\text{where } Z_T = Z_R + Z_C + Z_L$$

An important realization is that the impedance of an AC circuit is independent on the passing of time. Thus, we can directly calculate a coefficient for impedance of our circuit that can be used to calculate the current or voltage at any instance in time. For our purposes, we analyze an RLC Circuit, made up of three different independent impedances. An important characteristic of two of these components, the inductor and capacitor is that they are dependent upon the input driving voltage frequency. We can derive this through our solution for the voltage any point in time as the sum of voltage drops, through Kirchhoff's Voltage Law. Given that our driving voltage frequency is sinusoidal in nature, we can derive the frequency dependence of impedance as shown below:

Firstly, to understand the solution, Kirchhoff's Voltage Law must be understood. The Kirchhoff's voltage law follows that for any loop in a circuit, such that current flows from one end of a point in the circuit to the other end (the path) by the electric field in the circuit, the sum of voltage drops in the circuit is equal to zero.

$$\int Edl = \sum V_i(t) = 0$$

$$\sum V_i(t) = V_{input}(t) - V_R(t) - V_C(t) - V_L(t)$$

$$V_{input}(t) = V_R(t) + V_C(t) + V_L(t)$$

Given that the known voltage drop across each component is known through the following identities, we can further derive the second order differential equation.

$$V_L = \frac{\delta I(t)}{\delta t} * L$$
$$V_R = I(t) * R$$

$$V_C = \frac{Q}{C}$$

$$V_{input}(t) = \frac{\delta I(t)}{\delta t} * L + I(t) * R + \frac{Q}{C}$$

Taking the derivative with respect to time..

$$\frac{\delta V_{input}(t)}{\delta t} = \frac{\delta^2 I(t)}{\delta t^2} * L + \frac{\delta I(t)}{\delta t} * R + \frac{I}{C}$$

Given that we are driving the circuit with a sinusoidal voltage, we get:

$$V(t) = V_o e^{i\omega t} e^{i\alpha},$$

where α is the phase change between the current and voltage

$$I(t) = I_o e^{i\omega t}$$

$$i * \omega V_o e^{i\omega t} e^{i\alpha} = i^2 w^2 L I_o e^{i\omega t} + i\omega I_o * e^{i\omega t} R + \frac{Ioe^{i\omega t}}{C}$$

$$V_o e^{i\alpha} = iwLI_o + I_o R + \frac{Io}{iwC}$$

$$V_o e^{i\alpha} = I_o * (iwL + R + \frac{1}{iwC})$$

Thus, by our definition of voltage as the product of the current and impedance:

$$V_o e^{i\alpha} = I_o * (iwL + R + \frac{1}{iwC}) => V = I * Z_T$$

where,

$$R + iwL + \frac{1}{iwC} = R + iX = Z_T$$

Thus, our impedance depends upon a real value, Resistance, and a complex value, Reactance. This reactance is the linear combination of the inductance and capacitance vectors of the complex plan. In real theoretical terms, they are both in opposite directions, such that $X = \omega L - \frac{1}{\omega C}$.

Because we drive our circuit with a forced oscillation from the generating function, their exists a resonance. Resonance for an RLC circuit defines the frequency of oscillation that the magnitude of impedance in the reactance is zero. From our definition of the complex impedance $X = \omega L - \frac{1}{\omega C}$, based off of our inductance and capacitance their exists a resonance frequency ω_0 as derived below:

$$X = \omega_0 L - \frac{1}{\omega_0 C} = 0$$
$$\omega_0 L = \frac{1}{\omega_0 C}$$
$$\omega_0^2 = \frac{1}{LC}$$
$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Using the theoretical value of resonance from known inductance and capacitance, an experimental procedure can be done to replicate resonance of a circuit. Because the impedance of the complex plane made up of the impedances of the capacitor and inductor is zero, an experiment can measure the voltage drop across the two components to until it is zero, *resonance*. This is because the circuit acts as an short circuit across the two components causing a voltage drop of zero and a maximum amount of current to flow through the series circuit. For any other value of frequency, we can derive the impedance and current given that we know the voltage drop across as shown below:

$$Z(\omega) = \omega L - \frac{1}{\omega C}$$
$$Z(\omega) = L(\frac{\omega^2 - \omega_0^2}{\omega})$$

$$\text{By } V = I * Z$$
$$V(\omega) = Z(\omega)I(\omega)$$
$$I(\omega) = \frac{V(\omega)}{Z(\omega)}$$

where voltage is known and/or measured

The relationship of the voltage drop of the Inductor and Capacitor with the frequency is obvious by the behaviors of each impedance. As we decrease the frequency to zero, the capacitor takes up more of the complex impedance and the inductor becomes zero, and vice versa. Additionally, as stated before our impedance overall goes to zero as we approach the resonance. Thus, given a known voltage we find that the voltage drop across the two components is quadratic as seen graphically below.

Lastly we can infer that the through the current equation, using the impedance of the two components and voltage drop across the two, we can further relate the current with the resonance. Because the impedance is inversely related to the current, the relation simply the opposite, as shown below.

The final important note is on the derivation of Inductance. Inductance, as known by electromagnetic when its characteristics such as length and area is known, is:

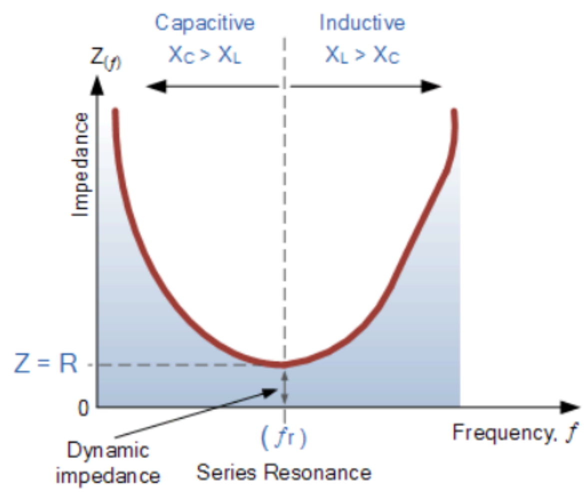


Figure 1: Graphical Representation of the Voltage Drop across the Inductor and Capacitor in series as a function of frequency.

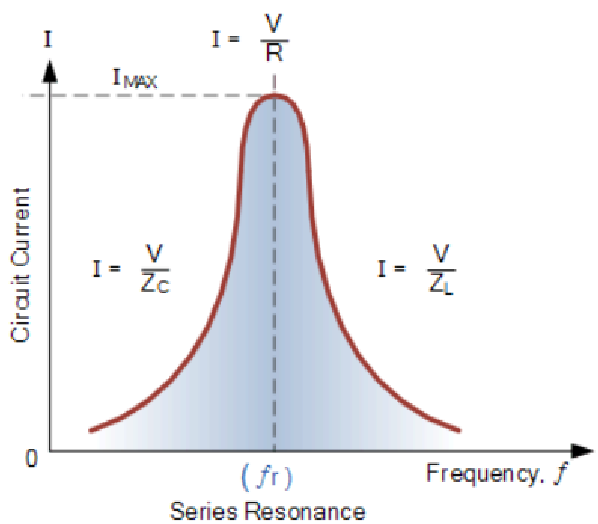


Figure 2: Graphical Representation of the Current through the circuit as a function of frequency.

$$L = \frac{\mu n^2 A}{l}$$

Given this derivation, with a known inductance, area, length, and μ_0, we can derive the probable amount of turns in the inductor. This can be useful in both manufacturing procedures, but also electromagnetism understanding. What is notable for our purposes is the permeability, μ. This permeability is affected by the the material inside the inductor, and in experiments can be varied for changing inductance. A higher permeability is related to a more dense material.

3 Experimental Procedure

To begin the lab, the proper equipment must be set up in order to proceed.

First, we need to address the necessary components for data collection. We will need a function generator to output a driving voltage/current, an oscilloscope to measure the voltage drop across a circuit, an LCR meter to evaluate the component impedances, a resistor, a capacitor, and several inductors of varying inductance for our independent variable. Next, we need to create the RLC circuit on the breadboard, a circuit designed for small circuit compositions, in any manner seen fit; the important thing is that the circuit must be closed and in a specific RLC design. Record the resistance of the resistor, found by identifying the colored bands on the resistor and looking at a resistor data sheet. Be sure to record the capacitance of the capacitor and the inductance of the inductor using the LCR meter, and the voltage supplied from the function generator using an oscilloscope. The circuit must now be attached to the function generator with a sinuisoidal output voltage, and from there we will analyze the sinusoidal graph given to us using the oscilloscope. When we have adjusted the input voltage into the circuit to be at 5V, we must next measure the voltage drop across the capacitor/inductor series using the oscilloscope. By adjusting the frequency by the output voltage of the function generator, we can evaluate the resonance of the circuit by the point of minimum voltage as described.

For Components, we used a 180 ohm resistor, 22μF and 15μF capacitor for two different data sets for each level, and varying inductors of different inductance. The first inductor is a referred to as the red inductor, noted by its distinct red color. This 13.5 cm length inductor was used in two levels, one with only air in the inductor and the other with three iron bars inside. The second inductor used is referred to as the pink inductor, noted by its distinct pink color. This inductor was of length 8.6 and presumed to be filled with a solid iron core for calculation of the amount of loops in the inductor. The third and final inductor used is referred to as the red-pink inductor. This 11.3 cm length inductor is filled with air, as seen with its hollow core. Below, we see the three inductors used.

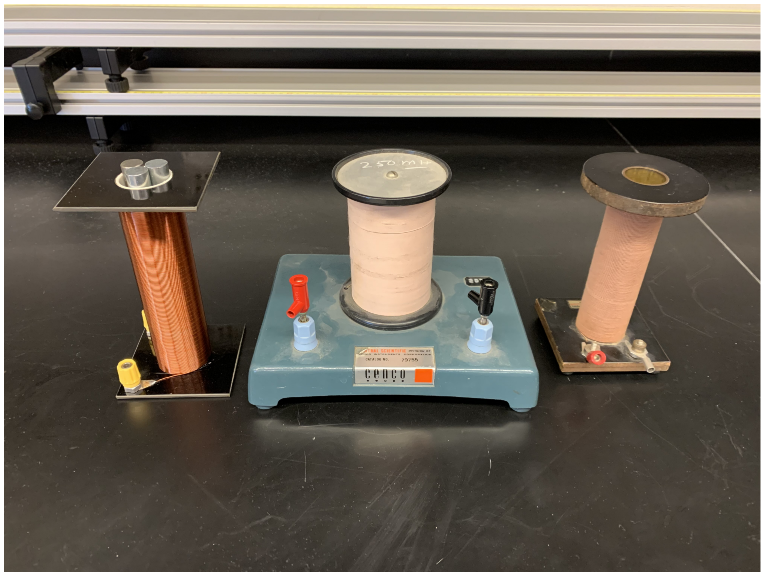


Figure 3: The three inductors used for the experiment. From right to left, [1] Red Inductor, [2] Pink Inductor, [3] Red-Pink Inductor

Below represents the ideal RLC Circuit design, allowing us to measure the voltage drop across the inductor-capacitor in series effectively.

Furthermore, a demonstration of the ideal oscilloscope trace is shown at resonance. In this graphing of the voltage drops across the circuit, the green represents that of the reals across the resistor, the blue represents that of the complex across the inductor, the grey represents that of the complex across the capacitor, and the pink represents the linear combination of the inductor and capacitor.

Once we find the resonance point of the wave, we will measure the frequency of resonance, and create a range of 20 intervals before and after this frequency. This will allow us to collect a precise measurement of data at a range of 20 values. These intervals will for most experiments be 20 Hz. Next, repeat these steps for resonance with multiple types of inductors Once all the data is collected, we will estimate the number of loops that each inductor has that provide the individual inductances. Lastly, we will utilize the data from our theoretical resonance with the experimental resonance, again for each inductor.

An important aspect of any paper is the accuracy of it. Thus, systematic errors will now be addressed.

Systematic error accounted for much of the uncertainty in our calculations. For voltage, we used the systematic error associated with measurements of the device, which is half the smallest incremental measurement. This means our uncertainty in voltage is 20mV. Furthermore, for frequency and Inductor we have a systematic error associated with the data collection, which is to the

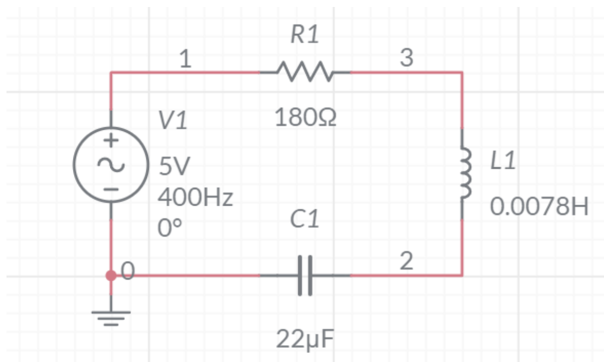


Figure 4: The theoretical RLC Circuit for the Red Inductor

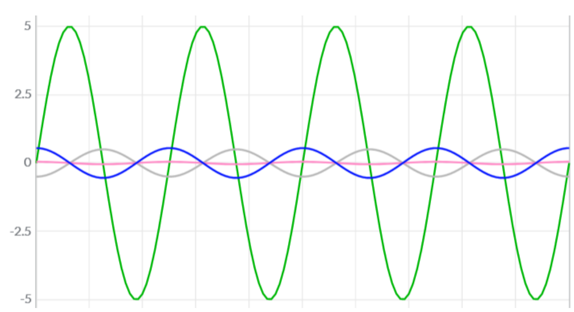


Figure 5: Graphical Representation of the RLC Circuit at resonance.

smallest digit. These associated values are:

$$\delta w = 0.628\frac{rad}{s}, \text{ where } \delta f = 0.1Hz- > \delta w = 2pi * \delta f$$

and

$$\delta L = 0.00001H$$

For Current, we utilized the quadratic sums of the independent uncertainties that make up its calculation, which are of the resonant and non-resonant frequencies, voltage, and inductance to calculate the systematic error. This is because, through our current equation stated before, we find that the uncertainty will be a quadratic sum. Below, we mathematically see this is true:

$$Z_{LC}(W) = L(\frac{w^2 - w_o^2}{w}), \text{ where } \delta Z_{LC} = \sqrt{\frac{\delta L}{|L|}^2 + \frac{\delta w}{|w|}^2 + \frac{\delta wo}{|w_o|}^2}$$

$$I(W) = \frac{V(w)}{Z(w)}$$

Random error for the data set would be calculated by a non-linear error function, but this could not be approximated well given that we did not have the knowledge to calculate an error function for this scenario of non-linear characteristics. But by collecting up to fifteen data points of approximations of component values, we have incorporated random error into our systematic error uncertainty calculations for the individual components.

4 Results

Firstly, we must state our component values and the output resonance values. For purposes of redoing the experiment if needed, the component values and the expected outcomes are shown below.

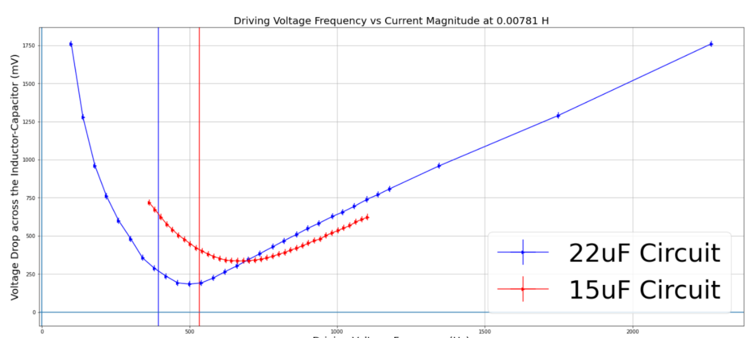
		22uF (+/- .00001 nF)			15uF (+/- .00001 nF)		
Inductance (+/- .00001H)	Estimated Number of Coils	Experimental Resonant Frequency (1 Hz +/- .1 Hz)	Theoretical Resonant Frequency	Resonant Voltage (1000mV +/- .20mV)	Resonant Frequency (1Hz +/- .1 Hz)	Theoretical Resonant Frequency	Resonant Voltage (1000mV +/- .20mV)
0.00781	726.54	499.8	393.7	184.0	663.0	532.8	336.0
0.02677	18.99	184.8	212.7	544.0	256.0	287.8	816.0
0.21006	44.44	70.0	75.9	2140.0	87.0	102.8	2400.0
0.01225	1325.82	260.0	314.4	864.0	374.4	425.5	1040.0
Driving Voltage		Capacitance					
5V		22uF	15uF				
		21.238	11.458				
		20.696	11.526				
		20.543	11.286				
		21.047	11.405				
		21.038	11				
		22uF	15uF				
		20.9124	11.42				

Figure 6: Data Table of Constants and Calculations.

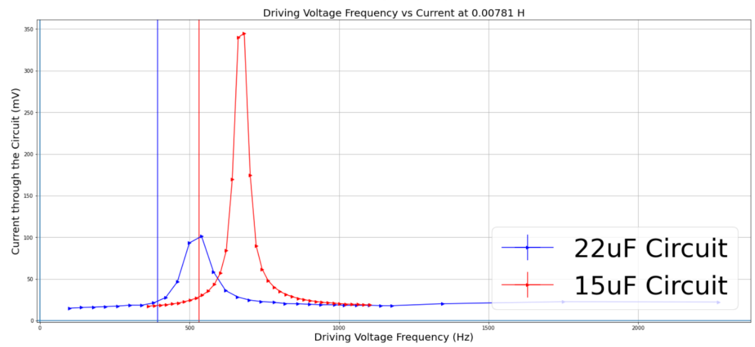
Given these results, we find that the experimental resonant frequency is within some bounds of the theoretical. Using the average fractional uncertainty of each resonant frequency, we find that our values are within a fractional uncertainty *on average* of 16

When we want to address the accuracy of the dependency of voltage across Inductor-Capacitor and current through the circuit on the angular frequency, such that the impedance change causes a change in either, we can use graphical analysis. Using graphical analysis, we come with the following graphical solutions. For each graph, the Blue represents the data set for the 22μF capacitor, the red for the 15μF capacitor, and the vertical lines represent the theoretical resonant frequency for each respective capacitor data set.

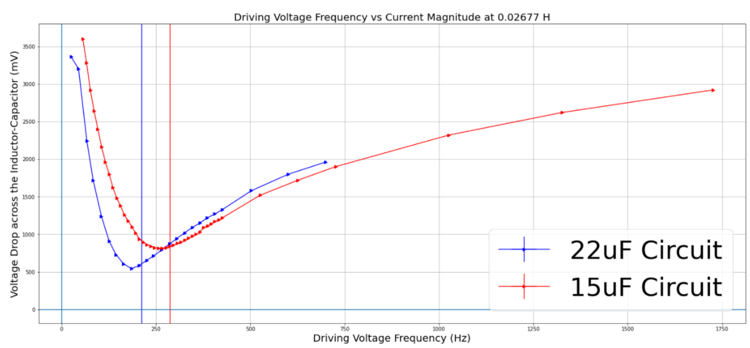
Red Inductor with no solid core



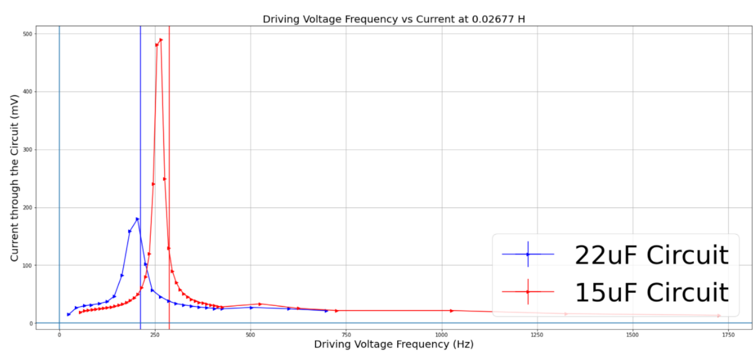
Red Inductor with a solid core



Driving Voltage Frequency vs Current at 0.00781 H
Current through the Circuit (mV)
Driving Voltage Frequency (Hz)
22uF Circuit
15uF Circuit

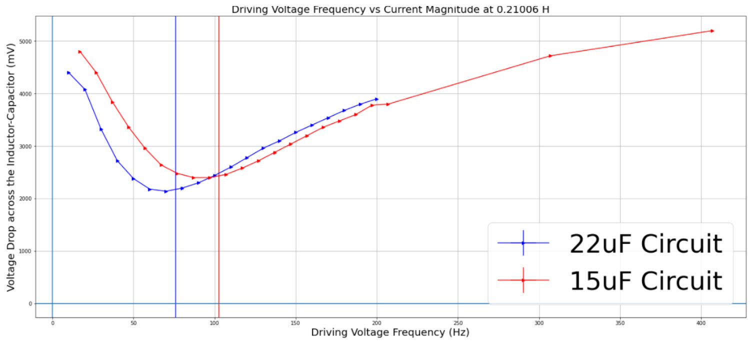


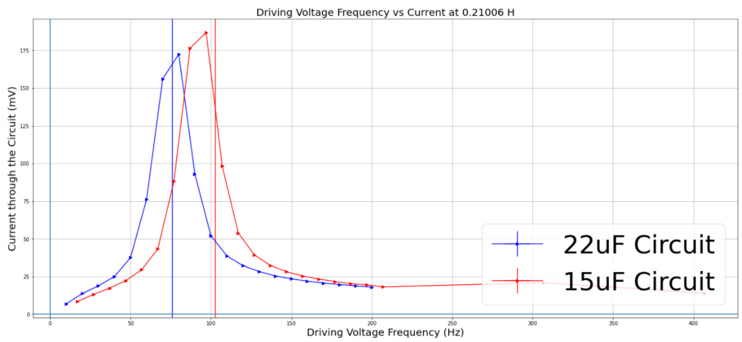
Driving Voltage Frequency vs Current Magnitude at 0.02677 H
Voltage Drop across the Inductor-Capacitor (mV)
Driving Voltage Frequency (Hz)
22uF Circuit
15uF Circuit



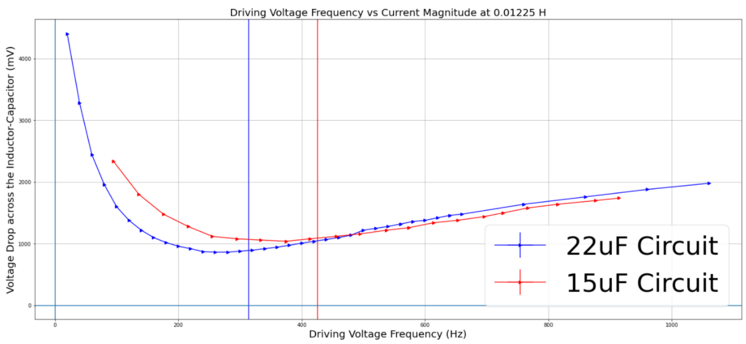
Driving Voltage Frequency vs Current at 0.02677 H
Current through the Circuit (mV)
Driving Voltage Frequency (Hz)
22uF Circuit
15uF Circuit

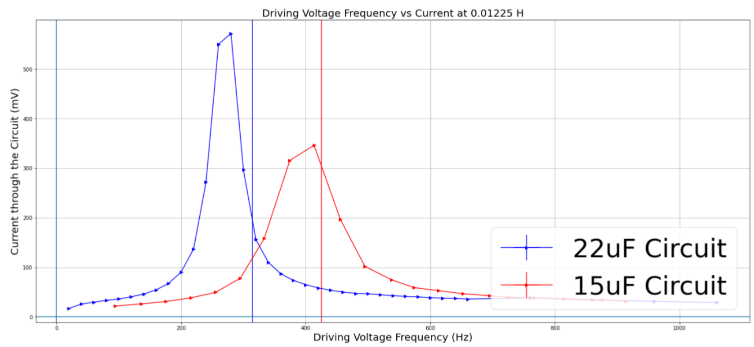
Pink Inductor





Red-Pink Inductor





5 Analysis

Upon Analysis of the data, it is readily apparent that the experimental procedure was successful in demonstrating the expected theoretical resonance behavior, with one minor slight deviation. This slight deviation is that the resonance voltage drop is not necessarly zero, but a positive value. For our expectations of the theoretical voltage drop as it approaches resonance, we have an accurate representation for the voltage at frequencies lower than the resonant frequency. An apparent problem appears when we approach an infinite frequency, such that it is not the expected quadratic rise for long. It appears that upon reaching some threshold, our rising voltage drop across the L-C becomes somewhat logrithmic and somewhat linear. Which fit is the true fit is not fully understood from the small amount of data taken in this region of frequency, but from our derived equation of $Z(\omega) = L(\frac{\omega^2-\omega_0^2}{\omega})$, we can infer that this change to a linearity directly relates causes:

$$Z(\omega) = L(\frac{\omega^2 - \omega_0}{\omega}) => Z(\omega) = L(\frac{\omega - \omega_0}{\omega})$$

This is inferred by the quadratic relationship that stems from the independent variable of frequency, f, becoming linear requires $(f * 2\pi)^2 => f * 2\pi$. Thus, beyond this threshold the linearity of impedance causes a linearity in voltage drop.

When we discuss the current behavior, we see a slight difference in the expectations vs reality of compared to the voltage drop. Using the derivation of current that we discussed earlier, the graphing of current was possible. These currents display the expected behavior of the resonance of a current through the circuit to precisely, with an average uncertainty across all of the data points of 0.0054A. In comparison to the graphical solution to the voltage drop, the current does not show a distinct difference in ampltitude at the threshold frequencies, maintaining the same polynomial fit it has. What is important to realize mathematically is that if the amplitude decreases to a point that the change in current amplitude is small, as is apparent past this threshold, the slope between points (a linear line) at the same interval as the data set becomes practically the same as that of the derivative of the function. This correlation directly corresponds to a linearity of the impedance, providing more information to the analysis found in the voltage drop. Thus, the linear solution of impedance past this threshold holds true.

Next to Last, upon analysis of the different resonance frequencies for each individual inductors the direct solution of resonant frequency is proven to be correct with a 16% deviation. We expect that as we increase the inductance, our resonant frequency decreases as a inverse root relationship. For instance, if we quadruple the inductance, as we do somewhat with adding the iron rods to the Red Inductor, $\sqrt{\frac{1}{4LC}} => \frac{1}{2}\sqrt{\frac{1}{LC}}$. Thus, should expect half the resonant frequency. Through analysis of the Red Inductor without and with iron rods, this relationship is proven. The direct factor from 0.00781 H to 0.02677 H is 3.43. Using the relationship, we should expect the resonant frequency of the Red Inductor without iron rods to be 1.85 times more than with iron rods. Given that the Resonant Frequencies are for the 22μF capacitor 499.8 Hz to 184.8 Hz and for the 15μF capacitor 663.0 Hz to 256.0 Hz respectively, we see that the factor is actually 2.70 given the data set. Because the resonance frequency deviations are highest in the Red Inductor with no rods, this relationship becomes inaccurate.

To compensate for this inaccuracy, the Red Inductor with iron rods and Pink Inductor relationship can be addressed. The Pink Inductor was found to be 7.85 times larger than that of the the Red Inductor with Rods, giving our resonant frequency of the Red Inductor to be 2.8. In analysis of the resonant frequencies of the Pink and Red Inductor, 70 Hz to 184.8 Hz for the 22μF Capacitor and 87 Hz to 256 Hz 15μ capacitor respectively, the relationship is proven to be correct with a range uncertainty for the factor of 0.16 of frequency.

Lastly, when we calculate the amount of coils in the inductors, we find that the values become faulty. The first point of error lies upon the number of expected coils in the Red Inductor. When we calculate it for both the with and without iron rods, we have a difference in coil number of 707.55! Because the exact value of permeability cannot be determined accurately, given that we do not know the exact composition of the iron rods, this could be the leading error in this calculation. For the other two inductors, we can not presume the same errors. The pink inductor, the highest inductance inductor, leads us to believe that it with its larger area, smaller length, and unknown permeability of presumed iron, it must have a small number of coils. The Red-Pink Inductor, with its larger area and smaller length to that of the red inductor, but almost twice as large inductance, is presumed to be a factor of nearly $\sqrt{2}$ larger. From our values of $726.54- > 1325.82$, this relationship is correct.

6 Conclusion

In conclusion, when we analyze the solutions of an RLC Circuit, we find that the theoretical solutions deviate from the experimental by an acceptable 16% percent deviation using the industry made components. This deviation causes us to believe that the inductors we used follows the laws of an inductor well, and further this can be extended onto the capacitor and resistor. This is because when evaluating the resonance point for the different levels of inductances and level of capacitance, the corresponding path of voltage drop across the L-C and current across the circuit as we independently change the input frequency describe the exact expected relationship we see when the frequency is below the resonance. As discussed, their exists a slight deviation in the theoretical voltage drop, which is that the voltage drop should approach zero at resonance, but is positive experimentally. Because circuits are not perfect and are a loop, this could be explained by the experiment being influenced by the system and its surroundings. What is unexpected is when the frequency goes beyond the expected, given that the resonant behavior of the voltage drop is expected to be quadratic. What we find is that the relationship of voltage drop becomes linear, due to the linearity of the impedance. Thus, we see a change of

$$Z(\omega) = L(\frac{\omega^2 - \omega_0^2}{\omega}) => Z(\omega) = L(\frac{\omega - \omega_0}{\omega})$$

This is further supported by the various levels of capacitance used, inductance used, and the relationship of frequency with current through the circuit. In the future, this unknown exact reasoning for this could be looked at more by more data collection.

Lastly, when we want to reverse engineer the properties of the inductors used, given that we measured the Inductance, length, cross-sectional area, and presumed the permeability to be iron or air, we find that the number of coils in the inductors follows an accurate relationship given that the resonant frequency calculation through $w_0 = \sqrt{(\frac{1}{LC})}$ can be used to find the factor between the inductors, and thus the factor between the number of loops. What could be done in the future to improve the accuracy of these loops is the measurements of the value of this parameter of the iron rods with air in the Red Inductor and the measurement of the permeability of the Pink Inductor. Both changes would give a valuable insight into understanding inductance and possible factors that are in play.

7 Bibliography

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