Tiled maps of multivariate data

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Mapping multi-attribute data is a challenging map design problem. We are developing an approach based on ‘tiling’ a map to enable simultaneous choropleth colour-based symbolisation of several attributes. This work aims to generalise earlier work on ‘woven maps’ reported by O’Sullivan and Bergmann (2021).

The idea is that large enough elements (the tiles) in a repeated pattern are present in every polygonal area in the map to ‘carry’ colour information conveying attribute values, but at the same time combinations of attribute values are ‘blended’ by the tiling pattern to give an overall impression of different attribute combinations. At the same time, the orientation, shape, and position of tiles in ordered patterns may enable a reader to see similarities among places based on their attribute values.

We first consider tiling theory, before considering their application to mapping including examples of tiled maps. Finally we note areas for further research.

Elements of tiling theory

In their encyclopedic survey, Grünbaum and Shephard (1987) repeatedly focus on tilings with specific properties to make progress in saying anything useful about tiling at all. More recent computational tiling theory indirectly validates this approach. Systematic enumeration of all tilings up to a certain complexity yields a ‘galaxy’ of 2.4 billion tilings (Zeller et. al. 2021), confirming the unmanageable variety of possible tilings! It is beyond our scope to even attempt an overview of tiling. Instead we present some key ideas and consider their applicability in mapping.

The symmetries of tiles and tilings

Symmetry is central to the study of tilings. A symmetry of a tile is a ‘rigid’ transformation (or isometry) of the plane that preserves its shape and size. The *symmetry group* of a tile is the set of isometries that preserve its shape. These fall into three groups:

- Those containing only reflections and/or rotations;
- Those containing translations in only one direction; and
- Those containing translations in more than one direction to give the 17 *wallpaper* groups.

The symmetry group of a whole tiling is the set of symmetries under which any tile T_i is mapped onto some other tile T_j. In effect, a symmetry of a tiling is an isometry that 'shuffles' its tiles. The symmetries of a tiling are not the same as those of its tiles, but must include some of them.

The wallpaper symmetry group is particularly important in tiling as it yields *periodic* tilings which repeat regularly over the plane. Any periodic tiling can be produced from a 'tileable unit' and two of the translation vectors that are symmetries of the tiling. This matters because it means that even seemingly complex tilings can be rendered by identifying a tileable unit. There is no unique tileable unit of a tiling. Instead, we can choose a unit convenient to the task at hand.

Tile shapes and transitivity groups

A *k*-hedral tiling is formed from *k* distinct shapes of tile (including under reflection and rotation). The tilings in Figure 1 are monohedral.

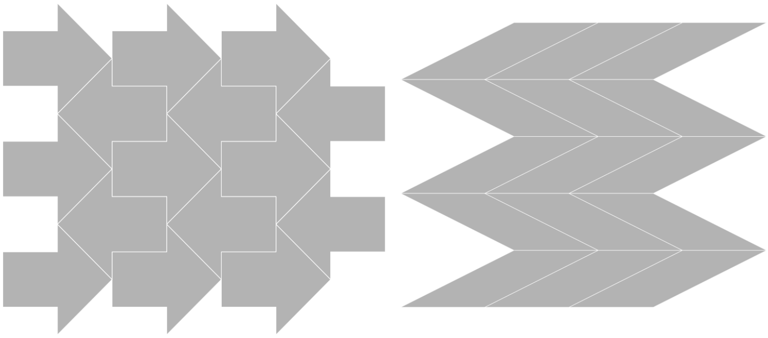

Figure 1 Two monohedral tilings.

The symmetries of a tiling map various tiles onto other tiles. The sets of tiles mapped on to one another this way are the *transitivity groups* of the tiling. An *isohedral* tiling has one transitivity group. One way to think about this is that tiles in an isohedral tiling are locally indistinguishable from one another: the tiling looks the same from the perspective of any tile. An isohedral tiling is necessarily monohedral. The concept extends naturally to *k*-isohedrality where a tiling has *k* transitivity groups.

Tiling theory and cartography

The relevance of the above for designing tilings for multivariate mapping remains unclear. *Directionality* or *orientation* is not relevant to mathematicians because tiles and tilings are considered identical subject to rotation. However, it is important for mapping. Many monohedral, isohedral tilings contain directionally distinguishable tiles (see Figures 1 and 2) because in mapping we apply a tiling at a specific orientation. Thus we may need a different (related) notion of the transitivity groups of a tiling where different tile orientations 'count' as different.

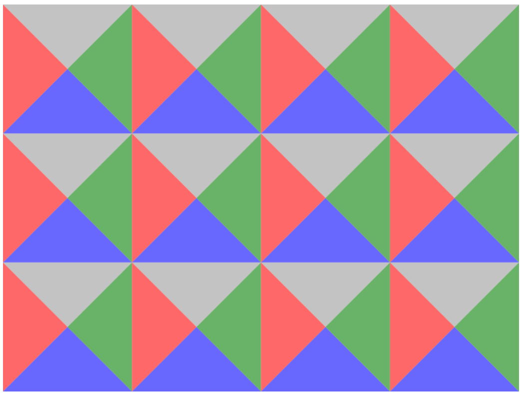

Figure 2 An isohedral tiling that could allow symbolisation of four different variables.

The notion of *periodicity* in tilings, i.e., that they must have at least two independent (not necessarily orthogonal) translational symmetry vectors is important. There are tilings that have no such symmetry (spiral tilings, for example) which may hold limited cartographic interest. For our implementation of tiled maps, periodicity is central.

Making tiled maps

All tilings we consider are periodic, and are rendered by repeatedly copying and translating a tileable unit (which may include repeated polygon elements) across the map area (see Figure 3).

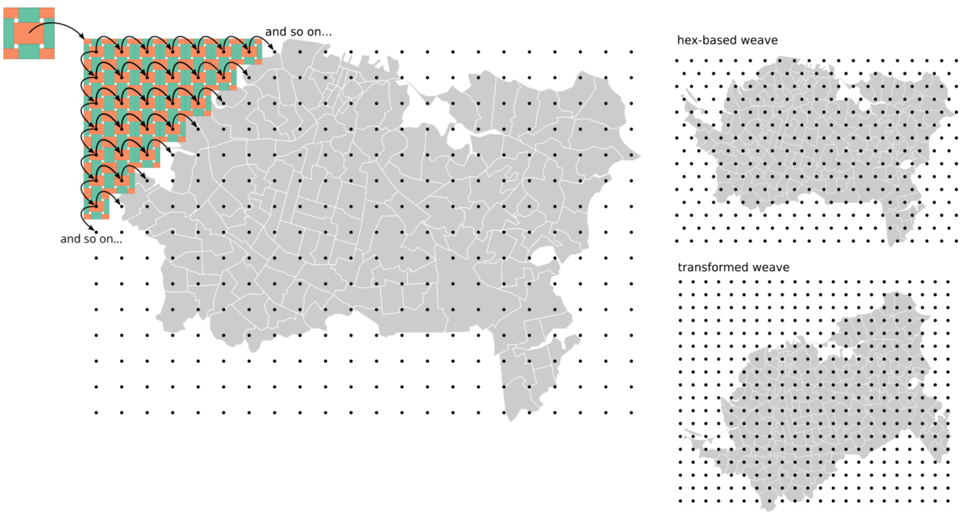

Figure 3 Schematic illustration of the method.

We have developed code (in python) that given a `TileUnit` and a 'region' to be tiled produces a geospatial data layer that is a tiling of the region. Using the code consists of the steps below:

```
Unit = TileUnit()  # or a WeaveUnit()
Tiling = Tiling(unit, region, id_var)
Tiled_map = tiling.get_tiled_map(rotation=0)
```

A `TileUnit` contains `tile` and `elements` datasets. The `tile` is the repeatable tileable shape, either rectangular or hexagonal. The `elements` dataset contains 'subtiles' each with an `element_id` attribute. The `TileUnit` also has a `tile_shape` attribute indicating which kind of grid of tile centres is required to tile a map with the `elements`.

A `Tiling` object is constructed from a supplied `TileUnit` and some region to be tiled (usually a geospatial polygon dataset). Based on the `TileUnit` a grid of tile centres is generated and used to translate copies of the `TileUnit`'s `elements` across the region to be tiled. A larger tiled area than required is stored so that subsequent calls to the `get_tiled_map` method can return the tiling at different rotations without regenerating the tiling from scratch.

We have begun to explore three categories of tiled map as discussed below.

Dissections of the regular tilings

Because any periodic tiling can be generated from two non-parallel translation vectors, all periodic tilings can be based on dissections of a parallelogram-shaped fundamental block. Here we are more interested in the possibilities offered by dissections into ‘subtiles’ of the regular tilings by triangles, squares or hexagons.

Figure 4 shows some possibilities up to ten subtiles. A challenge in using these dissections is to enable map readers to ‘indexically’ distinguish subtiles. One approach is to ‘inset’ subtiles in their containing main tile as in Figure 5, for 3- and 7-subtile dissections of a hexagon. A map based on the latter is shown in Figure 6.

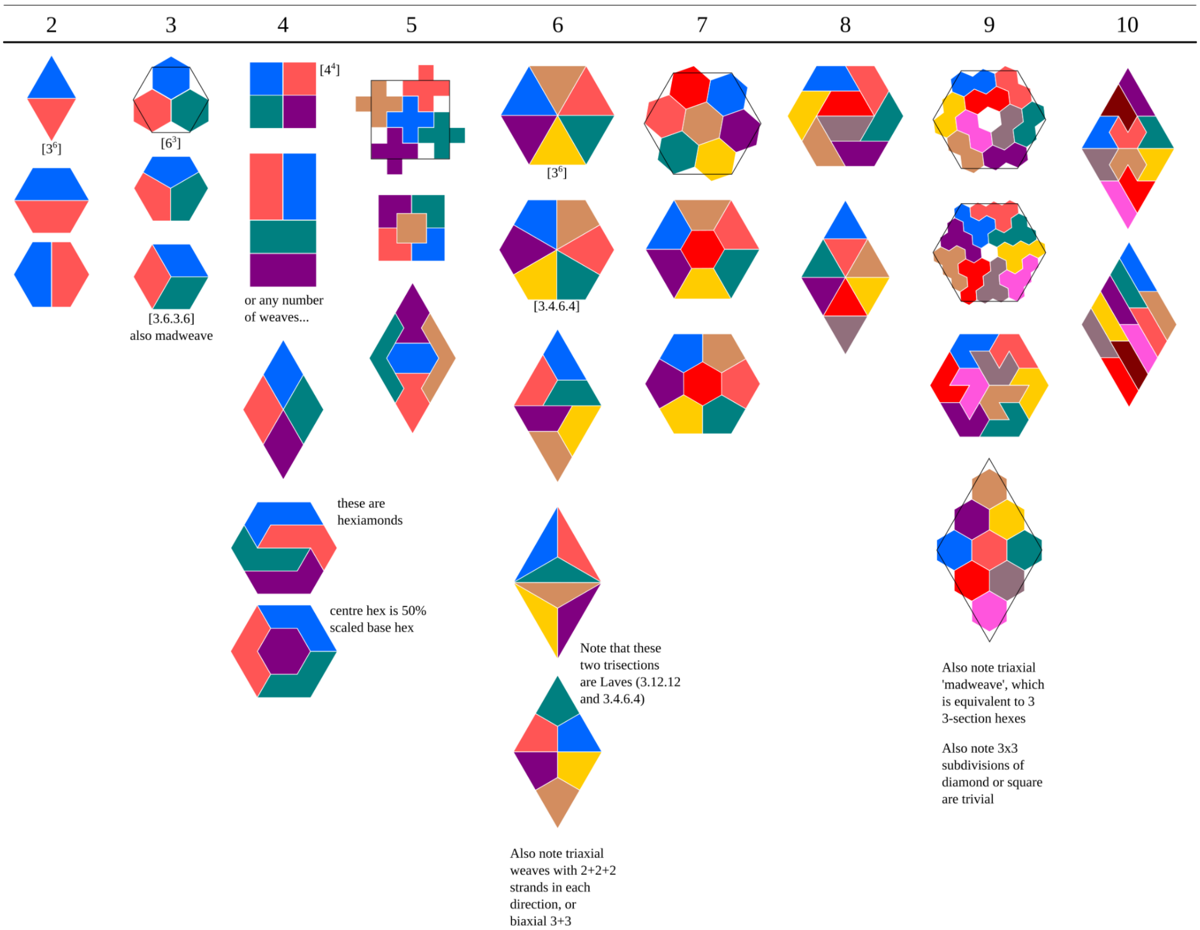

Figure 4 Tileable units from dissections of the regular tilings.

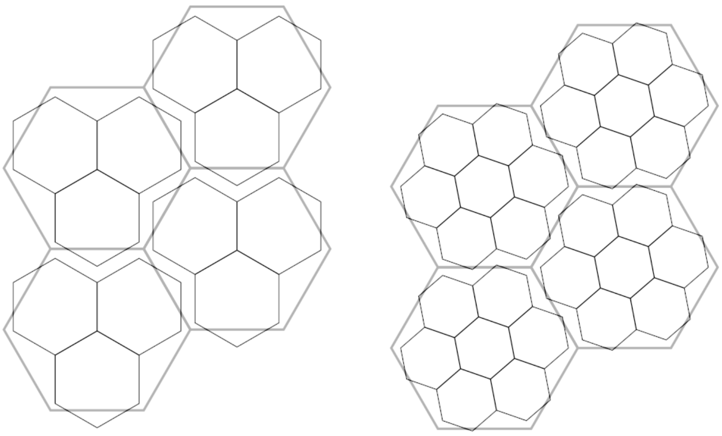

Figure 5 ‘Insetting’ 3- and 7-subtile dissections of the hexagon.

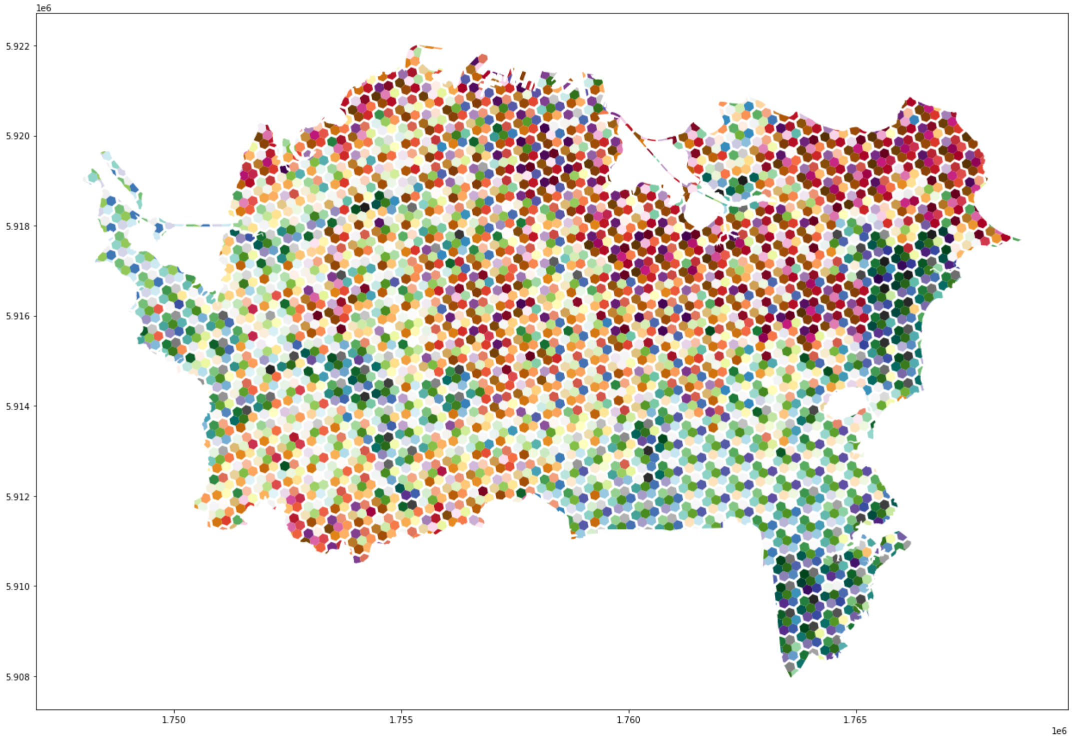

Figure 6 A map of seven attributes using a 7-subtile dissection of (6^3).

Woven maps

Another class of tilings with potential for mapping is 'woven' patterns. Weave patterns are considered by Grünbaum and Shephard (1988) who designate such tilings *isonemal fabrics*.

More useful for our purposes is a matrix mathematics for conventional (biaxial) weaving presented by Andrew Glassner (2002, see also Albaugh 2018). We have extended this to represent triaxial weaves as three intersecting biaxial weaves. It appears that triaxial weaves offer fewer options for variation than conventional weaving (Mooney 1984), although they are visually distinctive enough that it is useful to support them.

Weaves provide a way to dissect square and hexagonal tileable units with any desired number of distinct elements. The flexibility of the approach is apparent in Figure 7. A map of the data in Figure 6 is shown in Figure 8, using a weave-based tiling.

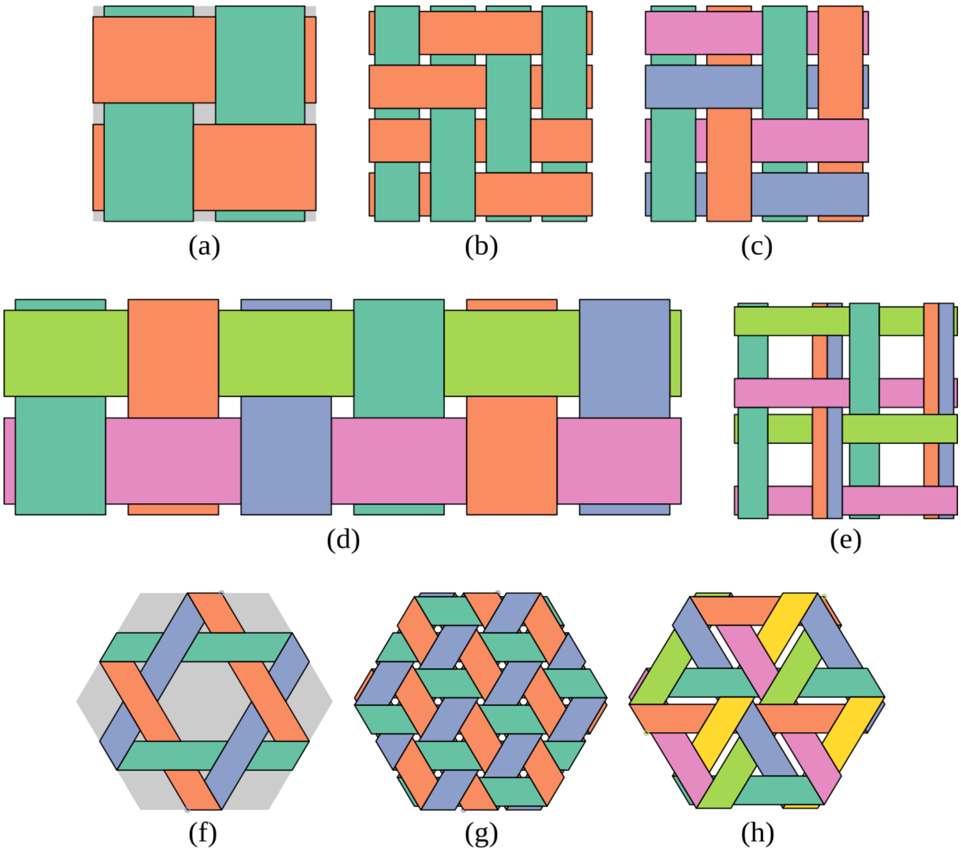

Figure 7 (a) plain weave, (b) two by two twill, (c) two by two basket weave with two strands in each direction, (d) plain weave where three attributes can be symbolised vertically, and two horizontally, (e) biaxial weave with 'skipped' strands creating 'holes', and some strands 'sliced' to carry more attributes, (f) open hexagon triaxial weave, (g) 'mad weave' (see Gailiunas 2017), and (h) mad weave with skipped strands. Colours denote distinct strands usable to carry different attribute data. Background grey shading in (a) and (f) shows the weave unit 'tile'.

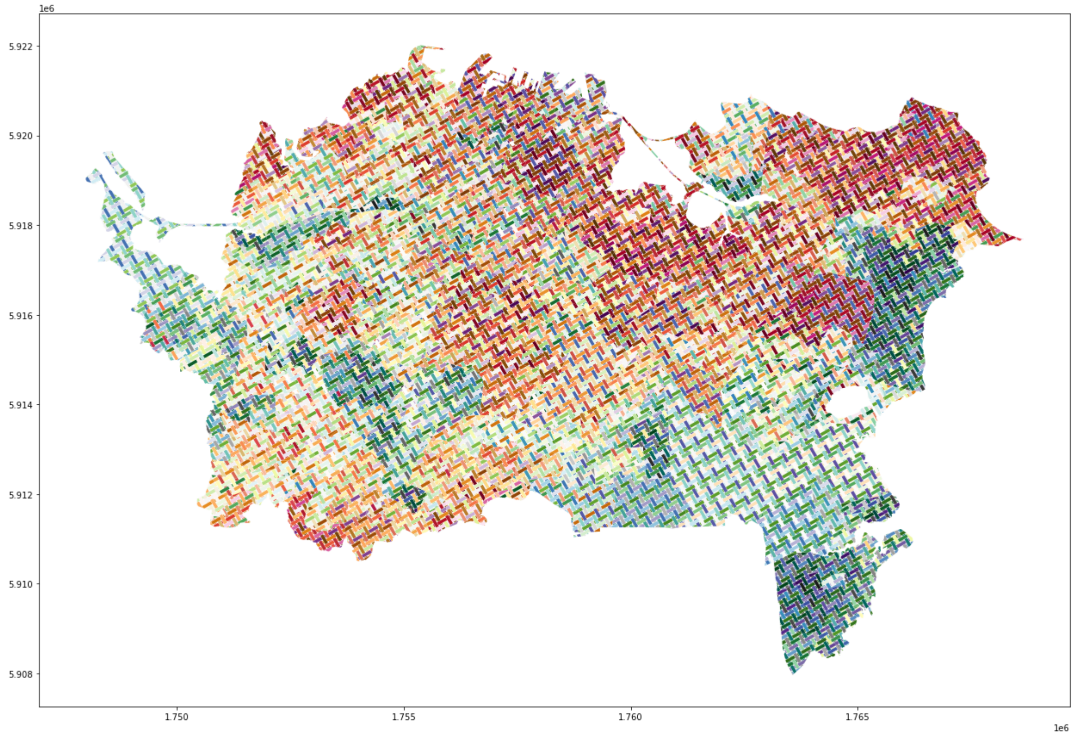

Figure 8 A map showing seven different attributes using a woven pattern.

Other potentially interesting tilings

The possibilities for tiled maps are endless. In the remainder of this section we show examples to illustrate the potential.

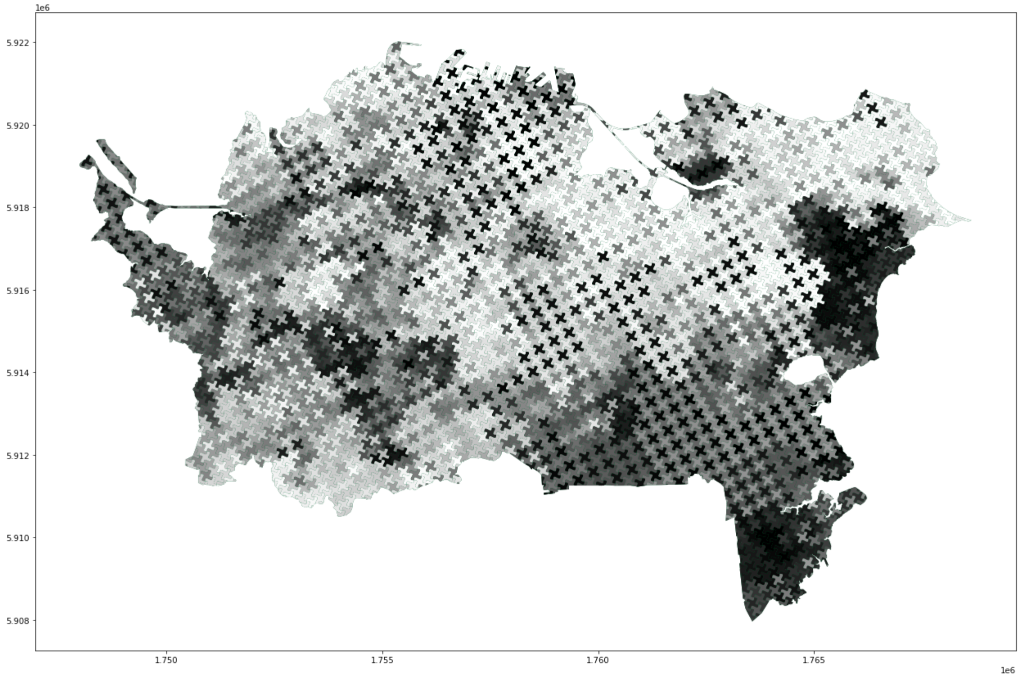

Figure 9 'Escheresque' tiled map with four variables symbolised on the same colour ramp.

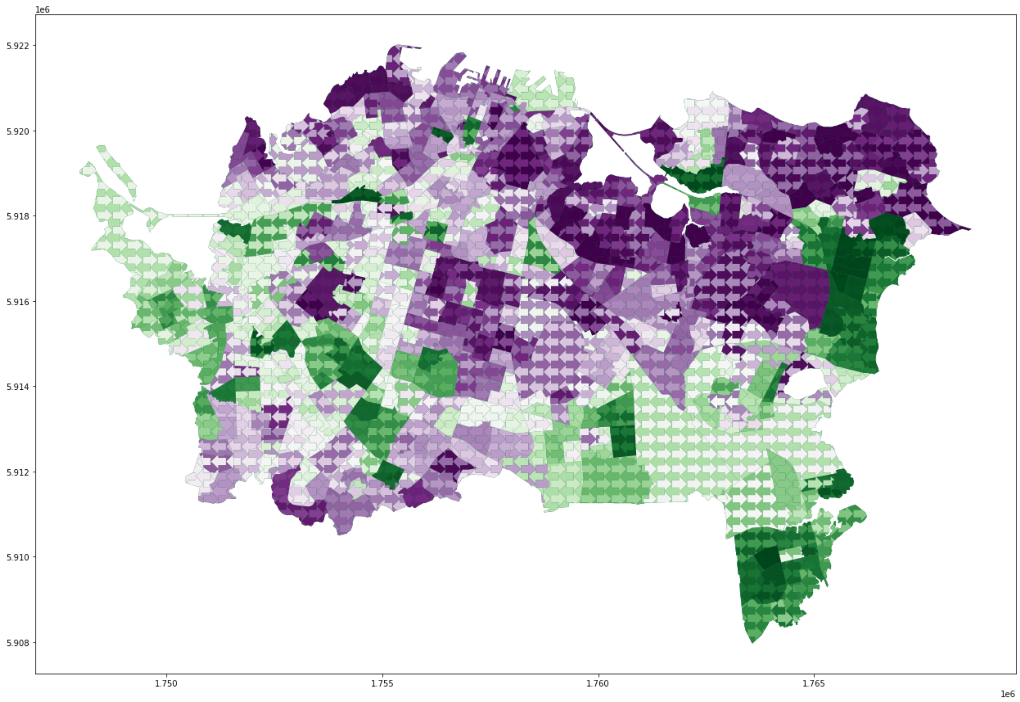

Figure 10 Opposed arrow shaped tiles make clear that they symbolise different attributes, even though the tiling is formally isohedral.

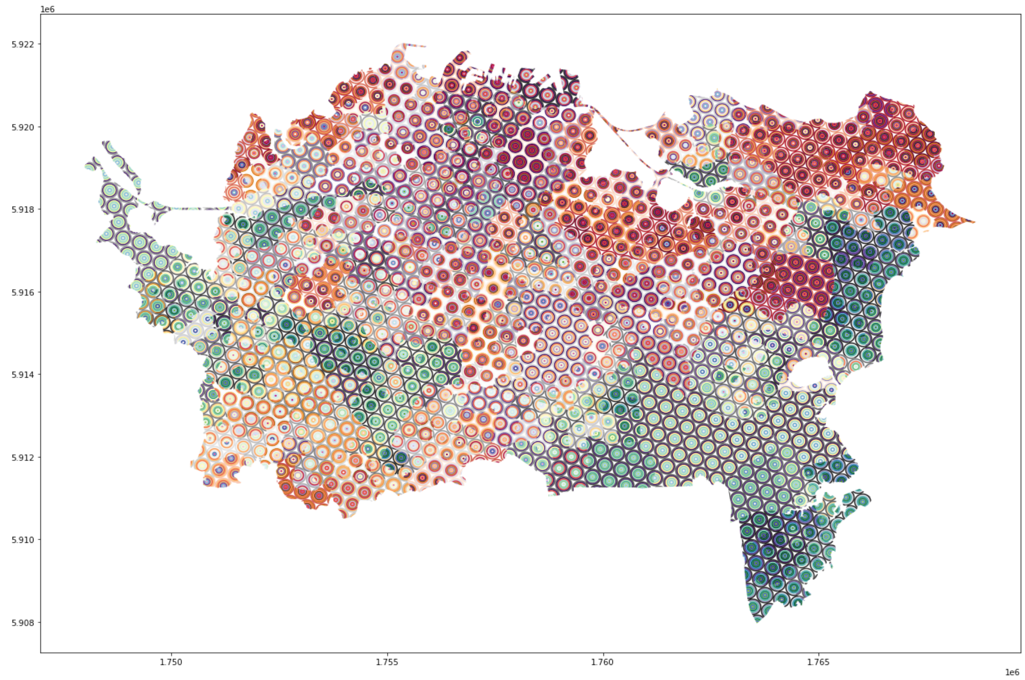

Figure 11 Circular annuli used to symbolise several attributes and tiled on a hexagonal grid. This shows a very different approach where the tile is a 'bearer' of a motif.

Conclusion

This work is at a preliminary stage. Code for making tiled maps is available at https://github.com/DOSull/weaving-space but is under development and subject to rapid change.

Many questions remain around the design of tiled maps, among them

- How does colour work in this setting? What kinds of colour combinations are usable, and how does what is usable or not depend on data distributions and relations among attributes?
- What symbolisation schemes work? Are continuous colour ramps, classified colour ramps workable, or is the approach more suited to categorical data?
- Tiled maps demand a better understanding of how ‘orientation’ in particular works as a visual variable and more work is required to develop such an understanding.

Tiled maps are an exploration of pattern and texture as visual variables (Bertin 1983) and we hope this work can contribute to an improved understanding of the many possibilities for creative cartographic design in this space.

References

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