Nonlinear Spacecraft Attitude Control Based on Lyapunov Stability

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1 INTRODUCTION

Athree-axis attitude control design, based on Lyapunov stability criteria, to stabilize spacecraft and orient it to point towards the Earth along its orbit, is presented in this report. This attitude control system is assumed to have three reaction wheels with optimal arrangement. The reaction wheels are aligned with spacecraft's axes. Control system inputs are the attitude error between the current and desired attitude presented in the quaternion form, and the angular velocity error. The controller output is the torque required to eliminate error. It analyzes the control torques required to meet the mission nadir pointing requirement within an orbit and discusses the performance of the model, namely control torque demands, angular momentum build-up in the wheels, body and wheels angular spin rates; and potential future developments.The remainder of this report is organized as follows. Section 2 describes the mathematical modeling of the spacecraft attitude dynamics, including the spacecraft body, reaction wheels, and kinematics formulation. Section 3 presents the Lyapunov-based nonlinear control design and discusses the stability analysis of the proposed controller. Section 4 details the MATLAB/Simulink implementation and simulation setup, including system parameters and test scenarios. Section 5 discusses the obtained simulation results for different cases, highlighting the controller's performance in terms of attitude tracking, control torque, and wheel momentum. Finally, Section 6 concludes the study and outlines recommendations for future work.

2 Equation of motion for spacecraft attitude

This section develops the mathematical model governing the spacecraft's rotational motion. The model consists of four systems: satellite dynamics, reaction wheels, satellite kinematics, a controller, and orbit propagation . The following flowchart demonstrates the data flow through subsystems in Fig. 1

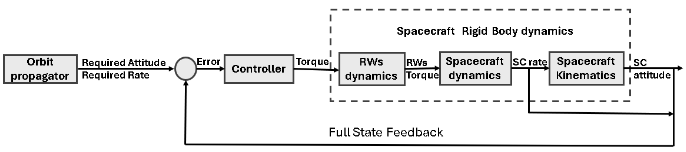

Figure 1: Overall simulation flowchart.

2.1 Spacecraft dynamics

The satellite dynamics is modeled using Euler's equation of motion as follows [1]:

$$\mathbf{T} = \dot{\mathbf{h}}_t + \mathbf{w} \times \mathbf{h}_t, \tag{1}$$

where $\dot{\boldsymbol{h}}_t$ is the time derivative of the spacecraft angular momentum in N.m, and $\boldsymbol{h}_t$ is the total angular momentum acting on the spacecraft body in N.m.s, $\boldsymbol{T}$ is the sum of all external moments applied to the spacecraft in N.m, $\boldsymbol{\omega}_s$ is the angular rate of the spacecraft in rads. The totla angular momentum is a combination of spacecraft angular momentum $\boldsymbol{h}_s$ and reaction wheels angular momentum $\boldsymbol{h}_w$ while $\boldsymbol{S}_w$ is the rotation matrix for three wheels aligned orthogonally in the directions of the axes of the spacecraft body frame as expressed in Eq. 2, in this case $\boldsymbol{S}_w$ is an identity matrix

$$\mathbf{h}_t = \mathbf{h}_s + \mathbf{S}_w \mathbf{h}_w \tag{2}$$

Therefore, Eq. (1) can be written as follows:

$$\mathbf{T} = \dot{\mathbf{h}}_s + \mathbf{S}_w \dot{\mathbf{h}}_w + \boldsymbol{\omega}_s \times \mathbf{h}_s + \boldsymbol{\omega}_s \times \mathbf{S}_w \mathbf{h}_w \tag{3}$$

Total external torque includes drag torque, magnetic torque, gravity gradient torque, solar radiation torque, and the torque produced by reaction wheel and acting on the spacecraft body. Assuming environmental disturbances are zero, then Eq. 1 becomes:

$$\tau_b = \mathbf{I}_s \dot{\boldsymbol{\omega}}_s + \mathbf{S}_w \mathbf{I}_w \dot{\boldsymbol{\omega}}_w + \boldsymbol{\omega}_s \times \mathbf{h}_s + \boldsymbol{\omega}_w \times \mathbf{S}_w \mathbf{h}_w \tag{4}$$

$$\dot{\boldsymbol{\omega}}_s = (\tau_b - \mathbf{S}_w \mathbf{I}_w \dot{\boldsymbol{\omega}}_w - \boldsymbol{\omega}_s \times \mathbf{h}_s - \boldsymbol{\omega}_w \times \mathbf{S}_w \mathbf{h}_w)/\mathbf{I}_s \tag{5}$$

where $\boldsymbol{I}_s$ is the inertia matrix of the spacecraft in kg.m^2,$\boldsymbol{I}_w$ is the inertia matrix of the spacecraft in kg.m^2, $\boldsymbol{\omega}_w$ is the angular rate of the reaction wheel in rads.

2.2 Reaction wheels dynamics

The angular rates of the wheels are governed by the control torque $\boldsymbol{\tau}_c$, demanded by the controller. The negative sign in Eq. 6 is interpreted as the reaction torque produced by the reaction wheels but acts on the spacecraft body. The mathematical model is based on the following equations:

$$\dot{\mathbf{h}}_w = -\mathbf{S}_w(\tau_c - \mathbf{b}_w \boldsymbol{\omega}_w) \tag{6}$$

$$\boldsymbol{\omega}_w = \mathbf{I}_w^{-1} \mathbf{h}_w - \mathbf{S}_w \boldsymbol{\omega}_s \tag{7}$$

Where $\boldsymbol{\omega}_w$ is the angular velocity of the wheel relative to the spacecraft body in rads^{-1}, and $\mathbf{b}_w$ is the damping coefficient of the wheels.

2.3 Spacecarft kinematics

The nonlinear spacecraft attitude kinematics equations of motion can be represented by using various attitude parameters, but representation through quaternion parameter has the property of non-singularity and it is free from the trigonometric component. Therefore, this representation is widely used to study the attitude behaviour of spacecraft. The kinematics of the spacecraft model is the part which expresses the relation between the attitude and angular velocities of the body and can be described by

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = 0.5 \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{8}$$

For the purpose of this study, quaternions are expressed as scalar first. Where ω is the spin rate of the satellite, and $\boldsymbol{q}$ is the orientation expressed in quaternion.

The quaternion dynamics can be expressed as:

$$\dot{\mathbf{q}} = \frac{1}{2}\omega\mathbf{q} + \frac{1}{2}q_0\omega, \tag{9}$$

$$\dot{q}_0 = -\frac{1}{2}\omega^T\mathbf{q}, \tag{10}$$

$$\omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{11}$$

where $\mathbf{q} = [q_1, q_2, q_3]^T$ is the vector part, q_0 is the scalar part.

3 The control design based on stability criteria Lyapunov

In order to design an attitude pointing control and guarantee the attitude stability of the spacecraft within the orbit, consider the following candidate Lyapunov function [2]:

$$V = \frac{1}{2}\omega^T I_s\omega + K_d q_{\mathrm{E}}^T q_{\mathrm{E}} + K_d(1 - q_{4\mathrm{E}})^2, \tag{12}$$

where q_{E} is the error attitude quaternion, $\boldsymbol{K}_d$ is a positive number, and V is a candidate Lyapunov function, which is positive definite and radially unbounded. In Eq. 13 , the error is defined from Quaternion multiplication which is denoted by $\otimes$ and the inverse of a unit quaternion is $q^{-1} = [\eta,\ -\epsilon^\top]^\top$.

Define the *error quaternion* by

$$q_E = q_T^{-1} \otimes q = \begin{bmatrix} s \\ \vec{v} \end{bmatrix} \begin{bmatrix} s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2 \\ s_1\vec{v}_2 + s_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \eta_E \\ \epsilon_E \end{bmatrix} \tag{13}$$

where $q_{i\mathrm{T}}$ and $q_{i\mathrm{S}}$ are components of the target attitude quaternion and the component of the spacecraft attitude in quaternion form, respectively. The first time derivative of V is given by:

$$\dot{V} = \omega^T I_s\dot{\omega} + 2K_d q_{\mathrm{E}}^T \dot{q}_{\mathrm{E}} - 2K_d(1 - q_{4\mathrm{E}})\dot{q}_{4\mathrm{E}}. \tag{14}$$

$$\dot{V} = \omega^T I_s\dot{\omega} + K_d(q_{\mathrm{E}}\omega^T + q_{4\mathrm{E}}\omega^T)q_{\mathrm{E}} + K_d(1 - q_{4\mathrm{E}})\omega^T q_{\mathrm{E}}. \tag{15}$$

Since ω is a skew symmetric matrix, $q_{\mathrm{E}}\omega^T q_{\mathrm{E}} = 0$ can be shown, and Eq. 14 can be simplified as:

$$\dot{V} = \omega^T(I_s\dot{\omega} + K_d q_{\mathrm{E}}). \tag{16}$$

Substituting Eq. 1 into Eq. 17, results in Eq. 15:

$$\dot{V} = \omega^T(\mathbf{w} \times \mathbf{h}_t - T_{\mathrm{c}} + K_d q_{\mathrm{E}}). \tag{17}$$

In order to make $\dot{V}$ positive, The attitude control law is expressed in the follows:

$$T_{\mathrm{c}} = -K_d q_{\mathrm{E}} - K_p\omega_d + \omega \times h_\omega. \tag{18}$$

4 Simulation

To evaluate the performance of the proposed nonlinear attitude control law, a numerical simulation was carried out using MATLAB/Simulink environment. The spacecraft model includes three orthogonal reaction wheels mounted along the body axes, with the system dynamics and control laws implemented in Simulink blocks. The detailed Simulink model is presented in Fig. 2.

For this simulation the spacecraft model is designed with an inertia matrix is given as

$$\mathbf{I}_s = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \mathbf{I}_w = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix} \tag{19}$$

Furthermore, the control gains are tuned as

$$\boldsymbol{K}_p = 0.5\,\mathbf{I}_{3\times3}, \quad \boldsymbol{K}_d = 5\,\mathbf{I}_{3\times3},$$

where $\mathbf{I}_{3\times3}$ denotes the 3×3 identity matrix. These chosen reaction wheels store 40mN.m.s af angular momentum at 47 rpm with a maximum torque of 1N.m.

The simulation scenarios were designed to assess the controller under different operating conditions.

Case 1: the spacecraft is left uncontrolled to observe its natural motion due to the wheel dynamics.

Case 2: the controller is activated to align the spacecraft with a fixed desired attitude direction.

Case 3: tests the ability of the controller to track a desired constant angular velocity.

Case 4: the controller is evaluated during the tracking of a complete orbital attitude profile, where the desired attitude and angular velocity vary continuously according to the orbital frame.

5 Results and Discussion

Simulation results demonstrate the effectiveness and robustness of the proposed nonlinear attitude controller.

Case 1: The Reaction Wheels are given an initial spin rate of 1 rad/s each along spacecraft's axes. The controller is turned off, thus there is no control torque. Fig. 3 and Fig. 4 show 200s of the reaction wheels and the body angular velocities, respectively. They spin in opposite directions but not equal in magnitude due to the difference in inertia. These results are expected as the system follows Newton's Third Law (Action–Reaction).

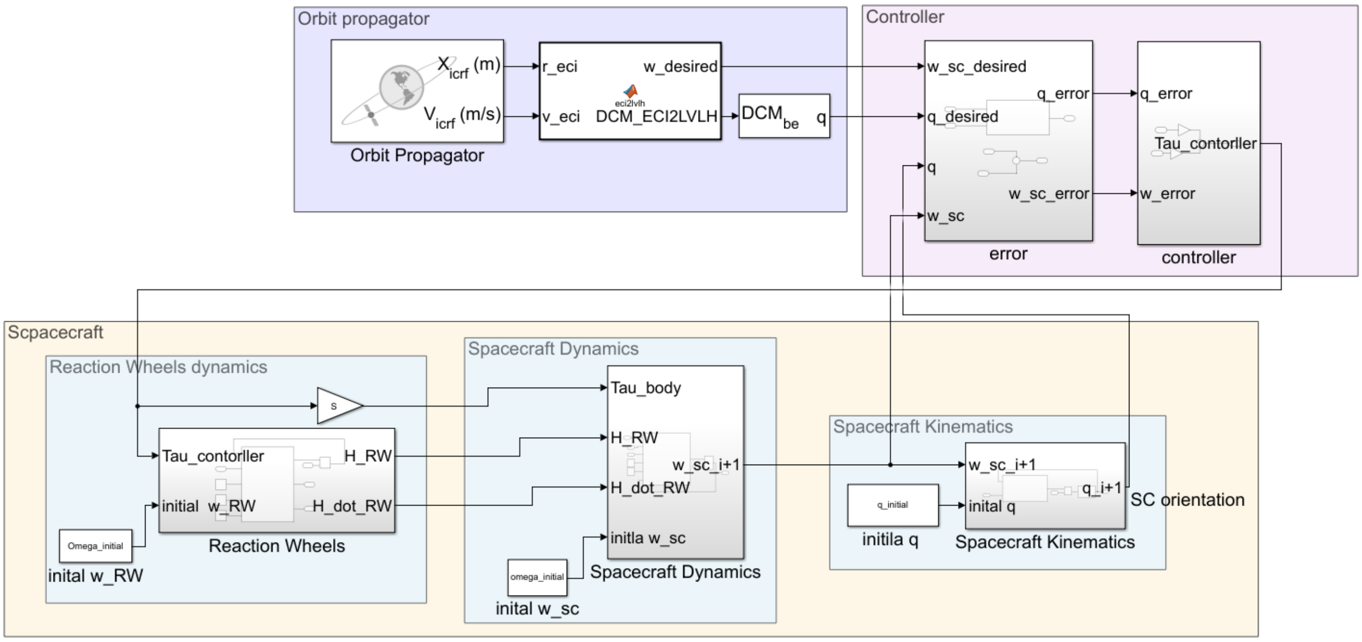

Figure 2: Simulink model of the spacecraft attitude control system.

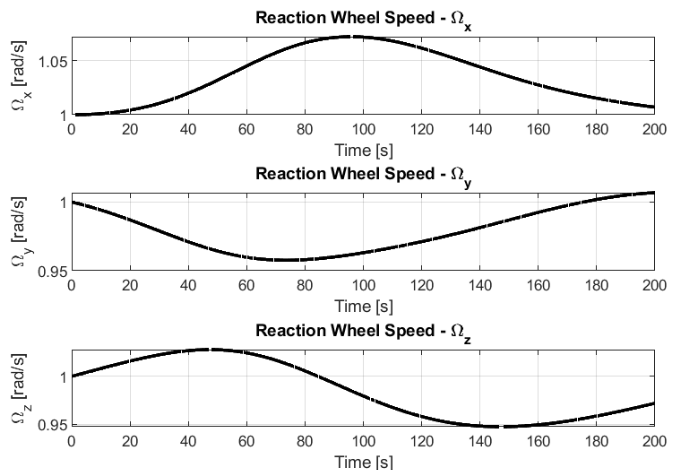

Figure 3: Reaction wheel angular velocities (No Control).

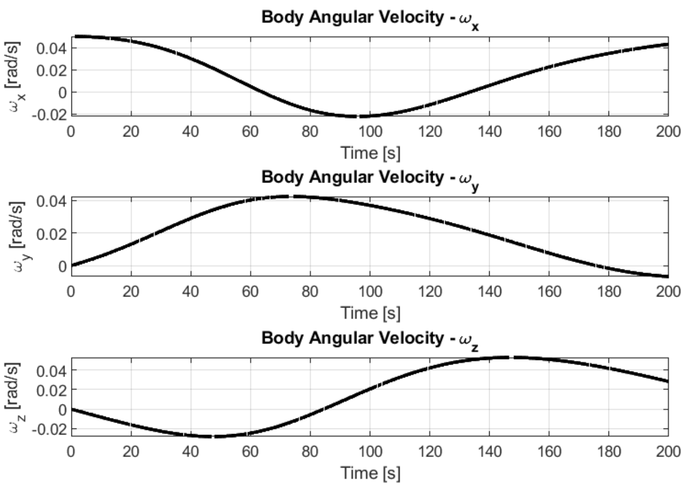

Figure 4: Spacecraft body angular velocities (No Control).

Case 2: The controller is applied for this scenario, the spacecraft is desired to reach the orientation $\boldsymbol{q_d} = [\mathbf{0.5}, \mathbf{0.5}, \mathbf{0.5}, \mathbf{0.5}]$, where its initial attitude is $\boldsymbol{q_0} = [\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}]$. Meanwhile the wheels and spacecraft have zero initial rates. The desires rate is set to zero $\boldsymbol{w_d} = \mathbf{0}$. As shown in Fig. 5 and Fig. 5 the attitude converges smoothly to the desired orientation within a short settling time 100s. Further analysis of the Fig 7 reveals that the control torque required by the system is approximately 0.17 N.m which is within the limits discussed in section 4. Analysis of Fig. 9 shows that the wheels go up to a maximum of 17.786 rads s^{-1} which is well within the saturation limits. Furthermore, Fig. 10 showcases a slow maneuver rate which is ideal.

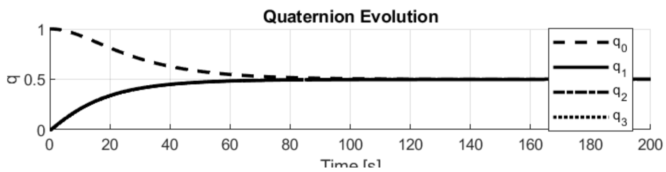

Figure 5: Spacecraft Attitude profile for case 2

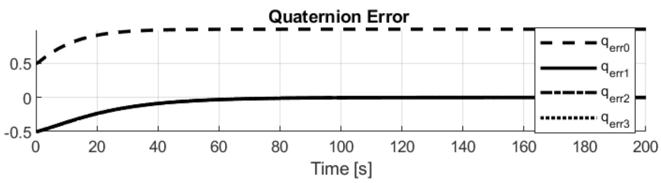

Figure 6: Quaternion error for case 2

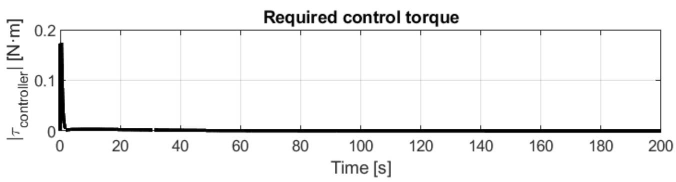

Figure 7: Required control torque for case 2

Case 3: This scenario is an orbit-following case where the

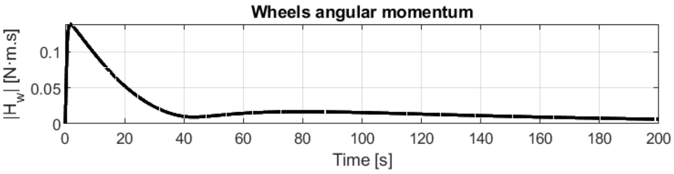

Figure 8: Angular momentum of wheels for case2

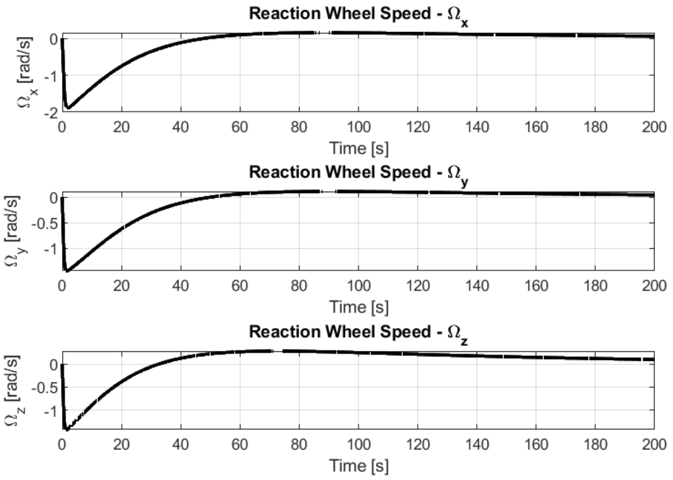

Figure 9: Wheels angular rate for case 2

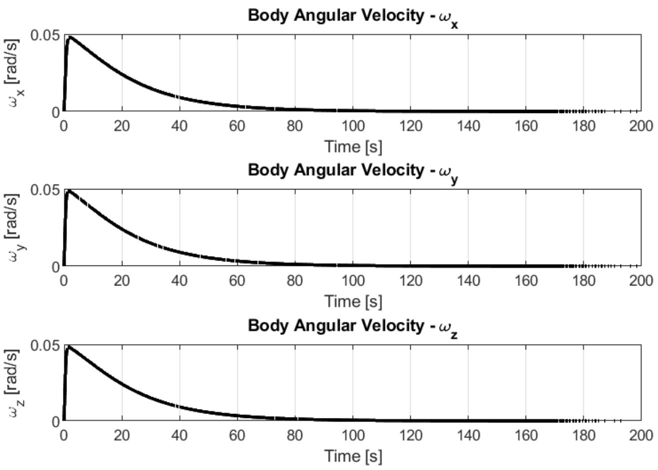

Figure 10: Spacecraft angular rates for case 2

spacecraft is required to point towards the earth. In this model the contorller tries to align spacecraft body frame with LVLH frame. Furthermore, The reference values of orientations and rates during a full orbit are calculated for the LVLH frame. First $\mathbf{r}_{\mathrm{ECI}}$,$\mathbf{v}_{\mathrm{ECI}}$ of the orbit was calculated using the orbit propagator block in simulink aerospace tool box by providing Keplerian elements High 359.954km , Eccentricity 0.00066761, Inclination 54.6146 deg, RAAN 247.4626 deg, Argument of Periapsis 130.2877 deg, and True Anomaly 325.2332 deg. After that unit vectors of LVLH frame can be calculated as follows[3]:

- Z-axis points towards Nadir:

$$\mathbf{Z}_{\mathrm{ORF}} = -\frac{\mathbf{r}_{\mathrm{ECI}}}{\|\mathbf{r}_{\mathrm{ECI}}\|}$$

- Y-axis is normal to the orbital plane in the anti-momentum direction:

$$\mathbf{Y}_{\mathrm{ORF}} = -\frac{\mathbf{r}_{\mathrm{ECI}} \times \mathbf{v}_{\mathrm{ECI}}}{\|\mathbf{r}_{\mathrm{ECI}} \times \mathbf{v}_{\mathrm{ECI}}\|}$$

- X-axis completes a right-handed system:

$$\mathbf{X}_{\mathrm{ORF}} = \mathbf{Y}_{\mathrm{ORF}} \times \mathbf{Z}_{\mathrm{ORF}}$$

Finally, the Fig. 11 shows the orbit quaternion profile during nearly one orbit. Fig 12 demonistrates the the error is only at the first few seconds due to the initial condition but it converges to zero afterward. Scpacecraft's rates with respect to inertial frame are shown in Fig. 13, this rates are the resultant of the spacecraft rates with respect to LVLH and the rates of LVLH with respect to inertial frame. As the orbital rate is in the scale of 10e-3 it is not very clear in this results. Reaction wheels' rates are in Fig. 14 the value is 40rpm which is far from saturation. Reaction wheels store up to 0.4 N.m.s angular momentum as appears in Fig. 15. The control torque required for this case is within the limits as shown in Fig. 16.

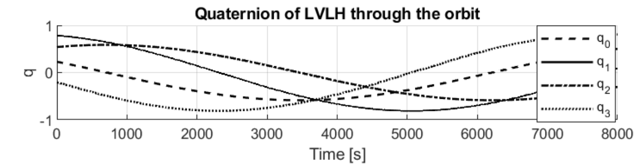

Figure 11: Quaternion profile of LVLH during the orbit

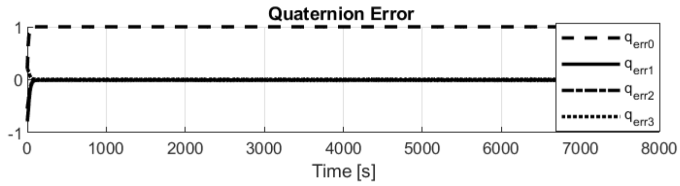

Figure 12: Quaternion error for case 3

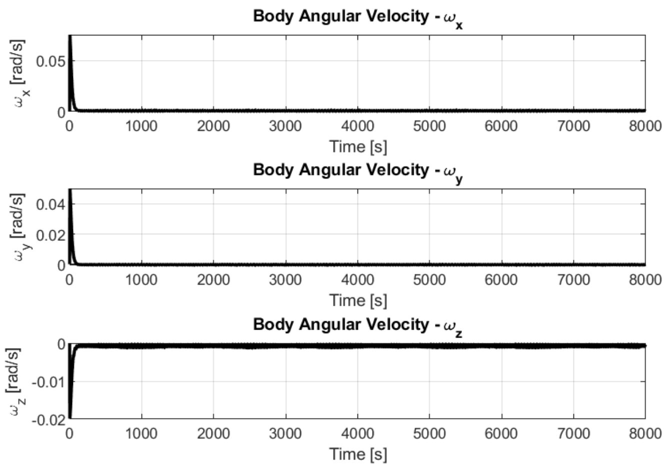

Figure 13: Spacecraft rates with respect to LVLH frame for case 3

6 Conclusion

In this report, a nonlinear Lyapunov-based attitude control system for a three-axis stabilized spacecraft

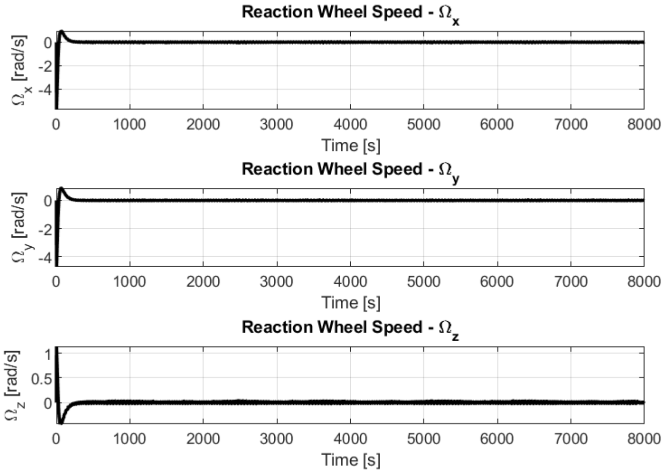

Figure 14: Reaction wheels rates for case3

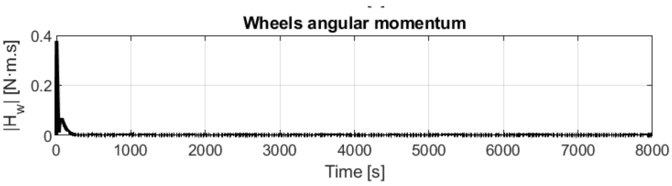

Figure 15: Reaction wheels angular momentum for case3

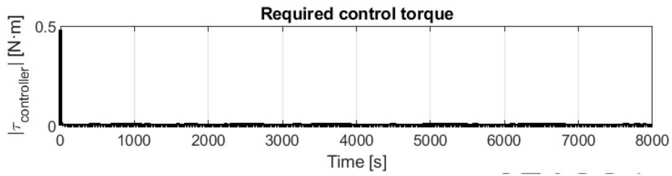

Figure 16: Required control torque for case 3

equipped with reaction wheels was presented. The simulation results demonstrates the controller's ability to achieve precise pointing ans stable tracking under various operational scenarios. The designed controller effectively minimized attitude and angular velocity errors while maintaining reaction wheel speeds and control torques within practical limits. Future work include extending the model to account for external disturbances and actuator saturation, as well as implementing real-time testing for hardware validation.

References

[1] A. M. Elbeltagy, A. Youssef, A. M. Bayoumy-Aly, and Y. Z. Elhalwagy, "Modeling and simulation of spacecraft pointing modes using quaternion-based nonlinear control laws," in *AIAA Guidance, Navigation, and Control Conference*, 2016.

[2] M. Bagheri, M. Kabganian, and R. Nada, "Three-axis attitude control design for a spacecraft based on lyapunov stability criteria," *Research Note*, 2010.

[3] Y. Yang, *Spacecraft Modeling, Attitude Determination, and Control: Quaternion-Based Approach.* Springer, 2019.