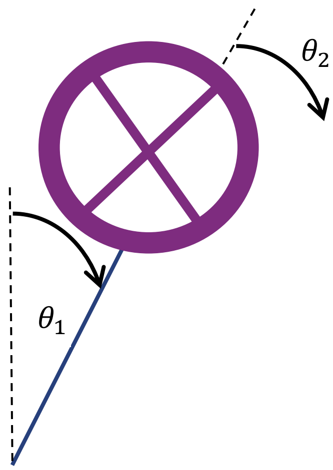
Inverted Pendulum on a cart



1. System Modeling

   1.1. Non-Linear Model

$$\ddot{\theta}_1 = \frac{g\sin(\theta_1)\,(m_1\,L_g + m_2\,L\,)\ \ - C_1\dot{\theta}_1 + \dot{\theta}_2\left(C_2 + \frac{n^2K_TK_v}{R}\right) - \frac{nK_T}{R}V}{(I_1 + m_1{L_g}^2 + m_2L^2)}$$

$$\ddot{\theta}_2 = \frac{-g\sin(\theta_1)\,(m_1\,L_g + m_2\,L\,)\ \ + C_1\dot{\theta}_1 - \dot{\theta}_2\left(1 + \frac{I_1}{I_2} + \frac{m_1{L_g}^2}{I_2} + \frac{m_2L^2}{I_2}\right)\left(C_2 + \frac{n^2K_TK_v}{R}\right) + V\,\frac{nK_T}{R}\left(1 + \frac{I_1}{I_2} + \frac{m_1{L_g}^2}{I_2} + \frac{m_2L^2}{I_2}\right)}{(I_1 + m_1{L_g}^2 + m_2L^2)}$$

Where,

C_1 the pendulum damping coeffecient, C_2 is thewheel damping coeffecient, and $\mathbf{V}$ is motor input voltage

   1.2. Linear Model

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\,\boldsymbol{x} + \boldsymbol{D}\,\boldsymbol{u}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}, \boldsymbol{x} = \begin{bmatrix}\theta_1\\ \theta_2\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{g(m_1\,L_g + m_2\,L\,)}{(I_1+m_1{L_g}^2+m_2L^2)} & 0 & \frac{-C_1}{(I_1+m_1{L_g}^2+m_2L^2)} & \frac{C_2+\frac{n^2K_TK_v}{R}}{(I_1+m_1{L_g}^2+m_2L^2)}\\ \frac{-g(m_1\,L_g + m_2\,L\,)}{(I_1+m_1{L_g}^2+m_2L^2)} & 0 & \frac{C_1}{(I_1+m_1{L_g}^2+m_2L^2)} & \frac{-\left(C_2\left(1+\frac{I_1}{I_2}+\frac{m_1{L_g}^2}{I_2}+\frac{m_2L^2}{I_2}\right)+\frac{n^2K_TK_v}{R}\right)}{(I_1+m_1{L_g}^2+m_2L^2)} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0\\ \frac{-\frac{nK_T}{R}}{(I_1+m_1{L_g}^2+m_2L^2)}\\ 0\\ \frac{\frac{nK_T}{R}\left(1+\frac{I_1}{I_2}+\frac{m_1{L_g}^2}{I_2}+\frac{m_2L^2}{I_2}\right)}{(I_1+m_1{L_g}^2+m_2L^2)} \end{bmatrix}$$

2. Controller

   2.1. Energy-Based Swing -up controller

$$E_{total} = T + v$$

$$E_{total} = \frac{1}{2} I_1 \dot{\theta}_1{}^2 + \frac{1}{2}(m_1 L_g{}^2 + m_2 L^2)\dot{\theta}_1{}^2 + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + (m_1 L_g + m_2 L) g\, cos(\theta_1)$$

$$E_{UP} = \frac{1}{2}(m_1 L_g{}^2 + m_2 L^2 + I_1)$$

$$E_{error} = E_{total} - E_{UP}$$

$$\tau_{swing} = -K_e (E - E_{UP}) sign(\cos(\theta_1))$$

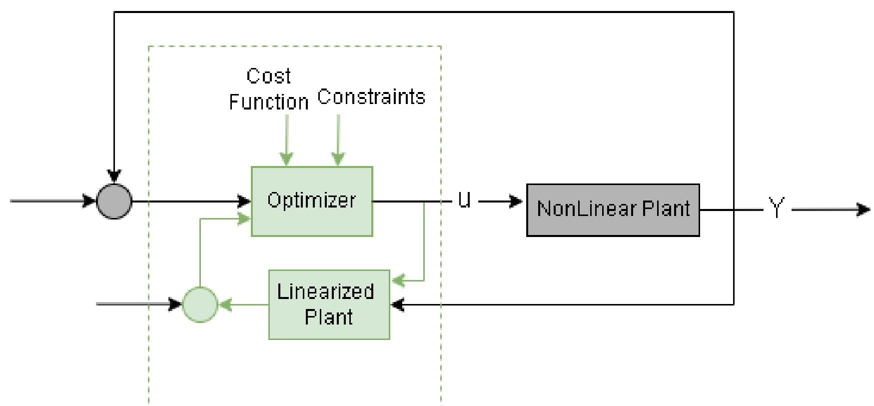
$$\tau = k_e((E - E_{target}) sign(\dot{\theta})$$

$$or\ \tau = k_e((E - E_{target}) sign(\dot{\theta}_1 \cos(\theta_1))$$

$$V = R \times \left(\frac{\tau}{K_t}\right) + K_v \times \dot{\theta}_2$$

2.2. Up-right controller

2.2.1. Model predictive control



$$\min J = |\boldsymbol{y_t} - \boldsymbol{y}|$$

$$s.t. \qquad \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

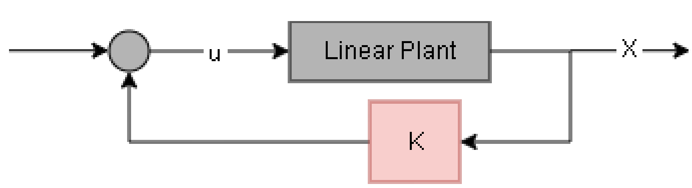
$$\boldsymbol{u_{low}} < \boldsymbol{u} < \boldsymbol{u_{high}} \qquad \boldsymbol{y_{t_{low}}} < \boldsymbol{y_t} < \boldsymbol{y_{t_{high}}}$$

2.2.2. Linear Quadratic Regulator

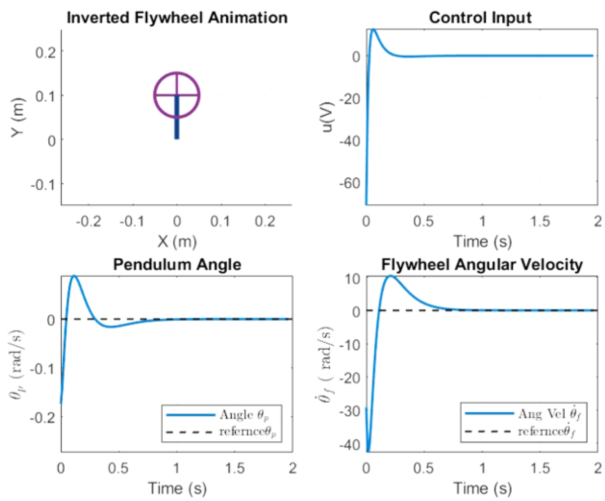
By minimizing cost function and solving Riccatti equation, K is derived

$$J = \int_0^\infty (x^T Q\, x + u^T R u) dt$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\, \boldsymbol{x} + \boldsymbol{B}\, \boldsymbol{u}, \boldsymbol{u} = -\boldsymbol{K}\, \boldsymbol{x}$$



3. Simulation results



Figur.1 LQR contoller for small angle error

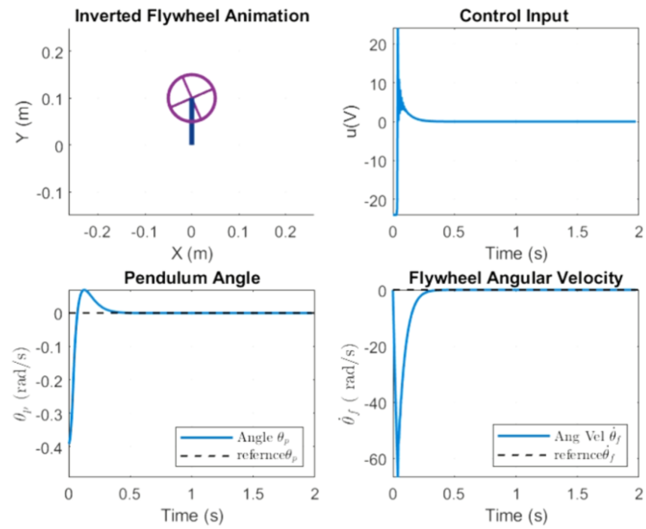


Figure.2 MPC controller for small angle error

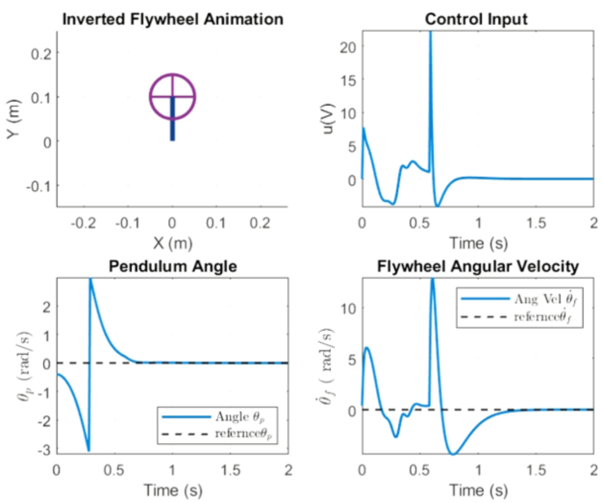


Figure.3 Swing-up with LQR for large angle error

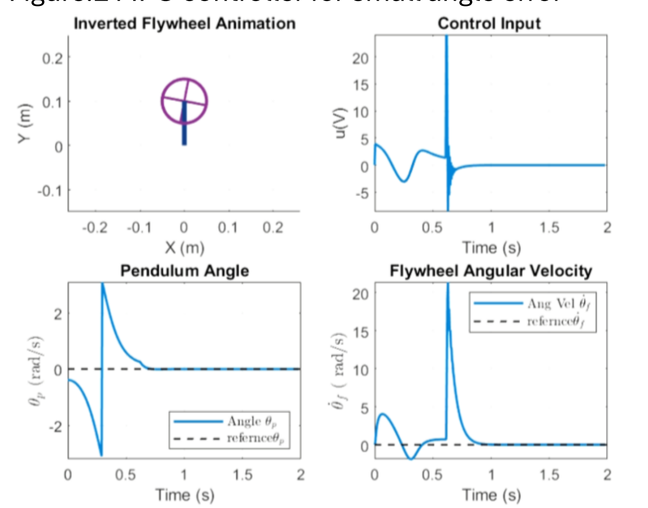


Figure.4 Swing-up with MPC for large angle error