Preliminaries

```
In[1]:= mix1 := NormalDistribution[0, 1]
mix2 := NormalDistribution[u, 1]
mixture := q * PDF[mix2, x] + (1 - q) * PDF[mix1, x]
norm := PDF[NormalDistribution[q * u, 1], x]
mixturetra := mixture /. x → q * u + x
stdnorm := PDF[NormalDistribution[0, 1], x]
logtermtra := Log[stdnorm / mixturetra]
```

mixturetra is the denominator in $f_x(u)$:

```
In[8]:= mixturetra
```

Out[8]= $$\frac{e^{-\frac{1}{2}(q\,u+x)^2}(1-q)}{\sqrt{2\pi}}+\frac{e^{-\frac{1}{2}(-u+q\,u+x)^2}\,q}{\sqrt{2\pi}}$$

logtermtra is $f_x(u)$ from the paper:

```
In[9]:= Simplify[logtermtra]
```

Out[9]= $$\mathrm{Log}\left[\frac{e^{-\frac{x^2}{2}}}{-e^{-\frac{1}{2}(q\,u+x)^2}(-1+q)+e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,q}\right]$$

The Taylor Approximation of $f_x(u)$

Lemma A.2, the Taylor approximation of $f_x(u)$ at 0:

```
In[10]:= Simplify[Series[logtermtra, {u, 0, 4}]]
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Out[10]= $$\frac{1}{2}\times(-1+q)\,q\,(-1+x^2)\,u^2-\frac{1}{6}\times((-1+q)\,q\,(-1+2\,q)\,x\,(-3+x^2))\,u^3+\frac{1}{24}\,q\,(-3+6\,x^2-x^4-12\,q^2\,(2-4\,x^2+x^4)+6\,q^3\,(2-4\,x^2+x^4)+q\,(15-30\,x^2+7\,x^4))\,u^4+O[u]^5$$

Lemma A.3:

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In[11]:= Integrate[Series[logtermtra, {u, 0, 4}] * stdnorm, {x, -Infinity, Infinity}]
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Out[11]= $$\frac{1}{4}(-1+q)^2\,q^2\,u^4$$

Derivatives of $f_x(u)$ without setting $u = 0$:

```
In[12]:= Simplify[D[logtermtra, {u, 1}]]
```

Out[12]= $$\frac{(-1+q)\,q\left(-e^{\frac{1}{2}(q\,u+x)^2}((-1+q)\,u+x)+e^{\frac{1}{2}((-1+q)\,u+x)^2}(q\,u+x)\right)}{e^{\frac{1}{2}((-1+q)\,u+x)^2}(-1+q)-e^{\frac{1}{2}(q\,u+x)^2}\,q}$$

In[13]:=
```
Simplify[D[logtermtra, {u, 2}]]
```

Out[13]=

$$\left((-1+q)\,q\left(e^{((-1+q)\,u+x)^2}\,(-1+q)\,q+e^{(q\,u+x)^2}\,(-1+q)\,q+e^{\frac{1}{2}\left(((-1+q)\,u+x)^2+(q\,u+x)^2\right)}\left(-1+u^2+q^2\left(-2+4\,u^2\right)-2\,u\,x+x^2+q\left(2-4\,u^2+4\,u\,x\right)\right)\right)\right)\Big/\left(e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)-e^{\frac{1}{2}(q\,u+x)^2}\,q\right)^2$$

In[14]:=
```
Simplify[D[logtermtra, {u, 3}]]
```

Out[14]=

$$\frac{1}{\left(e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)-e^{\frac{1}{2}(q\,u+x)^2}\,q\right)^3}\,e^{\frac{1}{2}\left(((-1+q)\,u+x)^2+(q\,u+x)^2\right)}\,(-1+q)\,q\,((-1+2\,q)\,u+x)\left(e^{\frac{1}{2}(q\,u+x)^2}\,q\left(3+u^2+4\,q^2\,u^2-2\,u\,x+x^2+q\left(-6-4\,u^2+4\,u\,x\right)\right)+e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)\times\left(-3+u^2+4\,q^2\,u^2-2\,u\,x+x^2+q\left(6-4\,u^2+4\,u\,x\right)\right)\right)$$

In[15]:=
```
Simplify[D[logtermtra, {u, 4}]]
```

Out[15]=

$$\frac{1}{4\left(e^{-\frac{1}{2}(q\,u+x)^2}\,(-1+q)-e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,q\right)^4}$$

$$\Bigg(24\,(-1+q)^4\,q^4\left(e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,((-1+q)\,u+x)-e^{-\frac{1}{2}(q\,u+x)^2}\,(q\,u+x)\right)^4-$$

$$16\,(-1+q)^2\,q^2\left(e^{-\frac{1}{2}(q\,u+x)^2}\,(-1+q)-e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,q\right)^2$$

$$\left(-e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,((-1+q)\,u+x)+e^{-\frac{1}{2}(q\,u+x)^2}\,(q\,u+x)\right)$$

$$\Big(-3\,e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)^2\,((-1+q)\,u+x)+e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)^2\,((-1+q)\,u+x)^3+$$

$$3\,e^{-\frac{1}{2}(q\,u+x)^2}\,q^2\,(q\,u+x)-e^{-\frac{1}{2}(q\,u+x)^2}\,q^2\,(q\,u+x)^3\Big)-4\times(1-q)\,q$$

$$\left(-e^{-\frac{1}{2}(q\,u+x)^2}\,(-1+q)+e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,q\right)^3\Big(-3\,e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)^3+3\,e^{-\frac{1}{2}(q\,u+x)^2}\,q^3+$$

$$6\,e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)^3\,((-1+q)\,u+x)^2-e^{-\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)^3\,((-1+q)\,u+x)^4-$$

$$6\,e^{-\frac{1}{2}(q\,u+x)^2}\,q^3\,(q\,u+x)^2+e^{-\frac{1}{2}(q\,u+x)^2}\,q^3\,(q\,u+x)^4\Big)+$$

$$12\,e^{-2\left(((-1+q)\,u+x)^2+(q\,u+x)^2\right)}\,(-1+q)^2\,q^2\left(e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)-e^{\frac{1}{2}(q\,u+x)^2}\,q\right)^2$$

$$\Big(e^{\frac{1}{2}(q\,u+x)^2}\,(-1+q)\times\left(-1+(-1+q)^2\,u^2+2\times(-1+q)\,u\,x+x^2\right)-$$

$$e^{\frac{1}{2}((-1+q)\,u+x)^2}\,q\left(-1+q^2\,u^2+2\,q\,u\,x+x^2\right)\Big)^2-$$

$$48\,e^{-2\left(((-1+q)\,u+x)^2+(q\,u+x)^2\right)}\,(-1+q)^3\,q^3\left(e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(-1+q)-e^{\frac{1}{2}(q\,u+x)^2}\,q\right)$$

$$\left(e^{\frac{1}{2}(q\,u+x)^2}\,((-1+q)\,u+x)-e^{\frac{1}{2}((-1+q)\,u+x)^2}\,(q\,u+x)\right)^2$$

$$\Big(-e^{\frac{1}{2}(q\,u+x)^2}\,(-1+q)\times\left(-1+(-1+q)^2\,u^2+2\times(-1+q)\,u\,x+x^2\right)+$$

$$e^{\frac{1}{2}((-1+q)\,u+x)^2}\,q\left(-1+q^2\,u^2+2\,q\,u\,x+x^2\right)\Big)\Bigg)$$