Modeling of 2D and 3D Gantry Systems

DPMA

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1 Dynamic Model of the 2D Gantry System

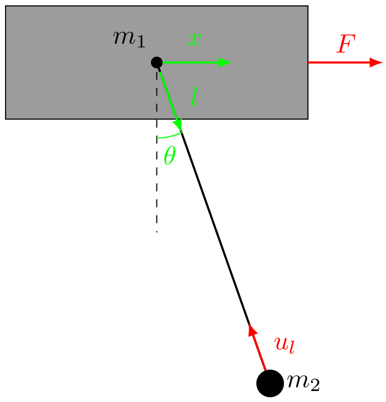

Figure 1: Diagram of the 2D-system

1.1 Kinematics

The generalized coordinates and external forces are $q^T = \begin{bmatrix} x & l & \theta \end{bmatrix}$, $Q^T = \begin{bmatrix} F, u_l, 0 \end{bmatrix}$. The position of the load can then be represented as $p(q)$:

$$p(q) = \begin{bmatrix} x_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x + l\sin\theta \\ -l\cos\theta \end{bmatrix}.$$

Such that the Jacobian $J(q)$ is:

$$J(q) = \begin{bmatrix} \frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial l} & \frac{\partial x_2}{\partial \theta} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial l} & \frac{\partial z_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & \sin\theta & l\cos\theta \\ 0 & -\cos\theta & l\sin\theta \end{bmatrix}.$$

The payload velocity $\dot{p}(q)$ is given by:

$$\dot{p} = J\dot{q} = \begin{bmatrix} \dot{x}_2 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} + \dot{l}\sin\theta + l\dot{\theta}\cos\theta \\ -\dot{l}\cos\theta + \dot{\theta}l\sin\theta \end{bmatrix}.$$

1.2 Energies

The cart with mass m_1 only moves in x:

$$T_1 = \frac{1}{2}m_1\dot{x}^2.$$

While the payload moves with velocity $\dot{p}$:

$$T_2 = \frac{1}{2}m_2\dot{p}^T\dot{p} = \frac{1}{2}m_2\left(J\dot{q}\right)^T\left(J\dot{q}\right) = \frac{1}{2}m_2\dot{q}^T(J^TJ)\dot{q}.$$

Total kinetic energy:

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{q}^T(J^TJ)\dot{q}.$$

The only potential energy in this system comes from the payload:

$$V = m_2 g(-l\cos\theta) = -m_2 g l\cos\theta.$$

1.3 Compact Matrix Setup for Euler-Lagrange

$$L = T - V$$
$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = Q_i$$

Since V has no dependence on $\dot{q}$:

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}}$$
$$\frac{\partial T_1}{\partial \dot{q}} = \begin{bmatrix} m_1\dot{x} \\ 0 \\ 0 \end{bmatrix}$$
$$\frac{\partial T_2}{\partial \dot{q}} = m_2 J^T J\dot{q} = m_2 J^T \dot{p}$$
$$\frac{\partial T}{\partial \dot{q}} = \begin{bmatrix} m_1\dot{x} \\ 0 \\ 0 \end{bmatrix} + m_2 J^T \dot{p}$$

Taking the time derivative:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = \begin{bmatrix} m_1\ddot{x} \\ 0 \\ 0 \end{bmatrix} + m_2(\dot{J}^T\dot{p} + J^T\ddot{p}) = \begin{bmatrix} m_1\ddot{x} \\ 0 \\ 0 \end{bmatrix} + m_2(\dot{J}^T J\dot{q} + J^T\ddot{p})$$

Using $\ddot{p} = \frac{d}{dt}(J\dot{q}) = \dot{J}\dot{q} + J\ddot{q}$:

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) &= \begin{bmatrix} m_1\ddot{x} \\ 0 \\ 0 \end{bmatrix} + m_2\big(\dot{J}^T J\dot{q} + J^T(\dot{J}\dot{q} + J\ddot{q})\big) \\ &= \begin{bmatrix} m_1\ddot{x} \\ 0 \\ 0 \end{bmatrix} + m_2\big(\dot{J}^T J\dot{q} + J^T\dot{J}\dot{q} + J^T J\ddot{q}\big)\end{aligned}$$

For computing $\dfrac{\partial L}{\partial q}$, only T_2 and V depend on q:

$$\frac{\partial T_2}{\partial q} = \frac{1}{2}m_2\frac{\partial}{\partial q}(\dot{p}^T\dot{p}).$$

Using $\dot{p}^T\dot{p} = \dot{q}^T J^T J\dot{q}$:

$$\frac{\partial T_2}{\partial q} = \frac{1}{2}m_2\Big(\dot{q}^T\frac{\partial}{\partial q}(J^T J)\dot{q}\Big) = \frac{1}{2}m_2\dot{q}^T\Big[\Big(\frac{\partial J}{\partial q}\Big)^T J + J^T\Big(\frac{\partial J}{\partial q}\Big)\Big]\dot{q}.$$

Knowing that $\dfrac{\partial T_2}{\partial q_i}$ is a scalar quadratic form in $\dot{q}$, symmetry $a^T = a$ can be used:

$$\Big[\dot{q}^T\Big(\frac{\partial J}{\partial q}\Big)^T J\dot{q}\Big]^T = \dot{q}^T J^T\Big(\frac{\partial J}{\partial q}\Big)\dot{q},$$
$$\frac{\partial T_2}{\partial q} = \frac{1}{2}m_2 2\begin{bmatrix} \dot{q}^T\Big(\frac{\partial J}{\partial x}\Big)^T J\dot{q} \\ \dot{q}^T\Big(\frac{\partial J}{\partial l}\Big)^T J\dot{q} \\ \dot{q}^T\Big(\frac{\partial J}{\partial \theta}\Big)^T J\dot{q} \end{bmatrix} = m_2\begin{bmatrix} \dot{q}^T\Big(\frac{\partial J}{\partial x}\Big)^T J\dot{q} \\ \dot{q}^T\Big(\frac{\partial J}{\partial l}\Big)^T J\dot{q} \\ \dot{q}^T\Big(\frac{\partial J}{\partial \theta}\Big)^T J\dot{q} \end{bmatrix}.$$

Finally for the potential energy component:

$$\frac{\partial V}{\partial q} = \begin{bmatrix} 0 \\ -m_2 g\cos\theta \\ m_2 gl\sin\theta \end{bmatrix}.$$

So:

$$\frac{\partial L}{\partial q} = m_2 \begin{bmatrix} \dot{q}^T \left(\frac{\partial J}{\partial x}\right)^T J\dot{q} \\ \dot{q}^T \left(\frac{\partial J}{\partial l}\right)^T J\dot{q} \\ \dot{q}^T \left(\frac{\partial J}{\partial \theta}\right)^T J\dot{q} \end{bmatrix} - \begin{bmatrix} 0 \\ -m_2 g\cos\theta \\ m_2 gl\sin\theta \end{bmatrix}$$

1.4 Explicit Expressions for the system

To write the equations in their explicit form with the addition of a damping term Q_D:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Q - Q_d\dot{q}, \tag{1}$$

Definitions for the following are required:

$$\dot{J} = \begin{bmatrix} 0 & \dot{\theta}\cos\theta & \dot{l}\cos\theta - \dot{\theta}l\sin\theta \\ 0 & \dot{\theta}\sin\theta & \dot{l}\sin\theta + \dot{\theta}l\cos\theta \end{bmatrix},$$

$$M_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} 0 \\ -m_2 g\cos\theta \\ m_2 gl\sin\theta \end{bmatrix}.$$

As well as:

$$A_1 = \frac{\partial J}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \frac{\partial J}{\partial l} = \begin{bmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \end{bmatrix},$$

$$A_3 = \frac{\partial J}{\partial \theta} = \begin{bmatrix} 0 & \cos\theta & -l\sin\theta \\ 0 & \sin\theta & l\cos\theta \end{bmatrix}.$$

Now defining:

$$\Gamma(q,\dot{q}) = m_2 \begin{bmatrix} \dot{q}^T {A_1}^T J\dot{q} \\ \dot{q}^T {A_2}^T J\dot{q} \\ \dot{q}^T {A_3}^T J\dot{q} \end{bmatrix},$$

and forming the matrices

$$M_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Writing it out fully gives:

$$M_1\ddot{q} + m_2 J^T J\ddot{q} + m_2(\dot{J}^T J + J^T\dot{J})\dot{q} - (\Gamma - G) = Q - Q_d\dot{q},$$

and finally:

$$\underbrace{(M_1 + m_2 J^T J)}_{M(q)}\ddot{q} + \underbrace{(m_2\dot{J}^T J + m_2 J^T\dot{J})\dot{q} - \Gamma}_{C(q,\dot{q})\dot{q}} + G = Q - Q_d\dot{q}.$$

2 2D Gantry Optimal Control Problem

2.1 Reformulating Dynamics

For usage in optimal control, a nonlinear state space model should be formed with the chosen state space and controls vector: $x^T = \left[x, l, \theta, \dot{x}, \dot{l}, \dot{\theta}\right], u^T = \left[F, u_l\right]$

$$\dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}\big(Q - (Q_d - C(q, \dot{q}))\dot{q} - G(q)\big) \end{pmatrix} = f(x, u)$$