1) Find ZT of $x(n) = [n(-0.5)^n u(n) * 4^n u(-n)]$

$$x_1(n) = n(-0.5)^n u(n)$$

$$X_1(z) = Z(n(-0.5)^n u(n))$$

$$= -z \frac{d}{dz}(Z((-0.5)^n u(n)))$$

$$= -z \frac{d}{dz}\left[\sum_{-\infty}^{\infty} (-0.5)^n u(n) z^{-n}\right]$$

$$= -z \frac{d}{dz}\left[\frac{z}{z+0.5}\right]$$

$$= -z \left(\frac{z+0.5 - z(1)}{(z+0.5)^2}\right)$$

$$= \frac{-z(0.5)}{(z+0.5)^2}$$

$$x_2(n) = 4^n u(-n)$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} 4^n u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} 4^n z^{-n}$$

Let $n = -n$

$$= \sum_{n=0}^{\infty} 4^{-n} z^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$$

$$X_2(z) = \frac{4}{4-z}$$

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$$X(z) = X_1(z) \cdot X_2(z)$$

$$= \frac{-z(0.5)}{(z+0.5)^2} \times \frac{4}{4-z}$$

$$= \frac{-2z}{(z+0.5)^2(4-z)}$$

$$= \frac{-2z}{(z^2+0.5^2+z)(4-z)}$$

$$= \frac{-2z}{4z^2+1+4z-z^3-0.25z-z^2}$$

$$= \frac{2z}{z^3-3z^2-3.75z-1}$$

2) Find IZT of $X(z) = \frac{z^4+z^2}{z^2-0.75z+0.125}$

ROC $|z|>0.5$

$$X(z) = \frac{z^2(z^2+1)}{(z-0.5)(z-0.25)}$$

$$\frac{X(z)}{z} = \frac{z^3+z}{(z-0.5)(z-0.25)}$$

$$\frac{X(z)}{z} = \frac{z^3+z}{z^2-0.75z+0.125}$$

long division:-

$$
\begin{array}{r|l}
 & z+0.75 \\
\hline
z^2-0.75z+0.125 & z^3+z \\
 & z^3-0.75z^2+0.125z \\
\hline
 & 0.75z^2+0.875z \\
 & 0.75z^2-0.5625z+0.09375 \\
\hline
 & 1.4375z-0.09375
\end{array}
$$

$$\frac{X(z)}{z} = (z+0.75) + \frac{(1.4375z-0.09375)}{(z-0.5)(z-0.25)}$$

Partial fraction:-

$$\frac{1.4375z-0.09375}{(z-0.5)(z-0.25)} = \frac{A}{(z-0.5)} + \frac{B}{(z-0.25)}$$

$$1.4375z-0.09375 = A(z-0.25)+B(z-0.5)$$

Let $z = 0.25$

$$1.4375(0.25)-0.09375 = B(0.25-0.5)$$

$$0.265625 = -0.25B$$

$$B = -1.0625$$

Let $z = 0.5$

$$1.4375(0.5)-0.09375 = A(0.25)$$

$$A = 2.5$$

$$\frac{X(z)}{z} = (z+0.75) + \frac{2.5}{(z-0.5)} - \frac{1.0625}{(z-0.25)}$$

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$$X(z) = z^2 + 0.75z + \frac{2.5z}{(z-0.5)} + \frac{(-1.0625)z}{(z-0.25)}$$

considering ROC :- $|z| > 0.5$

IZT :-

$$x(n) = \delta(n+2) + 0.75\,\delta(n+1) + 2.5(0.5)^n u(n) - 1.0625(0.25)^n u(n)$$

3) A DT-LTI system represented by difference equation $y(n) = y(n-1) + y(n-2) + x(n-1)$

(i) Find system function.

(ii) Indicate ROC if system is stable.

(iii) Indicate ROC if system is causal.

(iv) Obtain impulse responses in both cases.

(i) $y(n) = y(n-1) + y(n-2) + x(n-1)$

Z transform of the above eq^n.

$$Y(z) = z^{-1}(Y(z)) + z^{-2}Y(z) + z^{-1}X(z)$$

$$Y(z) - z^{-1}Y(z) - z^{-2}Y(z) = z^{-1}X(z)$$

$$Y(z)[1 - z^{-1} - z^{-2}] = z^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

$$H(z) = \frac{z^{-1}}{z^{-2}(z^2 - z^1 - 1)}$$

$$H(z) = \frac{z}{z^2 - z^1 - 1}$$

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If the system is causal the ROC lies outside the circle.

$$H(z) = \frac{z}{z^2 - z - 1}$$

$$H(z) = \frac{z}{\left[z - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z - \left[\frac{1-\sqrt{5}}{2}\right]\right]}$$

$$\frac{H(z)}{z} = \frac{1}{\left[z - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z - \left[\frac{1-\sqrt{5}}{2}\right]\right]}$$

$$\cancel{\frac{H(z)}{z}} \; \frac{H(z)}{z} = \frac{A}{\left[z - \left(\frac{1+\sqrt{5}}{2}\right)\right]} + \frac{B}{\left[z - \left[\frac{1-\sqrt{5}}{2}\right]\right]}$$

$$1 = A\left[z - \tfrac{1}{2} + \frac{\sqrt{5}}{2}\right] + B\left[z - \tfrac{1}{2} - \frac{\sqrt{5}}{2}\right]$$

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Let $z = \frac{1}{2} - \frac{\sqrt{5}}{2}$.

$1 = A(0) + B\left(-\frac{\sqrt{5}}{2} - \frac{\sqrt{5}}{2}\right)$

$\therefore B = \frac{-1}{\sqrt{5}}$

Let $z = \frac{1}{2} + \frac{\sqrt{5}}{2}$

$1 = A\left[\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}\right] + B(0)$

$\therefore A = \frac{1}{\sqrt{5}}$

$$\frac{H(z)}{z} = \frac{1/\sqrt{5}}{\left[z - \left[\frac{1+\sqrt{5}}{2}\right]\right]} - \frac{1/\sqrt{5}}{\left(z - \left[\frac{1-\sqrt{5}}{2}\right]\right)}$$

$$H(z) = \frac{1/\sqrt{5}\, z}{\left(z - \left[\frac{1+\sqrt{5}}{2}\right]\right)} - \frac{1/\sqrt{5}\, z}{\left(z - \left[\frac{1-\sqrt{5}}{2}\right]\right)}$$

If $H(z)$ is causal then

ROC $= |z| > \left|\frac{1+\sqrt{5}}{2}\right|$

(iv) Impulse response if the system is stable:-

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Impulse response if the system is causal:-

$$h(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u(n) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u(n)$$

4) Determine whether the system $H(z) = \frac{2z+1}{z^2+z-5/16}$ is causal and stable.

$$H(z) = \frac{2z+1}{z^2+z-5/16}$$

$$\frac{H(z)}{z} = \frac{2z+1}{z(z^2+z-5/16)}$$

$$\frac{H(z)}{z} = \frac{2z+1}{z(z-1/4)(z+5/4)} = \frac{A}{z} + \frac{B}{z-1/4} + \frac{C}{z+5/4}$$

$$2z+1 = A(z-1/4)(z+5/4) + Bz(z+5/4) + Cz(z-1/4)$$

let $z=0$

$1 = A(-1/4)(5/4)$

$\frac{-16}{5} = A$

let $z = -5/4$

$-\frac{5}{2}+1 = C(-5/4)(-5/4-1/4)$

$C = \frac{-4}{5}$

let $z = 1/4$

$\frac{1}{2}+1 = B\,3/2(1/4)$

$B = 4$

$$\frac{H(z)}{z} = \frac{-16/5}{z} + \frac{4}{z-1/4} - \frac{4/5}{z+5/4}$$

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$$H(z) = \frac{-16}{5} + \frac{4z}{z-1/4} - \frac{4}{5}\left[\frac{z}{z+5/4}\right]$$

$$h(n) = \frac{-16}{5}\delta(n) + 4(1/4)^n u(n) - \frac{4}{5}(-5/4)^n u(n)$$

for $n<1$
$h(n) = 0$
∴ The system is causal.

Stability:-

$$\sum_{n=\infty}^{+\infty} h(n) = \sum_{n=\infty}^{+\infty} \frac{-16}{5}\delta(n) + 4(1/4)^n u(n) + \left(-\frac{4}{5}\right)\left(\frac{-5}{4}\right)^n u(n)$$

$$= \sum_{n=0} \frac{-16}{5}\delta(0) + \sum_{n=0}^{\infty} 4(1/4)^n + \sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)\left(\frac{-5}{4}\right)$$

$$= \frac{-16}{5} + 4\left(\frac{1}{1-1/4}\right) - \frac{4}{5}\left(\frac{1}{1+5/4}\right)$$

= Finite.
∴ It is stable.

5) Find DTFT of $x(n) = \left(\frac{1}{4}\right)^n \cdot \sin\left(\frac{n\pi}{4}\right) \cdot u(n-1)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n \sin\frac{n\pi}{4} \cdot u(n-1)\, e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \sin\frac{n\pi}{4}\, e^{-j\omega n}$$

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$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore X(e^{j\omega}) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \left[\frac{e^{\frac{jn\pi}{4}} - e^{-\frac{jn\pi}{4}}}{2j}\right] e^{-j\omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n e^{\frac{jn\pi}{4}} e^{-j\omega n} - \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\frac{n\pi}{4}} e^{-j\omega n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=1}^{\infty} \left(\frac{1}{4} \cdot e^{j(\pi/4 - \omega)}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{4} e^{-j(\pi/4 + \omega)}\right)^n\right].$$

$$= \frac{1}{2j}\left[\frac{\frac{1}{4} e^{j(\pi/4 - \omega)}}{1 - \frac{1}{4} e^{j(\pi/4 - \omega)}} - \frac{\frac{1}{4} e^{-j(\pi/4 + \omega)}}{1 - \frac{1}{4} e^{-j(\pi/4 - \omega)}}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{4}\right]\left[\frac{e^{\frac{j\pi}{4}} e^{-j\omega}}{1 - \frac{1 e^{j\pi/4 - j\omega}}{4}} - \frac{e^{-\frac{j\pi}{4}} e^{-j\omega}}{1 - \frac{e^{-j\pi/4} e^{-j\omega}}{4}}\right]$$

$$= \frac{e^{-j\omega}}{8j}\left[\frac{4 e^{j(\pi/4)}}{4 - e^{j\pi/4} e^{-j\omega}} - \frac{4 e^{-j\pi/4}}{4 - e^{-j\pi/4} e^{-j\omega}}\right]$$

6) Find frequency response and impulse response of the system having input and output as following.

$x(n) = \left(\frac{1}{2}\right)^n u(n)$; $y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (e^{-j\omega})^n.$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \times \frac{1}{e^{j\omega}}\right)^n.$$

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 1/2}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)\right] e^{-j\omega n}.$$

$$Y(e^{j\omega}) = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^n e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n}.$$

$$Y(e^{j\omega}) = \frac{1}{4}\left[\frac{e^{j\omega}}{e^{j\omega} - 1/2}\right] + \frac{e^{j\omega}}{(e^{j\omega} - 1/4)}$$

$$H(e^{j\omega}) = \frac{Y(z)}{X(z)}$$

$$H(e^{j\omega}) = \frac{\frac{1}{4}\left(\frac{e^{j\omega}}{e^{j\omega} - 1/2}\right) + \frac{e^{j\omega}}{e^{j\omega} - 1/4}}{\frac{e^{j\omega}}{e^{j\omega} - 1/2}}$$

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$H(e^{jw}) = \frac{1}{4} + \frac{e^{jw}-1/2}{e^{jw}-1/4}$ ⇒ Frequency response.

$H_1(e^{jw}) = \frac{e^{jw}-1/2}{e^{jw}-1/4}$

$\frac{H_1(e^{jw})}{e^{jw}} = \frac{e^{jw}-1/2}{(e^{jw}-1/4)e^{jw}} = \frac{A}{e^{jw}} + \frac{B}{e^{jw}-1/4}$

$A(e^{jw}-1/4) + Be^{jw} = e^{jw}-1/2$.

Let $e^{jw}=0$. Let $e^{jw}=1/4$.

$A(-1/4) = -1/2$ $\quad$ $B(1/4) = (1/4 - 1/2)$

$A = 2$ $\quad$ $B = -1$

$\therefore \frac{H_1(e^{jw})}{e^{jw}} = \frac{2}{e^{jw}} + \frac{-1}{e^{jw}-1/4}$.

$H_1(e^{jw}) = 2 + \frac{-1e^{jw}}{e^{jw}-1/4}$.

$\therefore H(e^{jw}) = \frac{1}{4} + 2 - \frac{e^{jw}}{e^{jw}-1/4}$

$\therefore H(e^{jw}) = \frac{9}{4} - \frac{e^{jw}}{e^{jw}-1/4}$

Taking IDTFT:-

$h(n) = \frac{9}{4}\delta(n) - \left(\frac{1}{4}\right)^n u(n)$ ⇒ impulse response.