IS THIS OBJECT A CIRCLE?

Given a simple closed curve in the plane S, we seek simple algorithms that will identify S as not being a circle. Specifically, we wish to identify noncircles via algorithms that measure chords.

We'll identify $C \in \mathbb{R}^2$ as a circle if

$$C \subset \{x^2 + y^2 = 1 + \epsilon\}$$

where ϵ is a numerical tolerance on the circle. Given

$$S \not\subset \{x^2 + y^2 = 1 + \epsilon\}$$

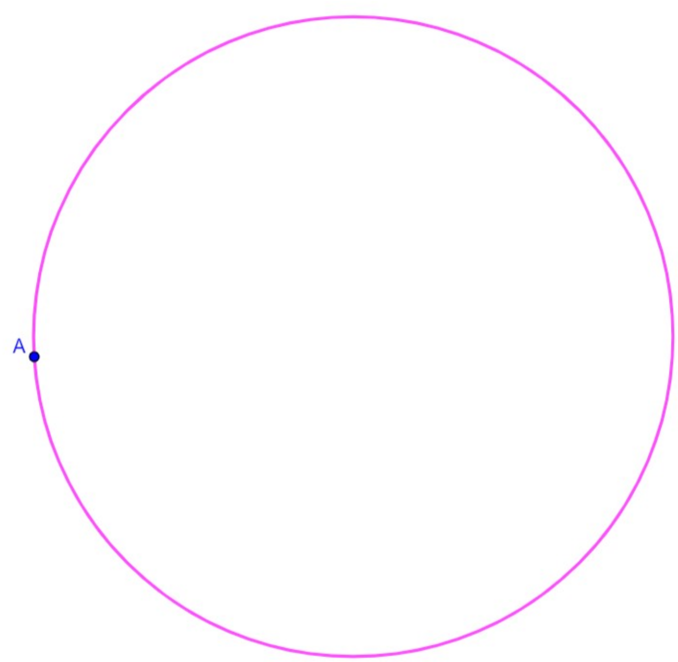
we wish to examine the methods of determining this non-inclusion by

(1) Measuring relative arc length.
(2) Measuring angles.
(3) Meauring chords.
(4) Measuring widths.

Suppose we fish some device out of the ocean with a face whose perimeter is given by a simple closed curve, S. To what degree can we determine if S is a circle by measuring chords and diameters?

If S ends up being a circle, an important piece of information about S would be its radius. If S is a circle, we can obtain the radius of S as follows.

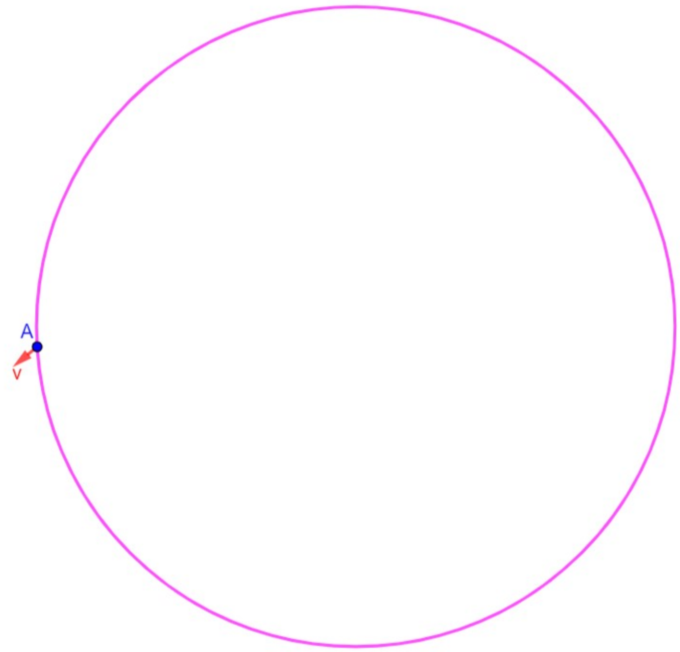
1. Pick a point on S, label this point A.

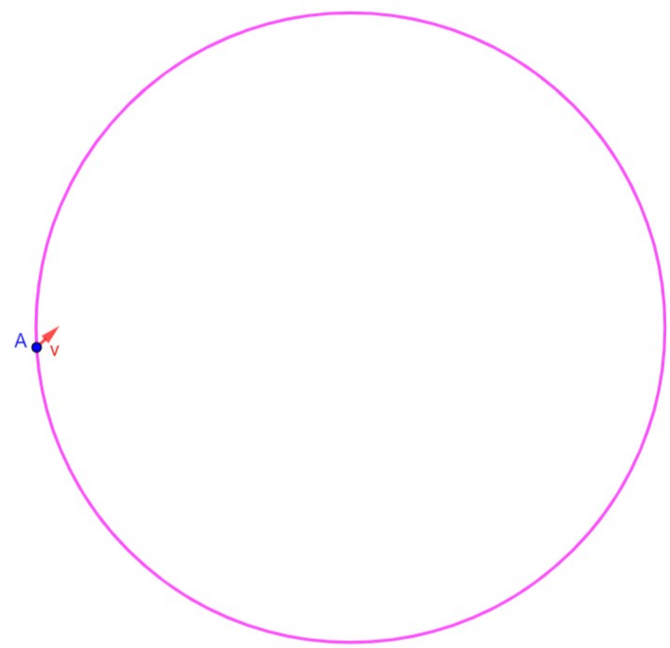


If S is a circle, then S is a simple closed curve that exists in a 2-dimensional subspace of $\mathbb{R}^3$. Since a 2D subspace of $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$, the Jordan curve theorem implies S divides this subspace into two regions, a region interior to S and a region exterior to S.

2. Pick a direction, that is, a vector, v, that lands you in the region interior to S and construct the ray with origin A eminating in the direction of v. For example,

is not a valid direction, whereas



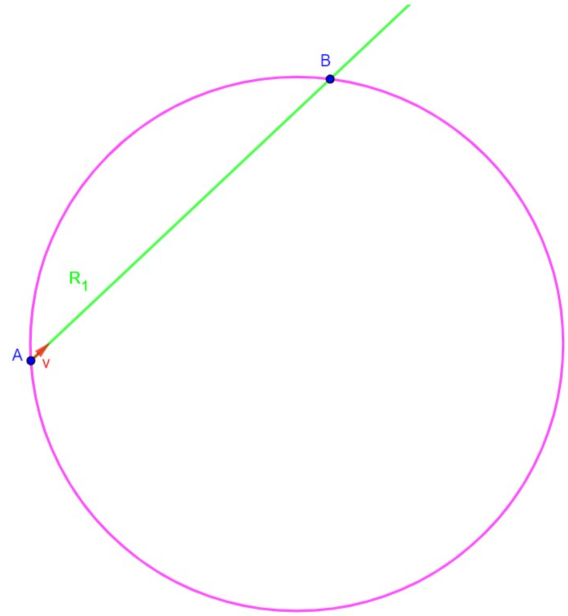


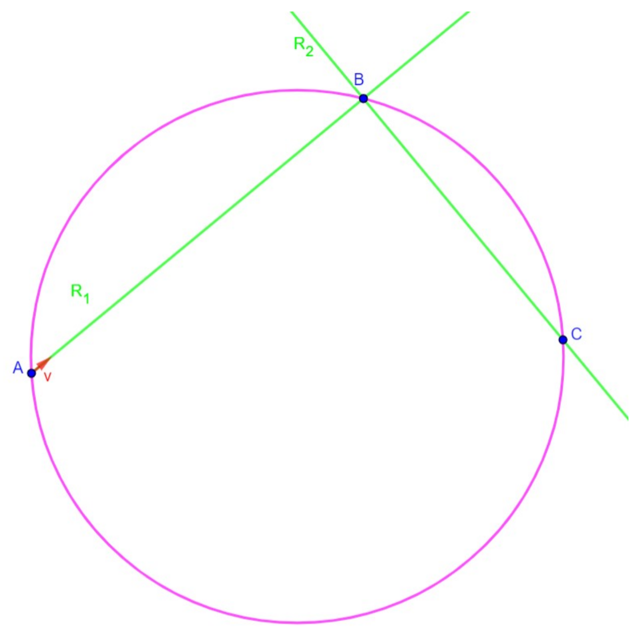
is a valid direction.

We then obtain the ray, R_1, and the intersection point, B.

3. At the point B, construct the line, R_2, passing through B, perpendicular to R_1. Label the point of intersection of R_2 and S that is not equal to B, C.

In the event that there is not another point, C, intersecting S, different from B, than R_2 is tangent to S at B. In this case, AB gives a diameter of S, and $r = \frac{1}{2}length(AB)$ gives the radius of S.

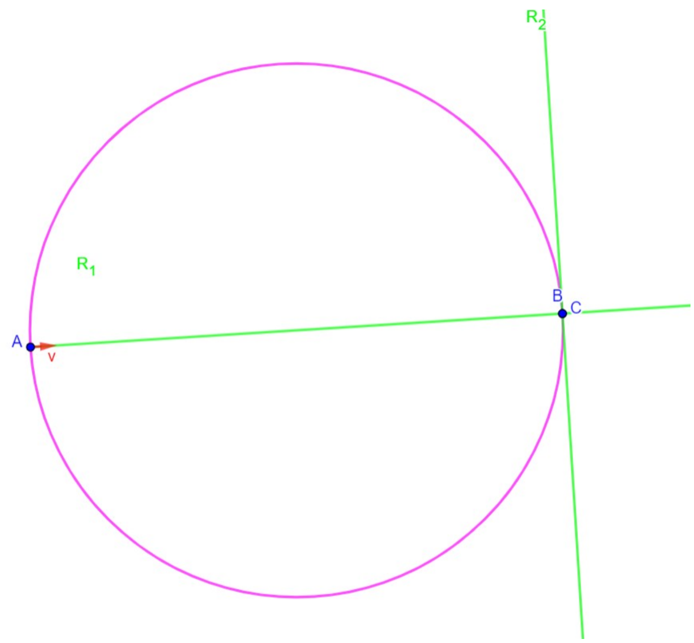


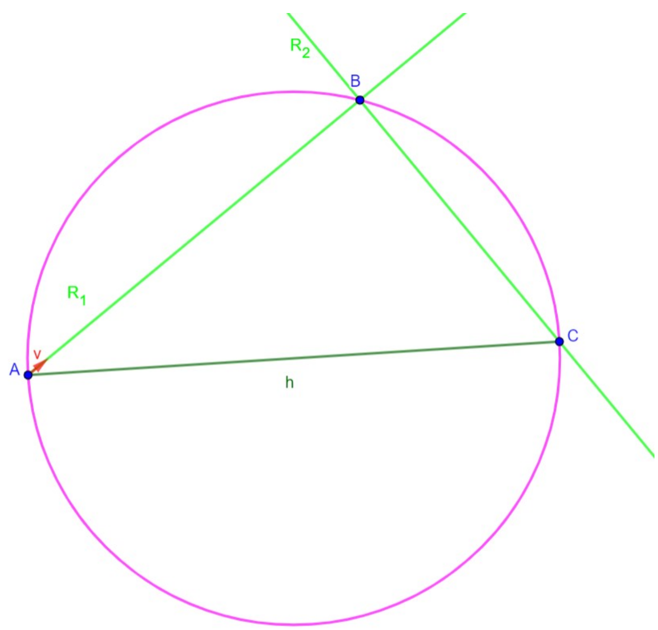


Otherwise, construct the line segment AC. Since the inscribed angle, $\angle ABC$, is $90°$ by construction, the central angle with inscribed points A and C must be $180°$. Hence AC is a diameter of S.

We can then take $r = \frac{1}{2}length(AC)$ as the radius of S.

Another way we could find the radius of our supposed circle, S, is to pick a point, A, on S, construct the line tangent to S and A and then construct the line perpendicular to this tangent line at A. We can then label the second intersection of S with this perpendicular line as B. Since lines tangent to a circle are perpendicular to radial lines at the point of tangency, AB is a diameter of S and we may take $r = \frac{1}{2}length(AB)$ as the radius for S. We know such a point, B, will exist since S is compact and the distance function is continuous.





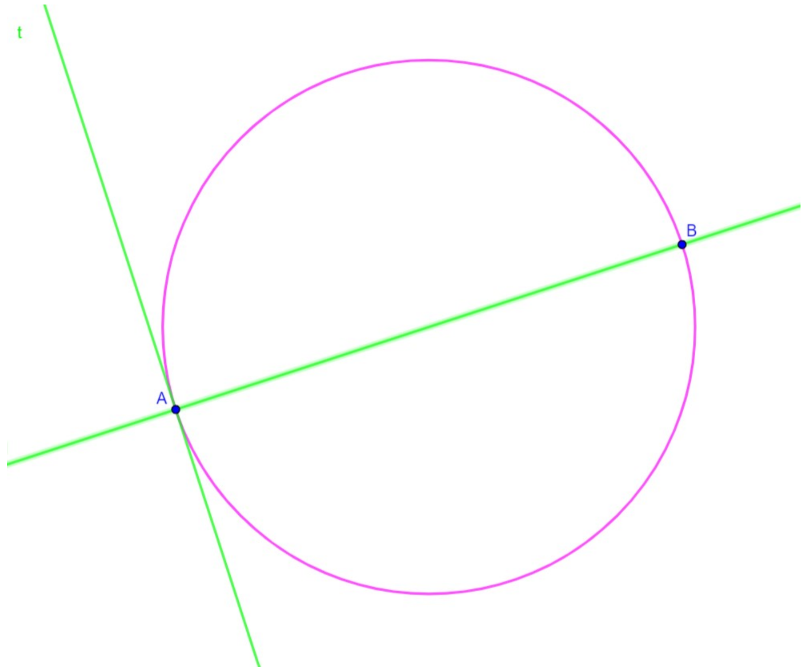
Once we have our radius, r, we then use inscribed, regular n-gons to get approximations of our circle. We can inscribe a regular n-gon in a circle, S, of radius r as follows:

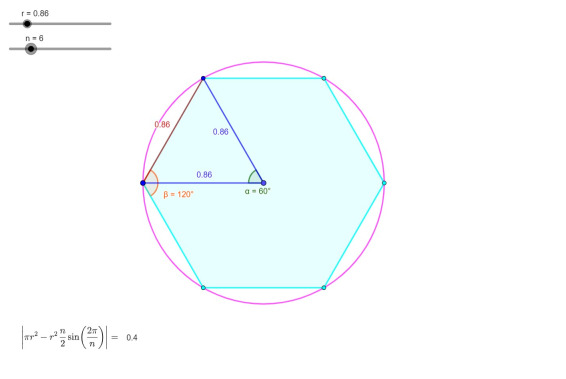
1. Pick a point on S. Using the fact that an interior angle of a regular n-gon has measure $\alpha = ((\frac{n-2}{n}) \cdot 180)^\circ$, from the diameter constructed in obtaining the radius of S, rotate $(\frac{1}{2}(\frac{n-2}{n}) \cdot 180)^\circ$ in either direction.

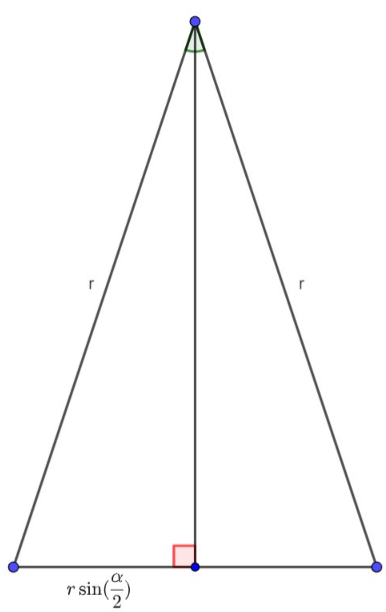
We aim to construct a side of our regular n-gon after performing this angle rotation. Note, a regular n-gon can be divided up into n equal isosceles triangles with the equal side lengths equal to the radius of our circle, and angle between these sides equal to $\frac{360}{n}^\circ$

In an isosceles triangle, we can find the length of the third side according to

So that the length of each side in our regular n-gon is $2r\sin\left(\frac{\pi}{n}\right)$.



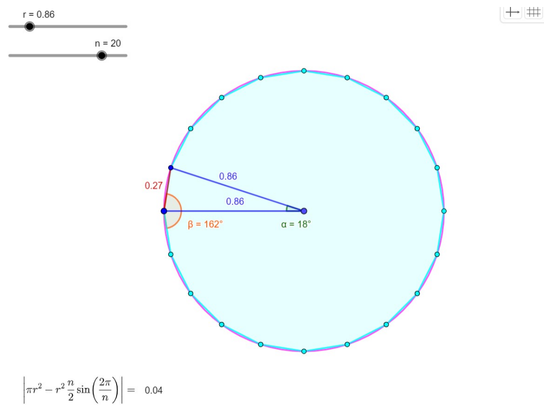




2. After rotating $(\frac{1}{2}(\frac{n-2}{n}) \cdot 180)^\circ$ from the diameter of S, construct a line segment of length $2r\sin\left(\frac{\pi}{n}\right)$. If S is a circle, this line segment should land back on S. If this line segment does not land back on S, you can conclude S is not a circle.

3. If the line segment constructed lands you back on S, rotate $((\frac{n-2}{n})\cdot 180)^\circ$ so that the terminal ray lies in the region interior to S. Construct a line segment of length $2r\sin\left(\frac{\pi}{n}\right)$. If S is a circle, this line segment should land back on S. If this line segment does not land back on S, you can conclude S is not a circle.

4. Repeat step 3 a total of $n-2$ more times. If at any point the line segment construction does not land back on S, or at any point the line segment constructed crosses S before fully constructed, you can conclude S is not a circle. If after completing n total side constructions you do not return to your starting point, you can conclude S is not a circle.



References

[label1] A. Ricotta, "Constant-Diameter Curves," The Mathematical Intelligencer **25**, no. 4, 2003, 4–5.