Homework 1

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This first problem set is to get you comfortable with computing probabilities, counting number of possibilities and extracting valuable information from probability distributions.

Problem 1.1. Smart gambler. You flip a coin 10 times and record the data in the form of head/tails or 1s and 0s
(i) What would be the probability of having 4 heads?
(ii) What would be the probability of HHHTTTHHHT sequence?
(iii) In how many ways can we have 2 head and 8 tails in this experiments.
(iv) Okay, now you got tired of flipping coins and decide to play some dice. You throw die 10 times what is the probability of never landing number 6?
(v) You throw a die 3 times what is the probability of obtaining a combined sum of 7?

Problem 1.2. Binomial, Gaussian and Poisson: The 3 distributions to know before heading to Las Vegas. During lecture we sketched a derivation of Gaussian distribution from binomial distribution: $P_N(n) = \frac{n!}{N!(N-n)!} p^n(1-p)^n$.
(i) I will ask you to make a step by step and a more careful derivation that leads to $P_N(n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{(n-\langle n\rangle)^2/2\sigma_n^2}$. Specifically, you will need Stirling approximation to get rid of factorials. Then you will carry out Taylor expansion around maximum of n and show that keeping second term in Taylor expansion is reasonable by comparing second and third terms. Finally you will normalize your distribution and obtain the desired result.
(ii) In the limit $N \to \infty$ but for very small values of $p \to 0$ such that $\lambda = pN = const$ there is another distribution that better approximates Binomial distribution: $p(x) = \frac{\lambda^k}{k!} e^{-\lambda}$ It is known as Poisson distribution. Poisson distribution is an excellent approximation for probabilities of rare events. Such as, infrequently firing neurons in the brain or rains in the desert. Derive Poisson distribution by taking the limit of $p \to 0$ in binomial distribution.
(iii) Using python and matplotlib on the same graph plot binomial probability distribution against Gaussian and Poisson distributions for different values of N=(10,100,1000,10000) and p=0.5 (iv) Do the same plot but for some small values of $p = 0.001$ confirming our statemets made above.

Problem 1.3. Getting cozy with the Gaussians. $P_G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(i) Compute the mean, showing that: $\langle x \rangle = \int_{-\infty}^{+\infty} dx x P_G(x) = \mu$

(ii) Compute the variance showing that: $\langle (x - \langle x \rangle)^2 \rangle = \int_{-\infty}^{+\infty} dx (x-\mu)^2 P_G(x) = \sigma^2$

(iii) Compute the moment generating function defined as: $G(k) = \int dx P_G(x) e^{kx} = e^{k\mu + \frac{\sigma^2 k^2}{2}}$
[Hint, a classic trick when dealing with Gaussian integrals is completing the square. You may have seen this in classes of quantum mechanics when showing that Gaussian distribution of momenta implies Gaussian distribution of position.]

(iv) Why is it called moment generating function? Expand in Taylor series of k and see why.

(iv) Compute the cumulant generating function defined as lnG(k):
Why is it called cumulant generating function? Make a taylor expansion in terms of k and see how many terms (cumulants) it produces.

Problem 1.4. The Monty hall and two kittens (i) Work out the Monty hall problem of 3 doors and 1 prize by making use of conditional probabilities and Bayes theorem.

(ii) In Monty hall problem when keeping single prize but adding more empty doors how would the probability of winning upon swithcing change?

(ii) You are working at a cat shelter in your spare time. You get informed that two kittens are collected from the streets of Ames. You are informed one of them is male. What is the probability that both kittens are males?

Problem 1.5. The most powerful tool of statistical mechanician. In the class we have obtained a powerful result that probability distributions of sums of random variables ($X = \sum_i X_i$) follow a general large deviation principle: $P_N(X) \sim e^{-NI(X)}$

(i) I know this is getting repetitive but please do make a plot of I(x) for p=1/2 case for and show that $I(x)$ can be Taylor expanded around minima up to second term justifying the use of gaussian distributions.

(ii) Comment on the reliability of measurement of an average of $\langle X \rangle = \frac{1}{N}\sum_i X_i$ with increasing number of measurements N. How fast our confidence of having obtained an accurate mean should increase with increasing N?

(iii) Imagine a 1D lattice of spins in the absence of magnetic field. Each site on a latice has equal probability to have $s_i = +1/2$ and $s_i = -1/2$. Hence on average we should expect no magnetization $\langle s \rangle = 0$. But at the same time we do know that because of thermal fluctuations spins can randomly flip up and down. Let us apply large deviation principle to quantify what is the likelihood of having spontaneous non zero magnetization due to thermal fluctuations: $s - \langle s \rangle = \delta = 10^{-6}$, e.g compute $P(\langle s \rangle + \delta s)/P(s\rangle)$. Consider cases of number of spins $N = 10^{23}$ and $N = 1000$.