Electric Field of a Charged Sphere

Introduction

COMSOL Multiphysics is a finite element package that can be used to solve a partial differential equation such as for example Poisson's equation as we discussed in EMT. For a scalar field $\psi(\mathbf{r})$ Poisson's equation looks like:

$$\nabla^2\psi = \nabla\cdot\nabla\psi = f(\vec{r}) \qquad [1]$$

f(**r**) represents a density of field sources and depends on the position vector r. Our version of COMSOL can solve one, two, or three dimensional problems. Poisson's equation is a partial differential equation and can be solved for some geometries by analytical techniques. In EMT and also in mechanics you might have studied already solutions of Poisson's equation for various geometries that have high symmetry. More complex problems can be solved by a computer. Note that the differential equation relates the 2nd derivative of the ψ field to the source field f. In Cartesian coordinates the one-dimensional Poisson equation becomes:

$$\frac{\partial^2\psi(x)}{\partial x^2} = f(x) \qquad [2]$$

Note that the special case where f(x) is equal to zero turns equation (2) into Laplace's equation, i.e.

$$\frac{\partial^2\psi(x)}{\partial x^2} = 0 \qquad [2b]$$

The two most frequently used numerical methods to solve Poisson's or Laplace's equation are:

1. The finite difference method
2. The finite element method

Comsol Multiphysics, uses the finite element method.

Minimum Energy Principles in Electrostatics.

It can be shown that Laplace's and Poisson's equation are satisfied when the total energy in the solution's area is minimized. This section will make this statement plausible for both cases. To keep the math simple we will restrict ourselves here to the one-dimensional case. For the 2D and the 3D case just double and triple the terms. Assume a one dimensional problem and assume that Poisson's equation is given by equation (2). Furthermore assume that u(x) is an approximation for

ψ(x), the solution of Poisson's equation. u(x) does not need to be a solution of Poisson's equation and differs from ψ(x), i.e.:

$$u(x)=\psi(x)+e(x)$$

where e(x) is the error in the approximate solution u(x).

For Laplace's equation the proof is rather straight forward. In the absence of charge the energy of the system is given by the integral of $\frac{1}{2}\varepsilon_o\left(\frac{du}{dx}\right)^2$ over the volume. For a 1D system without charge the energy F is given by:

$$F=\frac{1}{2}\varepsilon_o\int_0^1\left(\frac{du}{dx}\right)^2dx=\frac{1}{2}\varepsilon_o\left(\int_0^1\left(\frac{d\psi}{dx}\right)^2dx+2\int_0^1\frac{d\psi}{dx}\frac{de}{dx}dx+\int_0^1\left(\frac{de}{dx}\right)^2dx\right) \quad [3]$$

The 2nd term is zero because (notice that e=0 at the boundaries because the boundary conditions provide the exact value of ψ at x=0 and x=1):

$$\int_0^1\frac{d\psi}{dx}\frac{de}{dx}dx=\int_0^1\left[\frac{d}{dx}\left(e\frac{d\psi}{dx}\right)-e\frac{d^2\psi}{dx^2}\right]dx=\int_0^1\left[\frac{d}{dx}\left(e\frac{d\psi}{dx}\right)-e\frac{d^2\psi}{dx^2}\right]dx= \quad [4]$$
$$e\frac{d\psi}{dx}\Big|_0^1+\int_0^1e\cdot 0dx=0$$

For the 3rd step we used the Laplace's equation, i.e. $\frac{d^2\psi}{dx^2}=0$. Substituting equation (4) into

equation (3) gives for the electrostatic energy of our approximation u(x) [1]:

$$F=\frac{1}{2}\varepsilon_o\int_0^1\left(\frac{du}{dx}\right)^2dx=\frac{1}{2}\varepsilon_o\left(\int_0^1\left(\frac{d\psi}{dx}\right)^2dx+\int_0^1\left(\frac{du}{dx}-\frac{d\psi}{dx}\right)^2dx\right) \quad [5]$$

Note that both terms on the right hand side are positive. The first term represents the total energy of the system for ψ(x), the solution of Laplace's equation. The 2nd term will be small if du/dx (read approximated electric field) is close to dψ/dx (read electric field that is a solution of Laplace's equation). So the total energy of u(x) is close to the total energy of ψ(x) but always

larger [1]. So minimizing the total energy provides in a method to find u(x) whose electric field distribution is similar to the ψ(x). Because of the uniqueness theorem u(x)=ψ(x) if e(x)=0.

In the case space charge is present, this statement is still true but the proof is a little more complicated. The electrostatic energy is no longer given by equation (3). It is now necessary to subtract the contributions of the fixed charges, $\rho_s\psi$ from the field energy density [2]:

$$F=\int_0^1\left[\frac{1}{2}\varepsilon_o\left(\frac{du}{dx}\right)^2-\rho_s u\right]dx=\frac{1}{2}\varepsilon_o\left(\int_0^1\left(\frac{d\psi}{dx}\right)^2dx+2\int_0^1\frac{d\psi}{dx}\frac{de}{dx}dx+\int_0^1\left(\frac{de}{dx}\right)^2dx\right)-\int_0^1\rho_s u dx \qquad [6]$$

The 2nd term on the left hand side is no longer zero but can be simplified:

$$\begin{aligned}&\int_0^1\frac{d\psi}{dx}\frac{de}{dx}dx=\int_0^1\left[\frac{d}{dx}\left(e\frac{d\psi}{dx}\right)-e\frac{d^2\psi}{dx^2}\right]dx=\int_0^1\left[\frac{d}{dx}\left(e\frac{d\psi}{dx}\right)-e\frac{d^2\psi}{dx^2}\right]dx=\\&e\frac{d\psi}{dx}\Big|_0^1+\int_0^1 e\cdot\frac{\rho}{\varepsilon_o}dx=\int_0^1 e\cdot\frac{\rho}{\varepsilon_o}dx\end{aligned} \qquad [7]$$

In the 3rd step we used the information contained in Poisson's equation, i.e. $\frac{d^2\psi}{dx^2}=\frac{\rho}{\varepsilon_o}$.

Substituting equation (7) into equation (6) provides us with the following expression for the electrostatic energy:

$$F=\int_0^1\left[\frac{1}{2}\varepsilon_o\left(\frac{du}{dx}\right)^2-\rho_s u\right]dx=\frac{1}{2}\varepsilon_o\left(\int_0^1\left(\frac{d\psi}{dx}\right)^2dx+\int_0^1\left(\frac{du}{dx}-\frac{d\psi}{dx}\right)^2dx\right)-\int_0^1(u-e)\rho_s dx \qquad [8]$$

This can be further simplified using u-e=ψ:

$$\begin{aligned}&F=\frac{1}{2}\varepsilon_o\left(\int_0^1\left(\frac{d\psi}{dx}\right)^2dx+\int_0^1\left(\frac{du}{dx}-\frac{d\psi}{dx}\right)^2dx\right)-\int_0^1\psi\rho_s dx=\\&\frac{1}{2}\varepsilon_o\int_0^1\left(\frac{d\psi}{dx}\right)^2dx-\int_0^1\psi\rho_s dx+\frac{1}{2}\varepsilon_o\int_0^1\left(\frac{du}{dx}-\frac{d\psi}{dx}\right)^2dx\end{aligned} \qquad [9]$$

The first two terms on the right hand side give the electrostatic energy of the solution of Poisson's equation. F is minimum if du/dx (read the electric field of the approximated solution) is equal to dψ/dx (electric field of the solution of Poisson's equation). Or in other words the solution of Poisson's equation has the lowest electrostatic energy.

So summarizing, the above shows that the spatial distribution of the electric field given by a solution of Poisson's or Laplace's equations correspond to a state of minimum field energy integrated over the system's volume. We call this the minimum energy principle and it will be shown in the next section how this principle can be used to find the solution of Poisson's or Laplace's equations.

Finite Element Method.

To find a good approximation for the solution of Poisson's equation we will:

1. Parameterize u(x), i.e. $u(x) \rightarrow u_{u1,u2,..u(n-1)}(x)$
2. Derive an expression for the total energy F of the system in terms of parameters u_1, u_2, …u_{n-1}.
3. Find values for the parameter that will minimize the system energy F.

Parameterize u(x): To do so we divide the space up in small intervals (1D), surface areas (2D), or volumes (3D), called finite elements. For each element the solution of Poisson's equation is approximated by a polynomial function; for example for element i the function u is given by $u_i(x)$. For 1D-problems this polynomial function could be a simple straight line, a parabola, or a more complicated polynomial function. For 2D-problems this polynomial function could be a simple plane, a 2-dimensional parabola, or a more complicated curved surface, etc. To keep the math simple we will limit ourselves here to a 1D problem with the following boundary conditions, $\psi=0$ at $x=0$ and $\psi=1$ at $x=1$. Furthermore we will assume that the space charge is zero and we will approximate the solution of Laplace's equation in each element by a first order polynomial.

We divide the space between 0 and 1 up in N finite elements and approximate in each finite element ψ by a linear function u i.e.

$$u=b_1x+a_1 \text{ for } 0<x<x_1 \quad [10]$$

$$u=b_2x+a_2 \text{ for } x1<x<x_2 \quad [11]$$

…….

$$u=b_nx+a_n \text{ for } x_{n-1}<x<x_n \quad [12]$$

Note that a_1, a_2, ….a_n, and b_1, b_2, …b_n are unknowns that still have to be determined. The objective is to determine values for a_1, a_2, ….a_n, and b_1, b_2, …b_n so the piece wise linear function u is as close as possible to the real solution of Poisson's equation. From the boundary condition at $x=0$ we can conclude that $a_1=0$. Since ψ is continuous across the space, also u should be continuous. If we assume that u_1 is the approximation of ψ at the boundary between element 1 and 2 (i.e. at position x_1), and u_2 is the approximation of ψ at the boundary between element 2 and

3 (i.e at position x_2), etc., we can determine good estimates for a_1, a_2, ….a_n, and b_1, b_2, …b_n from good estimates of ψ at the boundaries between the elements, i.e. from u_1, u_2, … u_n. This leads to the following equations:

$$b_1 = \frac{u_1}{x_1}$$

$$a_2 = u_1 - \frac{u_2 - u_1}{x_2 - x_1} x_1 \qquad b_2 = \frac{u_2 - u_1}{x_2 - x_1}$$

.....

$$a_n = u_{n-1} - \frac{u_n - u_{n-1}}{x_n - x_{n-1}} x_{n-1} \qquad b_n = \frac{u_n - u_{n-1}}{x_n - x_{n-1}} = \frac{1 - u_{n-1}}{1 - x_{n-1}} \qquad [13]$$

Where in the last equation the boundary condition at x=1 is used and in the first expression the boundary condition at x=0 is applied. In addition to using the boundary conditions, we also divided the space up in small intervals (defined a mesh) and approximated $\psi(x)$ by a piece-wise linear function using the best estimations of u(x) at the boundaries of each interval as parameters.

Find an expression for F: An expression for the total energy of the system can be found from equation (5). The integral in equation (5) can be split in n integrals, one for each finite element, and then be evaluated using the equations (13). For this particular 1D case F is given by the following expression:

$$F = \int_0^{x_1} b_1^2 dx + \int_{x_1}^{x_2} b_2^2 dx + ... + \int_{x_{n-1}}^{1} b_n^2 dx = b_1^2 x_1 + b_2^2 (x_2 - x_1) + ... + b_n^2 (1 - x_{n-1}) = \frac{u_1^2}{x_1} + \frac{(u_2 - u_1)^2}{x_2 - x_1} + ... + \frac{(1 - u_{n-1})^2}{1 - x_{n-1}} \qquad [14]$$

Minimize F: The best estimate for $\psi(x)$ can be found by minimizing F, i.e. taking the partial derivatives towards u_1, u_2, ….. u_{n-1}, setting them equal to zero, and solving for u_1, u_2, …,u_{n-1}. This gives the following set of linear equations:

$$\frac{\partial F}{\partial u_1} = 0 \Leftrightarrow 2\frac{u_1}{x_1} - 2\frac{u_2 - u_1}{x_2 - x_1} = 0$$
$$\frac{\partial F}{\partial u_2} = 0 \Leftrightarrow 2\frac{u_2 - u_1}{x_2 - x_1} - 2\frac{u_3 - u_2}{x_3 - x_2} = 0 \qquad [15]$$
..
$$\frac{\partial F}{\partial u_{n-1}} = 0 \Leftrightarrow 2\frac{u_{n-1} - u_{n-2}}{x_{n-1} - x_{n-2}} - 2\frac{1 - u_{n-1}}{1 - x_{n-1}} = 0$$

So n-1 linear equations, which can be solved using Linear Algebra and matrices. How to solve such system of linear equation see Introductions to Mathematical Physics (PHYS 3320) or the Linear Algebra course of the Mathematics department (MATH 3377).

Above example assumes there is no space charge. If space charge is present equation (14) needs to be modified, but the math stays more or less the same.

More details on the Finite Element method can be found in references [3] and [4]. More details on how Comsol implemented the FEM in their software package can be found in reference [5].

Modeling Instructions.

In this exercise you will use Comsol to calculate the electric field and electric potential in and around a solid charged sphere. Open COMSOL Multiphysics from the Desktop by double clicking the Comsol Multiplhysics icon.

Model Wizard

As Comsol can be used to simulate various physics phenomena that are all described by a different differential equation, we first have to tell Comsol that we want to model an EMT stationary problem. To do so follow these instructions:

1) Find the **Model Wizard** window.
2) Click the **Next** arrow.
3) In the **Add Physics** tree, expand the **AC/DC** tree and right click on **Electrostatics (es)**.
4) Select **Add Physics**.
5) Click the **Next** arrow.
6) Under the **Preset Studies** tree, select **Stationary** (we are interested in finding the stationary solutions, not in solving transient effects).
7) Click the **Checkered Flag**.

Geometry

Next step is to define the geometry and geometrical objects of your simulation.

1) In the **Model Builder** window, under **Model 1** click **Geometry 1**.
2) In the **Geometry** settings window, locate the **Units** section.
3) From the **Angular unit** list, choose **Radians**.

Now we first define the space symmetry. This problem has definitely spherical symmetry. The geometry of this problem is defined by three concentric spheres. First of all a sphere with a small radius that defines the space taken by the charged spherical object. Then we need a sphere with a larger diameter which indicates the area immediately around the solid charged object. The program will calculate the electric field and electric potential in the space defined by the 2nd sphere. As in electrostatics one often chooses as boundary condition that the electric potential is zero at an infinite distance away from the charged object, this 2nd sphere ideally has an infinite radius. However this would mean that the number of elements will become infinite which means that it will take an infinite amount of calculation power to solve the problem. As we cannot define an infinite numbers of elements, we define a third sphere with a little larger radius. The space between the third and the 2nd sphere will be an Infinite Element Domain which condenses the distance from the 2nd sphere's surface to infinity in a finite space. So the surface of the 3rd sphere corresponds with infinity.

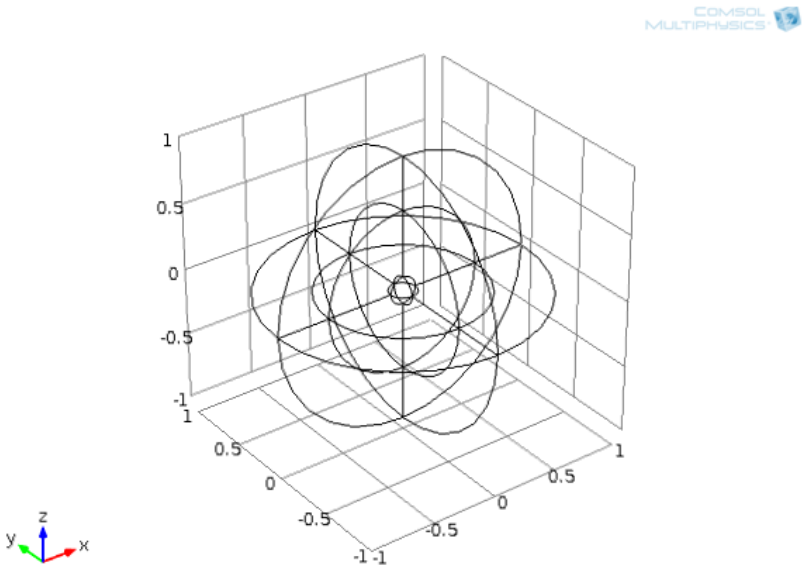
In the **Model Builder** window, under **Model 1** right-click **Geometry 1** and choose **Sphere**.

4) In the **Sphere** settings window, locate the **Size and Shape** section.
5) Click to expand the **Layers** section. In the table, enter the following settings:

Layer Name	**Thickness (m)**
Layer 1	.4
Layer 2	.5
Layer 3	.1

So the solid charged spherical object has a radius of 0.1 m, the immediate area around the object is a sphere with a radius of 0.6 m, and the infinity domain is defined as the area between the 0.6 meter radius sphere and the 1.0 m radius sphere.

6) Click the **Build All** button. This creates the three concentric spheres.
7) In the **Graphics** window, select the **Wireframe** button that is just left of the **Camera** button on the toolbar.

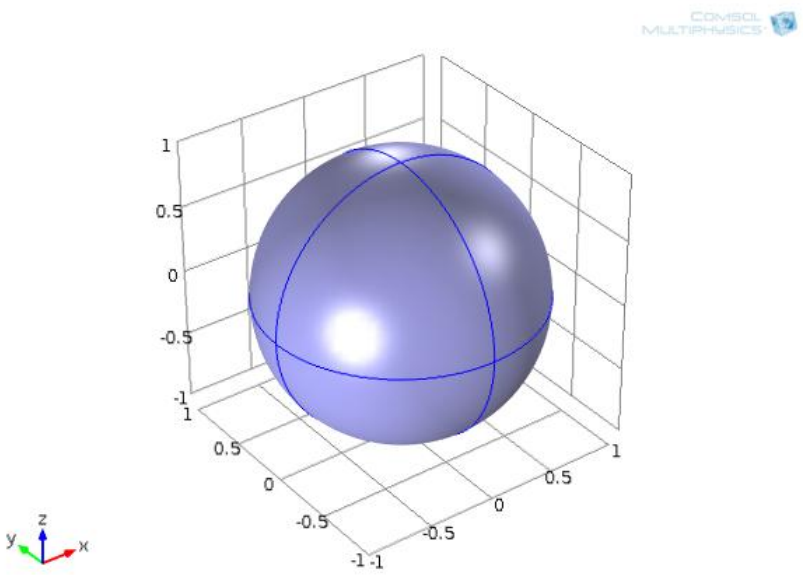


Definitions

Now we have to assign each spherical surface of our object. To select an element of the object, left click then right click in the graphics window. Left clicking it will make the object's segment orange. If you right click an orange segment, you will select it. Selected segments turn purple.

Infinite Element Domain 1

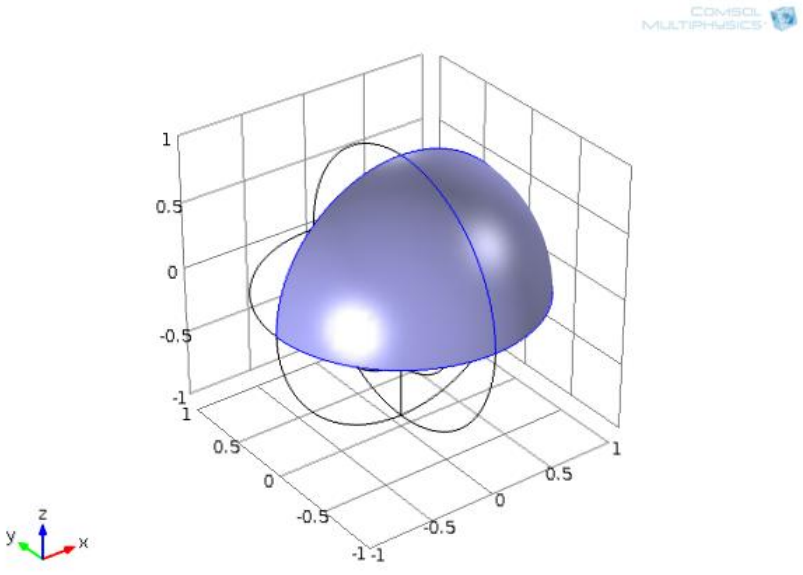
1) Right-click **Definitions** and choose **Infinite Element Domain**.
2) In the **Graphics** window, select Domains 1-4, 10, 11, 14, and 17. This action defines the area between the two largest spheres to be the infinite element domain.
3) Locate the **Geometry** section. From the **Type** list, choose **Spherical**.



View 1

Suppress some domains to get a better view when setting up the physics and reviewing the meshed results.

1) In the **Model Builder** window, under **Model 1,** expand **Definitions**, right-click **View 1** and choose **Hide Geometric Entities**.
2) Select Domains 2, 6, 11, and 13 only.



Materials

We will assume that the solid charged sphere has the same dielectric constant as vacuum. Also the space immediately around the solid charged object and the infinite element domain are assumed

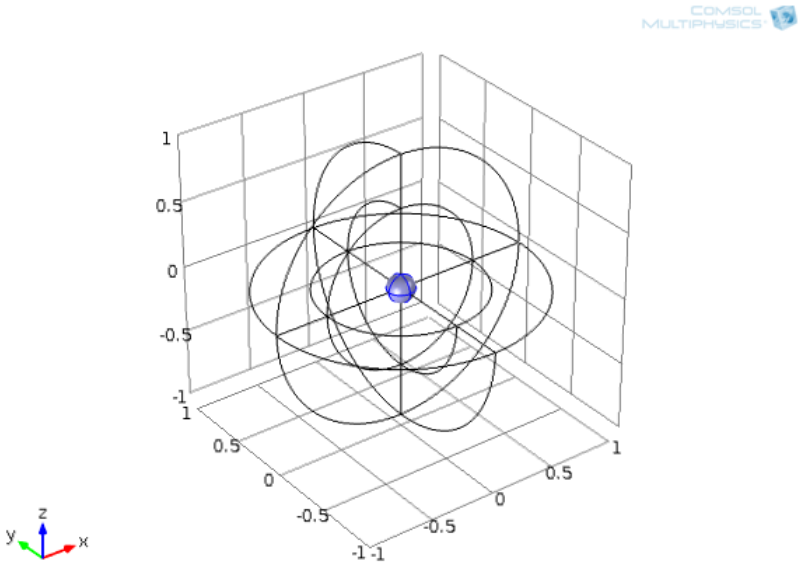
to have a dielectric constant equal to that of vacuum. So we only have one material. To define this material do the following:

1) In the **Model Builder** window, under **Model 1**, right-click **Materials** and choose **Material**.
2) Locate the **Geometric Entity** selection and select all domains.
3) Locate the **Material Contents** section. For the **Relative permittivity** property, type 1 in the **Value** column.

Electrostatics

Space Charge Density
Here we will set the core to have a volume charge density.

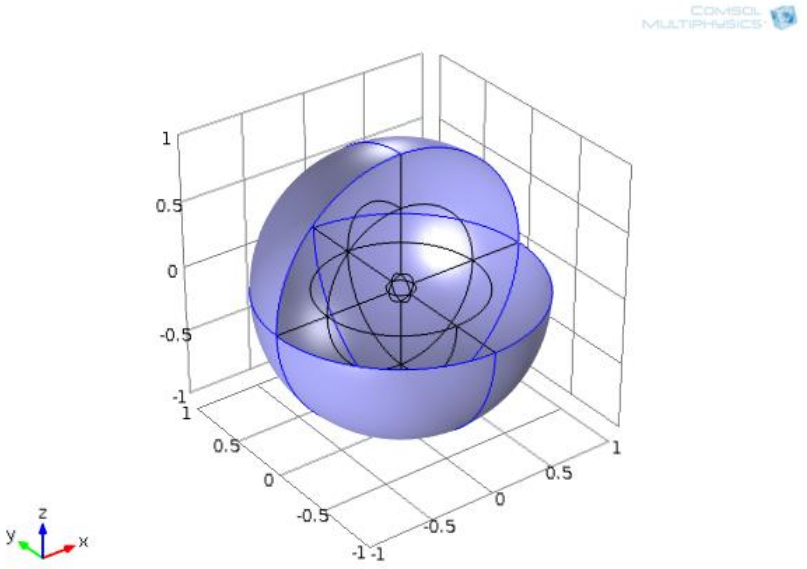
1) In the **Model Builder** window, under **Model 1**, right-click **Electrostatics(es)** and choose **Space Charge Density**.
2) Select Domain 9 only.
3) Locate the **Space Charge Density** section. Change the **Space Charge Density** from 0 to 1 C/m^3.



Electric Potential
Since this system is homogeneous through space, our boundary conditions tell us that the Electric Potential at r = ∞ is zero. So we must tell this to the program.

1) In the **Model Builder** window, under **Model 1**, right-click **Electrostatics(es)** and choose **Electric Potential**. Note that the electric potential default value is zero.
2) Select Boundaries 5-8, 24, 25, 35, and 46.

This action defines the outer surface of the infinite domain to have zero electric potential. This is the same boundary conditions that we used in class when deriving Coulomb's law for a point charge.

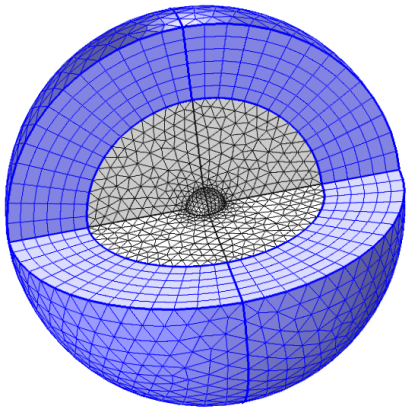


Mesh

Last of all we have to define the intervals, i.e. the mesh. Fortunately the program has an automatic mesh generator. This mesh generator will choose a smaller mesh close to the solid object where we want more detail.

1) In the **Model Builder** window, under **Model 1**, left-click **Mesh**.
2) Select **User-controlled mesh.**
3) In the Model Builder window right click the Mesh node and add a Free Tetrahedral node to the list.
4) Click the **Free Tetrahedral node** and add the following domains manually: 5, 6, 7, 8, 9, 12, 13, 15, and 16. This is the solid charged sphere in the center and the sphere around it.
5) Click the **Build Selected** button.
6) In the Model Builder window right click the Mesh node and add a **Swept Node**.
7) Select the Swept node and add the remaining domains to the swept node.

8) Click the **Build All** button. The program will automatically define a mesh. Note that the mesh is finer near the solid charged sphere in the center. .



Study

1) In the **Model Builder** window, under **Model 1**, right-click **Study**.
2) Select **Compute**. This will start the computational process. Depending on how fine you chose the mesh, the calculation will take more time. The progress of the calculation is reported in the text window on the right side of your screen under the graphics window. There is also a graph available that shows estimated error as a function of the iteration step. The calculation should not take more than 30 seconds on the computers in RFM3223.

Results

Now that the study has been computed, we need to tell the software how to present the calculated data. For this particular case we would like to see a 3D plot of the magnitude of the electric field and the magnitude of the electric potential. Furthermore it might be interesting to see a 1D-line plot that compares the electric field and potential relative to the distance from the center of the sphere.

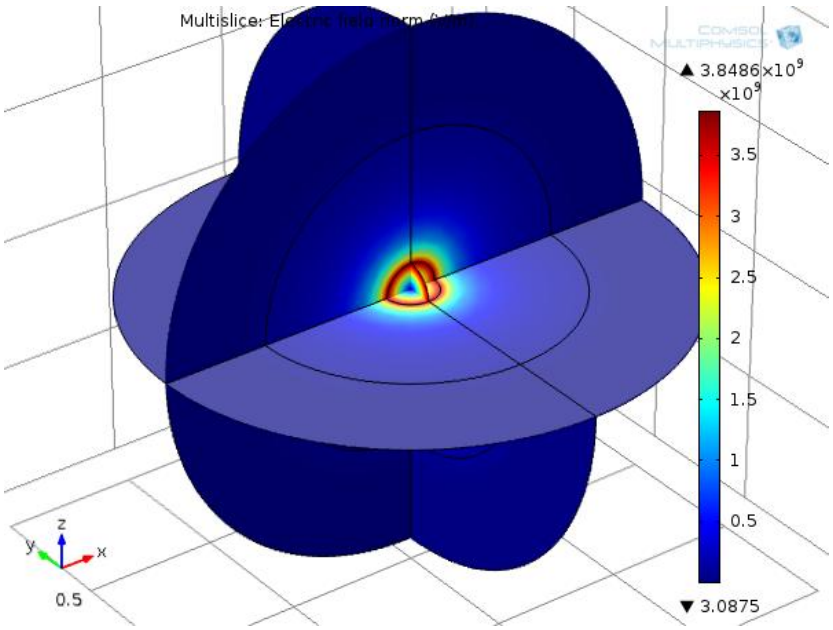
3D Electric Field

1) In the **Model Builder** window, right-click on **Results.**
2) Select **3D Plot Goup**.
3) A new 3D plot, named **3D Plot Group 2**, has been created under the **Results** tree.
4) Right-click it and rename it **3D Electric Field**.
5) Right-click on **3D Electric Field** and put the cursor over **More Plots**.
6) Select **Multislice**.
7) In the **Multislice** window, locate the **Expression** section.

8) Click the **Replace** button and select **Electrostatics**>**Electric**>**Electric field norm (es.normE)**.
9) Click the **Plot** button.
10) Once the Electric Field is plotted, click the **Zoom In** button.

Question1: Describe the plot in a few sentences. Where is the magnitude of the electric field the largest? Where is it zero? Is this in agreement with your expectations?

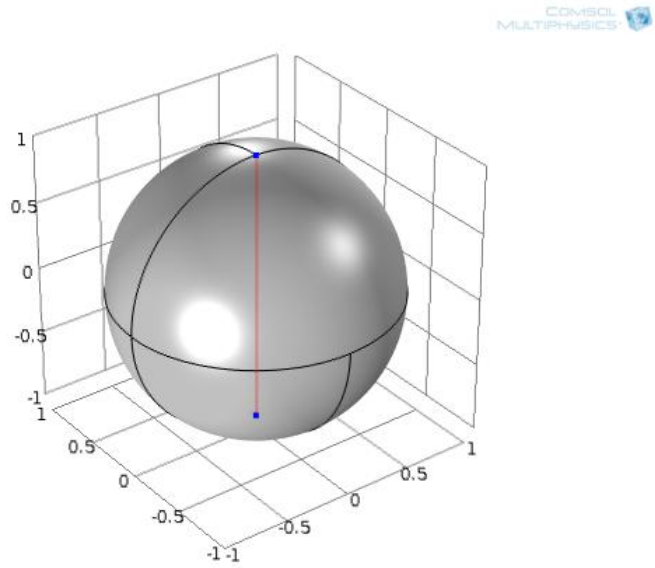
Question2: Now try to make a second 3-D plot of the electric potential. Describe the electric potential plot. Where is the electric potential maximum? Where is the potential minimum? Does this make sense?



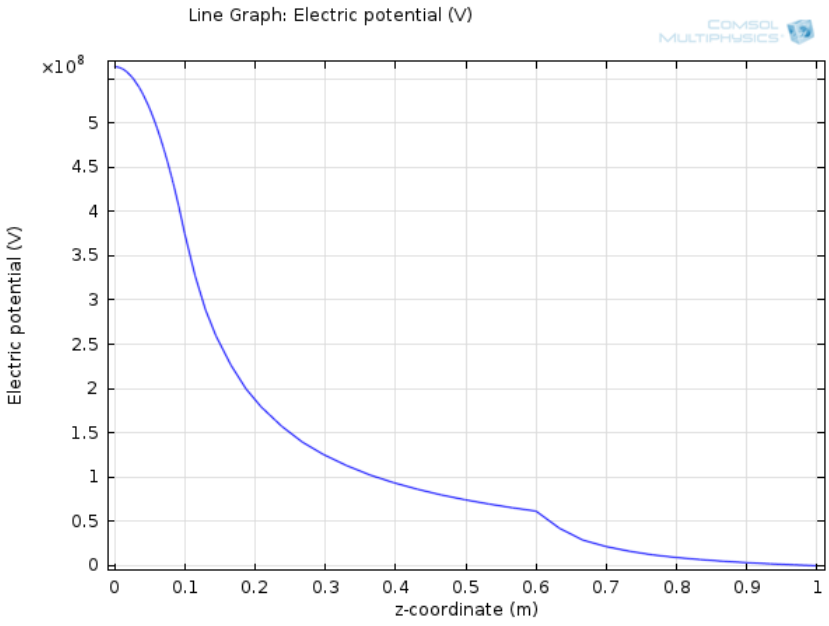
1D Electric Field v Electric Potential

1) In the **Model Builder** window, right-click on **Results.**
2) Select **1D Plot Group**.
3) A new 1D plot, named **1D Plot Group 1**, has been created under the **Results** tree.
4) Right-click it and rename it **Electric Field/Potential**.
5) Right-click on **Electric Field/Potential** and select **Line Graph**.
6) A **Line Graph 1** has been created under **Electric Field/Potential**.
7) Right-click and rename it **Potential**.
8) In the **Potential** window, locate the **x-Axis Data** section.
9) Change the **Parameter** from **Arc Length** to **Expression**.

10) Under **Expression** replace the V with a z.
11) Press the **Define Cut Line** button at the top of the window.

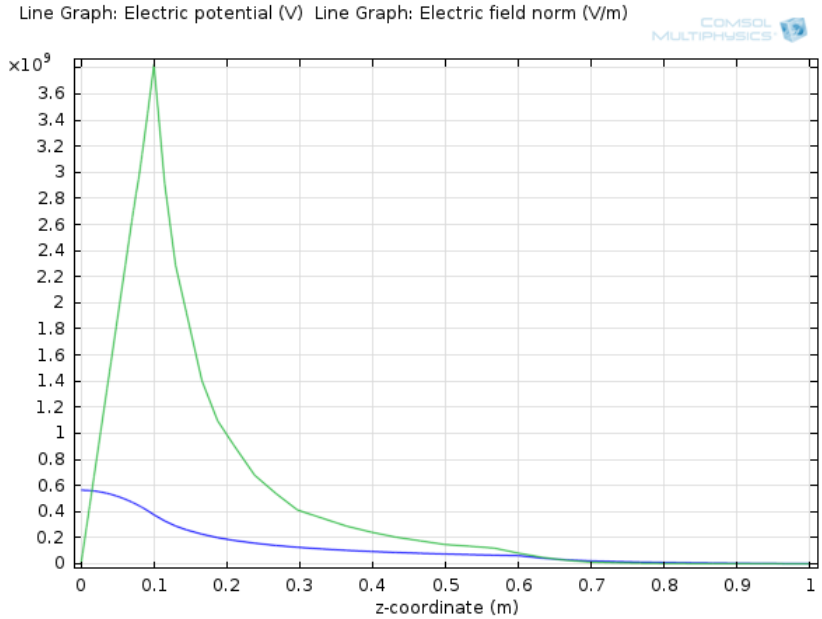


12) In the **Model Builder** window under **Results**, under **Data Sets**, select **Cut Line 3D 1**.
13) In the **Cut Line 3D 1** window, locate the **Line Data** section.
14) Change **Point 1** from -1 to 0 in the **z** column.
15) Click the **Plot** button.



16) Right-click on **Electric Field/Potential** and select **Line Graph** again.
17) A **Line Graph 2** has been created under **Electric Field/Potential**.
18) Right-click and rename it **Field**.

19) In the **Field** window, locate the **Data** section.
20) Change the **Data set** from **From Parent** to **Cut Line 3D 1**.
21) Now locate the **y-Axis Data** section.
22) Click the **Replace** button and select **Electrostatics**>**Electric**>**Electric field norm (es.normE)**.
23) Now locate the **x-Axis Data** section.
24) Change the **Parameter** from **Arc Length** to **Expression**.
25) Under **Expression** replace the V with a z.
26) Click the **Plot** button.



Question3: What happens at r = ∞?

Question4: What does the electric field and potential equation look like inside of the solid charged sphere? Is this in agreement with Gauss' law? Explain.

Question5: What if the core was copper? Change the model so that it has a copper solid object. Repeat the calculation. Plot the electric potential and the magnitude of the electric field as a function of the distance to the center of the sphere. Explain your result.

Question6: What if the charged object wasn't solid copper but consisted of a charged copper shell? Change the model to reflect this and calculate the magnitude of the electric field and the electric potential as a function of the distance to the center of the charged shell. Explain your results.

References:

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