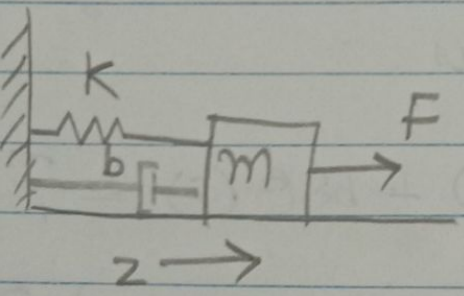
Design study D : Mass Spring Damper

Rita Abani 19244

Homework Problems:-

D.2 Kinetic Energy



To find an expression for the kinetic energy of the system, let's first find an expression for 'z'.

The kinetic enegy of the system is mainly dependent on the velocity of tha mass.

$$\therefore K = \frac{1}{2} m \dot{z}^2$$

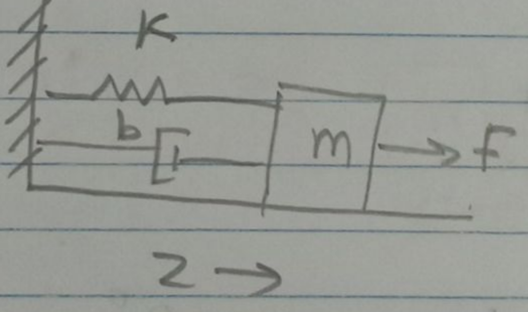
D.3 a) Potential Energy for the system

Potential energy, 'U' is the energy stored in the spring by virtue of its position.

$$U = \frac{1}{2} K (\text{displacement})^2$$

$$= \frac{1}{2} K z^2$$

b) Generalized coordinates are basically the minimum set of configuration variables required to define our system 'D'



As can be seen, we have only one generalized coordinate,

$$q = z$$

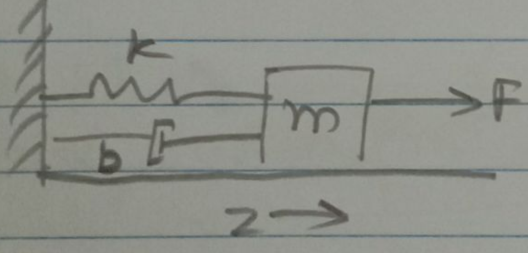
c) Generalized forces and damping forces.

Generalized forces are non conservative forces acting along each generalized coordinate.

here, force 'F' acts along 'z'

so, $\tau = (F)$

Damping forces are the forces applied due to dampers, from the diagram it is clear that the damping force acting along 'z' is due to 'b'



Damping force $= -B\dot{q}$

$= -b\dot{z}$

'$-$' because it acts in the opposite direction of generalized force.

d) For our system,

Lagrangian, $\mathcal{L} = K - U$

where K = Kinetic energy

U = Potential energy as found in D.2 and D.3 a) respectively.

$$L = K - U$$
$$= \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2$$

The Euler-Lagrange equation is.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$ where $q = z$ is our generalized coordinate.

$$\frac{\partial L}{\partial z} = -kz$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

$\therefore$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = m\ddot{z} + kz = 0$$

e) Code has been uploaded to my Git-Hub repository

D.4) a) Equilibria of the system.

We have the equation

$$m\frac{d^2z}{dt} + kz = F - b\dot{z} \rightarrow (1)$$

or $$m\frac{d^2z}{dt} + kz = F - b\frac{dz}{dt}$$

Defining $x = (z, \dot{z})^T$ and $u = F$, we get.

$$\dot{x} = \begin{pmatrix} \dot{z} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \frac{F}{m} - \frac{b\dot{z}}{m} - \frac{kz}{m} \end{pmatrix}$$

$$\triangleq f(x, u).$$

using

$$\left\{\begin{aligned} m\ddot{z} &= F - b\dot{z} - kz \\ \ddot{z} &= \frac{F}{m} - \frac{b\dot{z}}{m} - \frac{kz}{m} \end{aligned}\right\}$$

The equilibrium is when,

$f(x_e, u_e) = 0$

or when,

z_e = any expression

$\dot{z}_e = 0$

and $F_e = kz$.

Or at equilibrium, we can say there is no motion in the system,

$\ddot{z}_e = \dot{z}_e = 0$.

$\therefore$ (z_e, F_e) satisfy the conditions.

b) Jacobian linearization proceeds by replacing each term in the nonlinear DE by the first two terms in the Taylor's series expansion about the equilibrium point.

Defining.

$$\tilde{z} \triangleq z - z_e ; \quad \dot{\tilde{z}} \triangleq \dot{z} - \dot{z}_e = \dot{z}$$

$$\ddot{\tilde{z}} = \ddot{z} - \ddot{z}_e = \ddot{z}$$

and $\tilde{F} = F - F_e$

So $m\ddot{z} + kz = F - b\dot{z} \rightarrow (2)$

can be expanded about the equilibrium as follows:-

$$m\ddot{z} \approx m\ddot{\tilde{z}}$$

$$kz \approx kz_e + k \frac{\partial}{\partial z}(z)\Big|_{z_e} (z - z_e)$$

$$= kz_e + z_e \tilde{z} k$$

$$= k(z_e + z_e \tilde{z})$$

$$F = F_e + \frac{\partial F}{\partial F}\Big|_e (F - F_e) = F_e + \tilde{F}$$

$$b\dot{z} = b\dot{z}_e + b\frac{\partial \dot{z}}{\partial \dot{z}}\Big|_e (\dot{z} - \dot{z}_e) = b\dot{\tilde{z}}$$

Substituting these in ②,

$$\boxed{m\ddot{\tilde{z}} + k(z_e + z_e\tilde{z}) = F_e + \tilde{F} - b\dot{\tilde{z}}}$$

c) Feedback linearization proceeds.

using the feedback linearizing control.

$$F = kz + \tilde{F}$$

$$\boxed{m\ddot{z} = \tilde{F} - b\dot{z}}$$

$$F = F_e + \tilde{F} = kz_e + \tilde{F}$$

D.5. a) Use Laplace transform

Taking the linearized equation from D.4

$$m\ddot{z} = \tilde{F} - b\dot{z} \rightarrow (1)$$

Taking Laplace transform of (1) and setting all initial conditions to 0,

$$ms^2\theta(s) + bs\theta(s) = \tilde{F}(s)$$

Solving for $\theta(s)$ gives.

$$\theta(s) = \left(\frac{1}{ms^2 + bs}\right)\tilde{F}(s)$$

b) Transfer function :-

$$\left(\frac{1}{ms^2 + bs}\right) = \left(\frac{1/m}{s^2 + \frac{b}{m}s}\right)$$

$\therefore$ Transfer function $= \left(\frac{1/m}{s^2 + \frac{b}{m}s}\right)$

c) Block diagram.

$$\tilde{F}(s) \longrightarrow \boxed{\dfrac{1/m}{s^2 + \frac{b}{m} s}} \longrightarrow \theta(s)$$

D.6 If $y = z$, $u = F$ and $(z, \dot{z})^T = x$.

is given to us in the question.

Starting with the feedback linearized equation

$$\ddot{z} = \frac{\tilde{F}}{m} - \frac{b\dot{z}}{m}$$

$\therefore$

$$\dot{x} \triangleq \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \frac{\tilde{F}}{m} - \frac{b\dot{z}}{m} \end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ \frac{\tilde{u}}{m} - \frac{b}{m} x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\tilde{F}}{m} \end{pmatrix} \tilde{u}$$

Assuming measured output

$y = z.$

linearized output is :-

$$y = z = x_1 = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0\tilde{u}$$

$\therefore$ linearized state space equations are :-

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{\tilde{F}}{m} \end{pmatrix} \tilde{u}$$

$$y = (1 \quad 0) x$$