ECS 521 Spintronics and Nanomagnetics

HW # 1

Q 1) Determine the inverse of the matrices

a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A$

$AA^{-1} = I \Rightarrow A^{-1} = \frac{1}{|A|} \text{Adj } A = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} = \cancel{[-0.33 \;\; 0.667}$

$= \begin{bmatrix} -0.3333 & 0.6667 \\ 0.6667 & -0.3333 \end{bmatrix}$

Matlab command :- Y=inv(A) . where A = [1 2 ; 2 1]

b) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = B$

$BB^{-1} = I \Rightarrow B^{-1} = \frac{1}{|B|} \text{Adj } B$

In matlab, Z = inv(B) where B = [1 1 ; 2 2]

Matlab throws a warning: Matrix is singular to working precision yielding Z = Inf Inf
Inf Inf .

Q2) a) Determine the eigenvalues and eigenvectors of the matrices

a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

The matlab command

[V,D] = eig(A) is used

columns of V represent eigenvectors of A

The main diagonal of D represents the eigenvalues of A.

For.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

[V,D]= eig(A) yields $V = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$

and $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

eigenvalues are -1 and 3

eigenvectors are $\begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$ and $\begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$

On solving manually, we can verify this.

$\det \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda = -1$ or $\lambda = 3$

For $\lambda = -1$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, $x_1 = -x_2$ and $2 \cdot x_1^2 = 1$

$\Rightarrow x_1 = \frac{1}{\sqrt{2}}$

$x_2 = -\frac{1}{\sqrt{2}}$.

For $\lambda = 3$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$, $y_1 = y_2 = 0.7071$

For b) $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$(W, E) = eig(B)$

yields $W = \begin{bmatrix} -0.7071 & -0.4472 \\ 0.7071 & -0.8944 \end{bmatrix}$

$E = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

eigenvalues are 0 and 3

eigenvectors are $\begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$ and $\begin{bmatrix} -0.4472 \\ -0.8944 \end{bmatrix}$

Q3) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = 1$ or -1 are the eigenvalues.

For $\lambda = -1$, we have $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

one eigenvector is of the form $x_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ where $\sqrt{2}\, x_1 = 1$

or $x_1 = 0.7071$

and for $\lambda = 1$, we have $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$

or $y_1 = y_2$, the other eigenvector is of the form $y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

where $y_1 = y_2 = 0.7071$.

$\therefore$ The eigenvectors of σ_x are $\begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix}$ and $\begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}$

For $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

To find the eigenvalues,

$\begin{vmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + i^2 = 0$

or $\lambda^2 - 1 = 0 \Rightarrow \lambda = 1$ or $\lambda = -1$

For $\lambda = -1$

$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 - ix_2 = 0$

$\Rightarrow x_1 = ix_2$

One eigenvector is of the form $x_1 \begin{pmatrix} -i \\ -1 \end{pmatrix}$ where $x_1 = 0.7071$ since $x_1 = \frac{1}{\sqrt{2}}$

For $\lambda = 1$

$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \Rightarrow -y_1 - iy_2 = 0$

$y_1 = -iy_2.$

The other eigenvector is of the form $y_1 \begin{pmatrix} -i \\ 1 \end{pmatrix}$ where $y_1 = 0.7071$

The two eigenvectors of σ_y are $\begin{pmatrix} -0.7071i \\ -0.7071 \end{pmatrix}$ and $\begin{pmatrix} -0.7071i \\ 0.7071 \end{pmatrix}$

For $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

To find the eigenvalues,

$\left| \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \right| = 0 \Rightarrow (1-\lambda)(-1-\lambda) = 0$

$\lambda = -1$ or $\lambda = 1$

For $\lambda = -1$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{array}{c} 2x_1 + 0x_2 = 0. \\ x_1 = 0 \end{array}$$

One eigenvector is $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ after normalization $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda = 1$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \quad -2y_2 = 0$$

other eigenvector is $\begin{bmatrix} y_1 \\ 0 \end{bmatrix}$ after normalization $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The eigenvectors of σ_z are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Q4) a) $\det(\sigma_j) = -1$ for $j = x, y, z$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$|\sigma_x| = 0 - 1 = -1$ $\quad |\sigma_y| = 0 + i^2 = 1(-1) = -1$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|\sigma_z| = -1 - 0 = -1$

$\det(\sigma_j) = -1$

b) $Tr(\sigma_j) = 0 \qquad j = x, y, z$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Tr(\sigma_x) = 0 \qquad Tr(\sigma_y) = 0 \qquad Tr(\sigma_y) = 1 - 1 = 0.$$

c) $\sigma_x{}^2 = \sigma_y{}^2 = \sigma_z{}^2 = I$

$$\sigma_x \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_z \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d) $\sigma_x \sigma_y \sigma_z = iI$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

e) $\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$-\sigma_y \sigma_x = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$i\sigma_z = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\therefore \sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$$

f) $\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$

$$\sigma_y \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$-\sigma_z \sigma_y = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$i\sigma_x = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \therefore \sigma_y \sigma_z = \sigma_y \sigma_z = -\sigma_z \sigma_y$$
$$= i\sigma_x$$

g) $\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma_z \sigma_x$$

$$-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sigma_x \sigma_z$$

$$i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = i\sigma_y$$

$$\therefore \sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$$

h) $\sigma_p \sigma_q + \sigma_q \sigma_p = 0$ for $p \neq q$

$p, q = x, y, z$.

From e), f), g), we have.

$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_y \sigma_z = -\sigma_z \sigma_y$$

$$\sigma_z \sigma_x = -\sigma_x \sigma_z$$

$\Rightarrow$ for consecutive and $p \neq q$.

$$\sigma_p \sigma_q = -\sigma_q \sigma_p$$

$\therefore \sigma_p \sigma_q + \sigma_q \sigma_p$

$= -\sigma_q \sigma_p + \sigma_q \sigma_p = \underline{\underline{0}}$