ECS 521 Spintronics & Nanomagnetics

HW # 2

Problem 1

$\sigma_x, \sigma_y, \sigma_z$ applied to $\Psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$

$$\sigma_x \Psi(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} \psi_2(x) \\ \psi_1(x) \end{bmatrix}$$

$$\sigma_y \Psi(x) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} -i\psi_2(x) \\ i\psi_1(x) \end{bmatrix} = i \begin{bmatrix} -\psi_2(x) \\ \psi_1(x) \end{bmatrix}$$

$$\sigma_z \Psi(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} \psi_1(x) \\ -\psi_2(x) \end{bmatrix} = -1 \begin{bmatrix} -\psi_1(x) \\ \psi_2(x) \end{bmatrix}$$

Problem 2

Expected value along n^{th} coordinate axis at $r = (x, y, z)$ is

$\langle S_n \rangle (r,t) = [\Psi(r,t)]^{+} [S_n] [\Psi(r,t)]$; $S_n = (\hbar/2)\sigma_n$

$$\langle S_x \rangle = [\Psi(x)]^{+} [S_x] [\Psi(x)] = \begin{bmatrix} \psi_1^*(x) & \psi_2^*(x) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{bmatrix} \psi_2^*(x) & \psi_1^*(x) \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \frac{\hbar}{2} \left[\psi_2^*(x)\psi_1(x) + \psi_1^*(x)\psi_2(x) \right]$$

$$= \frac{\hbar}{2} \left(\psi_2^*(x)\psi_1(x) + \psi_1^*(x)\psi_2(x) \right)$$

$$\langle S_y \rangle = [\Psi(y)]^{+} [S_y] [\Psi(y)] = \begin{bmatrix} \psi_1^*(y) & \psi_2^*(y) \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \psi_1(y) \\ \psi_2(y) \end{bmatrix} \times \frac{\hbar}{2}$$

$$= \frac{\hbar}{2} \begin{bmatrix} i\psi_2^*(y) & -i\psi_1^*(y) \end{bmatrix} \begin{bmatrix} \psi_1(y) \\ \psi_2(y) \end{bmatrix} = \frac{\hbar}{2} \left(i\psi_2^*(y)\psi_1(y) - i\psi_1^*(y)\psi_2(y) \right)$$

$$\langle S_z\rangle = [\psi(z)]^{\dagger}[S_z][\psi(z)] = [\psi_1^*(z)\ \psi_2^*(z)]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\psi_1(z)\\ \psi_2(z)\end{bmatrix} \times \frac{\hbar}{2}$$

$$= [\psi_1^*(z)\ \ -\psi_2^*(z)]\begin{bmatrix}\psi_1(z)\\ \psi_2(z)\end{bmatrix} = \frac{\hbar}{2}\psi_1^*(z)\psi_1(z) - \psi_2^*(z)\psi_2(z)$$

For $|+\rangle_z = \begin{bmatrix}1\\0\end{bmatrix}$, we have

$$\langle S_x\rangle = \frac{\hbar}{2}[1\ \ 0]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}(0+0) = 0.$$

$$\langle S_y\rangle = \frac{\hbar}{2}(0-1) = 0$$

$$\langle S_z\rangle = \hbar/2$$

For $|-\rangle_z = \begin{bmatrix}0\\1\end{bmatrix}$, we have.

$$\langle S_x\rangle = 0.$$

$$\langle S_y\rangle = 0$$

$$\langle S_z\rangle = \frac{\hbar}{2}(-1) = -\frac{\hbar}{2}$$

Problem 3

$$[\sigma\cdot(p+eA)]^2 = (p+eA)^2 + 2m_0\mu_B B\cdot\sigma$$

To prove the above, let's first split the left hand side.

$$[\sigma\cdot(p+eA)]^2 = (\sigma p + e(\sigma\cdot A)]^2$$

$$= \underset{①}{(\sigma\cdot p)^2} + \underset{②}{e^2(\sigma\cdot A)^2} + \underbrace{e(\sigma\cdot p)(\sigma\cdot A) + e(\sigma\cdot A)(\sigma\cdot p)}_{③}$$

$$① = (\sigma\cdot p)^2 = p^2$$

$$② = e^2(\sigma\cdot A)^2 = e^2A^2$$

(3) $= (\sigma\cdot p)(\sigma\cdot A) + (\sigma\cdot A)(\sigma\cdot p)$

$(\sigma_x p_x + \sigma_y + \sigma_z)(P_x + P_y + P_z)(\sigma_x + \sigma_y + \sigma_z)(A_x + A_y + A_z)$

$+ (\sigma_x A \sigma_y + \sigma_z)(A_x + A_y + A_z)(\sigma_x + \sigma_y + \sigma_z)(P_x + P_y + P_z)$

$= \sigma_i \sigma_j (-i\hbar(\nabla_i \times A_j) + A_i P_j + A_j P_i)$

$= \sigma_i \sigma_j (A_i P_j + A_j P_i) - i\hbar \sigma_i \sigma_j (\nabla_i \times A_j)$

$= \sigma_i \sigma_j A_i P_j + \sigma_j \sigma_i A_j P_i - i\hbar (i \varepsilon_{ijk} \sigma_k)(\nabla_i \times A_j)$

$= 2\delta_{ij} A_j P_i + \hbar \sigma_k \varepsilon_{ijk} (\nabla_i A_j)$

$= 2AP + \hbar \sigma\cdot(\nabla \times A)$

$= 2AP + \hbar(\sigma\cdot B)$

$\therefore [\sigma\cdot(p+eA)]^2 = p^2 + e^2A^2 + 2AP + e\hbar(\sigma\cdot B)$

$\left(e\hbar = 2m_e \mu_B \text{ since } \mu_B = \frac{e\hbar}{2m_e}\right)$

$= (p+eA)^2 + 2m_e \mu_B B\cdot\sigma$

Problem 4

Prove that $e^{i\theta A} = \cos\theta\, I + i\sin\theta A$.

According to Taylor's series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\therefore e^{i\theta A} = \sum_k \frac{(i\theta A)^k}{k!}$$

$$= 1 + (i\theta A) + \frac{(i\theta A)^2}{2!} + \frac{(i\theta A)^3}{3!} \cdots$$

$$= \sum_{\substack{k=0,2 \\ \text{even}}} \frac{(i\theta A)^k}{k!} + i \sum_{\substack{k=1,3 \\ \text{odd}}} \frac{i^{k-1}(\theta A)^k}{k!}$$

$$= \left(I - \frac{I\theta^2}{2!} + \frac{I\theta^4}{4!} \cdots \right) + i\left(A\left(I\theta - \frac{I\theta^3}{3!} \cdots \right)\right)$$

$$= \cos\theta \, I + i \sin\theta \, A$$ (using expansion formula of sine & cosine).