Spintronics and Nanomagnetics

ECS 521/641

Instructor: Dr. Kuntal Roy

Electrical Engineering and Computer Science (EECS) Dept.

Indian Institute of Science Education and Research (IISER) Bhopal

Email: kuntal@iiserb.ac.in

Bloch sphere

Bloch sphere

- Useful tool to represent the actions of various quantum-mechanical operators on a spinor
- Link between the rather abstract concept of a spinor and more intuitive way of thinking of magnetic moment along a direction
- Describing the action of a spatially uniform (including time-dependent) magnetic field on the spin, e.g., Rabi formula
- Frequently invoked in spin-based quantum computing, e.g., quantum bit (qubit)

Spinor and Qubit

- The 2-component wavefunction representing an arbitrary spin state can be written as

$$[\psi(x)\ ] = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \phi_1 \begin{bmatrix}1\\0\end{bmatrix} + \phi_2 \begin{bmatrix}0\\1\end{bmatrix} = \phi_1|+\rangle_z + \phi_2\ |-\rangle_z$$

where

$$|\phi_1|^2 + |\phi_2|^2 = 1$$

- $\pm z$-polarized states $\Rightarrow$ classical bits 0 and 1

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \qquad\qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$|\chi\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\1\end{bmatrix} = \alpha|0\rangle + \beta\ |1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Bloch sphere concept

- A measurement of the spin component along an arbitrary direction characterized by a unit vector $\hat{\boldsymbol{n}}$

$$\boldsymbol{S}_{op} = \frac{\hbar}{2}\boldsymbol{\sigma} \qquad \boldsymbol{S}\cdot\hat{\boldsymbol{n}} \qquad \text{eigenvalues} \quad \pm\frac{\hbar}{2}$$

Exercise **Prove** $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = i\,\boldsymbol{\sigma}\cdot\ (\boldsymbol{a}\times\boldsymbol{b}) + (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{I}$

$$\boldsymbol{a} = \boldsymbol{b} = \hat{\boldsymbol{n}} \ \Rightarrow (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})^2 = \boldsymbol{I}$$

- The eigenvalues of $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ are $\pm\mathbf{1}$
- The eigenvalues of $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$ are $\pm\frac{\hbar}{2}$

Eigenvectors of $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$

- Consider the following operator $\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})$

  acting on arbitrary spinor or qubit $|\chi\rangle$

Exercise **Determine** $(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}) \left[\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})|\chi\rangle\right]$

$$= \frac{1}{2}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}|\chi\rangle \pm \frac{1}{2}(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})^2|\chi\rangle = \pm\left[\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})|\chi\rangle\right]$$

$$(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})^{\mathbf{2}} = \boldsymbol{I}$$

$$\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})$$
$$= \frac{1}{2}\left(1 \pm \sigma_z n_z \pm \frac{1}{2}(\sigma_x + i\sigma_y)(n_x - in_y) \pm \frac{1}{2}(\sigma_x - i\sigma_y)(n_x + in_y)\right)$$

Eigenvectors of $\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}}$

$$\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}})$$
$$= \frac{1}{2}\left(1 \pm \sigma_z n_z \pm \frac{1}{2}(\sigma_x + i\sigma_y)(n_x - in_y) \pm \frac{1}{2}(\sigma_x - i\sigma_y)(n_x + in_y)\right)$$

$$(n_x, n_y, n_z) = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$$

$$n_x \pm in_y = sin\theta e^{\pm \mathrm{i}\phi}$$

$$\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}}) = \frac{1}{2}\left(1 \pm cos\theta\sigma_z \pm \frac{1}{2}(sin\theta e^{-\mathrm{i}\phi}\sigma_+ \pm sin\theta e^{+\mathrm{i}\phi}\sigma_{\_})\right)$$

$$\sigma_{\pm} = \sigma_x \pm i\sigma_y$$

Eigenvectors of $\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}}$

$$\frac{1}{2}(1 \pm \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}}) = \frac{1}{2}\left(1 \pm cos\theta\sigma_z \pm \frac{1}{2}(sin\theta e^{-\mathrm{i}\phi}\sigma_+ \pm sin\theta e^{+\mathrm{i}\phi}\sigma\_)\right)$$

$$\sigma_\pm = \sigma_x \pm i\sigma_y$$

Acting with these operators on ket $|0\rangle$

$$\frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}})|0\rangle = \cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$\frac{1}{2}(1 - \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{n}})|0\rangle = \sin\frac{\theta}{2}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

Eigenspinors

$$\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\widehat{\boldsymbol{n}})|0\rangle = \cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$\frac{1}{2}(1-\boldsymbol{\sigma}\cdot\widehat{\boldsymbol{n}})|0\rangle = \sin\frac{\theta}{2}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

Normalizing

$$|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$|\xi_n^-\rangle = e^{i\gamma}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$|\xi_n^-(\theta,\phi)\rangle = |\xi_n^+(\theta\to\pi-\theta,\phi\to\phi+\pi)\rangle$$

Eigenspinors – Orthogonal?

$$|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$|\xi_n^-\rangle = e^{i\gamma}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$\langle\xi_n^+|\xi_n^-\rangle = 0$$

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix} \qquad |\xi_n^-\rangle = e^{i\gamma}\begin{bmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$\langle\xi_n^+|\xi_n^-\rangle = \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0$$

Connecting Bloch sphere and spinors

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix} \qquad |\xi_n^-\rangle = e^{i\gamma}\begin{bmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$\langle\xi_n^+|\sigma_x|\xi_n^+\rangle = \sin\theta cos\phi \qquad \langle\xi_n^-|\sigma_x|\xi_n^-\rangle = -\sin\theta cos\phi$$
$$\langle\xi_n^+|\sigma_y|\xi_n^+\rangle = \sin\theta sin\phi \qquad \langle\xi_n^-|\sigma_y|\xi_n^-\rangle = -\sin\theta sin\phi$$
$$\langle\xi_n^+|\sigma_z|\xi_n^+\rangle = \cos\theta \qquad \langle\xi_n^-|\sigma_z|\xi_n^-\rangle = -\cos\theta$$

Two orthogonal spinors on the Bloch sphere are NOT perpendicular to each other, they subtend an angle 180°

Relationship with Qubit

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$|\xi\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}|\alpha|e^{i\phi_\alpha}\\ |\beta|e^{i\phi_\beta}\end{bmatrix} = e^{i\phi_\alpha}\begin{bmatrix}|\alpha|\\ |\beta|e^{i(\phi_\beta-\phi_\alpha)}\end{bmatrix}$$

$$\gamma = \phi_\alpha$$

$$\theta = 2\tan^{-1}\left(\frac{\sqrt{(1-|\alpha|^2}}{|\alpha|}\right)$$

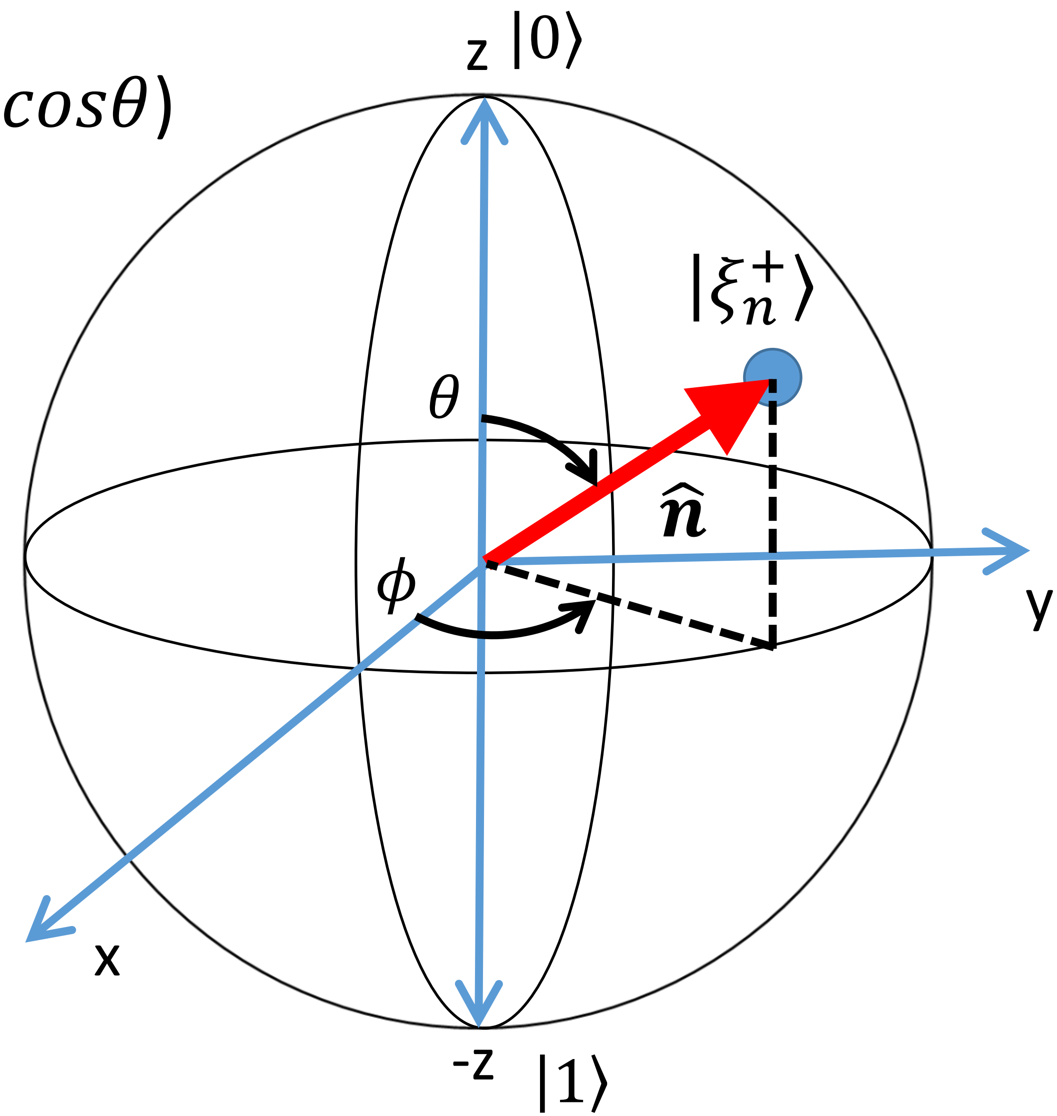
$$\phi = (\phi_\beta - \phi_\alpha)$$

Bloch sphere representation

$$\hat{\boldsymbol{n}} = (n_x, n_y, n_z)$$
$$= (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$$

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{bmatrix}$$

$$|0\rangle = |\xi_n^+(\theta = 0, \phi, \gamma)\rangle$$
$$|1\rangle = |\xi_n^+(\theta = \pi, \phi, \gamma)\rangle$$



Spin flip matrix

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix} = \boldsymbol{M}\,|\xi_n^-\rangle = \boldsymbol{M}e^{i\gamma}\begin{bmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix}0 & -e^{-i\phi}\\ e^{i\phi} & 0\end{bmatrix}$$

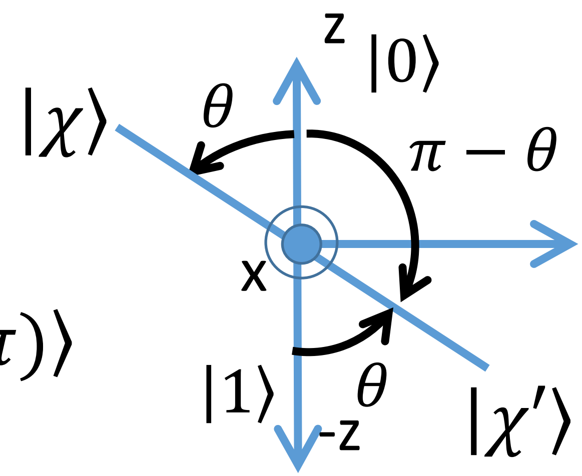
$$\boldsymbol{M} = \begin{bmatrix}1 & 0\\ 0 & e^{i\phi}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-i\phi}\end{bmatrix}$$

$$\boldsymbol{M} = e^{-i\frac{\pi}{2}}P(\phi)\sigma_y P(-\phi)$$

$P(\phi)$ is the phase shift matrix

Rotation matrices

$$|\chi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{-i\frac{\pi}{2}}|1\rangle = \begin{bmatrix} cos\frac{\theta}{2} \\ -isin\frac{\theta}{2} \end{bmatrix}$$



$$|\xi_n^-(\theta,\phi)\rangle = |\xi_n^+(\theta \to \pi-\theta, \phi \to \phi+\pi)\rangle$$

$$|\chi'\rangle = \cos\frac{\pi-\theta}{2}|0\rangle + \sin\frac{\pi-\theta}{2}e^{i\frac{\pi}{2}}|1\rangle$$

$$|\chi'\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{bmatrix} = i\begin{bmatrix} -i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}$$

$$A^2 = I$$

$$e^{i\theta A} = cos\theta\ I + i\ sin\theta\ A$$

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z}$$

$$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$