Problem 2

As calculated in HW # 1,

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ eigenvalues $= \pm 1$

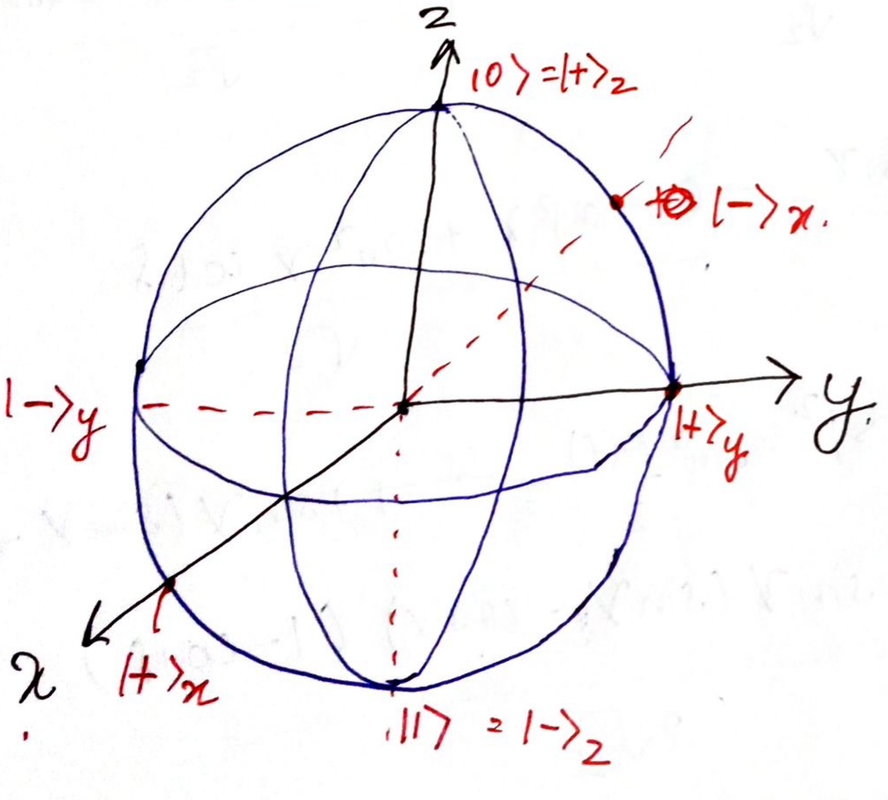
eigenvectors are $|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ eigenvalues $= \pm 1$

eigenvectors $|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ eigenvalues $= \pm 1$

eigenvectors $|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ . $\begin{vmatrix} 1-\lambda & 0 \\ 0 & i-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(i-\lambda) = 0$

$\Rightarrow$ eigenvalues are $1, i$ $\quad$ For $\lambda = 1$ $\begin{pmatrix} 0 & 0 \\ 0 & i-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

eigenvector $1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = i$, $\begin{pmatrix} 1-i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

eigenvector $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$, $\begin{pmatrix} 1-\lambda & 0 \\ 0 & e^{i\pi/4} - \lambda \end{pmatrix}$ $\quad \lambda_1 = 1$, $\lambda_2 = \frac{\sqrt{2}(1+i)}{2}$.

For $\lambda = 1$ $\begin{bmatrix} 1-\lambda & 0 \\ 0 & e^{i\pi/4} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ eigenvector $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = \frac{\sqrt{2}(1+i)}{2}$, $\begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eigenvector $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}$ $\qquad \lambda^2 - i = 0$

$\lambda = \pm 1.$

For $\lambda = -1$

$$\begin{bmatrix} -\lambda + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\lambda - \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

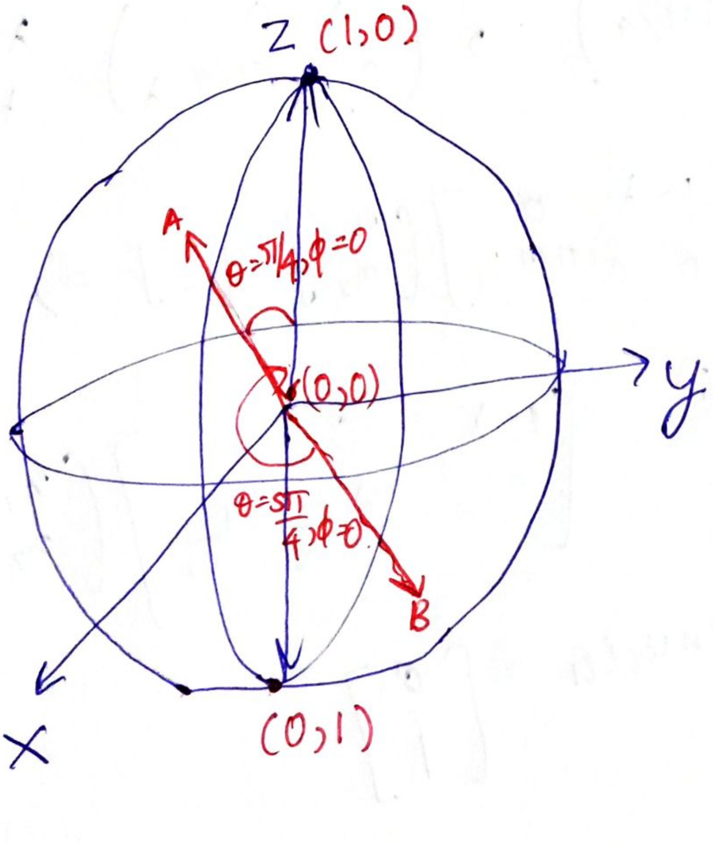
$A = \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix}$ is one eigenvector.

For $\lambda = 1$

$$\begin{bmatrix} -\lambda + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\lambda - \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$B = \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix}$ is another eigenvector

for H, these are $\phi = 0$ and $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$



Problem 3 

The eigenvalues of the Pauli spin matrices (as calculated in the previous homework) are $\pm 1$

For the spinor to be aligned along $+x$ axis, the eigenvector of $\sigma_x$ must satisfy

$$\sigma_x |+\rangle_x = +1 |+\rangle_x.$$

giving us $|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\therefore \quad \xi_{+x} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$\langle \xi_{+x} | \xi_n^{+} \rangle = \frac{1}{\sqrt{2}} [1 \quad 1]\, e^{i\gamma} \begin{bmatrix} \cos\theta/2 \\ e^{i\phi} \sin\theta/2 \end{bmatrix}$$

$$= \frac{e^{i\gamma}}{\sqrt{2}} [1 \quad 1] \begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

$$= \frac{e^{i\gamma}}{\sqrt{2}} \left( \cos\theta/2 + e^{i\phi} \sin\theta/2 \right)$$

$$|\langle \xi_{+x} | \xi_n^{+} \rangle|^2 = \frac{1}{2}\left( \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2e^{i\phi} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \right)$$

$= \frac{1}{2}\left(1 + \sin\theta\, e^{i\phi}\right)$ for the x-axis, $\phi = 0°$

$\therefore \quad e^{i\phi} = \cos\phi.$

$$= \frac{1}{2}(1 + \sin\theta\cos\phi) = \frac{1}{2}(1 + n_x) = \frac{1 + n_x}{2}$$ //

At the north pole, $t=0$ we have

$$\vec{n}(0) = (n_x(0), n_y(0), n_z(0)) = (0,0,1)$$

we need to find $n_x(t)$ which we can find from the following equation

$$n(t) = [U]\hat{n}(0) = \{[A]^{-1}(t)\left[[S](t)\, e^{[A](t)} [S]^{-1}(t)\right\} A(0)\}\hat{n}(0)$$

$$= \underset{3\times3}{\text{matrix } U} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ \cos^2\chi + \sin^2\chi\cos\delta \end{bmatrix}$$

here, $n_x(t) = 0$.
$n_y(t) = 0$
$n_z(t) = \cos^2\chi + \sin^2\chi\cos\delta$

$$\therefore \left|\langle \xi_{+x} | \xi_n^+ \rangle\right|^2(t) = \frac{1 + n_x(t)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

ii) If the spinor is the eigenstate of the hadamard matrix,

at $t=0$, $(n_x(0), n_y(0), n_z(0)) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

will be the unit vector

(5)

$$\therefore \langle 1|\varepsilon_{n}^{+}\rangle|^{2} = \left| [0 \quad 1] e^{i\gamma} \begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix} \right|^{2}$$

$$= \left(e^{i\gamma} e^{i\phi} \sin\theta/2\right)^{2} = \sin^{2}\left(\frac{\theta(t)}{2}\right)$$

$$= \frac{1-\cos\theta(t)}{2} \quad \text{or} \quad = \frac{1-n_z(t)}{2}$$

Using the equation

$$n(t) = [U]\hat{n}(0),$$

$$U_{3\times3} \times \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \Bigg[ \cancel{\frac{g(\delta,\chi)\sin\omega t + h(\delta,\chi)\cos\omega t + f(\delta)\cos\omega t \cos\chi}{\sqrt{2}}}$$

$$= \begin{bmatrix} \dfrac{g(\delta,\chi)\sin\omega t + h(\delta,\chi)\cos\omega t}{\sqrt{2}} + \dfrac{[f(\delta)\cos\omega t\cos\chi - \sin\delta\sin t]\sin\chi}{\sqrt{2}} \\ 0 \\ \dfrac{f(\delta)\cos\chi\sin\chi}{\sqrt{2}} - \dfrac{0 + \cos^{2}\chi + \sin^{2}\chi\cos\delta}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \langle 1|\varepsilon_{n}^{+}\rangle|^{2} = \sin^{2}\frac{\theta(t)}{2} = \cancel{\sqrt{2} - 1 + \sin\chi(\sin\chi - \cos}$$

$$= \frac{\sqrt{2} - 1 + \sin\chi(\sin\chi - \cos\chi)(1-\cos\delta)}{2\sqrt{2}}$$

At $\omega = \omega_0$, the probability is maximized

# Lecture 03. Bloch Sphere.

↳ Intuitive way to think of magnetic moment along a direction

Spinor and Qubit

a) $[\Psi(x)] = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \phi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \phi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \phi_1 |+\rangle_z + \phi_2 |-\rangle_z$

$|\phi_1|^2 + |\phi_2|^2 = 1$

b) $\pm z$-polarized state → classical bits 0 & 1

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|\chi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

Bloch sphere concept

- A measurement of spin component along an arbitrary direction characterized by a unit vector $\hat{n}$

$S_{op} = \frac{\hbar}{2}\sigma$ $\quad S.\hat{n}$ $\quad$ eigenvalues $\pm\frac{\hbar}{2}$

$(\sigma.a)(\sigma.b) = i\sigma.(a\times b) + (a.b)I$

Proof :- $a.(\sigma_x + \sigma_y + \sigma_z)\, b(\sigma_x + \sigma_y + \sigma_z)$

$= (a\sum_i \sigma_i)(b\sum_j \sigma_j) = \sum_{ij} a\, b\, \sigma_i \sigma_j$

If $a = b = n$, $(\sigma.n)^2 = I$

$$= \sum_{i,j} \frac{ab}{2} \times 2\sigma_i\sigma_j = \frac{ab}{2} \sum (\sigma_i\sigma_j + \sigma_j\sigma_i + \sigma_i\sigma_j - \sigma_j\sigma_i)$$

$$= \frac{ab}{2} \sum (\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j])$$

$$= \frac{ab}{2} (2\delta_{ij} + 2i\epsilon_{ijk}\sigma_k)$$

$$= \sum ab (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$$

$$= \sum_{ij} ab\,\delta_{ij} + i \sum_{i,j} ab\,\epsilon_{ijk}\sigma_k$$

$$= (a \cdot b) I + i \sum_{ij} ab\,\epsilon_{ijk}\sigma_k.$$

for $k = z$, $i(a_x b_y - b_x a_y)\sigma_z = i(a \times b)\sigma_z$

$$\Rightarrow (a \cdot b) I + i (a \times b) \cdot \sigma$$

$$\boxed{\begin{array}{l} \text{Eigenvalues of } \sigma \cdot \hat{n} \text{ are } \pm 1 \\ \text{Eigenvalues of } S \cdot \hat{n} \text{ are } \pm \frac{\hbar}{2} \end{array}}$$ ✯

Eigenvectors of $\sigma \cdot \hat{n}$ :-

- $\frac{1}{2}(1 \pm \sigma \cdot \hat{n})$ acts on $|x\rangle$ lets consider.

then $(\sigma \cdot n)\left[\frac{1}{2}(1 + \sigma \cdot n)|x\rangle\right] = ?$

Eigenvectors of $\sigma.\hat{n}$.

- Consider $\frac{1}{2}(1 \pm \sigma.\hat{n})$ acting on spinor $|\chi\rangle$

Calculate $(\sigma.\hat{n})\left[\frac{1}{2}(1 \pm \sigma.\hat{n})|\chi\rangle\right]$

$$= \frac{1}{2}(\sigma.\hat{n})|\chi\rangle \pm \frac{1}{2}(\sigma.\hat{n})^2|\chi\rangle$$

$$= \pm\left(\frac{1}{2}(1 \pm \sigma.\hat{n})|\chi\rangle\right]$$

$$\frac{1}{2}(1 \pm \sigma.\hat{n}) = \frac{1}{2}(1 \pm \sigma_z n_z \pm \frac{1}{2}(\sigma_x + \sigma_y)(n_x - in_y)$$
$$\pm \frac{1}{2}(\sigma_x - \sigma_y)(n_x + in_y))$$

$$= \frac{1}{2}\begin{bmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{bmatrix}$$

$$n_x \pm in_y = \sin\theta\, e^{\pm i\phi}$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\frac{1}{2}(1 \pm \sigma.\hat{n}) = \frac{1}{2}\left(1 \pm \cos\theta\, \sigma_z \pm \frac{1}{2}(\sin\theta\, e^{-i\phi}\sigma_+ \pm \sin\theta e\right.$$

$$\sigma_\pm = \sigma_x \pm i\sigma_y$$

$\frac{1}{2}(1+\sigma\cdot\hat{n})|0\rangle = \cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$

$\frac{1}{2}(1-\sigma\cdot\hat{n})|0\rangle = \sin\frac{\theta}{2}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$

On normalizing, we obtain

$$|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$|\xi_n^-\rangle = e^{i\gamma}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\phi}|1\rangle\right]$$

$$|\xi_n^-(\theta,\phi)\rangle = |\xi_n^+(\theta\to\pi-\theta, \phi\to\phi+\pi)\rangle.$$

$\langle\varepsilon_n^+| = ?$

$$|\varepsilon_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\theta/2 \\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$\langle\varepsilon_n^+| = e^{-i\gamma}\left[\cos\theta/2 \quad \sin\frac{\theta}{2}e^{-i\phi}\right]$$

$$|\varepsilon_n^-\rangle = e^{i\gamma}\begin{bmatrix}\sin\frac{\theta}{2} \\ -\cos\theta/2\, e^{i\phi}\end{bmatrix} \cancel{-\cos\frac{\theta}{2}e^{i\phi}}]$$

$$\langle\varepsilon_n^+|\varepsilon_n^-\rangle = \left[\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}e^{-i\phi}\right]\begin{bmatrix}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

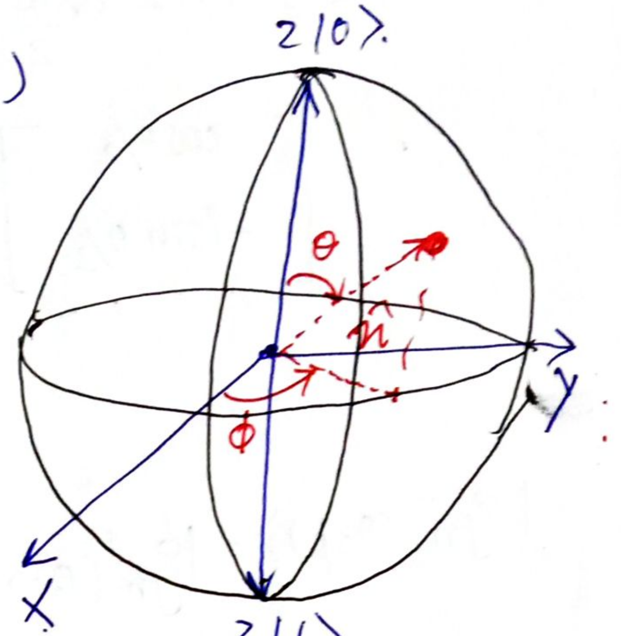
$$\therefore \boxed{\langle\varepsilon_n^+|\varepsilon_n^-\rangle = 0}$$

Bloch sphere representation :-

$\vec{n} = (n_x, n_y, n_z)$

$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}$




$|0\rangle = |\xi_n^+(\theta = 0, \phi, \gamma)\rangle$

$|1\rangle = |\xi_n^+(\theta = \pi, \phi, \gamma)\rangle$

Spin flip Matrix :-

$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{bmatrix} = M|\xi_n^-\rangle = M e^{i\gamma}\begin{bmatrix} \sin\theta/2 \\ -\cos\frac{\theta}{2} e^{i\phi} \end{bmatrix}$

$M = \begin{bmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{bmatrix}$

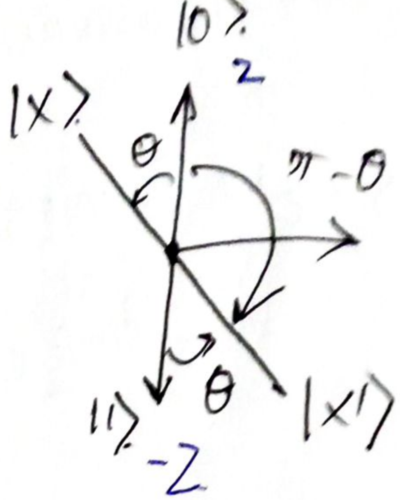
$M = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$

$M = e^{-i\pi/2} P(\phi)\sigma_y P(-\phi)$. $\quad P(\phi)$ = phase shift matrix

Rotation matrices

$$|x\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\, e^{-i\frac{\pi}{2}}\,|1\rangle$$

$$= \begin{bmatrix} \cos\theta/2 \\ -i\sin\theta/2 \end{bmatrix}$$




$$|\xi_n^-(\theta,\phi)\rangle = |\xi_n^+(\theta\to\pi-\theta,\ \phi\to\phi+\pi)\rangle$$

$$|x'\rangle = \cos\frac{\pi-\theta}{2}\,|0\rangle + \sin\frac{\pi-\theta}{2}\, e^{i\pi/2}\,|1\rangle$$

$A^2 = I$

$$|x'\rangle = \begin{bmatrix} \sin\theta/2 \\ i\cos\theta/2 \end{bmatrix} = i\begin{bmatrix} -i\sin\frac{\theta}{2} \\ \cos\theta/2 \end{bmatrix} \qquad A^2 = I$$

$$e^{i\theta A} = \cos\theta\, I + i\sin\theta\, A$$

$\cancel{|x'\rangle = [\sin}$

$$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

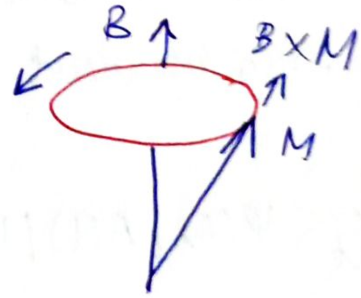
$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z}$$

Lecture 4 :- Evolution of spinor on Bloch Sphere

Larmor precession.




$$\frac{dM}{dt} = |\gamma| (B \times M) \qquad |\gamma| = \frac{g\mu_B}{\hbar}$$

$$\frac{d\langle S\rangle}{dt} = |\gamma| (B \times \langle S\rangle) \qquad S = \frac{\hbar}{2}\sigma$$

$$\frac{d\langle \sigma\rangle}{dt} = |\gamma| (B \times \langle \sigma\rangle) \qquad \langle \sigma_n\rangle = \langle \xi_{n^+} | \sigma_n | \xi_{n^+}\rangle$$

$$\frac{d}{dt}\begin{bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{bmatrix} = |\gamma| \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{bmatrix}$$

↳ this can be out.

$$B \times \langle\sigma\rangle = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ \langle\sigma_x\rangle & \langle\sigma_y\rangle & \langle\sigma_z\rangle \end{vmatrix}$$

$$\frac{d\langle\sigma_x\rangle}{dt} = |\gamma| (B_y\langle\sigma_z\rangle - B_z\langle\sigma_y\rangle)$$

$$\frac{d\langle\sigma_y\rangle}{dt} = |\gamma| (B_z\langle\sigma_x\rangle - B_x\langle\sigma_z\rangle)$$

$$\frac{d}{dt}\langle\sigma_z\rangle = |\gamma| (B_x\langle\sigma_y\rangle - B_y\langle\sigma_x\rangle)$$

# Ehrenfest theorem

- Time evolution of the expectation value of a time dependent observable for a q-mech system.

$$\frac{d}{dt}\langle \psi(t)|A(t)|\psi(t)\rangle$$

$$= \left[\frac{d}{dt}\langle \psi(t)|\right]A(t)|\psi(t)\rangle + \langle \psi(t)|A(t)\left[\frac{d}{dt}|\psi(t)\rangle\right]$$

$$+ \langle \psi(t)|\frac{dA(t)}{dt}|\psi(t)\rangle.$$

$$\frac{d}{dt}\langle \psi(t)| = -\frac{1}{i\hbar}H(t)\langle \psi(t)|$$

( from $E\psi = H\psi$ $\Rightarrow$ $\langle\psi| = \psi^*$

$i\hbar\frac{\partial \psi}{\partial t} = H\psi$

So, $-i\hbar\frac{\partial \psi^*}{\partial t} = H\psi^*$. )

$$\frac{d}{dt}|\psi(t)\rangle = +\frac{1}{i\hbar}H(t)|\psi(t)\rangle$$

$$\frac{d}{dt}\langle \psi(t)|A(t)|\psi(t)\rangle = \frac{1}{i\hbar}\langle \psi(t)|A(t)H(t)$$

$$-H(t)A(t)|\psi(t)\rangle + \langle \psi(t)|\frac{dA(t)}{dt}|\psi(t)\rangle$$

$$\frac{d\langle A(t)\rangle}{dt} = \frac{1}{i\hbar}\langle [A(t)H(t)]\rangle + \left\langle \frac{dA(t)}{dt}\right\rangle$$

# Deriving Larmor precession from Ehrenfest theorem.

$$\frac{d\langle A(t)\rangle}{dt} = \frac{1}{i\hbar}\langle [A(t)\, H(t)]\rangle + \left\langle \frac{dA(t)}{dt}\right\rangle \qquad H_B = \frac{-g\mu_B\, \sigma\cdot B}{2}$$

$$\frac{d\langle\sigma_x\rangle}{dt} = \frac{-1}{i\hbar}\frac{g\mu_B}{2}\left(B_y\langle[\sigma_x,\sigma_y]\rangle + B_z\langle[\sigma_x,\sigma_z]\rangle\right)$$

$$= |\gamma|\left(B_y\langle\sigma_z\rangle - B_z\langle\sigma_y\rangle\right)$$

$$\frac{d}{dt}\begin{bmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{bmatrix} = |\gamma| = \begin{bmatrix}0 & -B_z & B_y\\ B_z & 0 & -B_x\\ -B_y & B_x & 0\end{bmatrix}\begin{bmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{bmatrix}$$

$$\frac{d}{dt}\langle\sigma\rangle = |\gamma|\,(B\times\langle\sigma\rangle)$$

Precession angle and rate

$$|\gamma| = \frac{g\mu_B}{\hbar} \qquad \frac{dM}{dt} = |\gamma|\,(B\times M)$$

$$B\cdot\frac{dM}{dt} = 0 \Rightarrow \frac{d(B\cdot M)}{dt} = 0.$$

$$\frac{d\phi}{dt} = \frac{g\mu_B B_z}{\hbar}$$

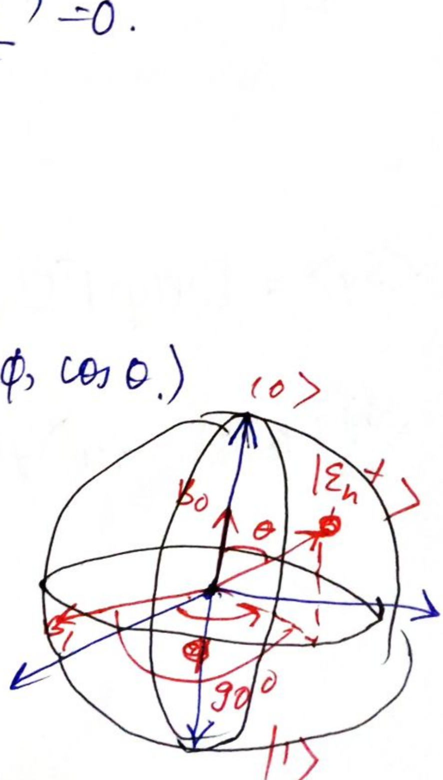
Rotation on Bloch sphere.

$$\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{bmatrix}$$

$B_0$ = time indep magnetic field along z

$B_1$ = rotating magnetic field in the $(x,y)$ plane chasing the spinor.



$= \cos\theta\, I + i \sin\theta A$ (using expansion formula of sine & cosine).

## Lecture 04

### Rotation on Bloch Sphere.

$\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ $\quad |\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}$

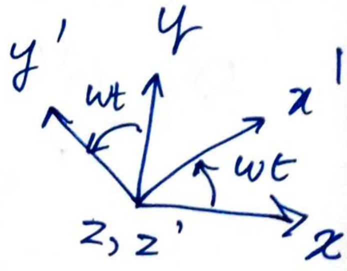
$B_0$ = time independent magnetic field along z-axis.
$B_1$ = rotating magnetic field in $(x,y)$ plane chasing the spinor.

$|0\rangle = |\xi_n^+ (\theta = 0, \phi, \gamma)\rangle$

$|1\rangle = |\xi_n^+ (\theta = \pi, \phi, \gamma)\rangle$

**Probability of spin flip**

We choose a new x and y axis upon rotation ~~translation~~




$\hat{x}' = \cos\omega t\, \hat{x} + \sin\omega t\, \hat{y}$

$\hat{y}' = -\sin\omega t\, \hat{x} + \cos\omega t\, \hat{y}$

$\hat{z}' = \hat{z}$

$\hat{n} = (n_x, n_y, n_z)$

$\hat{n}' = (n_x', n_y', n_z')$

$$\hat{n}' = \begin{bmatrix} n_x' \\ n_y' \\ n_z' \end{bmatrix} = [A]\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = [A]\hat{n}$$

$$[A] = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{n} = [A]^{-1}\hat{n}'$$

$$\frac{d\hat{n}}{dt} = [A]^{-1}\frac{d\hat{n}'}{dt} + \frac{d[A]^{-1}}{dt}\hat{n}' = [X]\hat{n} = [X][A]^{-1}\hat{n}'$$

Ralei formula:

$$[X'] = [A][X][A]^{-1} - [A]\frac{d[A]^{-1}}{dt}$$

$$\frac{d\hat{n}'}{dt} = [X']\hat{n}'$$

because $\frac{d\hat{n}}{dt} = [X]\hat{n}$ while $\frac{d\hat{n}'}{dt} = [X']\hat{n}'$

$$\therefore [A]^{-1}\frac{d\hat{n}'}{dt} + \frac{d[A]^{-1}}{dt}\hat{n}' = [A]^{-1}[X']\hat{n}' + \frac{d[A]}{dt}$$

$$\boxed{\therefore [X'] = [A][X][A]^{-1} - [A]\frac{d[A]^{-1}}{dt}}$$

$$\Rightarrow \boxed{\frac{d\hat{n}'}{dt} = \left\{ [A][X][A]^{-1} - [A]\frac{d[A]^{-1}}{dt} \right\}\hat{n}'}$$

$$\frac{d}{dt}\begin{bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{bmatrix} = \frac{g\mu_B}{\hbar}\begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix}\begin{bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{bmatrix}$$

because $\frac{d}{dt}\langle\sigma\rangle = |\gamma|(B \times \langle\sigma\rangle)$ $\quad |\gamma| = \frac{g\mu_B}{\hbar}$

$$\frac{d\hat{n}}{dt} = [K]\hat{n} \quad ; \quad B = (B_x, B_y, B_z)$$
$$= (B_1 \cos\omega t, B_1 \sin\omega t, B_0)$$

$$[A] = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 0 & \omega-\omega_0 & 0 \\ -(\omega-\omega_0) & 0 & -\omega_1 \\ 0 & \omega_1 & 0 \end{bmatrix}$$

$$\omega_0 = \frac{g\mu_B B_0}{\hbar} \qquad \omega_1 = \frac{g\mu_B B_1}{\hbar}$$

$$\boxed{\hat{n}'(t) = e^{[Q](t)}\hat{n}'(0)} \quad \Rightarrow \quad \boxed{\hat{n}(t) = [U](t)\,\hat{n}(0)}$$

$$[Q](t) = \int_0^t [X'](t')\,dt'$$

$$[Q](t) = \begin{bmatrix} 0 & (\omega-\omega_0)t & 0 \\ -(\omega-\omega_0)t & 0 & -\omega_1 t \\ 0 & \omega_1 t & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{bmatrix}$$

$$\alpha = (\omega-\omega_0)t \quad ; \quad \beta = -\omega_1 t$$

$$[P](t) = [S]^{-1}(t)\,[Q](t)\,[S](t) = \cancel{\mathrm{dig}(\lambda_1(t),\lambda_2(t),\lambda_3(t))}$$
$$= \mathrm{diag}(\lambda_1(t), \lambda_2(t), \lambda_3(t))$$

Let us find eigenvalues and eigenvectors of $[Q](t)$

$$\begin{vmatrix} -\lambda & \alpha & 0 \\ -\alpha & -\lambda & \beta \\ 0 & -\beta & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(\lambda^2+\beta^2) - \alpha(\alpha\lambda)$$
$$\Rightarrow -\lambda^3 - \lambda\beta^2 - \alpha^2\lambda$$
$$-\lambda^3 - \lambda(\beta^2+\alpha^2) = 0$$
$$\lambda(-\lambda^2 - 1(\beta^2+\alpha^2)) = 0 \Rightarrow \lambda = 0 \text{ or}$$
$$-(\lambda^2 + (\beta^2+\alpha^2)) = 0 \quad -\lambda^2 = (\beta^2+\alpha^2) \quad \text{or} \quad \lambda = i\sqrt{\alpha^2+\beta^2}$$
$$\text{or} -i\sqrt{\alpha^2+\beta^2}$$

eigenvectors $q_1 = \left[\frac{\beta}{\alpha}, 0, 1\right]$ ; $q_2 = \left[\frac{-\alpha}{\beta}, \frac{-i\sqrt{\alpha^2+\beta^2}}{\beta}, 1\right]$

$q_3 = \left[\frac{-\alpha}{\beta}, \frac{i\sqrt{\alpha^2+\beta^2}}{\beta}, 1\right]$

$$[S](t) = \begin{bmatrix} \frac{\beta}{\alpha} & -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ 0 & -\frac{i\sqrt{\alpha^2+\beta^2}}{\beta} & +\frac{i\sqrt{\alpha^2+\beta^2}}{\beta} \\ 1 & 1 & 1 \end{bmatrix}$$

$$[S]^{-1}(t) = \begin{bmatrix} \frac{\alpha\beta}{\alpha^2+\beta^2} & 0 & \frac{\alpha^2}{\alpha^2+\beta^2} \\ \frac{-\alpha\beta}{2(\alpha^2+\beta^2)} & \frac{i\beta}{2\sqrt{\alpha^2+\beta^2}} & \frac{0.5\beta^2}{\alpha^2+\beta^2} \\ \frac{-\alpha\beta}{2(\alpha^2+\beta^2)} & \frac{-i\beta}{2\sqrt{\alpha^2+\beta^2}} & \frac{0.5\beta^2}{\alpha^2+\beta^2} \end{bmatrix}$$

$$[Q](t) = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix} \qquad \begin{aligned} \alpha &= (\omega-\omega_0)t \\ \beta &= -\omega_1 t. \end{aligned}$$

$$e^{[Q](t)} = e^{[S](t)[P](t)[S]^{-1}(t)}$$

$$= [I] + [S][P][S]^{-1} + \frac{1}{2!}\left([S][P][S]^{-1}\right)^2 + \cdots$$

$$= [I] + [S]\left([P] + \frac{[P]^2}{2!} + \frac{[P]^3}{3!} + \cdots\right)[S]^{-1}$$

$$\boxed{e^{[Q](t)} = [S](t)\, e^{[P](t)}\, [S]^{-1}(t)}$$

$$\Rightarrow \hat{n}(t) = e^{[Q](t)}\, \hat{n}'(0)$$

$$n'(t) = [S](t)\, e^{[P](t)}\, [S]^{-1}(t)\, \hat{n}'(0)$$

$$\hat{n}' = [A]\hat{n} \quad ; \quad \hat{n}'(t) = [A](t)\, \hat{n}(t) \; ; \quad \hat{n}'(0) = [A](0)\, \hat{n}(0)$$

$$\hat{n}(t) = \underbrace{\left\{ [A]^{-1}(t) \left[ [S](t)\, e^{[P](t)}\, [S]^{-1}(t) \right] A(0) \right\}}_{[U](t).} \hat{n}(0)$$

$[U](t)$

$$= \begin{bmatrix} g(\delta,\chi)\sin\omega t + h(\delta,\chi)\cos\omega t & g(\delta,\chi)\cos\omega t - \cos\delta\sin\omega t & [f(\delta)\cos\omega t\cos\chi - \sin\delta\sin\omega t]\sin\chi \\ -g(\delta,\chi)\cos\omega t + h(\delta,\chi)\sin\omega t & g(\delta,\chi)\sin\omega t + \cos\delta\cos\omega t & [f(\delta)\sin\omega t\cos\chi - \sin\delta\cos\omega t]\sin\chi \\ f(\delta)\cos\chi\sin\chi & \sin\delta\sin\chi & \cos^2\chi + \sin^2\chi\cos\delta \end{bmatrix}$$

$$\delta = \sqrt{\alpha^2+\beta^2} = \sqrt{(\omega-\omega_0)^2 + \omega_1^2}\, t$$

$$f(\delta) = 1 - \cos\delta$$

$$h(\delta,\chi) = \cos\delta\cos^2\chi + \sin^2\chi$$

$$g(\delta,\chi) = \sin\delta\cos\chi$$

$$\chi = \tan^{-1}\frac{\omega_1}{\omega_0-\omega} \qquad \boxed{\sin^2\chi = \frac{\omega_1^2}{\omega_1^2 + (\omega_0-\omega)^2}}$$

$$\boxed{|\langle 1|\xi_n t\rangle|^2 = \frac{\sin^2\theta(t)}{2} = \frac{1-\cos\theta(t)}{2}} \qquad \boxed{\omega_0 = \frac{g\mu_B B_0}{\hbar}}$$

$$\boxed{|\langle 1|\xi_n t\rangle|^2 = \frac{1-n_z(t)}{2}}$$

$$\boxed{\hat{n}(t) = U(t)\, \hat{n}(0)} \qquad n(0) = (0,0,1)$$

$$\boxed{n_z(t) = \cos^2\chi + \sin^2\chi\cos\delta}$$

→ pos probability of spin flip in $|1\rangle$

$$|\langle 1|\xi_n^+\rangle|^2 = \frac{\sin^2\chi}{2}\left[1-\cos\delta(t)\right] \quad ; \quad n_z(t) = \cos^2\chi + \sin^2\chi \ldots$$

$\omega_0 = \frac{g\mu_B B_0}{\hbar}$ , Maximum when $\omega = \omega_0$

Spin-flip time :-

$$|\langle 1|\xi_n^+\rangle|^2 = \frac{\sin^2\theta(t)}{2} = \frac{1-\cos\theta(t)}{2} \quad ; \quad |\xi_n^+\rangle = e^{i\gamma}\left[\ldots\right]$$

$$|\langle 1|\xi_n^+\rangle|^2 = \frac{1-n_z(t)}{2} \quad ; \quad \omega_0 = \frac{g\mu_B B_0}{\hbar}$$

$$\hat{n}(t) = U(t)\hat{n}(0) \quad ; \quad \hat{n}(0) = (0,0,1)$$

$$t_s = \frac{T}{2} = \frac{\pi}{\omega_1} = \frac{\pi\hbar}{g\mu_B B_1}$$ which is maximum when $\omega = \omega_0$

---

Important points :-

Prove that $\frac{d\phi}{dt} = \frac{g\mu_B B_z}{\hbar}$

$$\frac{dM}{dt} = M\left(\frac{d\theta}{dt} + \sin\theta\frac{d\phi}{dt}\right) \quad ; \quad \frac{dr}{dt} = 0$$ because the tip is always on the sphere.

$$\frac{d}{dt}\left(|\gamma| B\times M\right) = |\gamma| B_c M\sin\theta.$$

$$\frac{d\phi}{dt} = |\gamma| B_z = \frac{g\mu_B B_z}{\hbar}.$$

Q) Show that $|\langle \xi_{+x}|\xi_n^+\rangle|^2 = \frac{1+n_x(t)}{2}$

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\theta/2 \\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix}$$