States, Observables, Measurements, and Dynamics in Quantum Mechanics

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Outline

- (I) A brief historical background of QM together with the notions of complementarity, uncertainty, and superposition principles
- (II) Axiomatic approach to QM
- (III) Most general descriptions for states, measurements, and dynamics in QM
- (IV) Examples
- (V) State preparation and identification; structure of state space
- (VI) von Neumann's measurement model and related issues
- (VII) Characterization of most general physical dynamics
- (VIII) Discussion

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for (I) + (II) + (III)

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(V) State preparation and identification; structure of state space

Bonafide states

- The Hilbert space structure $\mathcal{H}_S$ of the set of all pure states of **any** given quantum system S **guarantees** the very existence of any (normalized) linear superposition $|\psi\rangle = \sum_{j=1}^{d} \alpha_j |j\rangle$ (where $\{|j\rangle : j = 1, 2, \ldots d = \dim \mathcal{H}_S\}$ is an ONB for $\mathcal{H}_S$) as a **bonafide** pure state of S. .
- Existence of $|\psi\rangle$ as a bonafide state will be in question if there is some **superselection rule** which disallows some of these $|\psi\rangle$'s.
- As for example, in case S consists of two Fermions, those $|\psi\rangle$'s are not allowed each of which is not anti-symmetric under the exchange of the two Fermions.
- Evolution of any *Gaussian* state (*i.e.*, with Gaussian Wigner function) under a quadratic (in a and $a^\dagger$) Hamiltonian can never give rise to a *non-Gaussian* state.

State preparation (known input and pure output)

- **Pure state to pure state:** If one can prepare S in a particular pure state $|\psi_0\rangle$ (*e.g.*, preparating a single photon in a vertically polarized state $|V\rangle = a_V^\dagger|0\rangle$), by applying a suitable unitary evolution U on $|\psi_0\rangle$ – generated by some Hamiltonian $\hat{H}$ – one can **always** find S in **any** desired state $|\psi\rangle = U|\psi_0\rangle$, unless there is any restriction on the choice of $\hat{H}$.
- **Mixed to pure state:** Given **any** mixed state $\rho = \sum_{j=1}^{d} \lambda_j |\psi_j\rangle\langle\psi_j|$ (spectral decomposition), one can **always** perform the PVM $\mathcal{M} = \{P_j \equiv |\psi_j\rangle\langle\psi_j| : j = 1, 2, \ldots d\}$ on ρ – unless there is any restriction (*e.g.*, energy) in realizing such a measurement – to get the j-th post-measurement state $|\psi_j\rangle$ (with prob. $\mathrm{Tr}[P_j\rho] = \lambda_j$), which can then be changed to **any** desired state $|\psi\rangle = U_j|\psi_j\rangle$ via some unitary evolution U_j.

State preparation (known input and mixed output)

- **Pure to mixed state:** If S is prepared in **any** given state $|\psi_0\rangle$, one can apply an unitary evolution U_j with *apriori* probability λ_j – unless there is any restriction on realizing such unitaries – to prepare S in the state $|\psi_j\rangle = U_j|\psi_0\rangle$, so that the average state of S becomes the **desired** mixed state $\rho = \sum_{j=1}^{d} \lambda_j |\psi_j\rangle\langle\psi_j|$ (*not necessarily* the spectral decomposition of ρ).
- **Mixed to mixed state:** If S is prepared in **any** given mixed state ρ_0, it can **always** be changed into some (known) pure state $|\psi_0\rangle$ of S, and the latter can then be changed to **any** desired mixed state ρ of S – as mentioned above.

State preparation (unknown input and pure output): Quantum Erasure

- Given that S is prepared in **any** state ρ_0 (unknown) – pure or mixed – one can **always** perform the PVM $\mathcal{M} = \{P_0 \equiv |\psi\rangle\langle\psi|, P_1 \equiv |\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|, P_2 \equiv |\psi_2^{\perp}\rangle\langle\psi_2^{\perp}|, \ldots, P_{d-1} \equiv |\psi_{d-1}^{\perp}\rangle\langle\psi_{d-1}^{\perp}|\}$ on ρ_0 – unless there is any restriction in realizing such a PVM – to have either the 0-th output P_0 (in which case S has already been prepared in the **desired** state $|\psi\rangle$) or j-th output P_j (for $j = 1, 2, \ldots, d-1$), in which case, a suitable unitary U_j is applied to the collapsed state $|\psi_j\rangle$ to prepare S in the desired state $|\psi\rangle = U_j|\psi_j\rangle$.

Examples

- **Example 1:** Given that a bi-partite system $S = A + B$ (with $\dim\mathcal{H}_A = d = \dim\mathcal{H}_B$) is initially prepared in a maximally entangled state (MES) $|\psi_{max}\rangle = (1/\sqrt{d}) \times \sum_{j=1}^{d} |j\rangle_A \otimes |j\rangle_B$, by performing quantum operations **separately** on A and B, and communicating the outcomes via classical channel (*e.g.*, telephone) – the so-called 'LOCC' operations – one can **always** come up with **any** pure state $|\psi\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j}|\phi_j\rangle_A \otimes |\eta_j\rangle_B$ (Schmidt decomposition with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$) of S:
$(1/d, 1/d, \ldots, 1/d) \prec (\lambda_1, \lambda_2, \ldots, \lambda_d)$ (majorization).

- **Example 2:** From **no** mixed state ρ of S, the MES $|\psi_{max}\rangle$ of S can be obtained by LOCC – not even **probabilistically**.

Two-qubit examples

- Through the action of the dichotomic POVM $\mathcal{M} = \{E_A \equiv \lambda|0\rangle_A\langle 0| + (1-\lambda)|1\rangle_A\langle 1|, E_A^{\perp} \equiv (1-\lambda)|0\rangle_A\langle 0| + \lambda|1\rangle_A\langle 1|\}$ on A and followed by either no action (in case E_A 'clicks') or action of the unitary $U = |0\rangle\langle 1| + |1\rangle\langle 0|$ separately on both A as well as B (in case $E_A^{\perp}$ 'clicks'), $|\psi_{max}\rangle = (1/\sqrt{2}) \times (|00\rangle_{AB} + |11\rangle_{AB})$ gets transformed into $|\psi\rangle = \sqrt{\lambda}|00\rangle_{AB} + \sqrt{1-\lambda}|11\rangle_{AB}$ **deterministically**.

- A two-qubit mixed state even of the form $\rho = (1-\epsilon)|\phi^+\rangle_{AB}\langle\phi^+| + \epsilon|\phi^-\rangle_{AB}\langle\phi^-|$ (where ϵ is a very small positive no. and $|\phi^{\pm}\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$) can not be transformed into the state $|\phi^+\rangle$ with a **non-zero** probability, using LOCC only.

Impossibility of state transformation under LOCC even aymptotically

- Although a two-qubit mixed *entangled* state ρ can not be transformed into a two-qubit maximally entangled state with any non-zero probability, nevertheless, one can 'distill' a few copies of the two-qubit MES using LOCC when a large no. of copies of ρ is shared between two labs.
- But there do exist bi-partite entangled states from none of which – even if infinitely many copies of each of the states are shared between distant labs – one (or, more than one) copy of a bi-partite MES can be distilled, using LOCC only. These are the so-called 'bound entangled' states (BES).
- Note that LOCC preparation/formation of each such BES does require non-zero amount of 'free' entanglement (*i.e.* MES) – *non-zero entanglement cost.*

State identification

- In classical world, any set of *different* states are **perfectly** distinguishable. In the quantum world, the very existence of non-orthogonal states diminishes the power of distinguishability of different states.
- The notion of state distinguishability in the quantum world becomes meaningful *only* when the states (of a given quantum system) are **linearly independent** – as linearly dependent states will have *completely overlapping supports*.
- In case the operations are restricted to LOCC, even some pairwise orthogonal bi-partite (or, multi-partite) states remain indistinguishable – determinitically/probabilistically (*e.g.*, the set of four Bell states of two-qubits).

Unambiguous vs. minimum-error state discrimination

- Given a state ρ_j with *apriori* probability p_j from a set $\mathcal{S}_n = \{\rho_j, p_j | j = 1, 2, \ldots, n\}$ of n linearly independent states ρ_1, ρ_2, ..., ρ_n of a quantum system S of dim. d ($\geq n$), the **unambiguous** discrimination of the states would require existence of an $(n+1)$-element POVM $\mathcal{M}_{n+1} = \{E_j | j = 1, 2, \ldots, n+1\}$ on the states of S such that $\mathrm{Tr}[E_j \rho_k] = 0$ for all $j, k = 1, 2, \ldots, n$ with $j \neq k$, while $\mathrm{Tr}[E_j \rho_j] > 0$ for $j = 1, 2, \ldots, n$. To minimize the error prob. $P_{err} = \sum_{j=1}^{n} p_j \ \mathrm{Tr}[E_{n+1} \rho_j]$.

- For the **minimum-error discrimination** of the states in $\mathcal{S}_n$, there must exist an n-element PVM $\mathcal{M}_n = \{P_j | j = 1, 2, \ldots, n\}$ on the states of S such that the error prob. $P_{err} = \sum_{j=1}^{n} p_j \sum_{k(\neq j)=1}^{n} \ \mathrm{Tr}[P_k \rho_j]$ is minimum.

Minimum-error discrimination of two states

- Assume that the system S is prepared in one of the two states ρ_1, ρ_2 with respective prior probabilities p_1, p_2 (with $p_1 + p_2 = 1$).
- Let $p_1\rho_1 - p_2\rho_2 = \sum_{j=1}^{d_+} \lambda_j^+ |\psi_j^+\rangle\langle\psi_j^+| - \sum_{j=1}^{d_-} \lambda_j^- |\psi_j^-\rangle\langle\psi_j^-|$ (spectral decomposition) where $\lambda_j^\pm \geq 0$ and $d_+ + d_- = d$, the system dim.
- Apply now the PVM $\mathcal{M}_2 = \{P_+ \equiv \sum_{j=1}^{d_+} |\psi_j^+\rangle\langle\psi_j^+|, P_- \equiv \sum_{j=1}^{d_-} |\psi_j^-\rangle\langle\psi_j^-|\}$. If P_+ (P_-) clicks, *declare* the supplied state to be ρ_1 (ρ_2).
- The min. error prob. $P_{err} = p_1 \ \mathrm{Tr}[P_-\rho_1] + p_2 \ \mathrm{Tr}[P_+\rho_2] = (1/2) \times (1 - ||p_1\rho_1 - p_2\rho_2||_1)$, where the trace-norm $||p_1\rho_1 - p_2\rho_2||_1 = \sum_{j=1}^{d_+} \lambda_j^+ + \sum_{j=1}^{d_-} \lambda_j^-$.

State estimation

- For identifying a state $\rho(\vec{\theta})$ of a system S – supplied with *apriori* prob. $p(\vec{\theta})$ (where $\vec{\theta}$ is (generally) a vector $(\theta_1, \theta_2, \ldots, \theta_n) \in \Lambda$ of **continuous** parameters), one needs to choose an appropriate POVM $\mathcal{M} = \{E(\vec{\alpha})|\vec{\alpha} \in \Omega\}$ on the states of S such that clicking of $E(\vec{\alpha})$ would declare the state of S to be $\sigma(\vec{\alpha})$. Thus, the average post-measurement state (for given $\rho(\vec{\theta})$) is: $\sum_{\vec{\alpha}\in\Omega}\ \mathrm{Tr}[E(\vec{\alpha})\rho(\vec{\theta})] \times \sigma(\vec{\alpha}) \equiv \tau(\vec{\theta})$ (say).
- The overlap of the original input state $\rho(\vec{\theta})$ with average guessed state $\tau(\vec{\theta})$ is given by some 'fidelity' $F(\vec{\theta}|\mathcal{M})$. As for example, $F(\vec{\theta}|\mathcal{M}) = \ \mathrm{Tr}\sqrt{\sqrt{\rho(\vec{\theta})}\tau(\vec{\theta})\sqrt{\rho(\vec{\theta})}}$ (Bures metric).
- The 'average fidelity' for the estimation scheme is: $\overline{F}_{\mathcal{M}} \equiv \int_{\vec{\theta}\in\Lambda} d\vec{\theta} p(\vec{\theta}) F(\vec{\theta}|\mathcal{M})$ – needs to be maximized over all choices of $\mathcal{M}$ and $\sigma(\vec{\alpha})$.

State estimation: single qubit

- Choose $\rho(\vec{\theta}) = |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)|$ with $p(\vec{\theta}) = \frac{\sin\theta d\theta d\phi}{4\pi}$ and $|\psi(\theta,\phi)\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ (for $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$).
- Choose the PVM $\mathcal{M} = \{E_0 = |0\rangle\langle 0|, E_1 = |1\rangle\langle 1|\}$ with the corresponding declared states $\sigma_0 = |0\rangle\langle 0|$, and $\sigma_1 = |1\rangle\langle 1|$.
- Here $\tau(\vec{\theta}) = \sum_{j=0}^{1} |\langle j|\psi(\theta,\phi)\rangle|^2 |j\rangle\langle j|$ and $F(\vec{\theta}|\mathcal{M}) = \langle\psi(\theta,\phi)|\tau(\vec{\theta})|\psi(\theta,\phi)\rangle =$ $\langle\psi(\theta,\phi)|(\cos^2(\theta/2)|0\rangle\langle 0| + \sin^2(\theta/2)|1\rangle\langle 1|)|\psi(\theta,\phi)\rangle$ $= \cos^4(\theta/2) + \sin^4(\theta/2) = 1 - (1/2)\times\sin^2\theta.$
- Thus $\overline{F}_{\mathcal{M}} = \frac{1}{4\pi} \times \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} d\theta d\phi \sin\theta(1 - (1/2)\sin^2\theta) = 2/3$, better than simply the guessing fidelity $1/2$.

State estimation: parallel vs. anti-parallel qubits

- For $\rho(\vec{\theta}) = (|\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)|)^{\otimes 2}$ with $p(\vec{\theta}) = \frac{\sin\theta d\theta d\phi}{4\pi}$, the PVM $\mathcal{M} = \{E_j = |\Psi_j\rangle\langle\Psi_j| : |\Psi_j\rangle = (1/2)|\psi^-\rangle + (\sqrt{3}/2)|\psi(\theta_j,\phi_j)\rangle^{\otimes 2}$ for $j = 1,2,3,4\}$ (where $(\theta_1,\phi_1) = (0,0)$, $(\theta_2,\phi_2) = (\cos^{-1}(-1/3),0)$, $(\theta_3,\phi_3) = (\cos^{-1}(-1/3),2\pi/3)$, and $(\theta_4,\phi_4) = (\cos^{-1}(-1/3),4\pi/3)$) and the declared state $\sigma_j = (|\psi(\theta_j,\phi_j)\rangle\langle\psi(\theta_j,\phi_j)|)^{\otimes 2}$ corresponding to clicking of E_j ($j = 1,2,3,4$) would give rise to the optimal average fidelity $\overline{F}_{\mathcal{M}} = 3/4$.

- On the other hand, for $\rho(\vec{\theta}) = |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| \otimes |\psi(\pi-\theta,\pi+\phi)\rangle\langle\psi(\pi-\theta,\pi+\phi)|$ and $p(\vec{\theta}) = \frac{\sin\theta d\theta d\phi}{4\pi}$, the optimal average fidelity is $(1/3)\times(5\sqrt{3}+33)/(3\sqrt{3}-1)^2$ $(> 3/4)$.

Quantum Metrology

- A more general treatment of quantum state (or, channel) estimation problem – the parameter $\vec{\theta}$ may be encoded into the system states in an *arbitrarily given way* – is via quantum metrology, based on the quantum mechanical generalization of the classical **Fisher Information** and **Cramer-Rao** bound on it.
- In this case, the Cramer-Rao bound puts a **lower bound** on the amount of **uncertainty/error** – realized via Fisher Information – in persuing the estimation process.
- Quantum Cramer-Rao bound puts a **tighter** lower limit on the uncertainty/error compared to its classical counterpart. And the challenge is to achieve this tighter bound.

State tomography

- To determine a **completely unknown** state of a given quantum system S, **infinitely many** copies of the state needs to be supplied by the source (producing the state) and different measurent settings are used on different subgroups (each of infinite size) of copies of the state to determine **all** the state parameters.
- If a d-outcome PVM $\mathcal{M} = \{P_j | j = 1, 2, \ldots, d\}$ is measured on a state ρ of a d dim. quantum system S, then one gets $(d-1)$ no. of independent average values $\mathrm{Tr}[P_j \rho]$ (for $j = 1, 2, \ldots, d-1$) – as $\sum_{j=1}^{d} \mathrm{Tr}[P_j \rho] = 1$. We would then require a **min.** of $(d+1)$ no. of different such d-outcome PVMs in order to determine $(d-1) \times (d+1) = d^2 - 1$ no. of state parameters for ρ (as $\mathrm{Tr}[\rho] = 1$) [**MUB**].

Mutually unbiased bases (MUBs)

- n ($\leq d+1$) no. of d-outcome PVMs $\mathcal{M}_j = \{P_{jk} | k = 1, 2, \ldots, d\}$ (for $j = 1, 2, \ldots, d$) for a d dim. system form MUBs iff $\mathrm{Tr}[P_{jk} P_{j'k'}] = (1/d)$ for all $j, j' = 1, 2, \ldots, n$ with $j \neq j'$.
- For $d = p^n$ where p is any prime no. and n being any positive integer, **exactly** $(d+1)$ no. of such MUBs always exist.
- When $d = p_1^{n_1} \times p_2^{n_2} \times \ldots$ – where p_1, p_2, ..., are **different** prime nos. and $n_1, n_2, \ldots$ are positive integers – proof/disproof of existence of such $(d+1)$ no. MUBs is still *elusive*.
- For example, for $d = 6 = 2 \times 3$, one can always provide a set of **three** MUBs, and that no. seems to be the *maximum* achievable value.

Example of two MUBs in any dimension

- **Finite dim.:** Consider an ONB $\mathcal{B}_1 = \{|j\rangle : j = 0, 1, \ldots, d-1\}$ of a d dim. Hilbert space $\mathcal{H}$. Take $\omega = \exp[2\pi i/d]$. Consider now the Fourier transformed basis $\mathcal{B}_2 = \{|\phi_j\rangle \equiv (1/\sqrt{d}) \times \sum_{k=0}^{d-1} \omega^{jk}|k\rangle \ : \ j = 0, 1, \ldots, d-1\}$. Note that here $|\langle j|\phi'_j\rangle| = 1/\sqrt{d}$ for all $j, j' = 0, 1, \ldots, d-1$. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ are two MUBs.
- **CV case:** Consider now the following two bases: $\mathcal{B}_1 = \{|x\rangle : \langle x|x'\rangle = \delta(x - x')\}$ and $\mathcal{B}_2 = \{|p\rangle : \langle p|p'\rangle = \delta(p - p')\}$ – respectively the bases for position and momentum eigen states. As the position representation $\langle x|p\rangle$ of the momentum eigenstate $|p\rangle$ is given by: $\langle x|p\rangle = (1/\sqrt{2\pi\hbar}) \times e^{ipx/\hbar}$, therefore, for each pair (x, p), we have: $|\langle x|p\rangle| = 1/\sqrt{2\pi\hbar}$, and hence, the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ may be considered to be MUBs.

Structure of state space

- When $(d+1)$ no. of MUBs do not exist for a d dim. quantum system, how does one choose 'appropriate' measurement bases to perform tomography?
- In order to answer this question in an effective way, we need to understand the algebraic and/or geometric structure of the set of all states of a given quantum system.
- We will start with the simplest case – namely the case of a single qubit.

Bloch sphere representation for single qubit: pure states

- Consider the ONB $\{|0\rangle, |1\rangle\}$ for a two-level quantum system S, where $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$.
- Any pure state $|\psi\rangle$ of S can be taken as: $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \equiv |\psi(\theta,\phi)\rangle$ (say), where $\theta \in [0,\pi]$, and $\phi \in [0, 2\pi)$.
- In fact, one can verify that $(\hat{n}.\vec{\sigma})|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle$ while $(\hat{n}.\vec{\sigma})|\psi(\pi-\theta,\pi+\phi)\rangle = -|\psi(\pi-\theta,\pi+\phi)\rangle$, where $\hat{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ is a unit vector in $\boldsymbol{R}^3$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.
- $|\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| = \cos^2(\theta/2)|0\rangle\langle 0| + \sin^2(\theta/2)|1\rangle\langle 1| + (1/2)\times \sin\theta(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|) = (1/2)\times(I_{2\times 2} + \hat{n}.\vec{\sigma})$, $I_{2\times 2}$ being the 2 dim. identity matrix ($\hat{n}$: **bloch vector**).

Bloch sphere representation for single qubit: mixed states

- For any mixed state $\rho = \sum_{j=1}^{m} p_j |\psi(\theta_j, \phi_j)\rangle\langle\psi(\theta_j, \phi_j)|$ (where $0 \leq p_j \leq 1$ and $\sum_{j=1}^{m} p_j = 1$) of a qubit, we have: $\rho = (1/2) \times (I_{2\times 2} + (\sum_{j=1}^{m} p_j \hat{n}_j).\vec{\sigma})$ where $\hat{n}_j = (\sin\theta_j \cos\phi_j,\ \sin\theta_j \sin\phi_j,\ \cos\theta_j)$ for $j = 1, 2, \ldots, m$.
- Here $|\sum_{j=1}^{m} p_j \hat{n}_j| \leq \sum_{j=1}^{m} p_j = 1$, where equality holds iff $\hat{n}_1 = \hat{n}_2 = \ldots = \hat{n}_m$.
- Thus, for any single-qubit state (pure or mixed) ρ, we have: $\rho = (1/2) \times (I_{2\times 2} + \vec{n}.\vec{\sigma})$ where $\vec{n}$ is a real 3 dim. vector with $|\vec{n}| \leq 1$, and equality holds **iff** $\vec{n}$ is a unit vector (**equivalently**, ρ is a pure state). $\vec{n}$: **bloch vector** of ρ. Conversely, any 2×2 matrix of the form $(1/2) \times (I_{2\times 2} + \vec{n}.\vec{\sigma})$ is a bonafide single-qubit density matrix provided $|\vec{n}| \leq 1$.
- Thus state space of a single qubit is: $\{\vec{n} \in \mathbb{R}^3 : |\vec{n}| \leq 1\}$.

State space for higher dimensional quantum systems

- Basis of (d^2-1) no. of 'generalized Pauli' matrices: $\{\Lambda_{jk}^{(d)} \equiv |j\rangle\langle k| + |j\rangle\langle k| : 1 \leq j < k \leq d\} \bigcup \{\Gamma_{jk}^{(d)} \equiv -i|j\rangle\langle k| + i|j\rangle\langle k| : 1 \leq j < k \leq d\} \bigcup \{\Sigma_l^{(d)} \equiv \sqrt{2/(l(l+1))} \times (\sum_{j=1}^{l} |j\rangle\langle j| - l|l+1\rangle\langle l+1|) : 1 \leq l \leq d-1\}$, where $\{|j\rangle : j = 1, 2, \ldots, d\}$ is a ONB for a d dim. quantum system.
- $\rho = (1/d) \times (I_{d\times d} + \sum_{1\leq j<k\leq d} \lambda_{jk}\Lambda_{jk}^{(d)} + \sum_{1\leq j<k\leq d} \gamma_{jk}\Gamma_{jk}^{(d)} + \sum_{l=1}^{d-1} \sigma_l \Sigma_l^{(d)})$ where λ_{jk}'s γ_{jk}'s and σ_l's are real coefficients satisfying the positivity condition for ρ. The **bloch vector** $\vec{b} = (\lambda_{12}, \lambda_{13}, \ldots, \lambda_{(d-1)d}, \gamma_{12}, \gamma_{13}, \ldots, \gamma_{(d-1)d}, \sigma_1, \sigma_2, \ldots, \sigma_{d-1})$ of ρ must satisfy: $|\vec{b}| \leq \sqrt{(d(d-1))/2}$.
- $|\vec{b}| \leq \sqrt{d/(2(d-1))}$ **always** corresponds to a bonafide state.

(VI) von Neumann's measurement model

Measurement of an observable

- In order to measure an observable $\hat{A}$ of a quantum system S, when the latter is prepared in the state ρ, use the interaction Hamiltonian $\hat{H}_{SA} = g(\hat{A} \otimes \hat{X})$, acting on the joint system $S + A$, with A being the measuring aparatus and $\hat{X}$ being the position observable of A. g: interaction strength.
- We assume the initial state A to be $|\phi\rangle$, whose position wave function $\phi(x)$ is typically considered to be a Gaussian one with centre at the origin.
- Thus the joint initial state of $S + A$ is: $\eta_{SA}(0) = \rho_S \otimes |\phi\rangle_A\langle\phi|$.
- $\eta_{SA}(t) = \exp[-it\hat{H}_{SA}]\eta_{SA}(0)\exp[it\hat{H}_{SA}]$.
- Now measurement of the momentum observable $\hat{P}$ on A is done to detect its position displacement – through which, S will collapse into an eigenstate of $\hat{A}$.

Arthur-Keley's model of joint measurement

- In order to measure two system observables $\hat{A}$ and $\hat{B}$ jointly, consider the interaction Hamiltonian: $\hat{H}_{SA} = g_1(\hat{A} \otimes \hat{X}_1) + g_2(\hat{B} \otimes \hat{X}_2)$ for a measuring apparatus with two dim. configuration space (and hence, $[\hat{X}_1, \hat{X}_2] = 0$; $[\hat{P}_1, \hat{P}_2] = 0$). g_1, g_2: coupling strenghts.
- Initial state of $S + A$: $\eta_{SA}(0) = \rho_S \otimes |\phi\rangle_A\langle\phi|$, where ρ_S is arbitrary but $\phi(x_1, x_2) \equiv \langle x_1, x_2|\phi\rangle$ is considered (generally) to be Gaussian with origin as its centre.
- $\eta_{SA}(0)$ is now evolved under $\hat{H}_{SA}$ for some time t to get $\eta_{SA}(t)$.
- Now measurement of $\hat{P}_1$ and $\hat{P}_2$ is done simultaneously on A to get its position displacements, which, in turn, provides collapse of S into a joint eigenstate of $\hat{A}$ and $\hat{B}$ (provided they are **compatible**).

Question of joint measurement

- Two PVMs $\mathcal{M}_1 = \{P_j : j = 1, 2, \ldots, d_1\}$ and $\mathcal{M}_2 = \{Q_k : k = 1, 2, \ldots, d_2\}$ – acting on the states of a d dim. quantum system (with $d_1, d_2 \leq d$) – are **jointly measurable (or, compatible)** iff $[P_j, Q_k] = 0$ for all j, k. Otherwise, they are **not jointly measurable (or, incompatible)**.
- Two POVMs $\mathcal{M}_1 = \{E_j : j = 1, 2, \ldots, d_1\}$ and $\mathcal{M}_2 = \{F_k : k = 1, 2, \ldots, d_2\}$ – acting on the states of a d dim. quantum system – are jointly measurable (or, compatible) iff there exists another POVM $\mathcal{M} = \{G_{jk} | j = 1, 2, \ldots, d_1; k = 1, 2, \ldots, d_2\}$ – acting on the same system – such that $E_j = \sum_{k=1}^{d_2} G_{jk}$ and $F_k = \sum_{j=1}^{d_1} G_{jk}$. Otherwise, the POVMs are not jointly measurable (or, incompatible).

Joint measurement of unsharp observables

- By adding noise to any observable, one gets an **unsharp** observable: a PVM $\mathcal{M} = \{P_j : j = 1, 2, \ldots, d_1\}$ (with $d_1 \leq d$, the dim. of the system) changes into a POVM $\mathcal{M}(\lambda) = \{E_j \equiv (1-\lambda)P_j + (\lambda/d_1) \times I_{d\times d} : j = 1, 2, \ldots, d_1\}$ (say), where $0 \leq \lambda \leq 1$. (λ : **unsharpness parameter**).
- The unsharp versions of two (or, more) PVMs can become compatible for certain range of values of the unsharpness parameter(s) – even if the initial PVMs are incompatible. Similar is the case with POVMs.
- As for example, the unsharp versions $\mathcal{M}_x(\lambda_1) = \{E_x^+ \equiv (1-\lambda_1)P_x^+ + (\lambda_1/2)I_{2\times 2}, E_x^- \equiv (1-\lambda_1)P_x^- + (\lambda_1/2)I_{2\times 2}\}$ and $\mathcal{M}_z(\lambda_2) = \{E_z^+ \equiv (1-\lambda_2)P_z^+ + (\lambda_2/2)I_{2\times 2}, E_z^- \equiv (1-\lambda_2)P_z^- + (\lambda_2/2)I_{2\times 2}\}$ become compatible iff $\lambda_1^2 + \lambda_2^2 \leq 1$.

(VII) Characterization of most general physical dynamics

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Time-local most general open system dynamics

- When a quantum system S interacts with its environment E (often considered as a heat bath), the time-evolved state $\rho(t)$ will, in general, depend on its past: $\frac{d}{dt}\rho(t) = -i\int_0^t ds\mathcal{K}(t,s)[\rho(s)]$, where $\mathcal{K}(t,s)$ is the 'memory kernel' operator (Nakazima-Zwanzig master eqn.) – extremely difficult to solve for $\rho(t)$!
- On the other hand, the time-convolutionless time-local master eqn. for an 'open quantum system' is given by: $\frac{d}{dt}\rho(t) = -i\mathcal{G}(t,0)[\rho(t)]$ (Tokuyama and Mori master eqn.).
- The **most general time-local** master eqn. of an open quantum system can be cast into the 'Lindblad form': $\frac{d}{dt}\rho(t) = -i[\hat{H},\rho(t)] + \sum_\alpha \gamma_\alpha(t) \times [L_\alpha(t)\rho(t)(L_\alpha(t))^\dagger - (1/2) \times \{(L_\alpha(t))^\dagger L_\alpha(t), \rho(t)\}]$, where $\gamma_\alpha(t)$'s are decay rates (real nos.) and $L_\alpha(t)$'s are Lindblad operators.

(VIII) Discussion

Most general structures of states, measurements, and dynamics for quantum systems

- States: **density matrices** (can be considered as a reduced state of a pure state of system and ancilla)
- Measurements: **POVMs** (can be realized as a PVM on an extended system (system plus ancilla))
- Post-measurement state: depends on the **realization** of the POVM as a PVM on extended system
- Dynamics: **CPTP maps** (time-dependent), corresponding to some master equation (can be realized as a unitary dynamics on system plus ancilla, followed by tracing out the ancilla degree of freedom)
- With such a formalism, QM can now, in principle, handle realistic situations!

References

- Nielsen and Chuang's book on Quantum Information and computation (QIC)
- Asher Peres' book on Quantum Mechanics
- Preskill's Caltech lecture notes on QIC
- Book of Bengtsson and Zyczkowski on geometry of quantum states
- Look at different works of Heinosaari on joint measurements
- Breuer and Petruccione's book on open quantum systems