Lec 1: Mathematical Preliminaries. (Dr. PRABHA MANDAYAM)

Note Title 03-01-2022

Quantum Bit (Qubits) vs. Classical Bit

$\downarrow$ vector in a $\boxed{\text{linear vector space.}}$

$\downarrow$ $x \in \{0,1\}$ Binary-valued variable (field)

Points of discussion: (1) Linear Vector space
(2) Superposition
(3) Orthogonal
(4) Transform Quantum bits (Linear operators)

Bra-ket notation (Dirac notation)

x ———— x

(1) (Finite-dimensional) linear vector space (LVS) (complex field)

vector $|v\rangle \longrightarrow$ "ket" $- v$

(LVS) $\mathbb{V} \leftarrow$ set of vectors that satisfy:

(i) If $|v\rangle, |w\rangle \in \mathbb{V}$, $|v\rangle + |w\rangle \in \mathbb{V}$.

(ii) If $\alpha \in \mathbb{C}$ and $|v\rangle \in \mathbb{V}$, $\alpha|v\rangle \in \mathbb{V}$.

(complex scalar) (complex field)

(iii) If $|v\rangle, |w\rangle \in \mathbb{V}$, $\alpha, \beta \in \mathbb{C}$,

$\alpha|v\rangle + \beta|w\rangle \in \mathbb{V}$.

linearity. $\rightarrow$ complex linear combination

(iv) Zero-vector in $\mathbb{V}$. $\rightarrow |\phi\rangle$

$|v\rangle + |\phi\rangle = |v\rangle$, $\forall\ |v\rangle \in \mathbb{V}$.

Example: (1) Two-level quantum system. ≡ Quantum Bit (Qubit)

Two-dimensional complex linear vector space. $\mathbb{C}^2$

$|0\rangle, |1\rangle \in \mathbb{C}^2$.

(ket-0) (ket-1)

— Any element of $\mathbb{C}^2$ can be described as

$$\alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$$

$(\alpha, \beta \in \mathbb{C})$

— Representation? Column vectors / Row vectors.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times 1} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times 1}$$

⇒ $\mathbb{C}^2$ is the set of all (complex) column vectors of length-2

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle$$

* Every element can be represented as linear combinations of two vectors

$\boxed{|0\rangle, |1\rangle}$ → $\boxed{\text{Basis}}$ for the $\mathbb{C}^2$ space.

→ dimensionality! (2-dim complex LVS)

* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ → zero vector

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* Linear independence: $|v_1\rangle, |v_2\rangle \in \mathbb{V}$.

They are said to be linearly independent, if

$$\alpha|v_1\rangle + \beta|v_2\rangle = 0 \iff \alpha, \beta = 0.$$

(if and only if)

$\Downarrow$

linearly dependent ← $\alpha|v_1\rangle = -\beta|v_2\rangle$

$|v_1\rangle, |v_2\rangle, \dots |v_d\rangle \in \mathbb{V}$.

Linearly independent if $a_1|v_1\rangle + a_2|v_2\rangle + \cdots a_d|v_d\rangle = 0$ holds if and only if $a_1 = a_2 = \cdots a_d = 0$.

Example: (i) $|0\rangle, |1\rangle$:

Suppose $\alpha|0\rangle + \beta|1\rangle = 0$.

$$\Rightarrow \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = 0 \Rightarrow \alpha = \beta = 0.$$

(ii) $|+\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$, $|-\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

Exercise: (a) Are $|+\rangle, |-\rangle$ linearly independent?

(b) Are $\{|0\rangle, |+\rangle\}$, $\{|1\rangle, |-\rangle\}$ linearly independent?

$\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ linearly independent?

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(2) d-dimensional complex linear vector space :-

Representation : column vectors of length- d. $\equiv \mathbb{C}^d$

$$|a\rangle = \begin{pmatrix}a_1\\a_2\\\vdots\\a_d\end{pmatrix}_{d\times 1}$$

Elementary set of vectors for $\mathbb{C}^d$:-

$|0\rangle, |1\rangle, |2\rangle, \dots |d-1\rangle$.

$$|0\rangle = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \quad \dots \quad |d-1\rangle = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

Check: (i) $\{|0\rangle, |1\rangle, \dots |d-1\rangle\}$ is linearly independent.

(ii) Any $|v\rangle \in \mathbb{V}$ can be written as

$$|v\rangle = v_0|0\rangle + v_1|1\rangle + \cdots\ v_{d-1}|d-1\rangle.$$

$\{|0\rangle, |1\rangle, \ldots |d-1\rangle\}$ is a basis for $\mathbb{C}^d$.

↳ d elements $\Rightarrow$ d-dim. complex LVS.

$\Downarrow$

Qu(d)its

(3) A Vector space of Matrices!

Consider the set of 2×2 complex matrices $\equiv \mathbb{M}_{2\times 2}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2} \qquad a, b, c, d \in \mathbb{C}.$$

* $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = M_1$; $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = M_2$.

$\alpha M_1 + \beta M_2 = M_3 \in \mathbb{M}_{2\times 2}$.

Linear Vector Space!

* Dimensionality of this space $= 4$!

Elementary matrices: $\underset{E_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{E_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{E_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{E_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$.

$M \in \mathbb{M}$ can be written as

$$M = aE_1 + bE_2 + cE_3 + dE_4$$

$\downarrow$

Basis.

* check: E_1, E_2, E_3, E_4 are linearly independent!

(2) Inner-Product and orthogonality.

Dual of a ket $\equiv$ Bra

↳ column vector representation.

* d-dim complex LVS :

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}_{d\times 1}$$

Dual is defined as the complex conjugate - transpose of $|v\rangle$.

$$\left(|v\rangle^*\right)^T = \begin{pmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_d^* \end{pmatrix}^T_{d\times 1} = \begin{pmatrix} v_1^* & v_2^* & v_3^* & \cdots & v_d^* \end{pmatrix}_{1\times d}$$

Row vector!

$$= \langle v|$$

"bra"

$v_1 = \cancel{x} + iy$
$v_1^* = x - iy$

Transpose : $(i,j) \to (j,i)$

$$|v\rangle \xrightarrow[(*,T)]{\text{dual}} \langle v|$$

(ket vector) (bra-vector)

* Inner-product on the vectors :- $|v\rangle, |w\rangle \in \mathbb{V}$.

$$(|v\rangle, |w\rangle) = \langle v|w\rangle$$

(dual) (row vector)

bra-ket = scalar (number!)

$$|w\rangle = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} \qquad \langle v|w\rangle = \begin{pmatrix} v_1^* & v_2^* & \cdots & v_d^* \end{pmatrix}_{1\times d} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}_{d\times 1}$$

(matrix multiplication!) $= v_1^* w_1 + v_2^* w_2 + \cdots + v_d^* w_d$

$$= \sum_{i=1}^{d} v_i^* w_i \quad //.$$

* Orthogonal vectors: $|v\rangle, |w\rangle$ are orthogonal

iff $\langle v|w\rangle = 0 = \langle w|v\rangle$

Note: $\langle v|w\rangle$ and $\langle w|v\rangle$ are related as:

$$\boxed{\langle v|w\rangle = \langle w|v\rangle^*}$$

Examples: (i) $\{|0\rangle, |1\rangle\}$ are orthogonal vectors.

(ii) $\{|+\rangle, |-\rangle\}$ $\quad |+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Check: are orthogonal!

(iii) $\{|0\rangle, |+\rangle\} \rightarrow$ orthogonal ??

* Norm: "length" of a vector.

$$\| |v\rangle \| = \sqrt{\langle v|v\rangle} \geq 0.$$

$\langle v|v\rangle \longrightarrow$ real ✓

$$= \sum_i v_i^* v_i = \sum_i |v_i|^2 \geq 0.$$

$\||v\rangle\| = 0$ iff $|v\rangle = |\phi\rangle \rightarrow$ all-zero vector.

* Orthonormal Basis: A Basis is a set of vectors that spans the LVS.
(ON)

Any $|v\rangle \in \mathbb{V}$ can be represented as a linear combination of the basis vectors.

Basis $\quad \mathcal{B} = \{|b_1\rangle, |b_2\rangle, \ldots, |b_d\rangle\}.$

ON Basis $\Leftrightarrow$ $\quad \langle b_i|b_j\rangle = 0$ for all $i \neq j$.

$\langle b_i|b_i\rangle = 1$ for $i = 1, 2, \cdots d$.

Norm of $|b_i\rangle = 1$.

"Normalized" $\longrightarrow$ (unit-vectors) (in 3-d real vectors)

Example: In $\mathbb{C}^2$: $\{|0\rangle, |1\rangle\}$ forms an ON basis.

Check: $|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\longrightarrow$ Also an ON basis for $\mathbb{C}^2$!

$$\left.\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle . \\ |-\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{aligned}\right\} \quad \begin{aligned} |0\rangle &= a|+\rangle + b|-\rangle \\ |1\rangle &= c|+\rangle + d|-\rangle \end{aligned}$$

work out!

(3) Linear Transformations:

Example of $\mathbb{C}^2$: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times 1} \xrightarrow{??} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times 1}$ $(|1\rangle)$.

Matrices 2×2.

$\boxed{\text{Linear operators} \iff \text{Matrices}}$ Hermitian, $\boxed{\text{Unitary}}$.

(i) Flip operation $|0\rangle \longrightarrow |1\rangle$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2\times 2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(ii) Basis-transformations: $\{|0\rangle, |1\rangle\} \longleftrightarrow \{|+\rangle, |-\rangle\}$.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
(Hadamard)

(In the $\{|0\rangle, |1\rangle\}$ basis)

Check:
$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = |-\rangle. = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$
Superpositions!

* orthogonal vs. non-orthogonal

- classically, $\{0, 1\}$. are orthogonal! $\rightarrow$ distinct / distinguishable
- Quantumly, $|0\rangle, |+\rangle \rightarrow$ non orthogonal.

$$\langle 0|+\rangle = \frac{1}{\sqrt{2}}$$
$\rightarrow$ [cannot be distinguished perfectly!]

- ON basis. $\Longleftrightarrow$ Analogy to classical bits.

$\{|0\rangle, |1\rangle\} \Longleftrightarrow \{0, 1\}$.

* Linear Transformations:
$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$H(\alpha|0\rangle + \beta|1\rangle) = \alpha(H|0\rangle) + \beta(H|1\rangle)$$

Linearity-preserving