Completely positive maps

- Kraus operators :—

$$\mathrm{Tr}_E\left[U_{S+E}^{(t,0)}\left(\rho_S^{(0)}\otimes|0\rangle\langle 0|_E\right)U_{S+E}^{(t,0)\dagger}\right]=\rho_S(t)$$

$$\Longleftrightarrow \sum_{j=1}^{N} A_j(t)\,\rho_S(0)\,A_j(t)^{\dagger} = \rho_S(t)$$ Kraus representation

where $A_j(t) = {}_E\langle j|U_{S+E}^{(t,0)}|0\rangle_E$

with $\{|j\rangle_E : j=1,2,\ldots,N\}$ being an ONB for $\mathcal{H}_E$.

$A_j(t)$: Kraus operators

Trace preservation $\Longrightarrow \displaystyle\sum_{j=1}^{N} A_j(t)^{\dagger} A_j(t) = \mathbb{1}_{\mathcal{H}_S}$.

- Quantum dynamical evolutions :—

Any quantum mechanical evolution $\mathcal{N}$ should satisfy the following conditions :

① $\mathcal{N}$ should be linear (It should respect superposition principle)

② $\mathcal{N}$ should be hermiticity preserving (Observables to be mapped into observables)

③ $\mathcal{N}$ should be positivity preserving (density matrix should be mapped into a density matrix)

Let $\dim \mathcal{H}_S = d$. Let $\mathcal{B}(\mathcal{H}_S)$ be the Hilbert space of all bounded linear operators on $\mathcal{H}_S$ with the Hilbert-Schmidt inner product: $\langle A, B \rangle \equiv \mathrm{Tr}[A^\dagger B]$ for all $A, B \in \mathcal{B}(\mathcal{H}_S)$.

Thus $\mathcal{N}$ is a linear map on $\mathcal{B}(\mathcal{H}_S)$. So $\mathcal{N}$ can be considered as a $d^2 \times d^2$ matrix: $\mathcal{N} = (\mathcal{N}_{ij,kl})_{i,j,k,l=1}^{d}$.

So, we have here: $(\rho_s(t))_{ij} = \sum_{k,l=1}^{d} \mathcal{N}_{ij,kl}(t)(\rho_s(0))_{kl}$, where ρ_s is here considered as a column vector in d^2 dimension:

$$\rho_s = \big((\rho_s)_{11}, (\rho_s)_{12}, \dots, (\rho_s)_{1d}, (\rho_s)_{21}, (\rho_s)_{22}, \dots, (\rho_s)_{2d}, \dots,$$
$$\dots, (\rho_s)_{d1}, (\rho_s)_{d2}, \dots, (\rho_s)_{dd}\big)^T.$$

Thus we see that the linearity condition (1) is automatically satisfied.

Let $\mathrm{Tr}[\rho_s(t)] = \mathrm{Tr}[\rho_s(0)]$. So, we have here:

$$\sum_{i=1}^{d} (\rho_s(t))_{ii} = \sum_{i=1}^{d} (\rho_s(0))_{ii}, \quad \text{i.e.,}$$

$$\sum_{i=1}^{d} \sum_{k,l=1}^{d} \mathcal{N}_{ii,kl}^{(t)} (\rho_s(0))_{kl} = \sum_{i=1}^{d} (\rho_s(0))_{ii} = \sum_{k,l=1}^{d} (\rho_s(0))_{kl}\, \delta_{kl} \quad \forall \rho_s(0).$$

$$\Longrightarrow \sum_{i=1}^{d} \mathcal{N}_{ii,kl}^{(t)} = \delta_{kl} \quad \text{for all } k,l = 1,2,\ldots,d.$$

Thus the trace-preservation condition 2 here takes the form:

$$\sum_{i=1}^{d} \mathcal{N}_{ii,kl}^{(t)} = \delta_{kl} \quad \text{for all } k,l = 1,2,\ldots,d.$$

Hermiticity condition demands that $\left(\rho_s(t)\right)^\dagger = \rho_s(t)$ if $\left(\rho_s(0)\right)^\dagger = \rho_s(0)$.

$$\Rightarrow \left(\sum_{k,l=1}^{d} \mathcal{N}_{ij,kl}(t) \left(\rho_s(0)\right)_{kl}\right)^* = \sum_{k,l=1}^{d} \mathcal{N}_{ji,kl}(t) \left(\rho_s(0)\right)_{kl} \ \forall \rho_s(0) \text{ if}$$

$$\left(\rho_s(0)\right)^*_{kl} = \left(\rho_s(0)\right)_{lk}.$$

$$\Rightarrow \sum_{k,l=1}^{d} \mathcal{N}_{ij,kl}(t)^* \left(\rho_s(0)\right)_{lk} = \sum_{k,l=1}^{d} \mathcal{N}_{ji,lk}(t) \left(\rho_s(0)\right)_{lk} \ \forall \rho_s(0)$$

$$\Rightarrow \mathcal{N}_{ij,kl}(t)^* = \mathcal{N}_{ji,lk}(t) \text{ for all } i,j,k,l.$$

$\longrightarrow$ Hermiticity-preservation condition

Positivity preservation condition demands that:

$$\sum_{i,j=1}^{d} y_i^* \mathcal{N}_{ij,k\ell}^{(t)} \left(\rho_s(0)\right)_{k\ell} y_j \geq 0 \text{ for all } y_1, y_2, \dots, y_d \in \mathbb{C}$$

$$\text{if } \sum_{k,\ell=1}^{d} x_k^* \left(\rho_s(0)\right)_{k\ell} x_\ell \geq 0 \text{ for all } x_1, x_2, \dots, x_d \in \mathbb{C}.$$

$\hookrightarrow$ Positivity-preserving condition

<u>Defn</u>: Define now a $d^2 \times d^2$ matrix $V = \left(V_{ik,j\ell}^{(t)}\right)_{i,j,k,\ell=1}^{d}$

as: $$V_{ik,j\ell}^{(t)} = \mathcal{N}_{ij,k\ell}^{(t)} \text{ for all } i,j,k,\ell = 1, 2, \dots, d.$$

Then the hermiticity-preservation condition for $\mathcal{N}$ becomes:

$$V_{ik,jl}^{(t)\,*} = V_{jl,ik}^{(t)} \quad \text{for all } i,j,k,l = 1,2,\dots,d$$

$\Rightarrow$ V is a $d^2 \times d^2$ hermitian matrix

$\Rightarrow$ $V \underset{\text{decomposition}}{\overset{\text{spectral}}{=\!=\!=}} \sum_{m,n=1}^{d} \lambda_{mn} |\Psi^{(mn)}\rangle\langle\Psi^{(mn)}|$,

where λ_{mn} are real nos. and $|\Psi^{(mn)}\rangle = \sum_{p,q=1}^{d} a_{pq}^{(mn)} |pq\rangle$

with $\delta_{mm'}\delta_{nn'} = \langle \Psi^{(mn)} | \Psi^{(m'n')} \rangle$

$$= \sum_{p,q=1}^{d} a_{pq}^{(mn)*} a_{pq}^{(m'n')} .$$

Trace-preservation condition for $\mathcal{N}$ now becomes:

$$\sum_{i=1}^{d} V^{(t)}_{ik,il} = \delta_{kl} \text{ for all } k,l = 1,2,\dots,d.$$

Now $$V = \sum_{m,n=1}^{d} \lambda_{mn} |\Psi^{(mn)}\rangle\langle\Psi^{(mn)}|$$

$$= \sum_{m,n,p,q,p',q'=1}^{d} \lambda_{mn}\, a^{(mn)}_{pq}\, a^{(mn)*}_{p'q'} |pq\rangle\langle p'q'|$$

$$\Rightarrow V^{(t)}_{ik,jl} = \mathcal{N}^{(t)}_{ij,kl} = \sum_{m,n=1}^{d} \lambda_{mn}\, a^{(mn)}_{ik}\, a^{(mn)*}_{jl}$$

for all $i,k,j,l = 1,2,\dots,d$.

So trace-preservation $\Rightarrow$ $$\sum_{i,m,n=1}^{d} \lambda_{mn}\, a^{(mn)}_{ik}\, a^{(mn)*}_{il} = \delta_{kl} \text{ for all } k,l = 1,2,\dots,d.$$

Positivity preservation demands that:

$$\sum_{i,j=1}^{d} \sum_{k,l=1}^{d} y_i^* V_{ik,jl}^{(t)} \left(\rho_s(0)\right)_{kl} y_j \geq 0 \text{ for all } y_1, y_2, \dots, y_d \in \mathbb{C}$$

if $\sum_{k,l=1}^{d} x_k^* \left(\rho_s(0)\right)_{kl} x_l \geq 0$ for all $x_1, x_2, \dots, x_d \in \mathbb{C}$.

$$\Rightarrow \sum_{i,j,k,l,m,n=1}^{d} y_i^* \lambda_{mn} a_{ik}^{(mn)} a_{jl}^{(mn)*} \left(\rho_s(0)\right)_{kl} y_j \geq 0$$

for all $y_1, y_2, \dots, y_d \in \mathbb{C}$ if $\rho_s(0) \geq 0$

$$\Rightarrow \sum_{m,n=1}^{d} \lambda_{mn} \left[\sum_{k,l=1}^{d} \left\{ \sum_{i=1}^{d} y_i a_{ik}^{(mn)*} \right\}^* \left(\rho_s(0)\right)_{kl} \left\{ \sum_{i=1}^{d} y_i a_{il}^{(mn)*} \right\} \right.$$

≥ 0 for all $y_1, y_2, \dots, y_d \in \mathbb{C}$ if $\rho_s(0) \geq 0$.

$$\Rightarrow \sum_{m,n=1}^{d} \lambda_{mn}\, z_k^{(mn)*} \left(\rho_s(0)\right)_{k\ell}\, z_\ell^{(mn)} \geqslant 0 \text{ for all } z_1^{(mn)}, z_2^{(mn)}, \ldots, z_d^{(mn)} \in \mathbb{C}$$

if $\rho_s(0) \geqslant 0$ where $z_k^{(mn)*} = \sum_{i=1}^{d} y_i\, a_{ik}^{(mn)*}$.

<u>Assumption</u>: $\lambda_{mn} \geqslant 0$ for all $m, n = 1, 2, \ldots, d$.

This will automatically imply that $\mathcal{N} \geqslant 0$.

Here $\left(\rho_s(t)\right)_{ij} = \sum_{k,\ell=1}^{d} V_{ik,j\ell}(t) \left(\rho_s(0)\right)_{k\ell} = \sum_{k,\ell,m,n=1}^{d} \lambda_{mn}\, a_{ik}^{(mn)} \times$

$a_{j\ell}^{(mn)*} \left(\rho_s(0)\right)_{k\ell} = \sum_{m,n=1}^{d} \sum_{k,\ell=1}^{d} \left(A^{(mn)}\right)_{ik} \left(\rho_s(0)\right)_{k\ell} \left(A^{(mn)\dagger}\right)_{\ell j}$, where $\left(A^{(mn)}\right)_{ik} = \sqrt{\lambda_{mn}} \times a_{ik}^{(mn)}$ for $i, k = 1, 2, \ldots, d$. & for all $m, n = 1, 2, \ldots, d$.

Thus $A^{(mn)}$ is here a $d \times d$ matrix for all $m, n = 1, 2, \ldots, d$.

We therefore have here : $\rho_S(t) = \sum_{m,n=1}^{d} A^{(mn)} \rho_S(0) A^{(mn)\dagger}$ → Kraus representation

Now $\left(\sum_{m,n=1}^{d} A^{(mn)\dagger} A^{(mn)} \right)_{ij}$

$$= \sum_{m,n=1}^{d} \sum_{k=1}^{d} \left(A^{(mn)\dagger}\right)_{ik} \left(A^{(mn)}\right)_{kj} = \sum_{k,m,n=1}^{d} \left(A^{(mn)}\right)^{*}_{ki} \left(A^{(mn)}\right)_{kj}$$

$$= \sum_{k,m,n=1}^{d} \sqrt{\lambda_{mn}} \times a_{ki}^{(mn)*} \times \sqrt{\lambda_{mn}} \times a_{kj}^{(mn)} = \underline{\delta_{ji}} = \delta_{ij}$$

Follows from trace-preservation

$$\Rightarrow \boxed{\sum_{m,n=1}^{d} A^{(mn)\dagger} A^{(mn)} = \mathbb{1}_{\mathcal{H}_S}, \text{ the identity operator on } \mathcal{H}_S.}$$

Thus we see that the assumption that $V \geq 0$ (together with linearity, hermiticity-preservation and trace-preservation) gives rise to the following Kraus representation:

$$\rho_S(t) = \sum_{m,n=1}^{d} A^{(mn)}(t)\, \rho_S(0)\, A^{(mn)}(t)^\dagger$$

where $A^{(mn)}(t)$'s are $d \times d$ Kraus matrices, satisfying $\sum_{m,n=1}^{d} A^{(mn)}(t)^\dagger A^{(mn)}(t) = \mathbb{1}_{\mathcal{H}_S}$.

Here $\left(A^{(mn)}(t)\right)_{ij} = \sqrt{\lambda_{mn}} \times a^{(mn)}_{ij}$ for all $i, j, m, n = 1, 2, \ldots, d$, where the eigen states $|\Psi_{mn}\rangle$ of the positive operator V, corresponding to the eigenvalue λ_{mn}, is given, in terms of the standard product ONB $\{|ij\rangle : i,j = 1, 2, \ldots, d\}$ of $\mathbb{C}^d \otimes \mathbb{C}^d$, as:

$$|\Psi_{mn}\rangle = \sum_{i,j=1}^{d} a_{ij}^{(mn)} |ij\rangle .$$

- Does it really make any physically meaningful difference if we take $\mathcal{V} \geq 0$ instead of $\mathcal{V}$ to be hermitian, so far as the quantum dynamical map $\mathcal{N}$ is concerned?

We now see that it indeed makes a difference.

Consider the following bipartite pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$:

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |jj\rangle .$$

Note that, by using basis transformations on the individual subsystems, it is impossible to express $|\phi^+\rangle$ as $|\phi^+\rangle = |\psi\rangle \otimes |\eta\rangle$, where $|\psi\rangle, |\eta\rangle \in \mathbb{C}^d$. So $|\phi^+\rangle$ is an entangled state.

Consider now the transpose map $T : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$, acting on the ONB $\{ |i\rangle\langle j| : i, j = 1, 2, \dots, d \}$ of $\mathcal{B}(\mathbb{C}^d)$ as:

$$T(|i\rangle\langle j|) = |j\rangle\langle i| \text{ for all } i, j = 1, 2, \dots, d .$$

Thus here: $(\rho_s(t))_{ij} = \sum_{k,l=1}^{d} T^{(t)}_{ij,kl} (\rho_s(0))_{kl} \equiv (\rho_s(0))_{ji}$ for all $i, j = 1, 2, \dots, d$.

$$\Longrightarrow \quad T^{(t)}_{ij,kl} = \delta_{il}\,\delta_{jk} \quad \text{for all } i,j,k,l = 1,2,\dots,d.$$

→ Thus we see that T is a linear map.

$$\to \sum_{i=1}^{d} T^{(t)}_{ii,kl} = \sum_{i=1}^{d} \delta_{il}\,\delta_{ik} = \delta_{lk} = \delta_{kl}$$

$\Longrightarrow$ T is trace-preserving

$$\to T^{(t)}_{ji,lk} = \delta_{jk}\,\delta_{il} = \delta_{il}\,\delta_{jk} = \left(\delta_{il}\,\delta_{jk}\right)^* = \left(T^{(t)}_{ij,kl}\right)^*$$

$\Longrightarrow$ T is hermiticity-preserving

$$\to \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} y_i^* \, T_{ij,kl} \left(\rho_s(0)\right)_{kl} y_j = \sum_{i,j,k,l=1}^{d} y_i^* \, \delta_{il}\,\delta_{jk} \left(\rho_s(0)\right)_{kl} y_j$$

$$= \sum_{i,j=1}^{d} y_i^* \left(\rho_s(0)\right)_{ji} y_j = \sum_{j,i=1}^{d} \left(y_j^*\right)^* \left(\rho_s(0)\right)_{ji} y_i^* \geq 0$$

for all $y_1, y_2, \ldots, y_d \in \mathbb{C}$ as $\rho_s(0) \geq 0$.

$\Rightarrow$ $T \geq 0$ ✓

$\rightarrow$ What is the V matrix corresponding to T?

$$V^{(t)}_{ik,jl} = T^{(t)}_{ij,kl} = \delta_{il}\delta_{jk} \text{ for all } i,j,k,l = 1,2,\ldots,d$$ ✓

$\rightarrow$ Is this $V \geq 0$?

To check that, let us choose $|x\rangle = \sum_{p,q=1}^{d} x_{pq} |pq\rangle$ and let us consider the quantity $\langle x | V | x\rangle$.

Here $\langle X|V|X\rangle = \sum_{p,q,p',q'=1}^{d} x^*_{pq}\, x_{p'q'}\, V^{(t)}_{pq,p'q'}$

$= \sum_{p,q,p',q'=1}^{d} x^*_{pq}\, x_{p'q'}\, \delta_{pq'}\, \delta_{p'q} = \sum_{p,q=1}^{d} x^*_{pq}\, x_{qp}$

$= \mathrm{Tr}\left[X^* X \right]$, where $X = (x_{qp})_{q,p=1}^{d}$ is a $d\times d$ matrix.

Let us now choose $X = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| - |2\rangle\langle 1|)$ $\left[\text{i.e., } |X\rangle \equiv x_{12}|12\rangle + x_{21}|21\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)\right]$.

Then $\mathrm{Tr}\left[X^* X \right] = \frac{1}{2}\mathrm{Tr}\left[-|1\rangle\langle 1| - |2\rangle\langle 2|\right] = \underline{\underline{-1 < 0}}$

$\Longrightarrow V \not\geq 0$ ✓

$\rightarrow$ That $V \not\geq 0$ also follows from its spectral decomposition:

$$V = \sum_{i=1}^{d} |ii\rangle\langle ii| + \sum_{\substack{i,j=1 \\ i<j}}^{d} \left(|ij\rangle\langle ji| + |ji\rangle\langle ij|\right)$$

$$\stackrel{\text{S.D.}}{=} \sum_{i=1}^{d} |ii\rangle\langle ii| + \sum_{\substack{i,j=1 \\ i<j}}^{d} |\psi^{+}_{ij}\rangle\langle\psi^{+}_{ij}| - \sum_{\substack{i,j=1 \\ i<j}}^{d} |\psi^{-}_{ij}\rangle\langle\psi^{-}_{ij}|,$$

where $|\psi^{\pm}_{ij}\rangle = \frac{1}{\sqrt{2}}\left(|ij\rangle \pm |ji\rangle\right)$ for all $i,j = 1, 2, \dots, d$ with $i<j$.

Thus we see that the transpose map T is such a map which is linear, trace-preserving, hermiticity preserving as well as positivity preserving although the corresponding V map is not positive.

→ Thus T does not give rise to Kraus representation ✓

→ What does it mean physically?

Apply T on one side of $|\phi^+\rangle$, without disturbing the other side —— What do you expect? Do you expect to get a valid bipartite state (may be unnormalized) after this action? Had T been a physical map, one would have expected that! Unfortunately ~~that~~ is not the case!

Here $$(\mathbb{1}\otimes T)(|\phi^+\rangle\langle\phi^+|) = (\mathbb{1}\otimes T)\left(\frac{1}{d}\sum_{i,j=1}^{d}|i\rangle\langle j|\otimes|i\rangle\langle j|\right)$$

$$= \frac{1}{d}\sum_{i,j=1}^{d}|i\rangle\langle j|\otimes T(|i\rangle\langle j|) = \frac{1}{d}\sum_{i,j=1}^{d}|i\rangle\langle j|\otimes|j\rangle\langle i|$$

$$= \frac{1}{d}V \not\geq 0$$

$\Rightarrow$ Partial transposition (i.e., transposition of one subsystem) is not a physical process.

- A quantum dynamical process $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ should be such that: (i) it should be linear, (ii) trace-preserving, (iii) hermiticity-preserving, (iv) positivity-preserving, and (v) positivity-preserving when it acts on one side of a subsystem, irrespective of the dimension of the other subsystem.

$\Rightarrow$ Transpose map T is not a valid quantum dynamical process.

- Is a linear, trace-preserving, hermiticity-preserving, positivity-preserving map $\mathcal{N}: \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ also a quantum dynamical map whenever the corresponding $\mathcal{V}$ map ≥ 0 ?

The answer is yes.

Let $|\chi\rangle = \sum_{p=1}^{d'} \sum_{q=1}^{d} \chi_{pq} |pq\rangle$ be any pure state in $\mathbb{C}^{d'} \otimes \mathbb{C}^{d}$.

$$\left((\mathbb{1} \otimes \mathcal{N})(|\chi\rangle\langle\chi|) \right)_{ij,kl} = \Big((\mathbb{1} \otimes \mathcal{N}) \Big(\sum_{p,p'=1}^{d'} \sum_{q,q'=1}^{d} \chi_{pq} \chi^*_{p'q'} |p\rangle\langle p'| \otimes |q\rangle\langle q'| \Big) \Big)_{ij,kl} = \Big(\sum_{p,p'=1}^{d'} \sum_{q,q'=1}^{d} \chi_{pq} \chi^*_{p'q'} |p\rangle\langle p'| \otimes \mathcal{N}(|q\rangle\langle q'|) \Big)_{ij,kl}$$

$$= \left(\sum_{p,p'=1}^{d'} \sum_{q,q'=1}^{d} x_{pq} x^*_{p'q'} |p\rangle\langle p'| \otimes \left\{ \sum_{m,n=1}^{d} \left(\mathcal{N}(|q\rangle\langle q'|) \right)_{mn} |m\rangle\langle n| \right\} \right)_{ij,kl}$$

$$= \left(\sum_{p,p'=1}^{d'} \sum_{m,n,q,q'=1}^{d} x_{pq} x^*_{p'q'} \left(\sum_{k',l'=1}^{d} \mathcal{N}^{\times}_{mn,k'l'} \left(|q\rangle\langle q'| \right)_{k'l'} \right) \times |pm\rangle\langle p'n| \right)_{ij,kl}$$

$$= \left(\sum_{p,p'=1}^{d'} \sum_{m,n,q,q',k',l'=1}^{d} x_{pq} x^*_{p'q'} \mathcal{N}_{mn,k'l'} \delta_{k'q} \delta_{q'l'} |pm\rangle\langle p'n| \right)_{ij,kl}$$

$$= \left(\sum_{k',l',m,n=1}^{d} \sum_{p,p'=1}^{d'} x_{pk'} x^*_{p'l'} \mathcal{N}_{mn,k'l'} |pm\rangle\langle p'n| \right)_{ij,kl}$$

$$= \sum_{k',l'=1}^{d} x_{ik'} x^*_{kl'} \mathcal{N}_{jl,k'l'} = \sum_{k',l'=1}^{d} x_{ik'} x^*_{kl'} V_{jk',ll'}$$

$$= \sum_{k',l'=1}^{d} x_{ik'} x^*_{kl'} \left(\sum_{m,n=1}^{d} \lambda_{mn} \, a^{(mn)}_{jk'} \, a^{(mn)*}_{ll'} \right)$$

$$= \sum_{m,n=1}^{d} \lambda_{mn} \left(\sum_{k'=1}^{d} x_{ik'} \, b^{(mn)}_{k'j} \right) \left(\sum_{l'=1}^{d} x_{kl'} \, b^{(mn)}_{l'l} \right)^*$$

(where $b^{(mn)}_{k'j} = a^{(mn)}_{jk'}$)

$$= \sum_{m,n=1}^{d} \lambda_{mn} \left\{ X B^{(mn)} \right\}_{ij} \left\{ X^* B^{(mn)*} \right\}_{kl}$$

(where $X = (x_{ik'})_{i=1;k'=1}^{d';d}$ and $B^{(mn)} = (b^{(mn)}_{l'l})_{l',l=1}^{d}$)

Thus for any $|\Phi\rangle = \sum_{i=1}^{d'} \sum_{j=1}^{d} \Phi_{ij} |ij\rangle \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d}$, we have:

$$\langle \Phi | (\mathbb{1} \otimes \mathcal{N})(|X\rangle\langle X|) |\Phi\rangle = \sum_{i,k=1}^{d'} \sum_{j,l=1}^{d} \Phi^*_{ij} \Phi_{kl} \left((\mathbb{1} \otimes \mathcal{N})(|X\rangle\langle X|) \right)_{ij,kl}$$

$$= \sum_{i,k=1}^{d'} \sum_{j,l,m,n=1}^{d} \lambda_{mn} \left[\Phi^*_{ij} \left\{ X B^{(mn)} \right\}_{ij} \right] \times \left[\Phi^*_{kl} \left\{ X B^{(mn)} \right\}_{kl} \right]^*$$

$$= \sum_{m,n=1}^{d} \lambda_{mn} \left| \langle \Phi | \chi B^{(mn)} \rangle \right|^2 \geqslant 0 \quad \text{as } \lambda_{mn} \geqslant 0$$

for all $m, n = 1, 2, \ldots, d$ (where $|\chi B^{(mn)}\rangle = \sum_{i=1}^{d'} \sum_{j=1}^{d} \{\chi B^{(mn)}\}_{ij} |ij\rangle$)

$$\Rightarrow (\mathbb{1} \otimes \mathcal{N})(|\chi\rangle\langle\chi|) \geqslant 0 \text{ whenever } \mathcal{V} \geqslant 0$$

$\Rightarrow (\mathbb{1} \otimes \mathcal{N})(\rho) \geqslant 0$ whenever $\mathcal{V} \geqslant 0$, for all density matrices ρ of the bipartite system $\mathbb{C}^{d'} \otimes \mathbb{C}^{d}$

Note that here d' can be any positive integer.

- **Completely positive map**: Any linear, trace-preserving, hermiticity-preserving, positivity-preserving map $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$, which is also positivity-preserving when acts on a d-dim subsystem of any bipartite system, is known to be a trace-preserving completely positive map (TPCP map).

Thus we see that a trace-preserving quantum dynamical map is nothing but a TPCP map

- We have seen that for a linear, trace-preserving, hermiticity-preserving map $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$, if $\mathcal{V} \geq 0$, then $\mathcal{N}$ is TPCP.
Is it true that for any TPCP map $\mathcal{N}$, $\mathcal{V} \geq 0$?

Let $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ be a TPCP map.

Let $|\eta\rangle = \sum_{i,k=1}^{d} \eta_{ik} |ik\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$.

~~Then~~ $\langle \eta | V | \eta \rangle = \sum_{i,k,j,l=1}^{d} \eta^*_{ik} \eta_{jl} V_{ik,jl}$

$$= \sum_{i,k,j,l=1}^{d} \eta^*_{ik} \eta_{jl} \mathcal{N}_{ij,kl} = \sum_{i,j=1}^{d} \Big((\mathbb{1} \otimes \mathcal{N})(|\eta^*\rangle\langle\eta^*|) \Big)_{ii,jj}$$

$$= \Big(\sum_{i=1}^{d} \langle ii|\Big) \Big((\mathbb{1} \otimes \mathcal{N})(|\eta^*\rangle\langle\eta^*|) \Big) \Big(\sum_{j=1}^{d} |jj\rangle\Big)$$

$$= d \langle \phi^+ | \Big((\mathbb{1} \otimes \mathcal{N})(|\eta^*\rangle\langle\eta^*|) \Big) |\phi^+\rangle \quad \Big(\text{as } |\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle\Big)$$

$\geqslant 0$ as $(\mathbb{1} \otimes \mathcal{N})(|\eta^*\rangle\langle\eta^*|) \geqslant 0$ for all $|\eta\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

$\Longrightarrow V \geqslant 0$

- Thus we see that every TPCP map $\mathcal{N}: \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ will have a Kraus representation: $\rho_s(t) = \sum_{m,n=1}^{d} A_{mn}(t)\, \rho_s(0)\, A_{mn}(t)^\dagger$, where the Kraus operators $A_{mn}(t)$'s are given by $(A_{mn})_{ij} = \sqrt{\lambda_{mn}} \times a_{ij}^{(mn)}$ (for $i, j = 1, 2, \ldots, d$) where the positive $d^2 \times d^2$ matrix $V = (V_{ik,jl})_{i,k,j,l=1}^{d} = (\mathcal{N}_{ij,kl})_{i,j,k,l=1}^{d}$ has eigenvectors $|\Psi_{mn}\rangle = \sum_{i,j=1}^{d} a_{ij}^{(mn)} |ij\rangle$ corresponding to the eigenvalues λ_{mn}.
Here $\sum_{m,n=1}^{d} A_{mn}(t)^\dagger A_{mn}(t) = \mathbb{1}_{\mathbb{C}^d}$.

- Does every Kraus representation give rise to a TPCP map?

To see this, let us consider the following Kraus representation: $\rho_S(t) = \sum_{j=1}^{D} A_j(t)\,\rho_S(0)\,A_j(t)^\dagger$ with $\sum_{j=1}^{D} A_j(t)^\dagger A_j(t) = \mathbb{1}_{\mathcal{H}_S}$ for any $\rho_S(0) \in \mathcal{B}(\mathcal{H}_S)$.

Let $d = \dim \mathcal{H}_S$. Then $(\rho_S(t))_{kl} = \sum_{j=1}^{D} \sum_{m,n=1}^{d} (A_j(t))_{km} (\rho_S(0))_{mn} (A_j(t))^*_{ln}$

$$\equiv \sum_{m,n=1}^{d} \mathcal{N}_{kl,mn}(t) (\rho_S(0))_{mn},$$

where $\mathcal{N}_{kl,mn}(t) = \sum_{j=1}^{D} (A_j(t))^{\times}_{km} (A_j(t))^*_{ln}$ for $k, l, m, n = 1, 2, \ldots, d$

$\Longrightarrow$ $\mathcal{N}$ is linear ✓

Now $\sum_{i=1}^{d} \mathcal{N}_{ii,kl}(t) = \sum_{i=1}^{d}\sum_{j=1}^{D} (A_j(t))_{ik} (A_j(t))^*_{il} = \sum_{j=1}^{D} \sum_{i=1}^{d} (A_j(t)^\dagger)_{li} \times$

$(A_j(t))_{ik} = \left(\sum_{j=1}^{D} A_j(t)^\dagger A_j(t) \right)_{lk} = \delta_{lk} = \delta_{kl}$

$\Longrightarrow$ $\mathcal{N}$ is trace-preserving

Again $\left(\mathcal{N}_{ij,kl}(t)\right)^* = \sum_{\alpha=1}^{D} (A_\alpha(t))^*_{ik} (A_\alpha(t))_{jl}$

And $\mathcal{N}_{ji,lk}(t) = \sum_{\alpha=1}^{D} (A_\alpha(t))_{jl} (A_\alpha(t))^*_{ik} = \left(\mathcal{N}_{ij,kl}(t)\right)^*$

for all $i, j, k, l = 1, 2, \ldots, d$

$\Longrightarrow$ $\mathcal{N}$ is hermiticity-preserving

Let $\rho_s(0) \in \mathcal{B}(\mathcal{H}_s)$ and let $\rho_s(0) \geq 0$.

Let $|\psi\rangle = \sum_{p=1}^{d} \psi_p |p\rangle \in \mathcal{H}_s$.

Then $\langle\psi| \rho_s(t) |\psi\rangle = \sum_{p,q=1}^{d} \psi_p^* \psi_q \left(\rho_s(t)\right)_{pq} = \sum_{p,q,r,s=1}^{d} \psi_p^* \psi_q \mathcal{N}^{(t)}_{pq,rs} \times$

$\left(\rho_s(0)\right)_{rs} = \sum_{j=1}^{D} \sum_{p,q,r,s=1}^{d} \psi_p^* \psi_q \left(A_j(t)\right)_{pr} \left(A_j(t)\right)^*_{qs} \left(\rho_s(0)\right)_{rs}$

$= \sum_{j=1}^{D} \sum_{r,s=1}^{d} \left(\sum_{p=1}^{d} \left(A_j(t)^\dagger\right)_{rp} \psi_p \right)^* \left(\rho_s(0)\right)_{rs} \left(\sum_{q=1}^{d} \left(A_j(t)^\dagger\right)_{sq} \psi_q \right)$

$= \sum_{j=1}^{D} \underbrace{\langle\psi| A_j(t)}_{\equiv \langle\phi_j|} \rho_s(0) A_j(t)^\dagger |\psi\rangle$, $A_j(t)^\dagger |\psi\rangle \equiv |\phi_j\rangle$ (may not be normalized)

$= \sum_{j=1}^{D} \langle\phi_j| \rho_s(0) |\phi_j\rangle \geq 0$

$\Rightarrow$ $\mathcal{N}$ is positivity-preserving ✓

Here for any $|\phi\rangle = \sum_{i,k=1}^{d} \phi_{ik} |ik\rangle \in \mathcal{H}_S \otimes \mathcal{H}_S$, we have here:

$$\langle \phi | V | \phi \rangle = \sum_{i,j,k,l=1}^{d} \phi_{ik}^* V_{ik,jl} \phi_{jl} = \sum_{i,j,k,l=1}^{d} \phi_{ik}^* \mathcal{N}_{ij,kl} \phi_{jl}$$

$$= \sum_{\alpha=1}^{D} \sum_{i,j,k,l=1}^{d} \phi_{ik}^* \phi_{jl} (A_\alpha)_{ik} (A_\alpha)_{jl}^* = \sum_{\alpha=1}^{D} \left| \langle \phi | A_\alpha \rangle \right|^2 \quad \left(\text{where } |A_\alpha\rangle = \sum_{i,k=1}^{d} (A_\alpha)_{ik} |ik\rangle\right)$$

$$\geq 0$$

$\Rightarrow$ V is positive-semidefinite

$\Rightarrow$ $(\mathbb{1} \otimes \mathcal{N})(\rho) \geq 0$ for any positive operator ρ in $\mathcal{B}(\mathcal{H}_{S'} \otimes \mathcal{H}_S)$ where $\dim \mathcal{H}_{S'}$ can be any positive integer.

All these together shows that $\mathcal{N}$ is a TPCP map

- Given any linear, hermiticity-preserving map $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$, is it always necessary to apply $(\mathbb{1} \otimes \mathcal{N})$ on all the bipartite states ρ of $\mathbb{C}^{d'} \otimes \mathbb{C}^d$ with all possible positive integral values of d', in order to check for the complete positivity (and hence, positivity) of $\mathcal{N}$?

We will now see that it is enough to check for the positivity of $(\mathbb{1} \otimes \mathcal{N})(|\phi^+\rangle\langle\phi^+|)$ where $|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ — a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$.

Let $|\eta\rangle = \sum_{i,j=1}^{d} \eta_{ij} |ij\rangle$ be any state in $\mathbb{C}^d \otimes \mathbb{C}^d$.

Then $\langle \eta | (\mathbb{1} \otimes \mathcal{N})(|\phi^+\rangle\langle\phi^+|) | \eta \rangle$

$$= \sum_{i,j,k,l=1}^{d} \eta_{ij}^* \eta_{kl} \langle ij | \left((\mathbb{1} \otimes \mathcal{N})(|\phi^+\rangle\langle\phi^+|) \right) | kl \rangle$$

$$= \frac{1}{d} \sum_{i,j,k,l=1}^{d} \eta_{ij}^* \eta_{kl} \langle ij | \left[\sum_{p,q=1}^{d} |p\rangle\langle q| \otimes \mathcal{N}(|p\rangle\langle q|) \right] | kl \rangle$$

$$= \frac{1}{d} \sum_{i,j,k,l,p,q=1}^{d} \eta_{ij}^* \eta_{kl} \delta_{ip} \delta_{qk} \langle j | \mathcal{N}(|p\rangle\langle q|) | l \rangle$$

$$= \frac{1}{d} \sum_{i,j,k,l=1}^{d} \eta_{ij}^* \eta_{kl} \langle j | \mathcal{N}(|i\rangle\langle k|) | l \rangle = \frac{1}{d} \sum_{i,j,k,l,m,n=1}^{d} \eta_{ij}^* \eta_{kl} \mathcal{N}_{jl,mn} \delta_{mi} \delta_{kn}$$

$$= \frac{1}{d} \sum_{i,j,k,l=1}^{d} \eta_{ij}^* \eta_{kl} \mathcal{N}_{jl,ik} = \boxed{\frac{1}{d} \sum_{i,j,k,l=1}^{d} \eta_{ji}'^{*} V_{ji,lk} \eta_{lk}'}, \text{ where}$$

$\eta'_{lk} = \eta_{kl}$ for all $k, l = 1, 2, \ldots, d$

$$\equiv \frac{1}{d} \langle \eta' | V | \eta' \rangle \text{ with } |\eta'\rangle = \sum_{l,k=1}^{d} \eta'_{lk} |lk\rangle$$

Next let us consider any two states $|\phi\rangle = \sum_{i=1}^{d'} \sum_{j=1}^{d} \phi_{ij} |ij\rangle$ and $|\chi\rangle = \sum_{i=1}^{d'} \sum_{j=1}^{d} \chi_{ij} |ij\rangle$ in $\mathbb{C}^{d'} \otimes \mathbb{C}^{d}$.

Then $\langle\chi| (\mathbb{1} \otimes \mathcal{N}) (|\phi\rangle\langle\phi|) |\chi\rangle$

$$= \sum_{i,i'=1}^{d'} \sum_{j,j',k,l=1}^{d} \chi^*_{ij} \chi_{i'j'} \phi_{ik} \phi^*_{i'l} \mathcal{N}_{jj',kl}$$

$\mathcal{N}_{jj',kl} \equiv V_{jk,j'l}$ (Similar to the earlier derivation)

$$= \sum_{j,k,j',l=1}^{d} \left\{ \sum_{i=1}^{d'} (\chi^T)_{ji} \phi^*_{ik} \right\}^* V_{jk,j'l} \left\{ \sum_{i'=1}^{d'} (\chi^T)_{j'i'} \phi^*_{i'l} \right\}$$

$\left\{ \sum_{i=1}^{d'} (\chi^T)_{ji} \phi^*_{ik} \right\} \equiv \frac{1}{\sqrt{d}} \eta'_{kj} \equiv \frac{1}{\sqrt{d}} \eta_{kj}$, so that $\left\{ \sum_{i=1}^{d'} (\chi^T)_{ji} \phi^*_{ik} \right\}^* \equiv \frac{1}{\sqrt{d}} \eta'^*_{kj} \equiv \frac{1}{\sqrt{d}} \eta^*_{kj}$

$\left\{ \sum_{i'=1}^{d'} (\chi^T)_{j'i'} \phi^*_{i'l} \right\} \equiv \frac{1}{\sqrt{d}} \eta'_{j'l} \equiv \frac{1}{\sqrt{d}} \eta_{lj'}$

$$\equiv \langle\eta| (\mathbb{1} \otimes \mathcal{N}) (|\phi^+\rangle\langle\phi^+|) |\eta\rangle$$

Thus we see that it is enough to look for the positivity of $(\mathbb{1} \otimes \mathcal{N})(|\phi^+\rangle\langle\phi^+|)$, in order to prove complete positivity of $\mathcal{N}$.

Unitary dialations of CPTP maps

- We have seen earlier that considering the evolution of the closed quantum system S+E, consisting of the system and its environment, to be unitary and thereby tracing out the environment degrees of freedom, one comes up with a Kraus representation for the quantum dynamical evolution of the system. As quantum dynamical evolutions are represented by TPCP maps, therefore, it is pertinent to ask for the existence of a joint unitary operator U_{S+E} on $\mathcal{H}_S \otimes \mathcal{H}_E$, from which the TPCP map can be obtained by tracing out the environment.

Let $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ be a TPCP map.

$\Longrightarrow$ There exists Kraus operators $A_j : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ $(j = 1, 2, \dots, D)$ such that $\sum_{j=1}^{D} A_j^\dagger A_j = 1_{\mathbb{C}^d}$ and $\rho \xrightarrow{\mathcal{N}} \rho' = \sum_{j=1}^{D} A_j \rho A_j^\dagger$ for all $\rho \in \mathcal{B}(\mathbb{C}^d)$. Let $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$ be an ONB for $\mathbb{C}^d$.

Consider now the following d no. of column vectors in $\mathbb{C}^{dD}$:—

$$\vec{u}_{kD+1} = \big(\langle 1|A_1|k+1\rangle, \langle 1|A_2|k+1\rangle, \dots, \langle 1|A_D|k+1\rangle, \langle 2|A_1|k+1\rangle, \langle 2|A_2|k+1\rangle, \dots, \langle 2|A_D|k+1\rangle, \dots, \dots, \langle d|A_1|k+1\rangle, \langle d|A_2|k+1\rangle, \dots, \langle d|A_D|k+1\rangle\big)^T$$

for $k = 0, 1, \dots, d-1$.

$\rightarrow$ Note that $$\vec{u}_{\ell D+1}^{\,\dagger}\, \vec{u}_{kD+1} = \sum_{m=1}^{D} \langle \ell+1| A_m^\dagger \underbrace{\left(\sum_{\alpha=1}^{d} |\alpha\rangle\langle\alpha|\right)}_{= 1_{\mathbb{C}^d}} A_m |k+1\rangle$$

$$= \delta_{(\ell+1)(k+1)} = \delta_{\ell k} \text{ for all } \ell, k = 0, 1, \dots, d-1.$$

Thus we see that the d no. of column vectors $\vec{u}_1, \vec{u}_{D+1}, \vec{u}_{2D+1}, \ldots, \vec{u}_{(d-1)D+1}$ in $\mathbb{C}^{dD}$ are pairwise orthonormal.

$\Rightarrow$ These d pairwise orthonormal vectors can be extended (in infinitely many ways) to a complete ONB $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{dD}\}$ of $\mathbb{C}^{dD}$.

Let us now consider the $dD \times dD$ unitary matrix U, whose 1st, 2nd, $\ldots$, dD-th columns are respectively $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{dD}$.

non-unique

So here: $\langle \alpha j | U | \beta 1 \rangle = \langle \alpha | A_j | \beta \rangle$ for all $\alpha, \beta = 1, 2, \ldots, d$ and for all $j = 1, 2, \ldots, D$.

$$\Rightarrow \quad A_j = \langle j | U | 1 \rangle \text{ for all } j = 1, 2, \ldots, D.$$

$$\Rightarrow \quad \rho_S \xrightarrow{\mathcal{N}} \rho_S' = \mathrm{Tr}_E \left[U (\rho_S \otimes |1\rangle_E\langle 1|) U^\dagger \right] = \sum_{j=1}^{D} A_j \rho_S A_j^\dagger$$

Hence we have here a unitary realization of the TPCP map.

Non-uniqueness of Kraus operators

- Consider a trace-preserving quantum dynamical evolution (i.e., a TPCP map) $\mathcal{N} : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$.

$$\Longrightarrow \rho \xrightarrow{\mathcal{N}} \rho' = \sum_{j=1}^{D} A_j \rho A_j^{\dagger} \quad \text{with} \quad \sum_{j=1}^{D} A_j^{\dagger} A_j = \mathbb{1}_{\mathbb{C}^d}.$$

Let $U = (u_{j\alpha})_{j=1\,;\,\alpha=1}^{D\,;\,D'}$ be a $D \times D'$ isometry, i.e.,

$$U^{\dagger} U = \mathbb{1}_{\mathbb{C}^{D'}} = U^{T} U.$$

Let $A_j = \sum_{k=1}^{D'} u_{jk} B_k \quad (j = 1, 2, \ldots, D)$

Then $\sum_{j=1}^{D} A_j^\dagger A_j = \sum_{k,k'=1}^{D'} (U^\dagger U)_{kk'} B_k^\dagger B_{k'} = \sum_{k,k'=1}^{D'} \delta_{kk'} B_k^\dagger B_{k'}$

$= \sum_{k=1}^{D} B_k^\dagger B_k = \mathbb{1}_{\mathbb{C}^d}$.

Also $\sum_{j=1}^{D} A_j \rho A_j^\dagger = \sum_{k,k'=1}^{D} (U^T U^*)_{kk'} B_k \rho B_{k'}^\dagger$

$= \sum_{k,k'=1}^{D} \delta_{kk'} B_k \rho B_{k'}^\dagger = \sum_{k=1}^{D} B_k \rho B_k^\dagger = \rho'$

Thus we see that $\{B_1, B_2, \ldots, B_{D'}\}$ also forms a set of Kraus operators for the TPCP map $\mathcal{N}$.

$\Rightarrow$ Kraus representation is not unique.