ACM Winter School 2022

Quantum gates & circuits :

qubit gates & circuits.

Outline :-

- Scope of the field
- gate model / circuit model of quantum computation.
- Single / Two - qubit / three qubits.

- **Universal gate set / universal quantum computation.**
- Clifford heirarchy
- Two fundamental theorems of quantum computation.

X

Examples:-

- State teleportation
- gate teleportation
- Hadamard gate identities
- 1D & 2D clusters.
- Repetition code

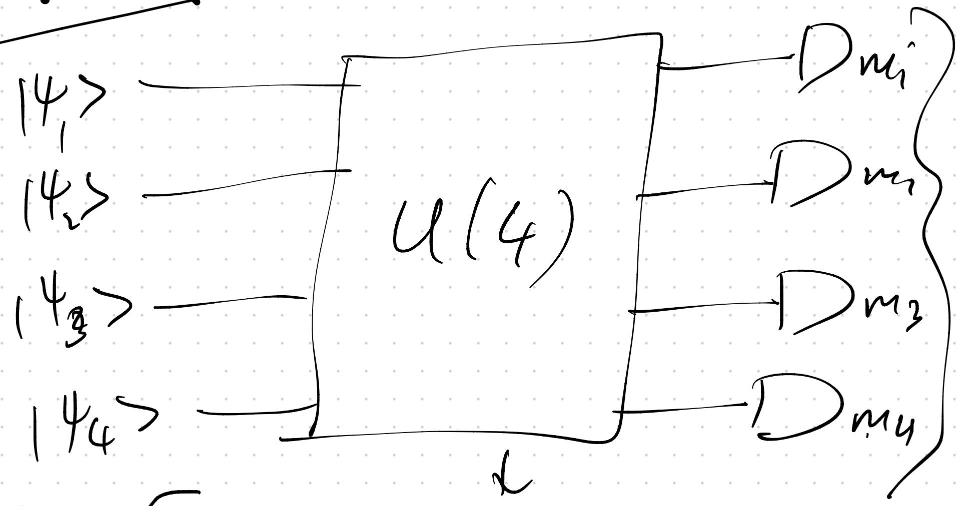
Imperfect gates

Scope:-

- Quantum Algorithms / QML.
- Quantum error correction circuits
- Quantum simulation
- Quantum cryptographic protocols.
- Preparation of ground states of many-body states
- Quantum advantage:
  - Random circuit sampling
  - Boson sampling

Gate Model:

4 qubit example



Readout / measurement

State initialization

gate implementation

gate $\equiv$ unitaries.

$|\psi_i\rangle \in \mathbb{C}^2$, $U(4) \in U_4$ group.

Single qubit gates:-

$|0_z\rangle, |1_z\rangle$: Pauli Z.

$A^{+}BA$

Bloch sphere :

pure states.

$$|\psi\rangle = a|0\rangle_z + b|1\rangle_z$$

$$P(\alpha)\,|\psi_{\theta,\varphi}\rangle = \cos\theta/2\,|0\rangle_z + e^{i(\varphi+\alpha)}\sin\theta/2\,|1_z\rangle.$$

Pauli matrices:-

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};\quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$|0\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$|1\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \qquad |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}.$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}.$$

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l . \quad (1)$$

$$[\sigma_j, \sigma_k]_+ = \{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbb{1}. \quad (2)$$

Identity:

- $$\boxed{\sigma_j \sigma_k = \delta_{jk}\mathbb{1} + i\epsilon_{jkl}\sigma_l .} \quad (3)$$
- $(\bar{a}\cdot\bar{\sigma})(\bar{b}\cdot\bar{\sigma}) = (\bar{a}\cdot\bar{b})\mathbb{1} + i(\bar{a}\times\bar{b})\cdot\bar{\sigma}$ where

$$\bar{a}\cdot\bar{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z .$$

$\bar{a} \in \mathbb{R}^3$.

$$e^{A} = \sum_{j=0}^{\infty} \frac{A^{j}}{j!}$$

In particular $A^2 = \mathbb{1}$:

$$e^{i\theta(\hat{n}\cdot\sigma)} = \cos\theta\,\mathbb{1} + i\sin\theta\,(\hat{n}\cdot\bar{\sigma})$$

$$\rho = \frac{1}{2}\left[\mathbb{1} + \bar{r}\cdot\bar{\sigma}\right] \quad \& \ |r| \leq 1.$$

pure state $|r| = 1$

$Tr(\sigma_i) = 0$.

- $\underset{2\times 2}{B} = \sum_{i=0}^{3} \bar{a}\cdot\bar{\sigma}$; $\sigma_0 = \mathbb{1}$; $\bar{a}_i \in$ Reals.

Other single qubit gates:-

$H^2 = \mathbb{1}$.

- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}[\sigma_z + \sigma_x]$.

$H = H^\dagger$.

$$\boxed{\begin{aligned} H^\dagger \sigma_x H &= \sigma_z \\ H^\dagger \sigma_z H &= \sigma_x . \end{aligned}}$$

$H \sigma_y H = -\sigma_y$.

$X \xleftrightarrow{H} Z$

$H|0\rangle = |+\rangle$.

$H|1\rangle = |-\rangle$.

$HH^\dagger = H^\dagger H = \mathbb{1}$

Conjugation: $H = H^\dagger$.

S gate : $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ diagonal gate.

$S^2 = Z$. : $SS^\dagger = S^\dagger S = \mathbb{1}$.

T-gate : $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$. : Diagonal gate

$TT^\dagger = \mathbb{1}$.

$T^2 = S$; $S^2 = Z \Rightarrow T^4 = Z$.

$T \xrightarrow{sq} S \xrightarrow{sq} Z$. [diagonal (1, phase).

Rotation gates : $\longrightarrow$ universality.

- $R_x(\theta) = e^{-i\theta\sigma_x/2}$

$$= \cos\theta/2\, \mathbb{1} - i \sin\theta/2\, \sigma_x .$$

$$R_x(\theta)^\dagger \sigma_x R_x(\theta) = \sigma_x .$$

$$R_x(\theta) : \begin{pmatrix} \sigma_y \\ \sigma_z \end{pmatrix} .$$

Rotation about x in the anti clock wise direction.

- $R_y(\theta) = e^{-i\theta\sigma_y/2}$
- $R_z(\theta) = e^{-i\theta\sigma_z/2}$: $\cos\theta/2\, \mathbb{1} - i \sin\theta/2\, \sigma_z$.

$\longrightarrow$ diagonal operator

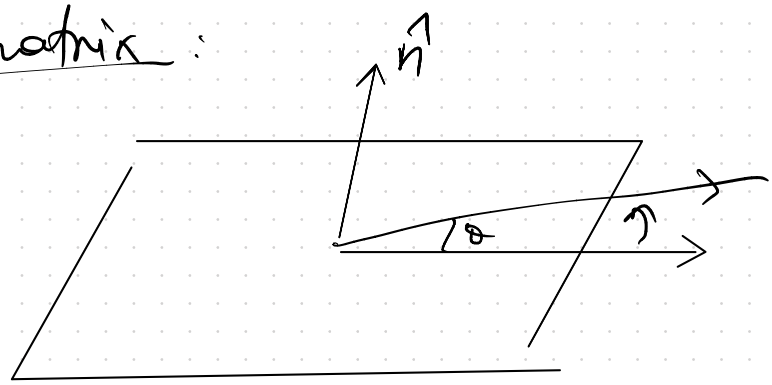
$$R_y(\theta) = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}.$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \rightarrow$$ diagonal in the Pauli z-basis.

general rotation matrix:

$$\boxed{R_{\hat{n}}(\theta) : = e^{-i\theta\, \hat{n}\cdot\sigma/2}}$$

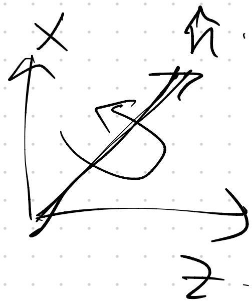
$(\theta, \hat{n})$



$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{\sqrt{2}}\{\sigma_x + \sigma_z\}$$

$$= \frac{1}{\sqrt{2}}\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\}$$

Consider the foll. operator :



$$e^{i\pi/2}\,\boxed{R_{\hat{n}}(\pi)} \quad ; \quad \hat{n} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$$

$$R_{\hat{n}}(\theta) = \cos\theta/2\ \mathbb{1} - i\sin\theta/2\ \hat{n}\cdot\vec{\sigma}$$

$$= e^{i\pi/2}\left\{-i\ \frac{1}{\sqrt{2}}[\sigma_x + \sigma_z]\right\} = \boxed{H}$$

$\mathcal{U}_{2\times 2}$ i.e any single qubit gate

$$\boxed{\mathcal{U} = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)}$$

decompose any

Angular momentum Chapter, Sakurai.

Adjoint representation:

$$R^{\dagger}_{\hat{n}}(\theta) (\bar{a}\cdot\bar{\sigma}) R_{\hat{n}}(\theta) = ? (\bar{b}\cdot\bar{\sigma})$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \Big[\quad R(\hat{n},\theta) \quad \Big]_{3\times 3} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

- $R_{\hat{n}}(\theta) \cdot R_{\hat{m}}(\theta') = R_{\hat{k}}(\varphi)$. [Ex].

- Infinitesimal Rotation:

$$e^{-i\delta\theta\,\hat{n}/2} \underset{\text{in } \delta\theta}{\overset{\text{linear}}{=\!=}} \cos\delta\theta\,\mathbb{1} - i\sin\delta\theta\,\hat{n}\cdot\sigma$$

$$\simeq \mathbb{1} - i(\delta\theta)\,\hat{n}\cdot\sigma.$$

Single qubit gates: - Pauli matrices
- Hadamard $[\sigma_x \longleftrightarrow \sigma_z]$.
- S, T, $P(\alpha)$ diagonal $\equiv \sigma_z$.

decomposition of any gate $\longleftarrow$ Rotation gates.

$$P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

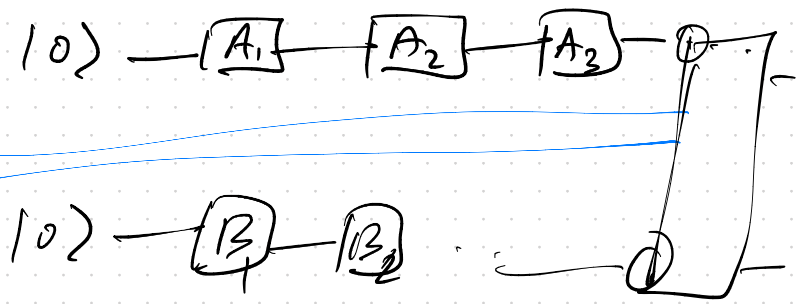
$Z = P(\pi)$; $S = P(\pi/2)$; $T = P(\pi/4)$.

$|0\rangle, |1\rangle \rightarrow |+\rangle, |-\rangle$

$$P(\theta)\left[a|0\rangle + b|1\rangle\right]$$
$$= a|0\rangle + b e^{i\theta}|1\rangle.$$

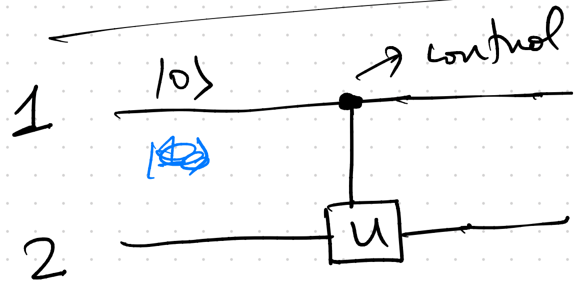
Two qubit gates :

↳ generating entanglement.



$$\boxed{|0\rangle \otimes |0\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}}}$$

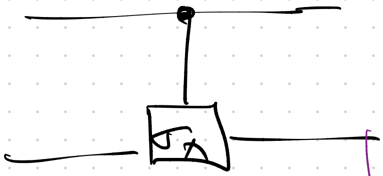
Controlled - U gates :-



$$CU = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U.$$

$$H |\psi\rangle \otimes |\varphi\rangle.$$

$CX \equiv CNOT$

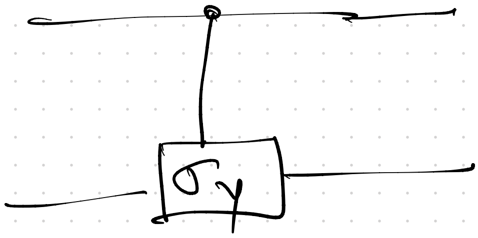


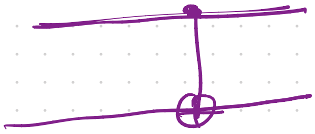
$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]_{4\times 4}$$

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$$

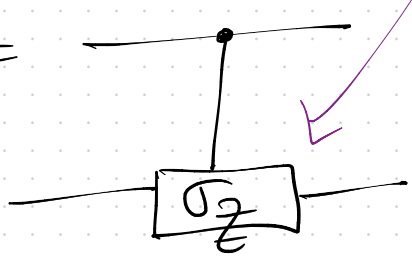
$CY, CZ:$

$CY:$

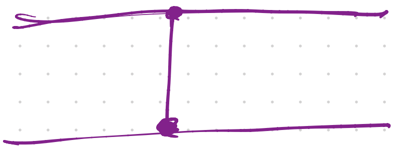




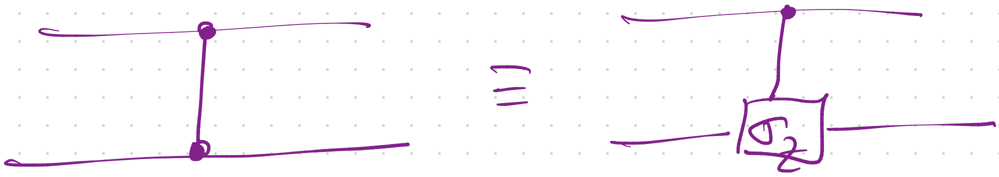
$CZ =$



$\equiv$



CZ :

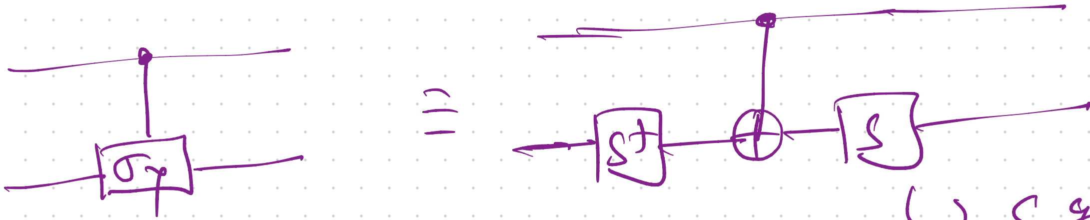


$H\sigma_x H = \sigma_z$; $H\sigma_z H = \sigma_x$.

(CZ)

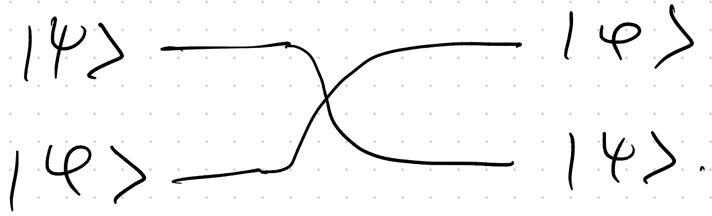
(CNOT)

CY :

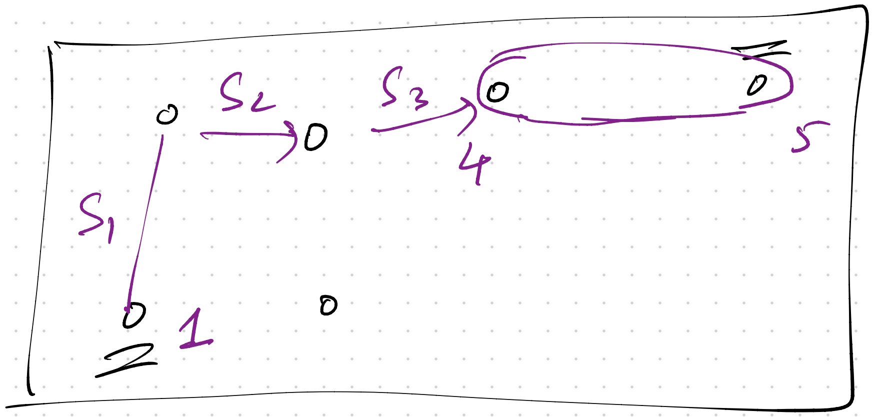


↳ S gate.

SWAP gate:-



①



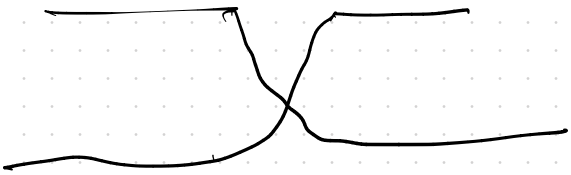
② Photonic computation: mux.

SWAP :- $= \Big[\, |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| \,\Big]$

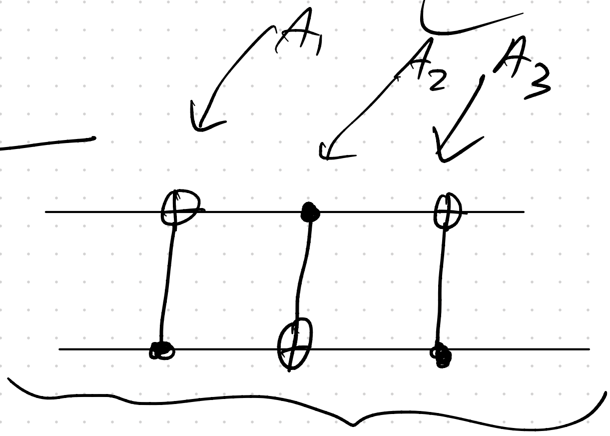
$\updownarrow$

CNOT's

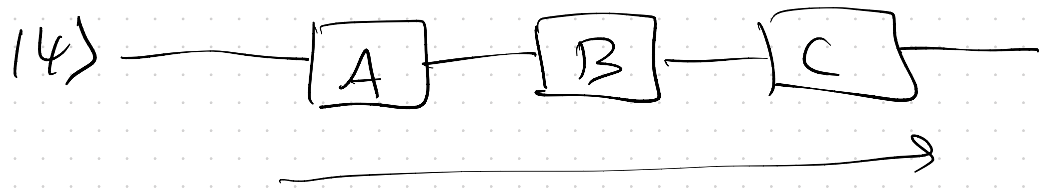
$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4}$$



$\equiv$



$A_3 \circ A_2 \circ A_1 \quad (\quad)$

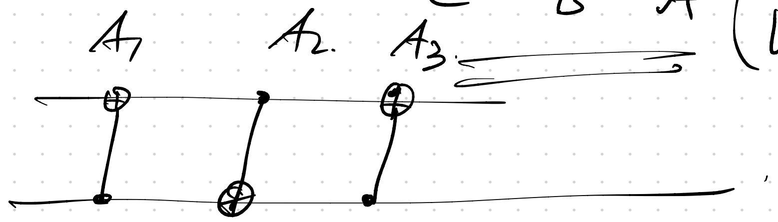


$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$

$= a|0\rangle + b|1\rangle$

$C \circ B \circ A \begin{pmatrix} a \\ b \end{pmatrix}$

Pf :-



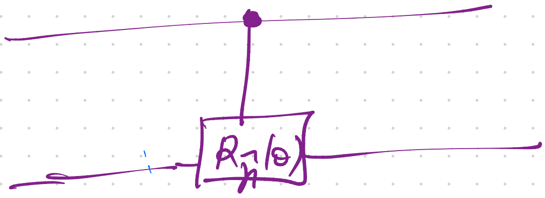
$$\underbrace{\left[|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x \right]}_{A_2} \circ \underbrace{\left[\mathbb{1} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| \right]}_{A_1}$$

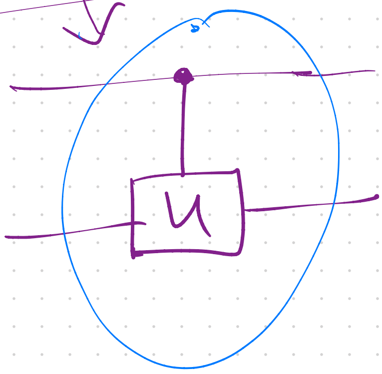
$$= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 1|$$

$$+ |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$

CROT:- controlled rotation gate.

superconducting.





$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

[ABC form]

ABC representation for single qubit unitaries:-

$$\underset{2\times2}{U} = e^{i\alpha} R_z(\beta)\, R_y(\gamma)\, R_z(\delta)$$

Define A, B, C operators :

$$A = R_z(\beta)\, \boxed{R_y(+\gamma/2)}$$

$$B = \boxed{R_y(-\gamma/2)}\, \boxed{R_z\left(-\left(\frac{\delta+\beta}{2}\right)\right)}$$

$$C = R_z\left(\frac{\delta-\beta}{2}\right).$$

$$A \circ B \circ C = \mathbb{1}.$$

$$A \cdot B \cdot C = R_z(\beta)\, R_z\left(-\frac{\delta+\beta}{2}\right) R_z\left(\frac{\delta-\beta}{2}\right).$$

$$= R_z(\beta)\, R_z(-\beta) = \mathbb{1}.$$

$$\sigma_x^2 = \mathbb{1}.$$

$$e^{i\alpha} A\, \sigma_x\, B\, \sigma_x\, C$$

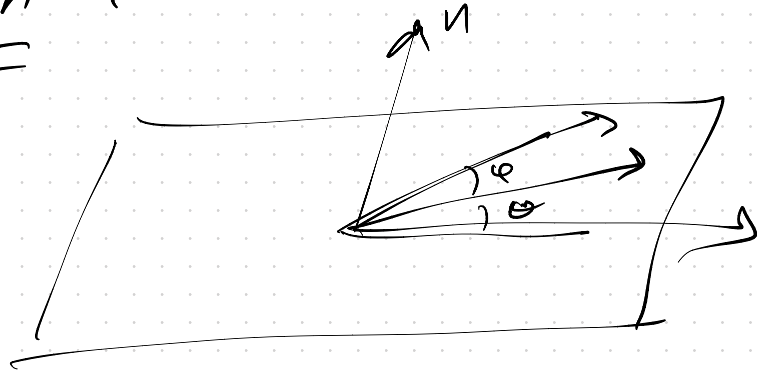
$$= R_z(\beta) \cdot R_y(\gamma/2)\, \boxed{\sigma_x\, R_y(-\gamma/2)\, \sigma_x \sigma_x}\, \boxed{R_z\left(-\frac{\delta+\beta}{2}\right) \sigma_x}\, R_z\left(\frac{\delta-\beta}{2}\right).$$

$$R_y(\gamma/2) \qquad \underbrace{R_z\left(\frac{\delta+\beta}{2}\right). \quad R_z\left(\frac{\delta-\beta}{2}\right).}_{R_z(\delta)}$$

$$R_y(\gamma)$$

$$= e^{i\alpha}\, R_z(\beta)\, R_y(\gamma)\, R_z(\delta) \equiv U_{2\times 2}.$$

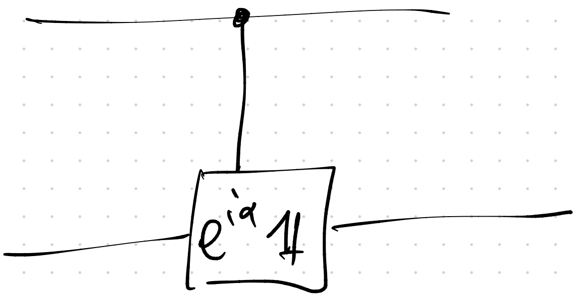
$$R_{\hat{n}}(\theta) \circ R_{\hat{n}}(\varphi) = R_{\hat{n}}(\theta + \varphi)$$



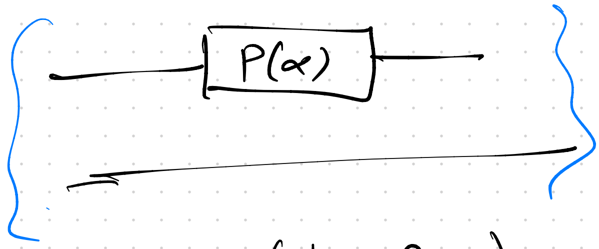
$$U = e^{i\alpha} \, A \, \sigma_x \, B \, \sigma_x \, C$$

$$\sigma_x \sigma_y \sigma_x = -\sigma_y$$

$$\sigma_x \sigma_z \sigma_x = -\sigma_z$$



$=$



$$P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$|0\rangle\langle 0| \otimes \mathbb{1}$$
$$+ |1\rangle\langle 1| \otimes e^{i\alpha}\mathbb{1}.$$

$|0\rangle|0\rangle$

$|0\rangle|1\rangle$

$e^{i\alpha}|1\rangle|0\rangle$

$e^{i\alpha}|1\rangle|0\rangle$.

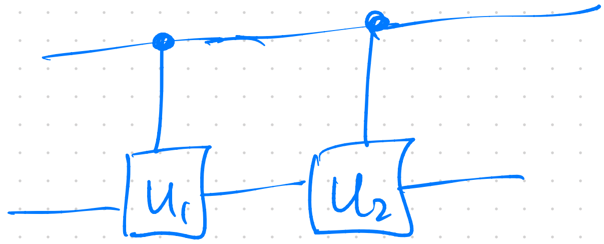
$|0\rangle|0\rangle \to |0\rangle|0\rangle$

$|0\rangle|1\rangle \to |0\rangle|1\rangle$

$|1\rangle|0\rangle \to e^{i\alpha}|1\rangle|0\rangle$

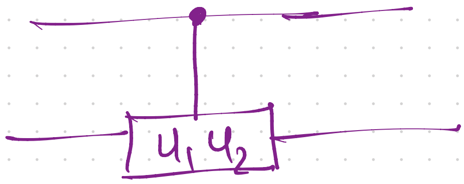
$|1\rangle|1\rangle \to e^{i\alpha}|1\rangle|1\rangle$.

Composing controlled unitaries :-



$(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_1)$

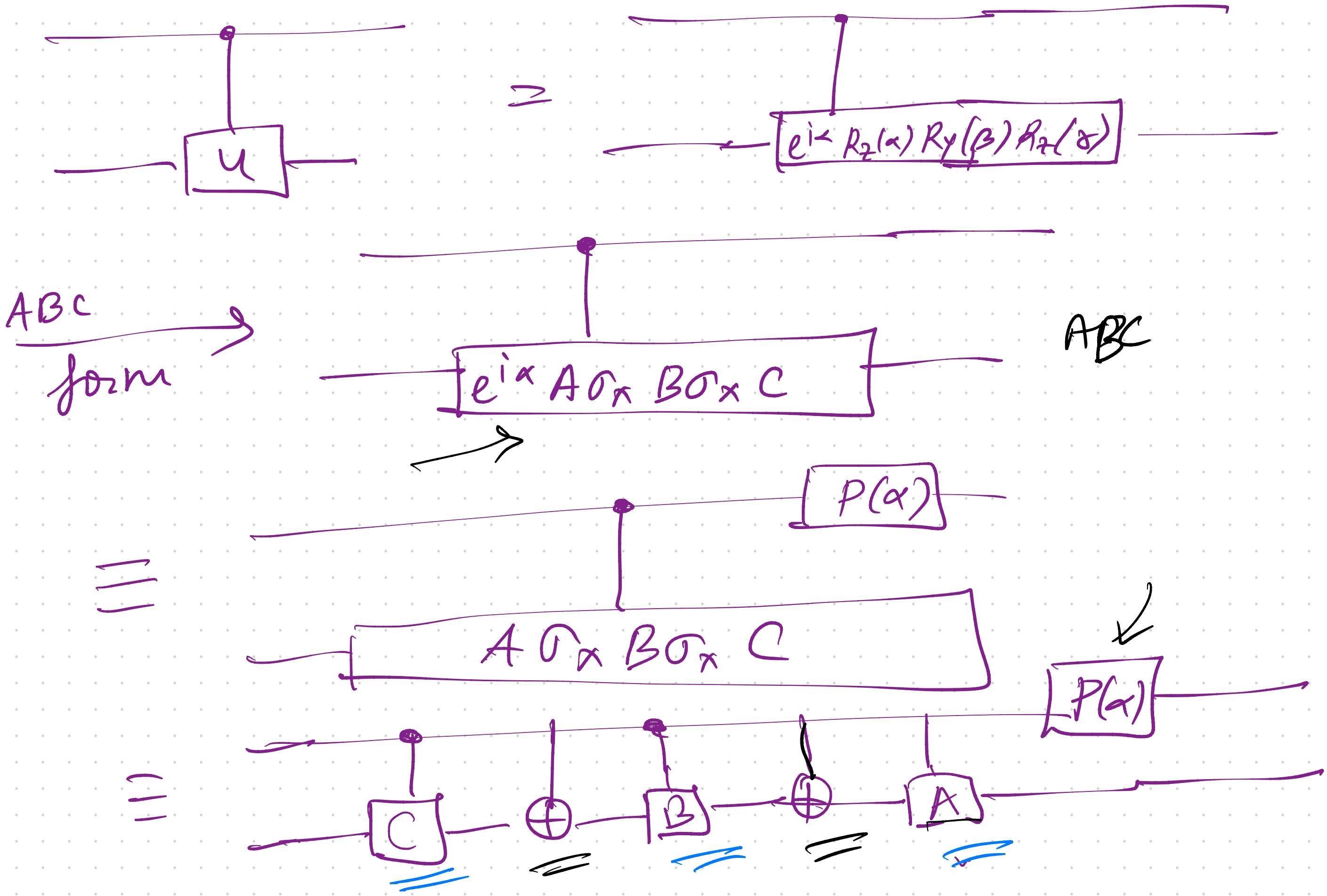
$\equiv$



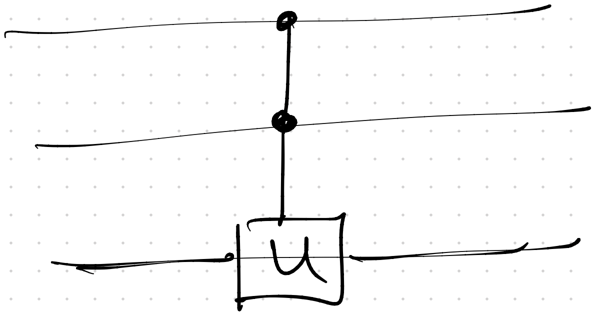
$$(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_2) \circ (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_1)$$

$$= |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes (U_2 U_1)$$

$\hookrightarrow$ is a controlled unitary.


U
=
e^{iα} R_z(α) R_y(β) R_z(γ)
ABC
form
e^{iα} A σ_x B σ_x C
ABC
P(α)
≡
A σ_x B σ_x C
P(α)
≡
C
B
A

Three qubit gates :-



C-C-U.

$\to$ CNOTs, T

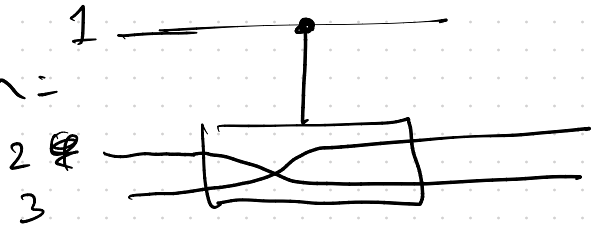
Toffoli: $U = \sigma_x$

CC-Z

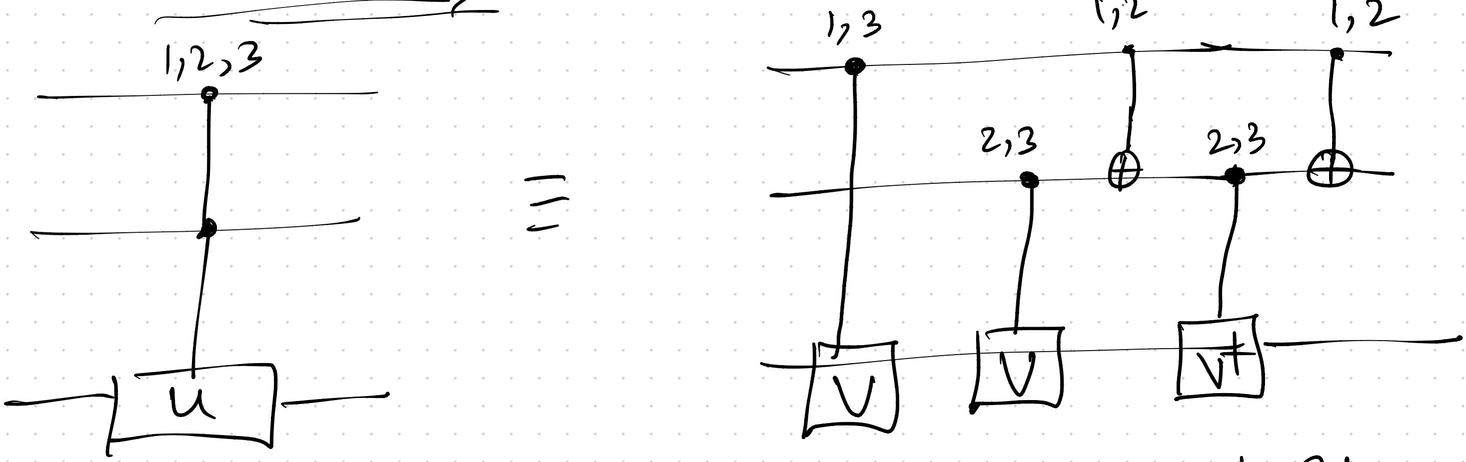
CC-Y

$\to$ CNOT's, C-U's $\Big\}$ 2 qubit gates

Fredkin:



$\mathcal{U} = V^2$; $\quad V V^\dagger = \mathbb{1}$.



→ 3 controlled $\mathcal{U}$.

→ 2 CNOT's.

$S = T^2$:

$Z = S^2$:

$\sqrt{X} = HSX$. : CCX : C-HSX.

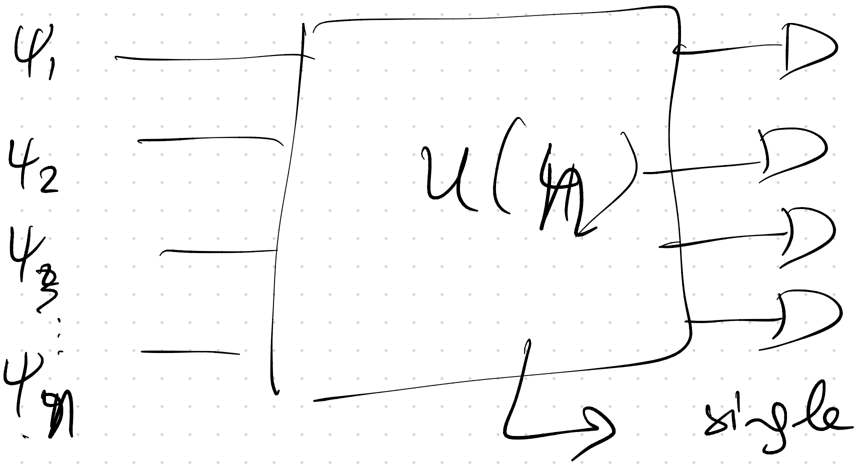
One ~~gay~~ can generalize this controlled gates to n-qubits:

$$(C - C - C \cdots\cdots)^{\otimes n} - \mathcal{U} \equiv \text{2 qubit } C\text{-}U\text{'s CNOT's.}$$

———— x ————

Universal gate sets:-

① $\mathcal{U}(n) \equiv \underbrace{e^{i\alpha} R_z(\cdot)\, R_y(\cdot) \cdot R_z(\cdot)}_{\text{continuous parameters}}$ + CNOT

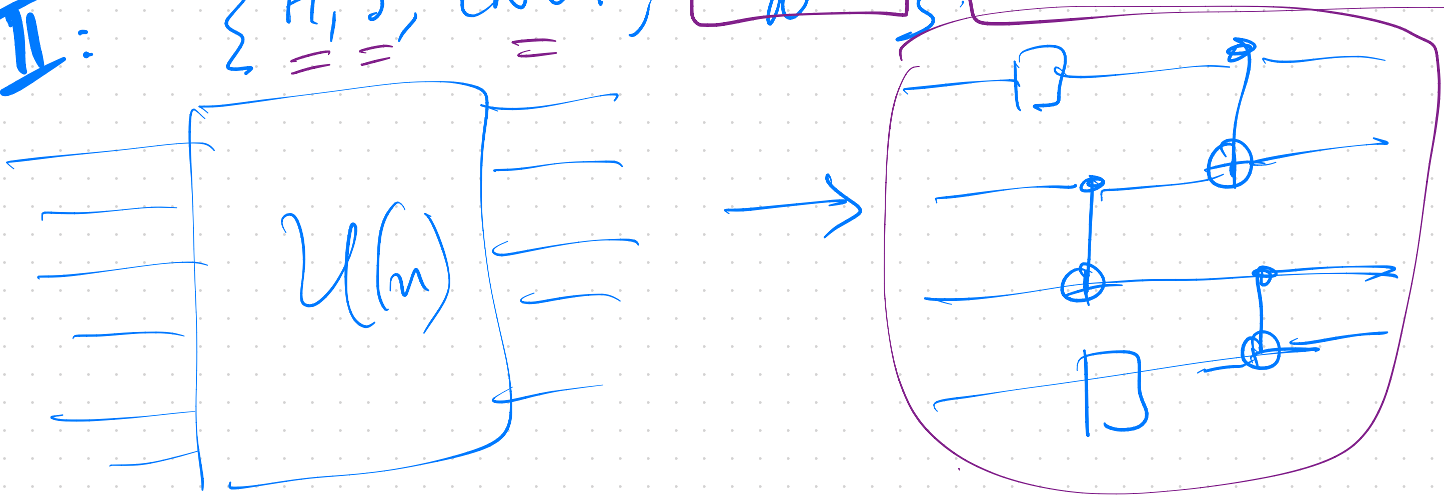


single + two qubit.

② Discrete sets:-

I: $\{H, S, \boxed{T}, CNOT\}$

II: $\{H, S, CNOT, \boxed{\text{Toffoli}}\}$



U(n)

- Approximation
- Accuracy.

Clifford heirarchy :

Pauli group :- $(\mathbb{P}_n)$

An n- qubit group is the group generated by $\hat{g} = \langle \pm \mathbb{1}, \pm i \mathbb{1}, \sigma_x, \sigma_y, \sigma_z \rangle$

$$g \in \mathbb{P}_n \;:\; g_1 \otimes g_2 \otimes g_3 \cdots$$

$$g_i \in \hat{g}.$$

Clifford group :-

- group of automorphisms of the Pauli group:

$$\mathcal{C}^{(2)} = \{ U / \, U P U^{\dagger} \in \mathbb{P}_n , \ \forall P \in \mathbb{P}_n \}$$

$$\mathbb{P}^n \subset \mathcal{C}^{(2)}$$

- Clifford group elements are insufficient for universal.

  → so we need elements beyond Clifford. T, Toffoli, infinitesimal rotation

- One set universal for Clifford alone are $\langle H, S, CNOT \rangle$.

- Clifford circuits
  - Stabilizer formalism in QECC.
  - Cluster state generation.
  - State & gate teleportation.

- Clifford circuits are classically simulable efficiently.

- $U = (\text{Clifford}) + $ one non Clifford element

classically simulable.

dictate

- Entanglement generated by Clifford alone is sufficient.

$P, \quad \underset{\mathbb{C}^{(2)}}{UPU^{\dagger}} \in \mathbb{P}_n \, ; \quad \underbrace{\underset{\mathbb{C}^{(3)}}{UPU^{\dagger}} \in \mathbb{C}^{(2)}}$

$=$

$\times$

Fundamental theorems of quantum computation

Thm 1:- Gottesman-Knill theorem (Clifford circuits).

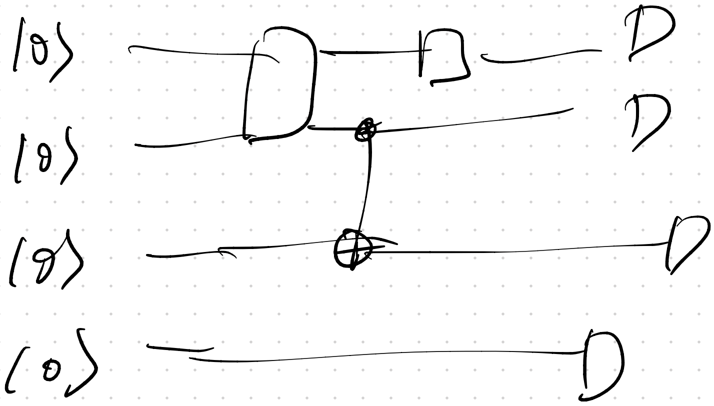
Any Q computer performing:

(a) Clifford group gates $\langle H, S, CNOT \rangle$.

(b) measurements of Pauli group operators

(c) Clifford gates conditional on the classical outcome of measurements.

can be perfectly simulated in polynomial time on a probabilistic classical computer.

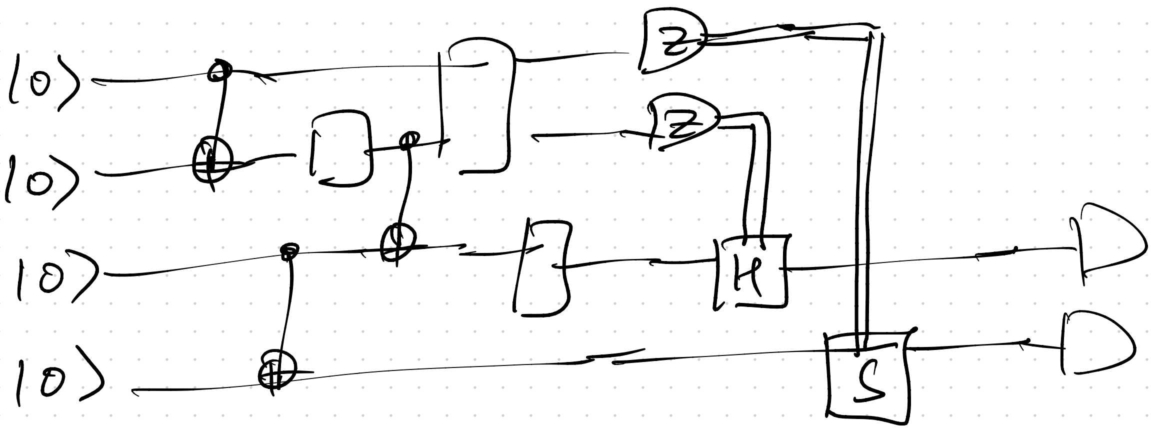


Efficiently Simulable

Clifford : $\langle H, S, CNOT \rangle$.

$\{T, \text{Toffoli}, \ldots\} =$

II Theorem: Solvay-Kitaev theorem (general universal circuits)



conditional feed-forward operation:

- eg:- Teleportation.

II Solvay - Kitaev thm:

Def:- INSTRUCTION set G for a qubit is a finite set of quantum gates that satisfies.

1: All gates in G are in $SU(2)$

2: $g \in G$, $g^{\dagger}$ also $\in G$.

3: G is universal in $SU(2)$.

Let G be an instruction set for $SU(2)$

Let $\varepsilon > 0$ (accuracy); $\exists$ 'c' : for any $U \in SU(2)$,

$\exists$ a finite sequence S from G of

length $O(\log^c(1/\varepsilon))$ and such that

$$\boxed{d(U, S) < \varepsilon.}$$

finite seq: S; finite set G;

$S = g_1 g_2 \dots, g_m$. $g_i \in G$.

{Clifford} + one NC element

{T, Toffoli, Fredkin
infinitesimal
Rotation ....}

Classical
simulable

Discrete set, finite sequence

Recap:

gate model
- Single
  - Pauli
  - H, T, S
  - Rotations
    - universal (?)
- Two
  - CNOT
  - SWAP
  - C-U (2-qubit)
- three
- - - -

ABC → universal (?)

ABC → C-U

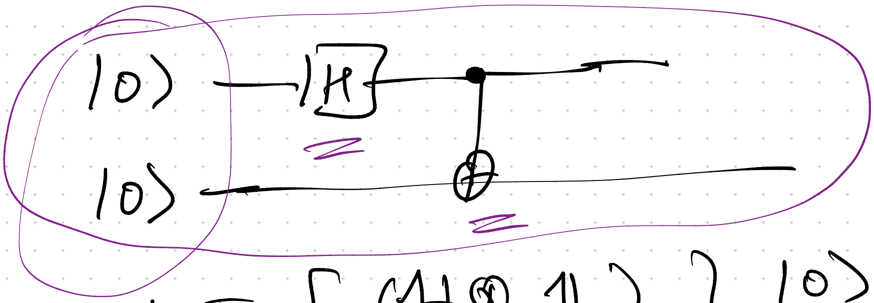
universal in n-qubit

Clifford heirarchy.
- G-K (Clifford)
- SK thm. U(n).

Examples:-

1: State preparation.

Bell state: $|\Phi^+_{AB}\rangle = \dfrac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$



$\equiv |\phi^+\rangle_{AB}$.

$H|0\rangle = |+\rangle$

$= \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$.

$$\text{CNOT}\left[\left(H \otimes \mathbb{1} \right) \right] |0\rangle |0\rangle$$

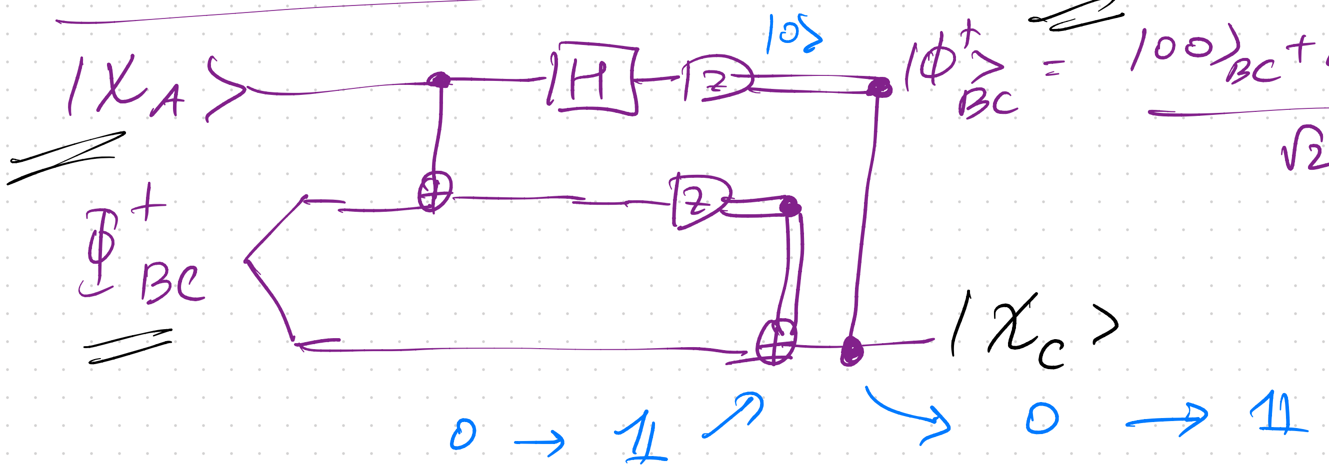
$$= \text{CNOT}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \right]$$

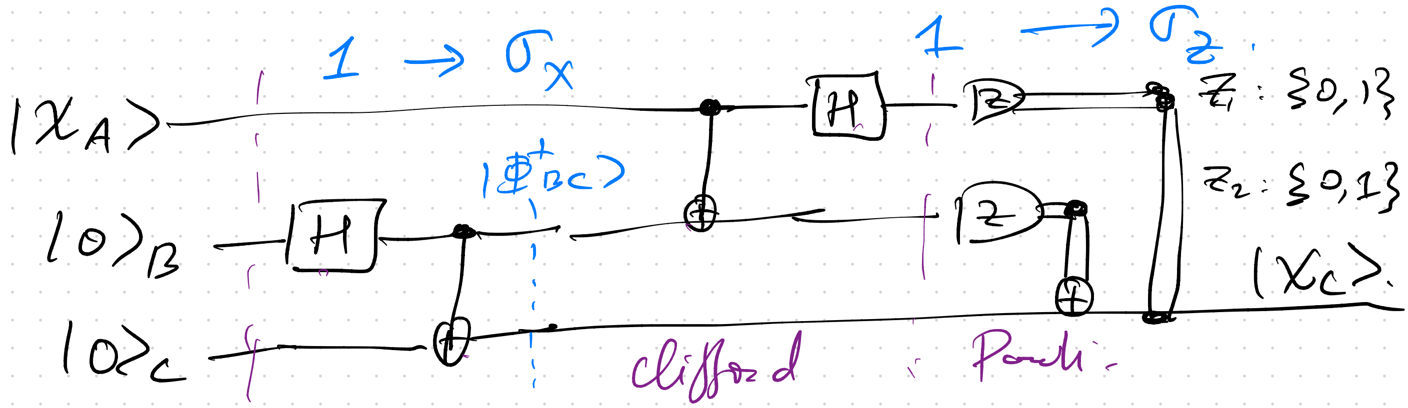
$$= \frac{1}{\sqrt{2}} \text{CNOT}\left[|00\rangle + |10\rangle \right] = \frac{1}{\sqrt{2}} \left[|00\rangle + |11\rangle \right]$$

State teleportation:-

$|\chi_A\rangle = a|0\rangle_A + b|1\rangle_A.$

$|\phi^+\rangle_{BC} = \dfrac{|00\rangle_{BC} + |11\rangle_{BC}}{\sqrt{2}}.$





Gate teleportation :-

- Implement, eg: non-clifford gate (N).

- $N(|+\rangle)$ $\rightarrow$ can be prepared.

  $\hookrightarrow$ resource state.

  + clifford circuits.

$\Rightarrow$

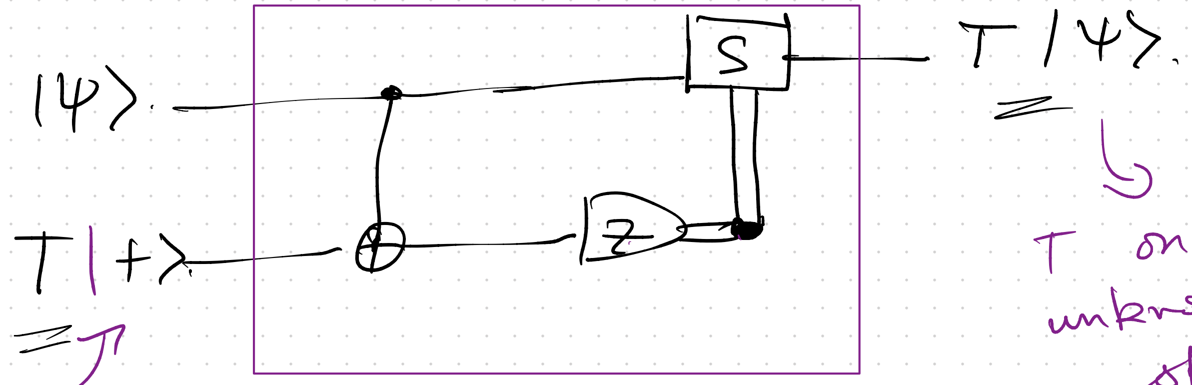
$N(|\psi\rangle)$.

T-gate teleportation:-

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$T|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{i\pi/4}|1\rangle\right] \longrightarrow \text{resource state.}$$

$a|0\rangle + b|1\rangle$



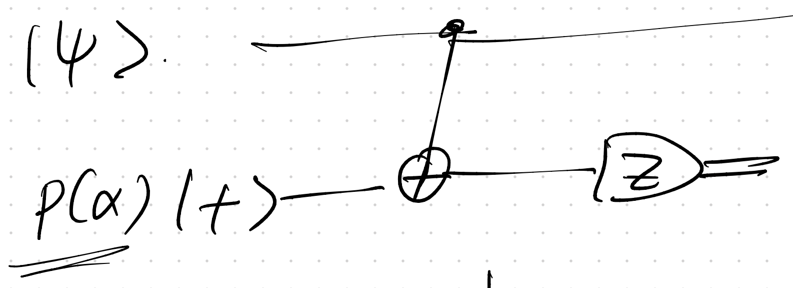
T on an unknown state.

resource state

Clifford gates + Pauli meas.

This gate teleportation idea can be generalized: $P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$.

$P(\alpha)|+\rangle$.



$= P(\alpha)|\psi\rangle$ $\quad$ $P^\dagger(\alpha)|\psi\rangle$

$z=0$ $\quad$ $z=1$

$T^2 = S$; $\quad T^\dagger = ST$.