Introduction to Quantum Computing

June 2021

Overview:

I.) From bits to qubits: Dirac notation, density matrices, measurements, Bloch sphere

II.) Quantum circuits: basic single-qubit & two-qubit gates, multipartite quantum states

III.) Entanglement: Bell states, Teleportation, Q-sphere

I. From bits to qubits

- classical states for computation are either "0" or "1"
- in quantum mechanics, a state can be in superposition, i.e., simultaneously in "0" and "1"
  - → superpositions allow to perform calculations on many states at the same time
    - ⇒ quantum algorithms with exponential speed-up

BUT: once we measure the superposition state, it collapses to one of its states

(→ we can only get one "answer" and not all answers to all states in the superposition)

⇒ it is not THAT easy to design quantum algorithms, but we can use interference effects

(→ "wrong answers" cancel each other out, while the "right answer" remai

Dirac notation & density matrices

- used to describe quantum states: let $a, b \in \mathbb{C}^2$. (→ 2-dimensional vector with complex entries)
  - ket: $|a\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$
  - bra: $\langle b| = |b\rangle^{+} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}^{+} = (b_1^* \; b_2^*)$ ← complex conjugated & transposed
  - bra-ket: $\langle b|a\rangle = a_0 b_0^* + a_1 b_1^* = \langle a|b\rangle^* \in \mathbb{C}$ (→ complex number) ← inner product
  - ket-bra: $|a\rangle\langle b| = \begin{pmatrix} a_0 b_0^* & a_0 b_1^* \\ a_1 b_0^* & a_1 b_1^* \end{pmatrix}$ (→ 2×2-matrix) ← not an "X"!!

- we define the pure states $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are orthogonal: $\langle 0|1\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$

  ↳ $|0\rangle\langle 0| = (1\;0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $|1\rangle\langle 1| = (0\;1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \rho_{00} \cdot |0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$$

- all quantum states can be described by density matrices, i.e., normalized positive Hermitian operators ρ : $tr(\rho) = 1$, $\rho \geq 0$, $\rho = \rho^+$

  $\Leftrightarrow$ for $\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$: $tr(\rho) = \rho_{00} + \rho_{11} \overset{!}{=} 1$, $\langle \psi|\rho|\psi\rangle \geq 0 \ \forall |\psi\rangle \Leftrightarrow$ all eigenvalues ≥ 0, $\rho^+ = \begin{pmatrix} \rho_{00}^* & \rho_{10}^* \\ \rho_{01}^* & \rho_{11}^* \end{pmatrix} = \rho$

- all quantum states are normalized, i.e., $\langle \psi|\psi\rangle = 1$, e.g. $|\psi\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

- spectral decomposition: for every density matrix ρ $\exists$ an orthonormal basis $\{|i\rangle\}_i$ s.t. $\rho = \sum_i \lambda_i |i\rangle\langle i|$, where $|i\rangle$: eigenstates, λ_i : eigenvalues, $\sum_i \lambda_i = 1$

- a density matrix is pure, if $\rho = |\phi\rangle\langle\phi|$, otherwise it is mixed

  $\rightarrow$ if ρ is pure, one eigenvalue equals 1, all others are 0,

  i.e. $tr(\rho^2) = \sum_i \lambda_i^2 = 1$ if ρ is pure, otherwise $tr(\rho^2) < 1$

- examples: (i) $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$ $\rightarrow$ pure , $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$ $\rightarrow$ pure

  (ii) $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ $\rightarrow$ mixed

  (iii) $\rho = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$

  $\hookrightarrow$ pure

Measurements

- we choose orthogonal bases to describe & measure quantum states ($\rightarrow$ projective measurement)

- during a meas. onto the basis $\{|0\rangle, |1\rangle\}$, the state will collapse into either state $|0\rangle$ or $|1\rangle$ $\rightarrow$ as those are the eigenstates of σ_z, we call this a z-measurement

- there are infinitely many different bases, but other common ones are

  $\{|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ and $\{|+i\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), |-i\rangle := \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\}$,

  corresponding to the eigenstates of σ_x and σ_y, respectively.

- Born rule: the probability that a state $|\psi\rangle$ collapses during a projective meas. onto the basis $\{|x\rangle, |x^\perp\rangle\}$ to the state $|x\rangle$ is given by

  $P(x) = |\langle x|\psi\rangle|^2$, $\sum_i P(x_i) = 1$

· examples: - $|\psi\rangle = \frac{1}{\sqrt{3}} \left(|0\rangle + \sqrt{2}\,|1\rangle\right)$ is meas. in the basis $\{|0\rangle, |1\rangle\}$:

$$\rightarrow P(0) = \left|\langle 0| \tfrac{1}{\sqrt{3}} \left(|0\rangle + \sqrt{2}\,|1\rangle\right)\right|^2 = \left| \tfrac{1}{\sqrt{3}} \underbrace{\langle 0|0\rangle}_{1} + \sqrt{\tfrac{2}{3}}\, \underbrace{\langle 0|1\rangle}_{0} \right|^2 = \underline{\underline{\tfrac{1}{3}}} \rightarrow \underline{\underline{P(1) = \tfrac{2}{3}}}$$

- $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right)$ is measured in the basis $\{|+\rangle, |-\rangle\}$:

$$\rightarrow P(+) = |\langle +|\psi\rangle|^2 = \left| \tfrac{1}{\sqrt{2}} \left(\langle 0| + \langle 1|\right) \cdot \tfrac{1}{\sqrt{2}} \cdot \left(|0\rangle - |1\rangle\right)\right|^2$$

$$= \tfrac{1}{4} \left| \underbrace{\langle 0|0\rangle}_{1} - \underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|0\rangle}_{0} - \underbrace{\langle 1|1\rangle}_{1} \right|^2 = 0 \rightarrow \text{expected, as } \langle +|\psi\rangle = \underbrace{\langle +|-\rangle}_{\text{orthogonal}} = 0$$

$\hookrightarrow P(-) = |\langle -|-\rangle|^2 =$

Bloch sphere:

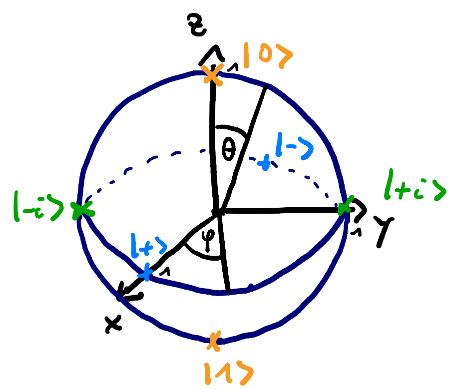
We can write any normalized (pure) state as $\quad |\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\varphi} \sin\frac{\theta}{2}\,|1\rangle$,

where $\varphi \in [0, 2\pi]$ describes the relative phase and $\theta \in [0, \pi]$ determines the probability to measure $|0\rangle$ / $|1\rangle$: $\quad p(|0\rangle) = \cos^2\frac{\theta}{2}$, $\; p(|1\rangle) = \sin^2\frac{\theta}{2}$.

⇒ all normalized pure states can be illustrated on the surface of a sphere with radius $|\vec{r}| = 1$, which we call the Bloch sphere

⇒ the coordinates of such a state are given by the Bloch vector: $\vec{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

<u>examples</u>:
- $|0\rangle$: $\theta = 0$, φ arbitrary $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- $|1\rangle$: $\theta = \pi$, φ arb. $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
- $|+\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = 0$ $\rightarrow \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $|-\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \pi$ $\rightarrow \vec{r} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$
- $|+i\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- $|-i\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \frac{3\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



<u>Be careful</u>: On the Bloch sphere, angles are twice as big as in Hilbert space, e.g. $|0\rangle$ & $|1\rangle$ are orthogonal, but on the Bloch sphere their angle is $180°$. For a general state $|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \ldots$ → θ is the angle on the Bloch sphere, while $\frac{\theta}{2}$ is the actual angle in Hilbert space!

⇒ Z-measurement corresponds to a projection onto the z-axis and analogously for X & Y!