Appendix A
Proof of potential game

Proof. We define player i's utility as $u_i(\alpha_i, \alpha_{-i}) = \sum_{k\in K} q_{i,k} - \sum_{k\in K} \alpha_{i,k} q_{i,k} + W\Big(1 - \frac{\sum_j \alpha_j \cdot \omega_j}{C}\Big)$, where $\omega_j \in \mathbb{R}^K$ is the resource requirement of each commodity, C is the system capacity.

We define potential function: $\phi(\alpha_i, \alpha_{-i}) = \sum_{j\in I, k\in K} q_{j,k} - \sum_{j\in I, k\in K} \alpha_{j,k} q_{j,k} + W\Big(1 - \frac{\sum_j \alpha_j \cdot \omega_j}{C}\Big)$.

To simplify, we substitute with $Q_i = \sum_{k\in K} q_{i,k}$, $A_i = \sum_{k\in K} \alpha_{i,k} q_{i,k}$, $A_{-i} = \sum_{j\in I, j\neq i, k\in K} \alpha_{j,k} q_{j,k}$, $B_i = \sum_k \alpha_{i,k} \omega_{i,k}$, $B_{-i} = \sum_{j\in I, j\neq i, k\in K} \alpha_{j,k} \omega_{j,k}$, and rewrite: $u_i(\alpha_i, \alpha_{-i}) = Q_i - A_i + W - \frac{W}{C}(B_i + B_{-i})$, $u_i(\alpha_i', \alpha_{-i}) = Q_i - A_i' + W - \frac{W}{C}(B_i' + B_{-i})$, $\phi(\alpha_i, \alpha_{-i}) = \sum_j Q_j - (A_i + A_{-i}) + W - \frac{W(B_i + B_{-i})}{C}$, $\phi(\alpha_i', \alpha_{-i}) = \sum_j Q_j - (A_i' + A_{-i}) + W - \frac{W(B_i' + B_{-i})}{C} \implies u_i(\alpha_i, \alpha_{-i}) - u_i(\alpha_i', \alpha_{-i}) = -(A_i - A_i') - \frac{W}{C}(B_i - B_i') = \phi(\alpha_i, \alpha_{-i}) - \phi(\alpha_i', \alpha_{-i})$ □

Since $\alpha_i \in \mathbb{R}^{|K|}$, the game under low contention is a finite potential game.

Appendix B
Second-price auction

We prove the theorem for $|M| = 2$ and $|K| = 1$. It is an extension from [1]. Unlike [1], we include in utility definition the second-price payment and cost for losing a bid. Based on [1], it can also be easily extended to multiple bidders.

A. Basic model

2 bidders receive continuously distributed valuations $v_i \in [l_i, m_i], i \in \{1, 2\}$ for 1 commodity, and choose their strategies $f_1(v_1), f_2(v_2)$ from the strategy sets F_1 and F_2. The resulting NE strategy pair is (f_1^*, f_2^*). Any strategy function $f(v)$ is increasing in v, with $f_1(l_1) = a$, and $f_1(m_1) = b$. We also assume the users have budgets (B_1, B_2), and that they cannot bid more than the budget. We define cost for losing the bid c_i. Furthermore, we define the inverse function of $f_1(v_1)$ to be: $h_1(y_1) = l$, if $y_1 \leq a_1$, $h_1(y_1) = f_1^{-1}(y_1)$, if $a_1 < y_1 < b_1$, and $h_1(y_1) = m$, if $y_1 \geq b_1$.

For a given f_1, if bidder 2 chooses a bidding function f_2, according to Eq. **??**, the expected utility for bidder 2 is $u_2(f_1, f_2) = \mathbb{E}_{v_1,v_2}[(v_2 + c_2) \cdot 1_{f_2(v_2)\geq f_1(v_1)}] - \mathbb{E}_{v_1,v_2}[f_1(v_1) \cdot 1_{f_2(v_2)\geq f_1(v_1)}] - c_2$, where $1_{f_2(v_2)\geq f_1(v_1)} = 1$, if $f_2(v_2) \geq f_1(v_1)$, otherwise 0. To simplify, we define $E_1 = \mathbb{E}_{v_1,v_2}[(v_2 + c_2) \cdot 1_{f_2(v_2)\geq f_1(v_1)}]$ and $E_2 = \mathbb{E}_{v_1,v_2}[f_1(v_1) \cdot 1_{f_2(v_2)\geq f_1(v_1)}]$. Hence, $u_2(f_1, f_2) = E_1 - E_2 - c_2$. E_2 is the expected second price payment when bidder 2 wins, and the payment should be no greater than $\min(b_2, B_2)$. Since to avoid overbidding, we assume $b_2 \leq B_2$, the set of feasible bidding functions for bidder 2 given f_1 is $S_2(f_1) = \{f_2 \in F_2 | u_2(f_1, f_2) \geq 0, E_2 \leq b_2\}$.

For the condition $u_2(f_1, f_2) \geq 0$ to hold, we can prove that at any point where $1_{f_2(v_2)\geq f_1(v_1)} = 1$, we have $v_2 \geq f_1(v_1)$, which is a sufficient condition of $u_2(f_1, f_2) \geq 0$. This is because f_2 is bidder 2's bidding signal, to avoid overbidding, $f_2(v_2) \leq \min(b_2, v_2)$, therefore $f_1(v_1) \leq v_2$. We thus simplify the above equation to: $S_2(f_1) = \{f_2 \in F_2 | E_2 \leq b_2\}$.

We formulate the problem into a utility maximization problem: $\max_{f_2 \in S_2(f_1)} u_2(f_1, f_2)$. We say f_2 is a best response of bidder 2, if $u_2(f_1, f_2) \geq u_2(f_1, f_2')$, $\forall f_2' \in S_2(f_1)$. A NE strategy pair (f_1^*, f_2^*) has the selected strategies as each other's best responses.

B. Form of the best response

Theorem B.1. Given bidder 1's bidding strategy $f_1 \in F_1, f_1(l_1) = a_1, f_1(m_1) = b_1$, bidder 2's best response has the form $\begin{cases} f_2(v_2) \leq a_1 & \text{for } v_2 \in [l_2, \theta_1] \\ f_2(v_2) = j_2 \cdot v_2 + d_2 & \text{for } v_2 \in [\theta_1, \theta_2] \\ f_2(v_2) \geq b_1 & \text{for } v_2 \in [\theta_2, m_2] \end{cases}$, where $\theta_1, \theta_2 \in [l_2, m_2]$ and $j_2\theta_1 + d_2 = a_1, j_2\theta_2 + d_2 = b_1$.

Proof. Given f_1 and bidder 2's bid y_2, probability that bidder 2 wins the bid is:

$P_2^{win}(y_2) = P(f_1(v_1) \leq y_2) = P(v_1 \leq h_1(y_2)) = \int_{l_1}^{h_1(y_2)} \mathbf{p}_1(v_1) \mathrm{d}v_1$, where $\mathbf{p}$ is the probability density function, and P is the cumulative function.

Bidder 2's optimization problem is: find a bidding function $y_2 = f_2(v_2)$ to maximize $E_1 - E_2$
$= \mathbb{E}_{v_1,v_2}[(v_2+c_2) \cdot 1_{f_2(v_2)\geq f_1(v_1)}] - \mathbb{E}_{v_1,v_2}[f_1(v_1) \cdot 1_{f_2(v_2)\geq f_1(v_1)}] = \int_{l_2}^{m_2} \int_{l_1}^{h(f_2(v_2))} \Big(v_2 + c_2 - f_1(v_1)\Big) \mathbf{p}_2(v_2) \mathbf{p}_1(v_1) \mathrm{d}v_1 \mathrm{d}v_2$, s.t. $E_2 \leq b_2$.

To solve the optimization problem, we write the Lagrangian function with multiplier λ:

$\mathcal{L}(v_2, \lambda) = E_1 - E_2 - \lambda(E_2 - b_2) = \int_{l_2}^{m_2} \Big[\int_{l_1}^{h_1(f_2(v_2))} V \mathbf{p}_1(v_1) \mathrm{d}v_1\Big] \mathbf{p}_2(v_2) \mathrm{d}v_2 - \lambda b_2$, where $V = v_2 + c_2 - (1 + \lambda) f_1(v_1)$.

Next, for each v_2, we find the f_2 that maximizes $\int_{l_1}^{h_1(y_2)} \Big(v_2 + c_2 - (1 + \lambda) f_1(v_1)\Big) \mathbf{p}_1(v_1) \mathrm{d}v_1$, $y_2 = f_2(v_2)$. $\max_{f_2}(\mathcal{L})$ is the equivalent of $\max_{f_2}(E_1)$.

For any given v_2, the above formula is the area below the function $z = v_2 + c_2 - (1 + \lambda) f_1(v_1)$, when v_1 moves in the range from l_1 to $h_1(y_2)$. As f_1 is monotonously increasing, z is monotonously decreasing. Therefore, to maximize the area below z, $h_1(y_2)$ should simply be chosen as the intersection of z and the x-axis, or $v_2 + c_2 - (1 + \lambda) f_1(h_1(y_2)) = 0$:

$y_2 = f_1(f_1^{-1}(y_2)) = f_2(v_2) = \frac{v_2 + c_2}{1 + \lambda}$, $\forall y_2 \in [a_1, b_1]$, or $v_2 \in [(1 + \lambda)a_1 - c_2, (1 + \lambda)b_1 - c_2]$.

Since $f_2(v_2)$ is monotonously increasing, $f_2(v_2) \leq a_1$, for $v_2 \in [l_2, (1 + \lambda)a_1 - c_2]$, and similarly, $f_2(v_2) \geq b_1$, for $v_2 \in [(1 + \lambda)b_1 - c_2, m_2]$.

Theorem B.1 implies that the best response of bidder 1 and 2 are both of the linear form. □

C. Existence of Nash equilibrium

Theorem B.2. When best response form is $f_1(v_1) = j_1 v_1 + d_1$ and $f_2(v_2) = j_2 v_2 + d_2$, we can always find a pair (j_1, j_2) such that both bidders' budget range $[a_i, b_i]$ would be satisfied in NE.

Proof. A NE exists if there is a pair (j_1, j_2) that satisfy the two constraints: $\mathbb{E}_{v_1,v_2}[f_1(v_1) \cdot 1_{f_2(v_2)\geq f_1(v_1)}] \leq b_2, \mathbb{E}_{v_1,v_2}[f_2(v_2) \cdot 1_{f_1(v_1)\geq f_2(v_2)}] \leq b_1$.

The following proves that such a pair exists. If we choose $c_1 = c_2 = c$, and given the linear best response forms, and given the bidders' bidding functions, we define $E_3 = \mathbb{E}_{v_1,v_2}[(v_1 - c) \cdot 1_{j_1 v_1 \geq j_2 v_2}]$ and $E_4 = \mathbb{E}_{v_1,v_2}[(j_2 v_2 - c) \cdot 1_{j_1 v_1 \geq j_2 v_2}]$.

Define bidder 1's feasible strategy set: $S_1(j_2) = \{j_1 \in [0,\infty) | E_4 \leq b_1\}$. Due to its linear form, and according to Eq. **??**, bidder 1's best response is: $\mathbf{b}_1(j_2) = \arg\max_{f_1 \in S_1(j_2)}(E_3 - E_4) = \arg\max_{y \in S_1(j_2)} \mathbb{E}_{v_1,v_2}[v_1 - j_2 v_2 \cdot 1_{y v_1 \geq j_2 v_2}]$. Utility $u_1(y) = \mathbb{E}_{v_1,v_2}[v_1 - j_2 v_2 \cdot 1_{y v_1 \geq j_2 v_2}]$ is a non-decreasing function of y defined on the set $S_1(j_2)$. To prove the existence of NE, we use Kakutani fixed point theorem.

Theorem B.3 (Kakutani fixed point theorem [2])**.** Let A be a non-empty, compact and convex subset of some Euclidean space R^n. Let $\varphi : A \to 2^B$ be an upper hemicontinuous set-valued function on A with the property that $\varphi(x)$ is non-empty, closed, and convex $\forall x \in A$. Then φ has a fixed point.

We prove Lemmas B.1-B.4 below, to show that our case meets the conditions of Theorem B.3. Hence, $\varphi : S_1 \to \mathbf{b}_1 \in 2^{S_1}$ has a fixed point, and there exists NE (Theorem B.2). □

Lemma B.1. Bidder 1 strategy set $A = S_1(j_2) = \{j_1 | \mathbb{E}_{v_1,v_2}[(j_2 v_2 + d_2) \cdot 1_{j_1 v_1 \geq j_2 v_2}] \leq b_1, j_1 \in [0,\infty)\}, \forall j_2 \in [0,\infty)$ is non-empty, convex, compact.

Proof. $S_1(j_2)$ is a strategy set and naturally non-empty. The product of all players' strategy sets are therefore also non-empty. For any given j_2, any combination of a feasible strategy's parameter still creates a feasible strategy (due to its linear form). Therefore $S_1(j_2)$ is convex. The set $S_1(j_2)$ contains all of its limits, therefore it is a closed set. Due to bidding range and budget, it is also bounded. The product of all players' strategy sets are therefore closed and bounded. According to Heine-Borel Theorem, the sets are compact. □

Definition B.1. : A set-valued function u defined on a convex set $S_1(j_2)$ is quasiconcave if every upper level set of u is convex, or $P_{j_1} = \{j_1 \in S(j_2) : u(j_1) \geq a\}$ is convex $\forall a \in \mathbb{R}$.

Lemma B.2. The correspondence $\varphi : S_1 \to 2^{S_1}$, where $\varphi(S_1) = \mathbf{b}_1$ is convex, $\forall s \in S_1$.

Proof. First, we prove utility u_i is quasiconcave.

Let $\sigma_i^1, \sigma_i^2 \in \mathbf{b}_i$, since they are best responses, we have utilities $u_i^1 = u_i(\sigma_i^1, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in S_i$, and $u_i^2 = u_i(\sigma_i^2, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in S_i$. Hence, $\lambda u_i^1 + (1-\lambda)u_i^2 \geq u_i(\tau_i, \sigma_{-i}), \lambda \in [0,1]$.

Given any $a \in \mathbb{R}$, if we create a upper level set p_a containing all $j_1 \in S(j_2)$ that meet the condition of having a utility $u_i \geq a$, and if p_a is always a convex set, then u_i is quasiconcave. This is apparent, as $u_1(j_1) = E_3 - E_4 = \mathbb{E}_{v_1,v_2}[(v_1 - j_2 v_2) \cdot 1_{j_1 v_1 \geq j_2 v_2}]$ is continuous and non-decreasing in j_1. If $j_1 v_1 \geq j_2 v_2$ and $j_1' v_1 \geq j_2 v_2$, we would always have $\lambda j_1 v_1 \geq \lambda j_2 v_2$ and $(1-\lambda) j_1' v_1 \geq (1-\lambda) j_2 v_2$ for any $\lambda \in [0,1]$. Adding both sides of the inequation respectively: $(\lambda j_1 + (1-\lambda) j_1') v_1 \geq j_2 v_2$, which means $\lambda j_1 + (1-\lambda) j_1'$ is also a member of p_a, or that any p_a is convex.

Since the utility function u_1 is defined on convex set S_1 and all of its upper level set is convex, the utility function is quasiconcave. Also, as u_i is quasiconcave, we have $u_i(\lambda\sigma_i^1 + (1-\lambda)\sigma_i^2, \sigma_{-i}) \geq \lambda u_i^1 + (1-\lambda)u_i^2 \geq u_i(\tau_i, \sigma_{-i})$. Therefore $\lambda\sigma_i^1 + (1-\lambda)\sigma_i^2$ is also a best response, it is in the $\mathbf{b}_i$ set. $\mathbf{b}_i$ is therefore convex-valued. Finally, φ is convex if and only if each $\mathbf{b}_i$ is convex. Any combination of best responses will still be a best response. □

Definition B.2 (Upper hemicontinuity [2])**.** Correspondence $S : \Psi \to \Xi$ is upper hemicontinuous, if for every $\psi_1 \in \Psi$ and $\epsilon > 0$, $\exists\delta > 0$ s.t.: if $\psi_2 \in \Psi$ and $||\psi_2 - \psi_1|| < \delta$, then $S(\psi_2) \subset B_\epsilon(S(\psi_1))$, where $B_\epsilon(x)$ denotes the ϵ-ball around x. Correspondence S is lower hemicontinuous, if for any open set $U \subset \Xi$ with $S(\psi_1) \cap U \neq \emptyset$, $\exists\epsilon > 0$, s.t. $\forall\psi_2 \in B_\epsilon(\psi_1)$, $S(\psi_2) \cap U \neq \emptyset$.

Lemma B.3. let bidder 2's feasible strategies j_2 be in a set Ψ, let bidder 1's strategies $A = S_1(j_2), j_2 \in \Psi$ be in a set Ξ. The correspondence: $S_1 : \Psi \to \Xi$ is continuous at all j_2.

Proof. $\forall j_2 \in \Psi$, and a ϵ-ball around $S_1(j_2)$, we can find a range δ around j_2, s.t. any $j_2' \in \Psi, ||j_2' - j_2|| < \delta$, has $S_1(j_2')$ within the ϵ-ball around $S_1(j_2)$. This is apparent, since for any given best response parameter j_2' in the neighborhood of j_2, the corresponding strategy set in $S_1(j_2)$ would be a set of j_1' that is in the neighborhood of j_1 (upper hemicontinuous). It is proven in [3] that if the graph $G(S_1)$ is convex when $S_1(j_2)$ is monotone increasing, then S_1 is lower hemicontinuous. In our case, due to the linear form, and according to Lemma B.1, S_1 is lower hemicontinuous. Therefore, S_1 is continuous [3]. □

Theorem B.4 (Berge's maximum theorem [2])**.** Let Ξ, Ψ be topological spaces, $u_1 : \Xi \times \Psi \to \mathbb{R}$ be a continuous function on the product space, and $S_1 : \Psi \to \Xi$ be a compact-valued correspondence s.t. $S_1(j_2) \neq \emptyset$, $\forall j_2 \in \Psi$. Define $u_1^*(j_2) = \sup\{u_1(j_1, j_2) : j_1 \in S_1(j_2)\}$, sup being the maximum operator of u, and the set of maximizers $S_1^* : \Psi \to \Xi$ by: $S_1^*(j_2) = \arg\sup\{u_1(j_1, j_2) : j_1 \in S_1(j_2)\} = \{j_1 \in S_1(j_2) : u_1(j_1, j_2) = u_1^*(j_2)\}$. If S_1 is continuous (i.e., both upper and lower) at j_2, then u_1^* is continuous and S_1^* is upper hemicontinuous with nonempty and compact values.

Lemma B.4. Correspondence $\varphi : S_1 \to 2^{S_1}$, where $\varphi(S_1) = S_1^* = \mathbf{b}_1$, is upper hemicontinuous with non-empty and compact values, and has a closed graph.

Proof. According to B.4, since S_1 is continuous (Lemma B.3), non-empty and compact (Lemma B.1), the correspondence φ is upper hemicontinuous with non-empty and compact values. It is apparent that best response set is a closed subset of the strategy set S on all $s \in S$. Therefore b_i is closed-valued. A closed-valued upper hemicontinuous correspondence has a closed graph. □

Lemmas B.1, B.2 and B.4 apply to the strategy sets of all players. According to the lemmas, we can prove that our setup meets the conditions of Theorem B.3, therefore the game has NE.

APPENDIX C
PARETO OPTIMALITY

Valuation of the service request is a linear function of the resource needed: $v_1 = g_1\omega_1 + k_1, v_2 = g_2\omega_2 + k_2$, g, k are constants, ω is amount of resource required. The allocation rule under NE is: $A^*_{v_1,v_2} = 1$, if $j_1 v_1 + d_1 \geq j_2 v_2 + d_2$, otherwise 2. Form of the condition is from best response form in appendix Sec. B-C. We also assume that both bidders have at least some access to the resources, as a form of fairness. We define the fairness constraint to be: $\mathbb{E}[\omega_1|_{A_{v1,v2=1}}]/\mathbb{E}[\omega_2|_{A_{v1,v2=2}}] = \gamma \in \mathbb{R}_{>0}$.

Theorem C.1. The allocation $A^*_{v_1,v_2}$ maximizes overall resource allocation $\omega_1 + \omega_2$, subject to the fairness constraint, when the valuations are linear functions of resources. Or, the NE of the game achieves optimal resource allocation.

Proof. Find the Lagrangian multiplier λ^* that satisfies the fairness constraint with NE allocation $A^*_{v_1,v_2}$. Define g, k as: $g_1 = (1{+}\lambda^*)/j_1$, $k_1 = -d_1/j_1$, and $g_2 = (1{-}\gamma\lambda^*)/j_2$, $k_2 = -d_2/j_2$. Then we can rewrite the allocation: $A^*_{\omega_1,\omega_2} = 1$, if $\omega_1(1{+}\lambda^*) \geq \omega_2(1-\gamma\lambda^*)$, otherwise 2. The rest of the proof is the same as in [1]. □

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