代数簇

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Definition 1.1. *Let k be a field. Define the affine space over k of dimension n to be $\mathbb{A}_k^n := k^{\oplus n}$.*

Definition 1.2. *Let $\mathbb{A}_k^n$ be affine space. Assume $S \subset k[x_1, \cdots, x_n]$ be a set of polynomials. Define $V(S) := \{x \in \mathbb{A}_k^n \big| f(x) = 0, \forall f \in S\}$. Such a set is called an algebraic set.*

Lemma 1.1. *Let $\mathbb{A}_k^n$ be affine space. Then algebraic sets have following properties*

- *For two subset $S_1, S_2 \subset k[x_1, \cdots, x_n]$, $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 = \{fg \big| f \in S_1, g \in S_2\}$*
- *For any collection $\{S_i\}$, $\cap_i V(S_i) = V(\cup_i S_i)$*

Remark 1.1. *With this lemma, we immediately have that algebraic sets give a topological structure on $\mathbb{A}_k^n$ as closed subsets, called the Zariski topology.*

1.2 Affine variety

From now on, we would rename algebraic variety in affine space as affine variety.

Definition 1.3. *Let $\mathbb{A}_k^n$ be affine space, $V \subseteq \mathbb{A}_k^n$ an affine subvariety. Define the associated ideal of V to be $I(V) := \{f \in k[x_1, \cdots, x_n] \big| f(x) = 0, \forall x \in V\}$*

Remark 1.2. *It is easy to see that $V \subseteq V(I(V))$.*

Theorem 1.1 (***Hilbert Nullstellensatz Theorem***)**.** *Let k be an algebraically closed field, $I \subseteq k[x_1, \cdots, x_n]$. Assume that $f \in k[x_1, \cdots, x_n]$ vanish at all points of $V(I)$. Then $f^r \in I$ for some $r > 0$.*

Corollary 1.1. *Let k be an algebraically closed field. Then there are some one-to-one correspondences*

$$\begin{aligned} \textit{radical ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \textit{affine varieties} \\ \textit{prime ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \textit{irreducible affine varieties} \\ \textit{maximal ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \textit{one-point affine varieties} \end{aligned}$$

Definition 1.4. *Let $V \subseteq \mathbb{A}_k^n$ be an affine variety, $I = I(V)$. Define the affine coordinate ring of V to be $A_V := k[x_1, \cdots, x_n]/I$.*

Corollary 1.2. *Every finitely generated reduced k-algebra is the coordinate function ring of some affine variety.*

1.3 Regular maps

Definition 1.5. *Let V, W be two affine varieties. Assume that $V \subseteq \mathbb{A}_k^n$ with coordinate $x_1, \cdots, x_n$ and $W \subseteq \mathbb{A}_k^m$ with coordinate $y_1, \cdots, y_m$. A regular map from V to W is a map $\phi : V \to W$ satisfying there exist polynomials $f_1, \cdots, f_m \in k[x_1, \cdots, x_n]$ such that $\phi = (f_i)_{1 \le i \le n}\big|_V$.*

Remark 1.3. *Given a regular map $\phi : V \to W$, there is a morphism $\phi^* : A_W \to A_V$ mapping f to $f \circ \phi$.*

Proposition 1.1. *Let V, W be two affine varieties. Assume that $V \subseteq \mathbb{A}_k^n$ with coordinate $x_1, \cdots, x_n$ and $W \subseteq \mathbb{A}_k^m$ with coordinate $y_1, \cdots, y_m$. The correspondence $\phi \leftrightsquigarrow \phi^*$ is one-to-one. In fact, $\phi = (\phi^*(y_1), \cdots, \phi^*(y_m))$.*

Definition 1.6. *Let V, W be two affine varieties, $\phi : V \to W$ a regular map. We say ϕ is isomorphic if there exists regular map $\psi : W \to V$ such that $\psi \circ \phi = \mathrm{id}_V$ and $\phi \circ \psi = \mathrm{id}_W$.*

Corollary 1.3. *Let V, W be two affine varieties, $\phi : V \to W$ a regular map. Then ϕ is isomorphic if and only if ϕ^* is isomorphic.*

Definition 1.7. *Let $\phi : V \to W$ be a regular map of affine varieties. We say that ϕ is dominant if $\phi(V)$ is dense in W.*

Proposition 1.2. *Let $\phi : V \to W$ be a regular map of affine varieties. Then ϕ is dominant if and only if ϕ^* is injective. In addition, if ϕ^* is surjective, then ϕ is injective.*

Proof. Assume that ϕ is dominant. Suppose that ϕ^* is not injective, then there exist different $f, g \in A_W$ such that $f \circ \phi = g \circ \phi$. Since there exist $y \in W$ and an open neighbourhood $W \supseteq U \ni y$ such that $f\big|_U \neq g\big|_U$ everywhere in U. But $\phi(V)$ is dense in W, we get $x \in V$ such that $\phi(x) \in U$, contradiction!

For the converse, assume that ϕ^* is injective. Suppose that ϕ is nor dominant. Then $\overline{\phi(V)} \subsetneq W$ so that $I(W) \subsetneq I(\overline{\phi(V)})$. Hence we can choose nonzero $f \in A_W$ s.t. f vanish at all points of $\phi(V)$. Thus $\phi^*(f) = 0 = \phi^*(0)$ contradicting to ϕ^* is injective.

Assume that ϕ^* is surjective. For any two points $x_1, x_2 \in V$, there exists $F \in A_V$ such that $F(x_1) \neq F(x_2)$. As ϕ^* is surjective, there exists $f \in A_W$ such that $F = f \circ \phi$. Thus $F(x_1) = f(\phi(x_1))$ and $F(x_2) = f(\phi(x_2))$ so that $\phi(x_1) \neq \phi(x_2)$. Hence ϕ is injective. $\square$

2 Projective Variety

2.1 Zariski topology

Definition 2.1. *Let k be a field. Define the projective space over k of dimension n to be $\mathbb{P}_k^n := \{k^{\oplus n+1} \setminus \{0\}\}/\sim$, where $x \sim y$ if and only if $x = ay$ for some $a \in k^\times$. Element in $\mathbb{P}_k^n$ is of the form $[a_0, a_1, \cdots, a_n]$.*

Remark 2.1. *To define algebraic set in $\mathbb{P}_k^n$, we cannot directly use set of polynomials since for general polynomial f and $x \sim y \in k^{\oplus n+1} \setminus \{0\}$, $f(x) = 0 \iff f(y) = 0$ is not always* *correct. However, if f is homogeneous, then $f(x) = a^{n+1}f(y)$ for some $a \in k^\times$ so that $f(x) = 0 \iff f(y) = 0$.*

Definition 2.2. *Let $\mathbb{P}_k^n$ be projective space. Assume $S \subset k[x_0, x_1, \cdots, x_n]$ be a set of homogeneous polynomials. Define $V(S) := \{x \in \mathbb{P}_k^n \big| f(x) = 0, \forall f \in S\}$. Such a set is called an algebraic set.*

Lemma 2.1. *Let $\mathbb{P}_k^n$ be projective space. Then algebraic sets have following properties*

- *For two subset $S_1, S_2 \subset k[x_0, x_1, \cdots, x_n]$ of homogeneous polynomials, $V(S_1) \cup V(S_2) = V(S_1S_2)$, where $S_1S_2 = \{fg \big| f \in S_1, g \in S_2\}$.*
- *For any collection $\{S_i\}$, $\cap_i V(S_i) = V(\cup_i S_i)$*

Remark 2.2. *With this lemma, we immediately have that algebraic sets give a topological structure on $\mathbb{P}_k^n$ as closed subsets, called the Zariski topology.*

2.2 Projective variety

From now on, we would rename algebraic variety in projective space as projective variety.

Definition 2.3. *Let $\mathbb{P}_k^n$ be projective space, $V \subseteq \mathbb{A}_k^n$ a projective subvariety. Define the associated homogeneous ideal $I(V)$ of V to be the ideal generated by $\{$homogeneous $f \in k[x_0, x_1, \cdots, x_n] \big| f(x) = 0, \forall x \in V\}$*

Similarly, by Theorem 1.1, we get some correspondences.

Corollary 2.1. *Let k be an algebraically closed field. Then there are some one-to-one correspondences*

$$\begin{aligned}
\text{radical homogeneous ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \text{projective varieties} \\
\text{prime homogeneous ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \text{irreducible projective varieties} \\
\text{maximal homogeneous ideals of } k[x_1, \cdots, x_n] &\leftrightsquigarrow \text{one-point projective varieties}
\end{aligned}$$

Lemma 2.2. *Let $\mathbb{P}_k^n$ be projective space over k with coordinate $x_0, x_1, \cdots, x_n$. Then for all $0 \leq i \leq n$, the open subset $D_+(x_i) := \{x_i \neq 0\}$ is homeomorphic to affine space $\mathbb{A}_k^n$ with Zariski topology.*

Hence, we can always embed affine variety into projective space. In particular, we have that $\mathbb{P}_k^0 \cong \mathbb{A}_k^1$. Thus after that we would generally state propositions of projective varieties without explanation for affine case.

In addition, the above lemma indicates that open subset in Zariski topology would also have some special properties. So we here give it a formal definition.

Definition 2.4. *Let k be a field. An open subset U of projective (resp. affine) space is called a quasi-projective (resp. affine) variety.*

2.3 Regular functions and regular maps

Definition 2.5. *Let X be a (quasi-)projective subvariety of $\mathbb{P}_k^n$. A function $f : X \to k$ is said to be regular at $x \in X$ if there exist homogeneous polynomials G, H of same degree such that $f = \frac{G}{H}$ in an open neighbourhood U of x and $H(p) \neq 0$ for all $p \in U$.*

We say that f is a regular function if it is regular at all points of X. Denote $A(X)$ to be the ring of regular functions on X.

Remark 2.3. *Regular functions are continuous map since locally they are defined as rational fractions.*

Theorem 2.1. *Let k be an algebraically closed, X irreducible (quasi-)affine subvariety of $\mathbb{A}_k^n$. Then a regular function $f : X \to k$ is defined by a polynomial i.e. regular function ring is just coordinate ring.*

Proof. Take open covering $\{U_i\}$ such that $f\big|_{U_i}$ is defined as $\frac{G_i}{F_i}$, where G_i and F_i are homogeneous polynomials of same degree. Hence $F_iG_j = F_jG_i$ on each $U_i \cap U_j$. Note that $U_i \cap U_j$ is nonempty open in X which is irreducible, so we have that $X = \overline{U_i \cap U_j} \subseteq V(F_iG_j - F_jG_i)$.

Now $F_iG_j = F_jG_i$ on X for all i, j. As U_i cover X and H_i has no root in U_i, there is no common root of all H_i. Thus by Hilbert Nullstellensatz, there exist polynomials a_i such that $\sum a_iH_i = 1$ on X. Take $g = \sum a_jG_j$, then

$$H_ig = H_i \sum a_jG_j = \sum a_jH_jG_i = G_i \tag{1}$$

get $\frac{G_i}{H_i} = g$ on X for all i so that f is defined by g. □

Remark 2.4. *With this theorem, now we can identify regular function ring and coordinate ring of affine varieties. From now on, we would use same notation $A(X)$ for them.*

In addition, the irreducible condition is not required. However, now we have not enough knowledge to give a proof.

Definition 2.6. *Let X, Y be (quasi-)projective varieties. Wa say a map $\varphi : X \to Y$ is regular if*

- *φ is continuous*
- *for all regular function $f : U \to k$ with $U \subseteq Y$ open, $f \circ \varphi : \varphi^{-1}(U) \to k$ is a regular function*

Example 2.1. *For homogeneous polynomial $F_0, \cdots, F_m$ of same degree with no common root, we can define a regular map $\mathbb{P}_k^n \longrightarrow \mathbb{P}_k^m$ defined by*

$$[x_0, x_1, \cdots, x_n] \longmapsto [F_0(x_0, \cdots, x_n), F_1(x_0, \cdots, x_n), \cdots, F_m(x_0, \cdots, x_n)] \tag{2}$$

In particular, if we take $F_0 = x_0^d$, $F_1 = x_0^{d-1}x_1$, $\cdots$ and $F_N = x_n^d$ where $N = \binom{n+d}{d} - 1$, the corresponding map $v_d : \mathbb{P}_k^n \to \mathbb{P}_k^N$ is called the dth Veronese map.

Lemma 2.3. *Let V, W be two affine varieties. Then the above definition of regular map from V to W is equivalent to our original definition.*

Reason 2.1. *By Proposition 1.1, it is clear that the two definitions are same.*

Proposition 2.1. *Let X be a (quasi-)projective subvariety of $\mathbb{P}_k^m$, $\varphi : X \to \mathbb{P}_k^m$ a regular map. Then for every $x \in X$, there exists affine open neighbourhood U of x and polynomials $F_0, \cdots, F_m \in A(U)$ such that $\varphi(y) = [F_0(y), \cdots, F_n(y)]$ for all $y \in U$.*

Proof. For $x \in X$, take affine open neighbourhood $D_+(x_i)$. Without loss of generality, we may assume that $x \in D_+(x_0) \cong \mathbb{A}_k^m$ with coordinate $x_1, \cdots, x_m$. In addition, we can also take affine open neighbourhood of $\varphi(x)$ in $\mathbb{P}_k^n$, say $\mathbb{A}_k^n$ with coordinate $y_1, \cdots, y_n$.

Now $U := X \cap D_+(x_0) \cap \varphi^{-1}(\mathbb{A}_k^n)$ is an (quasi-)affine subvariety of $\mathbb{A}_k^m$. By Theorem 2.1, $y_i \circ \varphi\big|_U : U \to \mathbb{A}_k^n \to k$ is defined by a polynomial F_i. Hence $\varphi\big|_U = [1, F_1, \cdots, F_n]$. □

Example 2.2. *Such local polynomials cannot glue up in general. Take $Z = V(xz - y^2)$. Consider the following map $Z \to \mathbb{P}_k^1$ defined by*

$$[x, y, z] \longmapsto \begin{cases} [x, y] & x \neq 0 \textit{ or } y \neq 0 \\ [y, z] & z \neq 0 \end{cases} \tag{3}$$

2.4 Rational maps

Definition 2.7. *Let X, Y be (quasi-)projective varieties. Consider pairs (U, f_U) where $U \subset X$ is an open dense subset and $f_U : U \to Y$ is a regular map. We $(U, f_U) \sim (V, f_V)$ if $f_U\big|_{U\cap V} = f_V\big|_{U\cap V}$. Such an equivalence class is called a rational map $X \dashrightarrow Y$. In particular, if Y is k, then rational map $X \dashrightarrow k$ is called a rational function.*

Remark 2.5. *For any rational map $f : X \dashrightarrow Y$, there exists maximal open subset U of X such that $f\big|_U : U \dashrightarrow Y$ is a regular map, called the domain of definition of f.*

Lemma 2.4. *Let X be an affine subvariety of $\mathbb{A}_k^n$. Then for any polynomial $f \in k[x_1, \cdots, x_n]$. the open set $D(f) \cap X$ is an affine subvariety of $\mathbb{A}_k^{n+1}$, which is defined by $I(X)$ and $x_{n+1}f - 1$, with coordinate ring $A(D(f) \cap X) = A(X)_f$.*

Remark 2.6. *The proof is easy. Note that by this lemma, it is not necessary for us to define the notion of quasi-affine variety. Hence from now on, we would omit the prefix of quasi-affine variety. Also, for convenience, we mean (quasi-)projective variety when we say variety.*

Proposition 2.2. *Let X be a variety. Then*
(a)$\{f : X \dashrightarrow k\}$ form a k-algebra, denoted by $K(X)$.
(b)$\forall U \subset X$ open dense subset, we have that $K(X) = K(U)$.
(c)If X is irreducible, then $K(X)$ is a field.
(d)If X is affine and irreducible, then $K(X)$ is the fraction field of $A(X)$.

Proof. (a), (b) are both obvious. Here we only prove (c) and (d). For (c), assume $f : X \dashrightarrow k$ is nonzero element in $K(X)$. Take its domain of definition U, then there exists open subset $U' \subseteq U$ such that $f\big|_{U'}$ is defnied by some polynomial P. Since X is irreducible, U' is dense so that $P : U' \to k$ also represents f.

Consider the open subset $W = \{x \in X \big| P(x) \neq 0\}$. As f is nonzero element, $W \cap U' \neq \varnothing$. Define $g : W \to k$ mapping x to $\frac{1}{P(x)}$, which is a regular function and hence a regular map. Thus we get inverse of f in $K(X)$. Conclude that $K(X)$ is a field.

For (d), assume that $f : X \dashrightarrow k$ can be represented by $P : U \to k$. By definition of Zariski topology, we can take open subset $U' \subseteq U$ of the form $D(H)$ for some $H \in A(X)$. Hence by Lemma 2.4, $f \in A(U') = A(X)_H$. Thus $K(X) \subseteq \cup_H A(X)_H = \mathrm{Frac}(A(X))$. The converse containment is obvious so that $K(X) = \mathrm{Frac}(A(X))$. $\square$

Remark 2.7. *For irreducible variety X, taking affine open dense subset $U \subseteq X$, by (c), $K(X) = K(U) = \mathrm{Frac}(A(U))$ is computable.*

Example 2.3. *Let $X \subseteq \mathbb{P}_k^m$ be a variety. Assume that homogeneous polynomials $F_0, \cdots, F_n \in k[x_0, \cdots, x_m]$ satisfy that $V(F_0, \cdots, F_n)$ does not contain any irreducible component of X, then $X \setminus V(F_0, \cdots, F_n)$ is dense in X and we have a rational map $X \dashrightarrow \mathbb{P}_k^n$ defined by*

$$x \longmapsto [F_0(x), \cdots, F_n(x)] \tag{4}$$

Definition 2.8. *Let X, Y be varieties. We say a regular map $f : X \to Y$ is dominant if image of f is dense in Y.*

We say a rational map $f : X \dashrightarrow Y$ is dominant if f can be represented by some dominant regular map $f\big|_U : U \to Y$.

Remark 2.8. *In general, we cannot directly define composition of two rational maps. But for a dominant rational map $f : X \dashrightarrow Y$ and any rational map $g : Y \dashrightarrow Z$, $g \circ f$ is a well defined rational map.*

In addition, for a dominant rational map $f : X \dashrightarrow Y$, it induces a canonical map $f^* : K(Y) \to K(X)$ mapping g to $g \circ f$, which is a homomorphism of k-algebras.

Proposition 2.3. *Assume that X, Y are irreducible varieties. Then*
(1)There is a one-to-one correspondence

$$f : X \dashrightarrow Y \text{ dominant rational map} \leftrightsquigarrow K(Y) \hookrightarrow K(X) \text{ extension of fields}$$

(2)If there exists an isomorphism $\iota : K(Y) \xrightarrow{\sim} K(X)$, then there exists open dense subset $U \subseteq X$ and $V \subseteq Y$ such that $U \xrightarrow{\sim} V$ under the corresponding dominant rational map.

2.5 Birational equivalence

Definition 2.9. *Let X, Y be varieties. We say that X, Y are birational if there exists open dense susbets $U \subseteq X$ and $V \subseteq Y$ such that $U \xrightarrow{\sim} V$.*

Definition 2.10. *Let $f : X \dashrightarrow Y$ be a dominant rational map. We say that f is birational if there exists a dominant rational map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.*

Corollary 2.2. *Let X, Y be varieties. Then X and Y are birational if and only if there exists a birational map $f : X \dashrightarrow Y$.*

Corollary 2.3. *Let X, Y be irreducible varieties. Then there exists one-to-one correspondence*

$$f : X \dashrightarrow Y \text{ birational map} \leftrightsquigarrow \text{isomorphism from } K(Y) \text{ to } K(X) \text{ over } k$$

Proposition 2.4. *Let X be an irreducible variety. Then there exists a hypersurface in $\mathbb{A}_k^{n+1}$ birational to X, where $n = \operatorname{trdeg}_k K(X)$.*

Proof. Let X be an irreducible variety. Since the problem is local, we can assume that X is affine. Then $A(X)$ is finitely generated k-algebra. By Noetherian Normalization Theorem, there exists transcendental basis $x_1, \cdots, x_n$ of $K(X)$ over k such that $K(X)/k(x_1, \cdots, x_n)$ is a finite extension.

Since every finite extension of fields is a simple extension, there exists some $\alpha \in K(X)$ such that

$$\begin{aligned} K(X) &= k(x_1, \cdots, x_n)[\alpha] \\ &\cong k(x_1, \cdots, x_n)[x_{n+1}]/(f_\alpha) \end{aligned} \tag{5}$$

where f_α is the minimal polynomial of α in $k(x_1, \cdots, x_n)$. Assume that $f_\alpha = x^d + P_1 x^{d-1} + \cdots + P_d$, where $P_i = \frac{A_i}{B_i}$ for some polynomial A_i, B_i.

Take $B = \prod_i B_i$, then f_α can be viewed as an element in $k[x_1, \cdots, x_n]_B[x_{n+1}] = k[x_1, \cdots, x_{n+1}]_B$. Now $K(X) \cong \operatorname{Frac}(k[x_1, \cdots, x_{n+1}]_B/(f_\alpha))$. Hence by Proposition 2.3, X is birational to the hypersurface $V(f_\alpha) \subset \mathbb{A}_k^{n+1}$. $\square$

3 Blow Up

Blow up is a tool to resolute the singular point i.e. for variety X with singularities, by blowing up, we can find variety Y with possibly less singularities and a regular map $Y \to X$.

3.1 Segre embedding

Definition 3.1. *For projective space $\mathbb{P}_k^m$ with coordinate $x_0, \cdots, x_m$ and $\mathbb{P}_k^n$ with coordinate $y_0, \cdots, y_n$, there is a map $\mathbb{P}_k^m \times \mathbb{P}_k^n \to \mathbb{P}_k^{mn+m+n}$ defined by*

$$([x_0, \cdots, x_m], [y_0, \cdots, y_n]) \longmapsto [x_i y_j]_{i,j} \tag{6}$$

called the Segre embedding.

Set coordinate z_{ij} for $\mathbb{P}_k^{mn+m+n}$. Then $[z_{ij}]$ is in the image of Segre embedding if and only if the following matrix is of rank 1

$$\begin{pmatrix} z_{00} & \cdots & z_{0n} \\ z_{10} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m0} & \cdots & z_{mn} \end{pmatrix} \tag{7}$$

which is equivalent to determinants of submatrix of the form $\begin{pmatrix} z_{ij} & z_{ij'} \\ z_{i'j} & z_{i'j'} \end{pmatrix}$ all vanish. Thus the image of Segre embedding is the subvariety cut out by $\{z_{ij}z_{i'j'} - z_{i'j}z_{ij'} \big| \forall i, j, i', j'\}$.

Definition 3.2. *Let $f \in k[x_1, \cdots, x_m, y_1, \cdots, y_n]$ be a polynomial. We say that f is bihomogenous of degree (d, e) if $\forall u, v \in k$, we have that*

$$f(ux_1, \cdots, ux_m, vy_1, \cdots, vy_n) = u^d v^e f(x_1, \cdots, x_m, y_1, \cdots, y_n) \tag{8}$$

In fact, f should be of the form $\sum_{i,j} a_{ij} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_n^{j_n}$, where $i_1 + \cdots + i_m = d$ and $j_1 + \cdots + j_n = e$ for all i, j.

Proposition 3.1. *Zariski closed subset of $\mathbb{P}_k^m \times \mathbb{P}_k^n$ is of the form $V(f_1, \cdots, f_r)$ with f_i bihomogenous. In particular, when $m = n$, the diagonal set $\Delta_{\mathbb{P}_k^n}$ defined by $\{x_i y_j - x_j y_i\}_{0 \leq i,j \leq n}$ is a Zariski closed subset.*

Remark 3.1. *In fact, since each closed subset of $\mathbb{P}_k^m \times \mathbb{P}_k^n$ under product topology can be written as $\cap_i (V_i \times \mathbb{P}_k^n \cup \mathbb{P}_k^m \times V_i')$, where $V_i \subseteq \mathbb{P}_k^m$ and $V_i' \subseteq \mathbb{P}_k^n$ are closed subsets, it is clear that Zariski topology is finer than product topology. In particular, $(\mathbb{P}_k^n \times \mathbb{P}_k^m$, product topology$) \to (\mathbb{P}_k^{mn+m+n}$, Zariski topology$)$ is a continuous map. Thus the image is also irreducible.*

Now embed general varieties into projective spaces, we can define product of varieties.

Definition 3.3. *Let X, Y be varieties. Assume $X \subseteq \mathbb{P}_k^n$ and $Y \subseteq \mathbb{P}_k^m$. Define the product $X \times Y$ with subspace topology of $\mathbb{P}_k^n \times \mathbb{P}_k^m$.*

Remark 3.2. *In particular, when $X = \mathbb{A}_k^n$ and $Y = \mathbb{A}_k^m$, we have that $\mathbb{A}_k^n \times \mathbb{A}_k^m \cong \mathbb{A}_k^{m+n}$.*

Proposition 3.2. *Let X, Y be varieties. Then*
(1)Projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are regular maps.
(2)Given a map $f : Z \to X \times Y$ of varieties, f is a regular map if and only if $p_1 \circ f$ and $p_2 \circ f$ are both regular maps.

Definition 3.4. *Let $f : X \to \mathbb{P}_k^n$ be a regular map. Define the graph $\Gamma_f \subseteq X \times \mathbb{P}_k^n$ of f to be $\{(x, f(x)) \big| x \in X\}$.*

Lemma 3.1. *Let $f : X \to \mathbb{P}_k^n$ be a regular map. Then Γ_f is closed.*

Reason 3.1. *In fact, if consider regular map $X \times \mathbb{P}_k^n \to \mathbb{P}_k^n \times \mathbb{P}_k^n$ mapping (x, y) to $(f(x), y)$. then Γ_f is preimage of $\Delta_{\mathbb{P}_k^n}$ and hence closed.*

Remark 3.3. *As any variety can be embedded into some projective space as a locally closed subset, we can also define graph for general regular map $X \to Y$.*

3.2 Blow up of a point

Given a point in $\mathbb{P}_k^n$. By coordinate transformation, without loss of generality, can assume the point is just origin $0 = [1, 0, \cdots, 0]$. Consider the projection $\pi_0 : \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ sending $[x_0, \cdots, x_n]$ to $[x_1, \cdots, x_n]$, which is a regular map outside 0.

Set $U = \mathbb{P}_k^n \setminus \{0\}$. Consider the closure of Γ_{π_0} in $\mathbb{P}_k^n \times \mathbb{P}_k^{n-1}$, called the blow up of $\mathbb{P}_k^n$ at 0 and denoted by $Bl_0\mathbb{P}_k^n$.

Lemma 3.2. *Denote coordinate of $\mathbb{P}_k^n$ by $x_0, \cdots, x_n$ and coordinate of $\mathbb{P}_k^{n-1}$ by $y_1, \cdots, y_n$. Then $Bl_0\mathbb{P}_k^n \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^{n-1}$ is defined by the bihomogenous polynomials $\{x_iy_j - x_jy_i \big| 0 \leq i, j \leq n\}$.*

Proof. Obviously, $Bl_0\mathbb{P}_k^n \subseteq V(\{x_iy_j - x_jy_i\})$. Want to show that there is a covering consisting of irreducible subsets $V_i \subseteq V(\{x_iy_j - x_jy_i\})$ such that $\Gamma_{\pi_0} \cap V_i$ contains some nonempty open subset of V_i.

Consider projection maps

$$\begin{tikzcd} & V(\{x_iy_j - x_jy_i\}) \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^{n-1} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}_k^n & & \mathbb{P}_k^{n-1} \end{tikzcd} \tag{9}$$

For every $\beta = [b_1, \cdots, b_n] \in \mathbb{P}_k^{n-1}$, its preimage $p_2^{-1}(\beta)$ is the closed subset cut out by $x_ib_j - x_jb_i$. It is clear that $p_1 \circ p_2^{-1}(\beta)$ is the line connecting 0 and $[0, b_1, \cdots, b_n]$.

For each $U_i = D_+(y_i) \subseteq \mathbb{P}_k^{n-1}$, since $x_iy_j - x_jy_i = y_i(x_i\frac{y_j}{y_i} - x_j)$ for all j, we have that

$$\begin{aligned} p_2^{-1}(U_i) &\cong p_1 \circ p_2^{-1}(U_i) \\ &= \{[x_0, x_i\frac{y_1}{y_i}, \cdots, x_i\frac{y_n}{y_i}]\} \\ &\cong \{[x_0, x_i] \times (\frac{y_1}{y_i}, \cdots, \frac{y_n}{y_i})\} \\ &= \mathbb{P}_k^1 \times U_i \end{aligned} \tag{10}$$

which is an open subset of $\mathbb{P}_k^1 \times \mathbb{P}_k^{n-1}$ hence irreducible. Thus $p_2^{-1}(U_i)$ is irreducible for all i.

For each $\alpha = [a_0, \cdots, a_n] \neq 0$, we have that $p_1^{-1}(\alpha) = \alpha \times [a_1, \cdots, a_n]$. Hence $p_1^{-1}(W_i) \xrightarrow{\sim} W_i$, where $W_i = D_+(x_i) \subseteq \mathbb{P}_k^n$. Note that $p_1^{-1}(W_i) \subseteq \Gamma_{\pi_0} \cap p_2^{-1}(U_i)$ is open in $\mathbb{P}_k^n \times \mathbb{P}_k^{n-1}$, done! $\square$

Remark 3.4. *Above argument also shows that p_1 is an isomorphism over $\mathbb{P}_k^n \setminus \{0\}$.*

Definition 3.5. *Let X be a subvariety of $\mathbb{P}_k^n$, $x \in X$. Define blow up of X at x to be $Bl_xX := \overline{p_1^{-1}(X \setminus \{x\})}$ with a commutative diagram*

$$\begin{tikzcd} Bl_xX \arrow[r, hook] \arrow[d, "p_X"] & Bl_x\mathbb{P}_k^n \arrow[d, "p_1"] \\ X \arrow[r, hook] & \mathbb{P}_k^n \end{tikzcd} \tag{11}$$

Similarly, p_X is an isomorphism over $X \setminus \{x\}$ and $p_X^{-1}(x) \subseteq x \times \mathbb{P}_k^{n-1}$.

3.3 Resoluting singularities

Example 3.1. *(1)Take curve $C = V(x_0x_1^2 - x_2^2(x_0 - x_2)) \subseteq \mathbb{P}_k^2$ with $0 \in C$. Consider $U_0 = D_+(x_0)$. Then $C \cap U_0 = V(x_{1/0}^2 - x_{2/0}^2 + x_{2/0}^3)$. It is clear that 0 is a singular point. By blow up at 0, we would get $p_1^{-1}(0)$ is nonsingular.*
(2)For rational map $u : \mathbb{P}_k^2 \to \mathbb{P}_k^2$ mapping $[x, y, z]$ to $[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}] = [yz, zx, xy]$. Then u is not

regular at $p_1 = [1,0,0]$, $p_2 = [0,1,0]$ *and* $p_3 = [0,0,1]$. *By blowing up one by one, we would get* $Bl_{p_1,p_2,p_3}\mathbb{P}^2_k$ *such that there exists regular birational map* ϕ *making the following diagram commutative*

$$\begin{tikzcd} Bl_{p_1,p_2,p_3}\mathbb{P}^2_k \arrow[d] \arrow[dr, "\phi"] & \\ \mathbb{P}^2_k \arrow[r, dashed, "u"] & \mathbb{P}^2_k \end{tikzcd} \tag{12}$$

Theorem 3.1 (***Hironaka***)**.** *Let* k *be an algebraically closed field of characteristic* 0. *Then (1)For all variety* X, *there exists a series of blow-ups*

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X \tag{13}$$

such that X_n *is smooth.*
(2)For all rational map $f : X \dashrightarrow Y$, *there exists a series of blow-ups*

$$\pi : X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X \tag{14}$$

such that $f \circ \pi : X_n \to Y$ *is regular.*

For the second example above, we can say more things about birational maps of $\mathbb{P}^2_k$. Set $Cr(\mathbb{P}^2_k) := \{f : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k \text{ birational}\}$, called Cremona group. By Corollary 2.3, there is a one-to-one correspondence between $Cr(\mathbb{P}^2_k)$ and $\mathrm{Aut}(K(\mathbb{P}^2_k)/k)$, where $K(\mathbb{P}^2_k) = K(\mathbb{A}^2_k) = k(x,y)$.

Then it is clear that $u \in Cr(\mathbb{P}^2_k)$ is an involution and $SL_k(3) \subseteq Cr(\mathbb{P}^2_k)$. In fact, by the following Theorem, $Cr(\mathbb{P}^2_k)$ is just generated by these elements.

Theorem 3.2 (***Noether-Castelnuovo***)**.** *Any birational map* $\varphi \in Cr(\mathbb{P}^2_k)$ *is of the form* $u^s \circ M_1 \circ u \circ M_2 \circ \cdots M_n \circ u^t$, *where* $M_i \in SL_k(3)$ *and* $s,t \in \{0,1\}$.

3.4 Images of regular maps

Theorem 3.3. *Assume that* X *is a variety and* Y *is a projective variety. Then the projection* $p : X \times Y \to X$ *is a closed map.*

Proof. Recall that resultant of polynomials $F = a_0 + a_1x + \cdots + a_sx^s$ and $G = b_0 + b_1x + \cdots + b_tx^t$ is a matrix of degree $s+t$

$$Res(F,G) = \begin{pmatrix} a_0 & a_1 & \cdots & a_s & & & \\ & a_0 & a_1 & \cdots & a_s & & \\ & & & & \ddots & & \\ & & & a_0 & a_1 & \cdots & a_s \\ b_0 & b_1 & \cdots & b_t & & & \\ & b_0 & b_1 & \cdots & b_t & & \\ & & & & \ddots & & \\ & & & b_0 & b_1 & \cdots & b_t \end{pmatrix} \tag{15}$$

Step 1: Take $Y = \mathbb{P}^1_k$ and $X = \mathbb{P}^m_k$, want to show that $\pi : \mathbb{P}^m_k \times \mathbb{P}^1_k \to \mathbb{P}^m_k$ is closed. Let $Z \subseteq \mathbb{P}^m_k \times \mathbb{P}^1_k$ be a closed subset, defined by bihomogenous polynomials $F_i(x_0, \cdots, x_m, y_0, y_1)$

for $1 \leq i \leq h$. Define $\overline{F_i} = F_i(x_0, \cdots, x_m, 1, Y) \in k[x_0, \cdots, x_m][Y]$. Claim that $\pi(Z) \subseteq \mathbb{P}^m_k$ is the closed subset defined by $Res(\overline{F_i}, \overline{F_j}) \in k[x_0, \cdots, x_m]$ for $1 \leq i, j \leq h$.

For $\alpha \in \pi(Z)$, there exists $[\beta_0, \beta_1] \in \mathbb{P}^1_k$ such that $\alpha \times [\beta_0, \beta_1] \in Z$ so that $F_i(\alpha, \beta_0, \beta_1) = 0$ for all i. If $\beta_0 \neq 0$, then replace $[\beta_0, \beta_1]$ by $[1, \frac{\beta_1}{\beta_0}]$ and hence $\frac{\beta_1}{\beta_0}$ is common root of all $\overline{F_i}(\alpha, Y)$. Thus $Res(\overline{F_i}, \overline{F_j}) = 0$ for all i, j.

If $\beta_0 = 0$, then leading coefficients of $\overline{F_i}(\alpha, Y)$ are all 0 and hence $Res(\overline{F_i}, \overline{F_j})(\alpha) = 0$ for all i, j. Thus we get $\pi(Z) \subseteq V(Res(\overline{F_i}, \overline{F_j}))$.

Conversely, assume that all resultants vanish at $\alpha \in \mathbb{P}^m_k$. We need to show that $\alpha \in \pi(Z)$. If α is common root of all the leading coefficients of $\overline{F_i}$, then obviously $\alpha \times [0, 1] \in Z$ and hence $\alpha \ni \pi(Z)$.

Suppose that there exists some $F \in I(Z)$ whose leading coefficient does not vanish at α. Note that $\overline{F}(\alpha, 1, Y)$ has only finitely many roots, denoted by $\gamma_1, \cdots, \gamma_N$. Since $Res(\overline{F}, \overline{G})(\alpha) = 0$ and leading coefficient of $\overline{F}(\alpha, Y)$ is nonzero, we have that $G(\alpha, 1, Y)$ vanish at least one point in $\gamma_1, \cdots, \gamma_N$.

Suppose that all $\alpha \times [1, \gamma_i]$ are not in Z. Then there exist G_i such that $G(\alpha, 1, \gamma_i) \neq 0$. Multiplied by some power of y and x_j, where $\alpha \in D_+(x_j)$, we may assume that all G_i are of same bidegree.

Then for all a_i, bihomogenous polynomial $\sum_i a_i G_i \in I(Z)$ has some root in $\{\gamma_1, \cdots, \gamma_N\}$. Consider linear map $\phi : k^N \to k^N$ defined by

$$(a_1, \cdots, a_n) \longmapsto (\sum_i a_i G_i(\alpha, 1, \gamma_1), \cdots, \sum_i a_i G_i(\alpha, 1, \gamma_N)) \tag{16}$$

Note that $\operatorname{im} \phi \subseteq \cup_i H_i$ where $H_i = \{a_i = 0\}$. While $\operatorname{im} \phi$ is vector space over infinite field k, it cannot be covered by finitely many proper subspaces so that $\operatorname{im} \phi \subseteq H_i$ for some i, while implies that all $G_j(\alpha, 1, \gamma_i) = 0$. In particular, we get that $G_i(\alpha, 1, \gamma_i) = 0$, contradiciton!

Step 2: For general variety X, want to show that $X \times \mathbb{P}^1_k \to X$ is closed. In this case, embed X into some $\mathbb{P}^m_k$ and consider a commutative diagram

$$\begin{tikzcd} X \times \mathbb{P}^1_k \arrow[r, "\pi_X"] \arrow[d, hook] & X \arrow[d, hook] \\ \mathbb{P}^m_k \times \mathbb{P}^1_k \arrow[r, "\pi"] & \mathbb{P}^m_k \end{tikzcd} \tag{17}$$

Let Z be a closed subset. Then there exists closed subset $Z' \subseteq \mathbb{P}^m_k$ such that $z' \cap X = Z$. Clearly, $\pi_X(Z) = \pi(Z') \cap X$ is still closed in X.

Step 3: For general variety X, want to show that $X \times \mathbb{P}^n_k \to X$ is closed. Recall that for all $x \in \mathbb{P}^n_k$, we can blow up at x. For all standard affine open subset $U_i \subseteq \mathbb{P}^{n-1}_k$, we have a commutative diagram

$$\begin{tikzcd} U_i \times \mathbb{P}^1_k \cong q^{-1}(U_i) \arrow[r, hook] \arrow[d] & Bl_x\mathbb{P}^n_k \arrow[r, "p"] \arrow[d, "q"] & \mathbb{P}^n_k \\ U_i \arrow[r, hook] & \mathbb{P}^{n-1}_k & \end{tikzcd} \tag{18}$$

Taking product by X over k, we get a commutative diagram

$$\begin{tikzcd} X \times U_i \times \mathbb{P}^1_k \arrow[r, hook] \arrow[d] & X \times Bl_x\mathbb{P}^n_k \arrow[r, "\mathrm{id} \times p"] \arrow[d, "\mathrm{id} \times q"] & X \times \mathbb{P}^n_k \arrow[d, "\pi_X"] \\ X \times U_i \arrow[r, hook] & X \times \mathbb{P}^{n-1}_k \arrow[r] & X \end{tikzcd} \tag{19}$$

For all closed subset $Z \subseteq \mathbb{P}^n_k$, set $\widetilde{Z} = (\mathrm{id} \times p)^{-1}(Z)$ which is a closed subset of $X \times Bl_x\mathbb{P}^n_k$. Then $Z_i := \widetilde{Z} \cap (\mathrm{id} \times q)^{-1}(X \times U_i)$ is closed in $(\mathrm{id} \times q)^{-1}(X \times U_i)$.

By step 1, $X \times U_i \times \mathbb{P}^1_k \to X \times U_i$ is closed so that $(\mathrm{id} \times q)(Z_i)$ is closed in $X \times U_i$ for all i. Thus $(\mathrm{id} \times q)(\widetilde{Z})$ is closed in $X \times \mathbb{P}^{n-1}_k$. By induction on n, $X \times \mathbb{P}^{n-1}_k$ is closed so that $\pi_X(Z)$ is closed.

Step 4: For general variety X and projective Y, want to show that $X \times Y \to X$ is closed. Since Y is projective, we can embed it into $\mathbb{P}^n_k$ as a closed subset for some n. Consider the following commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[d, hook] & X \arrow[d, equal] \\ X \times \mathbb{P}^n_k \arrow[r, "\pi"] & X \end{tikzcd} \tag{20}$$

As $X \times Y$ is a closed subset of $X \times \mathbb{P}^n_k$ and π is closed, it is clear that π_X is also closed. $\square$

Corollary 3.1. *Let Y be a projective variety. Then any regular map $f : Y \to Z$ is closed.*

Reason 3.2. *For all closed subset $W \subseteq Y$, $\Gamma_f \cap (W \times Z) \hookrightarrow Y \times Z \to Z$ has closed image.*

Corollary 3.2. *Any regular function on a connected projective variety X is constant.*

Proof. Let $f : X \to \mathbb{A}^1_k$ be a regular function. Then $\widetilde{f} : X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is a regular map. Hence $\widetilde{f}$ is closed and $\widetilde{f}(X)$ is closed in $\mathbb{P}^1_k$. Note that $\widetilde{f}(X) \neq \mathbb{P}^1_k$ since $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is not surjective. Thus $\widetilde{f}(X)$ is a finite set and by connectivity, f is constant. $\square$

Corollary 3.3. *Let $X \subseteq \mathbb{P}^n_k$ be a closed subvariety. Assume that X is connected and X is not a one-point set. Then for all hypersurface $H \subseteq \mathbb{P}^n_k$, X meets H.*

Proof. Say $H = V(F)$ where F is a homogeneous polynomial of degree d. Consider dth Veronese map $V_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1}_k$. Denote $N = \binom{n+d}{d} - 1$. Suppose that $F = \sum_I a_I x^I$, where I varies over $(a_0, a_1, \cdots, a_n)$ satisfying that $\sum a_i = d$ and $x_I = \prod_i x_i^{a_i}$. Take $\widetilde{H}$ to be the hypersurface in $\mathbb{P}^N_k$ defined by $\sum_I a_I x_I$. Then $v_d^{-1}(\mathbb{P}^N_k \setminus \widetilde{H}) = \mathbb{P}^n_k \setminus H$.

Suppose that $X \cap H = \varnothing$, then $X \subseteq \mathbb{P}^n_k \setminus H$. Now for all homogeneous polynomial G of degree d, $\frac{G}{F}\big|_X$ defines a regular function on X. As X is connected projective variety, by Corollary 3.2, $\frac{G}{F}\big|_X$ is constant. Note that $\frac{G}{F}\big|_X$ can also be viewed as a regular function on $v_d(X)$. Taking G varies over x^I for all I, it is clear that $v_d(X)$ is a point. While v_d is injective, contradiction! $\square$

4 Dimension Theory

4.1 Dimension

Definition 4.1. *Let X be a topological space. The dimension of X denoted by* $\dim X$ *is the supremum of n such that there exists a chain of irreducible closed sbusets of length n.*

Proposition 4.1. *Let X be a topological space, $Y \subseteq X$ subspace with induced topology. Then we have that*
(1)$\dim Y \leq \dim X$
(2)If X is irreducible of finite dimension and Y is proper subset, then $\dim Y < \dim X$.
(3)If $X = \cup_i X_i$ is union of finitely many closed subsets, where X_i are closed, then $\dim X = \max_i\{\dim X_i\}$.

Corollary 4.1. *Let X be a variety. Assume that* $\dim X = 0$*, then X is union of finitely many points.*

Reason 4.1. *Since X can be written as union of finitely many irreducible components, by previous proposition, we would get each component of X is of dimension* 0 *and hence is one-point set.*

Remark 4.1. *If each irreducible component of X is of dimension n, then we say that X is pure of dimension n (or say that X is equidimensional of dimension n).*

Given $x \in X$, define the dimension of X at x to be $\dim_X x := \sup\{n \big| \exists \{x\} \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$ *chain of irreducible closed subsets*$\}$. *Clearly,* $\dim X \geq \dim_X x$.

Proposition 4.2. *Let X be an irreducible affine variety, $A(X)$ coordinate ring. Then* $\dim X = \dim A(X)$.

Reason 4.2. *Immediately comes from the one-to-one correspondence between irreducible subvarieties and prime ideals.*

Theorem 4.1. *Let A be an integral domain over field k (not necessarily algebraically closed). Then* $\dim A = \mathrm{trdeg}(\mathrm{Frac}(A)/k)$ *the Krull dimension of $A(X)$.*

Corollary 4.2. $\dim \mathbb{A}_k^n = n$.

Theorem 4.2. *Let A be an integral domain of finite type over field k (not necessarily algebraically closed). Then for all prime ideal $\mathfrak{p}$ of A, we have that* $\mathrm{height}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A$.

Corollary 4.3. *Let X be an irreducible variety. Then for all irreducible subvariety $Y \subseteq X$,* $\dim Y + \mathrm{codim}\, Y = \dim X$. *In particular,* $\dim_X x = \dim X$ *for all $x \in X$.*

Reason 4.3. *By taking affine open subset U meeting Y, we can reduce to the case that X is affine. Then apply preceding theorem to $A(X)$, we are done.*

Proposition 4.3. *Let X be a variety. Then* $\dim X$ *is finite and for all open dense subset $U \subseteq X$, we have that* $\dim U = \dim X$.

Example 4.1. *(1)*$\dim \mathbb{P}_k^n = n$ *since* $D_+(x_i) \cong \mathbb{A}_k^n$ *is an open dense subset.*
(2)$\dim(\mathbb{P}_k^n \times \mathbb{P}_k^m) = n + m$*.*
(3)Assume that X *is irreducible, then* $\dim X = \operatorname{trdeg}(K(X)/k)$ *since we can take affine open dense subset* $U \subseteq X$ *and hence*

$$\dim X = \dim U = \operatorname{trdeg}(K(U)/k) = \operatorname{trdeg}(K(X)/k) \tag{21}$$

(4)Let $f : X \dashrightarrow Y$ *be a dominant rational map between irreducible varieties. Then* f *would correspond to a field extension* $K(Y) \hookrightarrow K(X)$*. Hence* $\dim Y = \operatorname{trdeg}(K(Y)/k) \leq \operatorname{trdeg}(K(X)/k) = \dim X$*.*

Though $K(X)$ may be not a field when X is not irreducible, we can generalize the last entry in Example 4.1.

Proposition 4.4. *Let* $f : X \dashrightarrow Y$ *be a dominant rational map. Then for all irreducible component* W *of* Y*, there is a dominant rational map* $\widetilde{f} : \cup_i V_i \dashrightarrow W$*, where* V_i *are all irreducible components of* X *meeting* $f^{-1}(W)$*. Moreover, we have that* $\dim W \leq \max_i\{\dim V_i\}$ *and in particular,* $\dim Y \leq \dim X$*.*

Proof. Assume U is the domain of definition of f. Then $f^{-1}(W)$ would be union of some irreducible components of U, say $f^{-1}(W) = \cup_i (V_i \cap U)$. And we get a dominant rational map $\widetilde{f} : \cup_i V_i \dashrightarrow W$.

Note that $W = \overline{\cup_i f(V_i \cap U)} = \cup_i \overline{f(V_i \cap U)}$. As W is irreducible, $W = \overline{f(V_i \cap U)}$ for some i. Then by same argument as (4) in Example 4.1, we get that $\dim W \leq \dim V_i \leq \max_i\{\dim V_i\}$. □

4.2 Dimension and defining equations

Theorem 4.3 (***Krull's Principal Ideal Theorem***)**.** *Let* A *be a noetherian domain,* $f \in A$ *nonzero. Then all minimal prime ideals in* $V(f)$ *are of height* 1*.*

Remark 4.2. *For noetherian ring* A *and* $f \in A$*, we also has Krull's Principal Ideal Theorem that all minimal prime ideals in* $V(f)$ *are of height at most* 1 *and take equality if* f *is not zero-divisor.*

Theorem 4.4. *Let* $X \subseteq \mathbb{P}_k^n$ *be a quasi-projective subvariety,* $F_1, \cdots, F_r$ *homogeneous polynomials. Then we have that*
(1)If X *is of pure dimension* n *and* $X \cap V(F_1, \cdots, F_r)$ *is nonempty, then each irreducible component of* $X \cap V(F_1, \cdots, F_r)$ *is of dimension* $\geq n - r$*.*
(2)If X *is closed of dimension* n *and* $r \leq n$*, then* $X \cap V(F_1, \cdots, F_r) \neq \varnothing$*.*

Remark 4.3. *For (2), recall that by Corollary 3.3, for projective* X *which is not one-point set,* $X \cap V(F) \neq \varnothing$*. This is just the* $r = 1$ *case.*

Corollary 4.4. *Hypersurfaces* $V(F)$ *of* $\mathbb{A}_k^n$ *(resp.* $\mathbb{P}_k^n$*) are exactly the closed subvarieties of pure dimension* $n - 1$*.*

Reason 4.4. *Assume that $F = \prod_i F_i$ is the irreducible factorization. Then $V(F) = \cup_i V(F_i)$ and by Krull's Principal Theorem and Proposition 4.1,* $\dim V(F) = \max_i\{\dim V(F_i)\} = \max_i\{n-1\} = n-1$.

Conversely, assume that $Y \subseteq \mathbb{A}^n_k$ is an irreducible closed subvariety of dimension $n-1$. Take some $F = \prod_i F_i \neq 0 \in I(Y)$. Then $Y \subseteq V(F) = \cup_i V(F_i)$. As Y is irreducible, we may assume that $Y \subseteq V(F_1)$ which is also of dimension $n-1$ so that $Y = V(F_1)$.

Example 4.2. *Let $\mathbb{P}^3_k$ be projective space with coordinate x, y, z, w, $F_1 = xw - yz$, $F_2 = xz - y^2$ and $F_3 = yw - z^2$. Then* $\dim V(F_1, F_2, F_3) = 1 > 3 - 3 = 0$. *This is because F_3 is zero-divisor in $k[x, y, z, w]/(F_1, F_2)$.*

4.3 Generically finite morphism

From now on, by a morphism of varieties, we mean a regular map.

Definition 4.2. *Let $f : X \to Y$ be a morphism of varieties. Assume $\mathcal{P}$ is a property of fibers. We say that a general fiber of f satisfies property $\mathcal{P}$ if there exists open dense subset $U \subseteq Y$ such that for all $y \in U$, fiber $f^{-1}(y)$ satisfies property $\mathcal{P}$.*

Definition 4.3. *Let $f : X \to Y$ be a morphism of varieties, $\sharp k$ uncountable. Assume $\mathcal{P}$ is a property of fibers. We say that a very general fiber of f satisfies property $\mathcal{P}$ if there exist countably many proper closed subset $V_i \subseteq Y$ such that for all $y \in \cup_i V_i$, fiber $f^{-1}(y)$ satisfies property $\mathcal{P}$.*

Theorem 4.5. *Let X, Y be irreducible affine varieties, $f : X \to Y$ a dominant morphism. Then a general fiber of f is finite if and only if the corresponding field extension $f^* : K(Y) \hookrightarrow K(X)$ is a finite extension.*

Proof. Assume $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$. Consider graph Γ_f of f, there is a commutative diagram

$$\begin{tikzcd} \Gamma_f \arrow[r, hook] \arrow[dr, "\sim"] & X \times Y \arrow[r, hook] \arrow[d] \arrow[dr] & \mathbb{A}^{n+m}_k \arrow[dr] & \\ & X \arrow[r, "f"] & Y \arrow[r, hook] & \mathbb{A}^m_k \end{tikzcd} \tag{22}$$

Replacing X and f by Γ_f and $\Gamma_f \xrightarrow{\sim} X \xrightarrow{f} Y$ respectively, then the new f is still a morphism induced by the canonical projection $\mathbb{A}^{n+m}_k \to \mathbb{A}^m_k$.

Induct on n. For $n = 1$, denote coordinate of $\mathbb{A}^{1+m}_k$ by $x, y_1, \cdots, y_m$. Then $A(X)$ is generated by $\overline{x}$ over $A(Y)$. If x is algebraic over $K(Y)$, assume $G(T) \in K(Y)[T]$ is the minimal polynomial of x. Write $G(T) = T^d + a_1T^{d-1} + \cdots + a_d$, where $a_i \in K(Y)$. Multiplied by product of dominators, we get $\widetilde{G}(T) = b_0T^d + b_1T^{d-1} + b_d$, where $b_i \in A(Y)$ and $b_0 \neq 0$.

For all $y \in Y$ and $(\alpha, y) \in f^{-1}(y)$, as soon as $b_0(y) \neq 0$, α would be zero of the equation $\widetilde{G}(T, y)$ hence $f^{-1}(y)$ is a finite set with $\sharp f^{-1}(y) \leq d = [K(X) : K(Y)]$. Moreover, if $K(X)$ is separable over $K(Y)$, then the equality holds.

If x is transcendental over $K(Y)$, then for all $F \in I(X)$, written as $b_0(y_1, \cdots, y_m)x^d + b_1(y_1, \cdots, y_m)x^{d-1} + \cdots + b_d(y_1, \cdots, y_m)$, we have that $b_j(y) = 0$ for all $y \in Y$ so that

$b_j(y_1, \cdots, y_m) \in I(Y)$ for all j. Thus for all $y \in Y$ and all $a \in k$, $F(a, y) = 0$. Conclude that $\mathbb{A}^1_k \times \{y\} \subseteq X$ and hence $f^{-1}(y) = \mathbb{A}^1_k \times \{y\}$.

For $n > 1$, consider following commutative diagram

$$\begin{tikzcd} \mathbb{A}^{n+m}_k \arrow[r] & \mathbb{A}^{n-1+m}_k \arrow[r] & \mathbb{A}^{n-2+m}_k \arrow[r] & \cdots \arrow[r] & \mathbb{A}^m \\ X = Y_0 \arrow[r] \arrow[u, hook] & Y_1 \arrow[r] \arrow[u, hook] & Y_2 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & Y_n = Y \arrow[u, hook] \end{tikzcd} \tag{23}$$

By induction, we complete the proof. □

Remark 4.4. *When we replace X by Γ_f, we in fact transfer information about f into $I(X)$ while the new f becomes trivial. In addition,* $\dim X = \dim Y$ *and for general y, $\sharp f^{-1}(y) \leq [K(X) : K(Y)]$ taking equality when* $\operatorname{Char} k = 0$.

Example 4.3. *For the case that* $\operatorname{Char} k = p$, *take morphism $f : X = \mathbb{A}^1_k \to Y = \mathbb{A}^1_k$ sending x to x^p. Then for all $\alpha \in \mathbb{A}^1_k$, $f^{-1}(\alpha) = \alpha^{\frac{1}{p}}$ is a unique point while $[K(X) : K(Y)] = p$.*

Lemma 4.1. *Let X, Y be irreducible varieties, $f : X \to Y$ a dominant morphism. Assume that U is an affine open dense subset of Y and $V \subseteq f^{-1}(U)$ is an affine open dense subset of X. Then a general fiber of f is finite if and only if a general fiber of $f\big|_V$ is finite.*

Proof. The " $\Rightarrow$" arrow is trivial since for all $y \in U$, $f\big|_V^{-1}(y) \subseteq f^{-1}(y)$. For the converse, since a general fiber of $f\big|_V$ is finite, by Theorem 4.5, $\dim X = \dim V = \dim U = \dim Y$. Take $W = V^c$, then W is a proper subset of X and hence $\dim W < dim X$. As $f\big|_W : W \to \overline{f(W)}$ is dominant, $\dim \overline{f(W)} \leq \dim W < \dim X = \dim Y$ so that $\overline{f(W)}$ is a proper subset. Now take $\overline{f(W)}^c \cap U$, we would see that for all $y \in \overline{f(W)}^c \cap U$, $f^{-1}(y) \subseteq V$, done! □

Corollary 4.5. *Let X, Y be irreducible varieties, $f : X \to Y$ a dominant morphism. Then a general fiber of f is finite if and only if the corresponding field extension $f^* : K(Y) \hookrightarrow K(X)$ is a finite extension.*

Proof. Take affine open dense subset $U \subseteq Y$ and affine open dense subset $V \subseteq f^{-1}(U)$. Then $f\big|_V : V \to U$ is a dominant morphism between affine varieties. Hence a general fiber of $f\big|_V$ is finite if and only if $K(U) = K(Y)$ is finite over $K(V) = K(X)$. Then by Lemma 4.1, done! □

Corollary 4.6. *Let X be an irreducible variety of dimension n. Then there exists a dominant morphism $f : X \to \mathbb{P}^n_k$ with all fibers finite.*

Proof. We may assume that $X \subseteq \mathbb{P}^N_k$ is a closed subset for some $N \geq n$. When $N = n$, the only closed subset of $\mathbb{P}^n_k$ of dimension n is itself hence we can take the identity map. Otherwise, when $N > n$, $X \subsetneq \mathbb{P}^N_k$. Pick $x \in \mathbb{P}^N_k \setminus X$, may assume $x = [1, 0, \cdots, 0]$. Take $\pi : \mathbb{P}^N_k \dashrightarrow \mathbb{P}^{N-1}_k$ to be the projection from x. Then $\pi\big|_X : X \to \mathbb{P}^{N-1}_k$ is a morphism.

For all $p \in \mathbb{P}^{N-1}_k$, $\pi\big|_X^{-1}(p) = X \cap L_p$, where L_p is the line connecting p and x. Since $L_p \not\subset X$, there exists $f \in I(X)$ which is not in $I(L_p)$, Hence by Krull's Principal Ideal Theorem, $V(f) \cap L_p$ is of codimension 1 in L_p so that $X \cap L_p \subseteq V(f) \cap L_p$ is a finite set.

Let $Y = \overline{\pi(X)} \subseteq \mathbb{P}_k^{N-1}$. By Theorem 4.5, $\dim Y = \dim X = n$. Hence by induction, there exists a dominant morphism $Y \to \mathbb{P}_k^n$ with all fibers finite. Now, $X \to Y \to \mathbb{P}_k^n$ is our desired morphism. $\square$

Corollary 4.7. *Let X, Y be varieties. Then $\dim X \times Y = \dim X + \dim Y$.*

Proof. May assume that X, Y irreducible. Then there exist $f : X \to \mathbb{P}_k^{\dim X}$ dominant with all fibers finite and $g : X \to \mathbb{P}_k^{\dim Y}$ dominant with all fibers finite. Consider $f \times g : X \times Y \to \mathbb{P}_k^{\dim X} \times \mathbb{P}_k \dim Y$, which is dominant with all fibers finite. Note that $\dim \mathbb{P}_k^{\dim X} \times \mathbb{P}_k \dim Y = \dim \mathbb{A}_k^{\dim X} \times \mathbb{A}_k \dim Y = \dim \mathbb{A}_k^{\dim X + \dim Y} = \dim X + \dim Y$. Thus by Theorem 4.5, $\dim X \times Y = \dim X + \dim Y$. $\square$

Corollary 4.8. *Let $X, Y \subseteq \mathbb{P}_k^n$ be varieties. Then*
(1)If X, Y are irreducible, then any nonempty irreducible component of $X \cap Y$ has dimension $\geq \dim X + \dim Y - n$.
(2)If X, Y are closed and $\dim X + \dim Y \geq n$, then $X \cap Y$ is nonempty.

Proof. (1)Consider $p : \mathbb{A}_k^{n+1} \to \mathbb{P}_k^n$ sending to $(x_0, \cdots, x_n)$ to $[x_0, \cdots, x_n]$. Then for all $x \in \mathbb{P}_k^n$, denote $C^0(X) = p^{-1}(X)$ of dimension $\dim X + 1$ and $C^0(X \cap Y) = C^0(X) \cap C^0(Y)$. Note that $C^0(X) \cap C^0(Y) = (C^0(X) \times C^0(Y)) \cap \Delta_{\mathbb{A}_k^{n+1}}$ and by Corollary 4.7, $\dim(C^0(X) \times C^0(Y)) = \dim X + \dim Y + 2$, we get that $\dim C^0(X \cap Y) \geq \dim X + \dim Y - n + 1$ so that $\dim X \cap Y \geq \dim X + \dim Y - n$.

(2)Take $\overline{C^0(X)} = C^0(X) \cup \{0\}$. Then

$$\begin{aligned} \dim(\overline{C^0(X)} \cap \overline{C^0(Y)}) &\geq \dim(C^0(X) \cap C^0(Y)) \\ &\geq n + 2 - n + 1 \\ &= 1 \end{aligned} \tag{24}$$

Thus $\overline{C^0(X)} \cap \overline{C^0(Y)}$ is not just $\{0\}$ so that $X \cap Y \neq \varnothing$. $\square$

4.4 Morphism and dimension

Example 4.4. *Recall blow up of $\mathbb{P}_k^n$ at x with projection $p : Bl_x\mathbb{P}_k^n \to \mathbb{P}_k^n$. Then $p^{-1}(\mathbb{P}_k^n \setminus \{x\}) \xrightarrow{p} \mathbb{P}_k^n \setminus \{x\}$ is an isomorphism and $p^{-1}(x) \cong \mathbb{P}^{n-1}$.*

Definition 4.4. *Let $\varphi : X \to Y$ be a morphism. Define $X_x := \varphi^{-1}(\varphi(x))$ to be the fiber of φ containing x.*

Lemma 4.2. *Let Y be an affine variety of pure dimension d. Then for all $y \in Y$, there exist $F_1, \cdots, F_d \in A(Y)$ such that $y \in V(F_1, \cdots, F_d)$ and $Y \cap V(F_1, \cdots, F_d)$ is a finite set.*

Proof. Induct on d. For $d = 0$, $Y = V(I(Y))$ satisfies condition. For $d > 0$, note that for all $y \in Y$ and $y' \neq y$, there exists hyperplane $H_{y,y'}$ through y but not y'. Hence the intersection of all hyperplanes through y is just $\{y\}$.

Similarly, since $d > 0$, for all irreducible component Y_i of Y, we can find $y_i' \in Y_i \setminus \{y\}$ and there exists hyperplane H_y such that H_y does not contain y_i' for all i. Hence $Y_i \cap H_y$ is of dimension $d-1$ for all Y_i containing y. Then take $Y' = \cup_{Y_i \ni y}(Y_i \cap H)$ which is of pure dimension $d-1$. By induction, done! $\square$

Remark 4.5. *In fact, with argument above, we get a stronger result that we can take $F_1, \cdots, F_d$ to be linear equations.*

Theorem 4.6. *Let $\varphi : X \to \mathbb{P}^n_k$ be a morphism. Consider the function $\delta : X \to \mathbb{N}$ sending x to $\dim_{X_x} x$. Then δ is upper semicontinuous i.e. for all r, $X(r) := \{x \in X \big| \delta(x) \geq r\}$ is closed. Moreover, if X is irreducible, then $\dim X = \dim \overline{\varphi(X)} + \min_{x\in X} \delta(x)$.*

Proof. Write $X = X_1 \cup \cdots \cup X_r$, where X_i are irreducible components. Denote $\varphi_i : X_i \hookrightarrow X \to \mathbb{P}^n_k$. Then $X_x = \cup_i (X_x \cap X_i)$ and $X(r) = \cup_i X_i(r)$. Thus we may assume X is irreducible.

Set $Y = \overline{\varphi(X)}$, then $\varphi : X \to Y$ is dominant between irreducible varieties. For all $x \in X$ mapping to some $y \in Y$, by Lemma 4.2, there exists affine open neighbourhood V of y and regular functions $F_1, \cdots, F_d$ on V vanishing at y and $\dim(V \cap V(F_1, \cdots, F_d)) = 0$, where $d = \dim Y$. Note that $\{y\}$ is a connected component of $V \cap V(F_1, \cdots, F_d)$, we get $X_x = \varphi^{-1}(y)$ is a union of some irreducible components of $\varphi^{-1}(V) \cap V(\varphi^* F_1, \cdots, \varphi^* F_d)$.

By Theorem 4.2 and Krull's Principal Ideal Theorem, each irreducible component has dimension $\geq \dim X - d$. Thus for all $x \in X$, $\delta(x) \geq \dim X - d = \dim X - \dim Y$. Claim that there exists open dense subset $U \subseteq X$ such that for all $x \in U$, $\delta(x) = \dim X - \dim Y$.

Firstly consider affine case. By similar argument as proof of Theorem 4.5, we replace X by graph Γ_f of f and there is a commutative diagram

$$\begin{array}{ccccccccc} \mathbb{A}^{n+m}_k & \longrightarrow & \mathbb{A}^{n-1+m}_k & \longrightarrow & \mathbb{A}^{n-2+m}_k & \longrightarrow & \cdots & \longrightarrow & \mathbb{A}^m \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\ X = Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n = Y \end{array} \tag{25}$$

For $Y_i \to Y_{i+1}$, if $K(Y_i)/K(Y_{i+1})$ is transcendental, then fiber of all $y_{i+1} \in Y_{i+1}$ is isomorphic to projective line of dimension $1 = \dim Y_i - \dim Y_{i+1}$. If $K(Y_i)/K(Y_{i+1})$ is algebraic, then a general fiber of $Y_i \to Y_{i+1}$ is finite hence is of dimension $0 = \dim Y_i - \dim Y_{i+1}$. Conclude that our claim holds for affine case.

For general case, take affine open subset $U \subseteq Y$ and $V \subseteq \varphi^{-1}(U)$ affine open subset of X. Then there exists open dense subset $V' \subseteq V$ such that for all $x \in V'$, $\delta(x) = \dim V - \dim U = \dim X - \dim Y$. Since V' is also dense in X, we complete proof of the claim.

For $r \leq \dim X - \dim Y$, we have seen that $X(r) = X$. For $r > \dim X - \dim Y$, there exists closed subvariety $F \subsetneq X$ such that $X(r) \subseteq F$. Consider $F(r)$, clearly $F(r) \subseteq X(r)$. Conversely, for all $x \in X(r)$, there exists some irreducible component X' of X_x containing x such that $\dim X' \geq r$. Hence $X' \subseteq X(r) \subseteq F$ so that $X' \subseteq F_x$ and $x \in F(r)$. Now $F(r) = X(r)$. Note that $\dim F < \dim X$, by induction, we are done! $\square$

Corollary 4.9. *Let X be an irreducible variety, $\varphi : X \to Y$ dominant morphism. Then*
(1)for all $y \in \operatorname{im}\varphi$, any irreducible component of $\varphi^{-1}(y)$ has dimension $\geq \dim X - \dim Y$.
(2)there exists nonempty open subset $U \subseteq Y$ such that for all $y \in U$, $\varphi^{-1}(y)$ is of pure dimension $\dim X - \dim Y$.

Proof. (1)For all irreducible component X' of $\varphi^{-1}(y)$, there exists $x' \in X'$ such that X' is the only irreducible component containing x'. Hence $\dim X' = \dim_{\varphi^{-1}(y)} x' = \dim_{X_{x'}} x' \geq \dim X - \dim Y$.

(2)We show that for all irreducible component Z of $X(\dim X - \dim Y + 1)$, the closure of its image in Y is a proper subset i.e. $\overline{\varphi(Z)} \subsetneq Y$. For $x \in Z$ such that x is not contained in any other irreducible component of $X(\dim X - \dim Y + 1)$, there is some irreducible component X' of X_x containing x have dimension $\geq \dim X - \dim Y + 1$.

Thus $X' \subseteq X(\dim X - \dim Y + 1)$ and $X' \subseteq Z$ as X' irreducible. If we take $U = X \backslash (\cup Z')$, where Z' varies all irreducible components of $X(\dim X - \dim Y + 1)$ except Z, then for all $x \in U$, we get $\dim_{Z_x} x = \dim_{X_x} x \geq \dim X - \dim Y + 1$.

Consider $\varphi\big|_Z : Z \to Y$. By claim in the proof of previous theorem, there exists open dense subset $U' \subseteq Z$ such that for all $x' \in U'$, $\delta(x') = \dim Z - \dim \overline{\varphi(Z)}$. As Z is irreducible, $U \cap U' \neq \varnothing$ hence $\dim \overline{\varphi(Z)} \leq \dim Z - \dim X + \dim Y - 1 < \dim Y$. Thus $\dim(\cup_Z \overline{\varphi(Z)}) = \max_Z\{\dim \overline{\varphi(Z)}\} < \dim Y$. And $W = (\cup_Z \overline{\varphi(Z)})^c$ is our desired set. ☐

Remark 4.6. *Part (2) in fact tells that for a dominant map from irreducible variety, there would be some nonempty open subset contained in the image.*

Proposition 4.5. *Let $\varphi : X \to Y$ be a closed morphism. Then $Y(r) := \{y \in Y \big| \dim \varphi^{-1}(y) \geq r\}$ is a closed subset of Y.*

Reason 4.5. *Note that $Y(r) = \varphi(X(r))$.*

Example 4.5. *Consider $\varphi : \mathbb{A}^3_k \to \mathbb{A}^3_k$ sending (x, y, z) to $(x, (xy-1)y, (xy-1)z)$. Easy to check that φ is dominant and a general fiber of φ is finite (the corresponding field extension is finite). Assume $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in Y(1)$. Then for all $(x, y, z) \in \varphi^{-1}(\alpha)$, we have that*

$$\begin{cases} x = \alpha_1 \\ (xy-1)y = \alpha_2 \\ (xy-1)z = \alpha_3 \end{cases} \tag{26}$$

Since $\varphi^{-1}(\alpha)$ has dimension ≥ 1, the above system of equations cannot only have finitely many solutions. Hence $Y(1) \subseteq \{(a, 0, 0) \big| a \neq 0\}$. In fact, the equality holds so that $Y(1)$ is not closed in $\mathbb{A}^3_k$.

Proposition 4.6. *Let $\varphi : X \to Y$ be a closed dominant morphism. Assume that Y is irreducible and all fibers of φ are irreducible of dimension r. Then X is irreducible and of dimension $\dim Y + r$.*

Proof. Assume X_i are irreducible components of X. For all $y \in Y$, let $d_i(y)$ be the dimension of the fiber of $\varphi\big|_{X_i}$ at y. When $y \notin \operatorname{im} \varphi\big|_{X_i}$, set $d_i(y) = -1$. Then $r = \max_i\{d_i(y)\}$ so that $Y = \cup_i\{y \in Y \big| d_i(y) \geq r\}$.

By Proposition 4.5, each $\{y \in Y \big| d_i(y) \geq r\}$ is closed. Since Y is irreducible, there exists some i such that $Y = \{y \in Y \big| d_i(y) \geq r\}$. Hence for all $y \in Y$, $\varphi^{-1}(y) = \varphi\big|_{X_i}^{-1}(y)$. Conclude that $X = X_i$ irreducible. For dimension, by the claim, we get $\dim X - \dim Y = r$. ☐

4.5 Applications

Lemma 4.3 (***Rigidity Lemma***). *Let X be an irreducible projective variety over Y, $y_0 \in Y$. Consider a morphism $\varphi : X \times Y \to Z$ between varieties. Assume that $\varphi(X \times \{y_0\})$ is one-point set $\{z_0\}$. Then for all $y \in Y$, $\varphi(X \times \{y\})$ is also one-point set.*

Example 4.6. *This example shows that projectiveness is necessary. For $\varphi : \mathbb{A}_k^1 \times \mathbb{A}_k^1 \to \mathbb{A}_k^1$ sending (x, y) to xy, φ contracts $\mathbb{A}_k^1 \times \{0\}$ to 0 but not contracts $\mathbb{A}_k^1 \times \{y\}$ to 0.*

Proof. Let Γ_f be the graph of φ. Then $\Gamma_\varphi \subseteq (X \times Y) \times Z$ is a closed subvariety. Consider another projection $(X \times Y) \times Z \to Y \times Z$. Since X is projective, q is a closed morphism so that $q(\Gamma_\varphi)$ is closed subset in $Y \times Z$.

Consider $\psi : q(\Gamma_\varphi) \to Y \times Z \xrightarrow{p} Y$ which is clearly a surjection. Note that $\psi^{-1}(y_0) = (y_0, z_0)$ of dimension $0 \geq \dim q(\Gamma_\varphi) - \dim Y$, we get that $\dim Y = \dim q(\Gamma_\varphi)$. For all $\alpha \in q(\Gamma_\varphi)$, we have that

$$\begin{aligned} \dim q\big|_{\Gamma_\varphi}^{-1}(\alpha) &\geq \dim \Gamma_\varphi - \dim q(\Gamma_\varphi) \\ &= \dim X + \dim Y - \dim Y \\ &= \dim X \end{aligned} \tag{27}$$

while for all $\alpha \in q(\Gamma_\varphi) \subseteq Y \times Z$, $q^{-1}(\alpha) = X \times \{\alpha\}$. Since $q^{-1}(\alpha)$ also has dimension $\dim X$, we get that $q^{-1}(\alpha) = q\big|_{\Gamma_\varphi}^{-1}(\alpha)$. In particular, for all y and $z = \varphi(0, y)$, take $\alpha = (y, z)$ and we get $q^{-1}(\alpha) = X \times \{y\} \times \{z\}$. Conclude that $X \times \{y\}$ is contracted to $\{z\}$. $\square$

Definition 4.5. *Let X be a variety. Assume that X admits a group structure $(X, \cdot, \iota, e)$, where multiplication $\cdot : X \times X \to X$, inverse $\iota : X \to X$ and identity e satisfy group structure. Then we call X an algebraic group.*

Example 4.7. *(1)$(\mathbb{A}_k^1, +)$*
(2)$({\mathbb{A}_k^1}^*, \cdot)$
(3)$(GL_n(k), \cdot)$
(4)$(O_n(k), \cdot)$, where $O_n(k) := \{A \in GL_n(k) \big| A^\dagger A = I_n\}$.

Definition 4.6. *Let X be an irreducible projective algebraic group. We call that X an abelian variety.*

Theorem 4.7. *The group operation of abelian variety is commutative.*

Proof. Let X be an abelian variety. Consider map $\varphi : X \times X \to X$ sending (x, y) to $x^{-1}yx.$, which is in fact a morphism as composition of morphisms

$$\begin{matrix} X \times X & \to & X \times X \times X & \to & X \times X \times X & \to & X \\ (x, y) & & (x, y, x) & & (x^{-1}, y, x) & & x^{-1}yx \end{matrix} \tag{28}$$

Take the identity element e_X. Then φ contracts $X \times \{e_X\}$. By rigidity lemma, for all y, $X \times \{y\}$ is contracted by φ. Hence $x^{-1}ax = e_X^{-1}ae_X = a$ and $ax = xa$. $\square$

Example 4.8. *Let $X \subseteq \mathbb{P}_k^2$ be a general cubic curve. Then X is a one-dimensional abelian variety whose group structure can be intuitively given as following. For $x_1, x_2 \in X$, the two points determines a line L which should meet X at three points counting multiplicity, denoted by x_1, x_2, x_3. We define the multiplication by sending (x_1, x_2) to x_3.*

Proposition 4.7. *Let X be an abelian variety, Y an abelian group. Assume that $\varphi : X \to Y$ is a morphism such that $\varphi(e_X) = e_Y$. Then φ is also a group homomorphism.*

Proof. Consider map $\psi : X \times X \to Y$ sending (x, x') to $\varphi(x)\varphi(x')\varphi(xx')^{-1}$, which is a morphism as composition of morphisms

$$\begin{matrix} X \times X \\ (x, x') \end{matrix} \to \begin{matrix} X \times X \times X \\ (x, x', xx') \end{matrix} \to \begin{matrix} Y \times Y \times Y \\ (\varphi(x), \varphi(x'), \varphi(xx')) \end{matrix} \to \begin{matrix} Y \\ \varphi(x)\varphi(x')\varphi(xx')^{-1} \end{matrix} \tag{29}$$

As $\psi(X \times \{e_X\}) = \{e_Y\}$, by rigidity lemma, $\psi(X \times \{x'\})$ is a one-point set for all $x' \in X$. Hence we have

$$\varphi(x)\varphi(x')\varphi(xx')^{-1} = \varphi(e_X)\varphi(x')\varphi(x')^{-1} = e_Y \tag{30}$$

Thus $\varphi(x)\varphi(x') = \varphi(xx')$, done! □

Definition 4.7. *Let X be a variety. A subset S of X is said to be constructible if it is a finite union of locally closed subsets i.e. $S = \cup_i (U_i \cap V_i)$, where U_i are open and V_i are closed.*

Remark 4.7. *Moreover, S can be written as $\cup_i S_i$, where S_i is open dense in $\overline{S_i}$ and $S_i = U_i \cap V_i$ for V_i irreducible.*

Theorem 4.8 (***Chevalley's Theorem***)**.** *Let $\varphi : X \to Y$ be a morphism. Then the image of X is a constructible subset of Y.*

Proof. We may assume X is irreducible and φ is dominant. By Corollary 4.9, there exists nonempty open subset $U \subseteq Y$ such that for all $y \in Y$, fiber at y is of dimension $\dim X - \dim Y \geq 0$ hence nonempty. Thus $U \subseteq \mathrm{im}(\varphi)$.

Let $F = Y \setminus U$. Then $\varphi(X) = U \cup \varphi(\varphi^{-1}(F))$ where $\varphi^{-1}(F) \subsetneq X$ is a subvariety of lower dimension. We finish the proof by induction on $\dim X$. For the case that $\dim X = 0$, image is just a one-point set hence constructible. □

4.6 Finite morphisms

Theorem 4.9. *Let $\varphi : X \to Y$ be a morphism. Assume that all fibers of φ is finite and X is projective. Then for all $y \in Y$, there exists affine open neighbourhood V of y such that $U := \varphi^{-1}(V)$ is affine and $A(U)$ is finite over $A(V)$.*

Remark 4.8. *More generally, in fact we have that quasi-finiteness + universally closedness would imply finiteness. In addition, finiteness is local on the base i.e. if there exists an affine open covering satisfying the condition, then all affine open cover would satisfy the condition.*

Example 4.9. *Let $\varphi : \mathbb{A}_k^1 \setminus \{0\} \hookrightarrow \mathbb{A}_k^1$ be the natural inclusion. Then φ is quasi-finite but not finite since $k[t] \to k[t, \frac{1}{t}]$ is not finite homomorphism.*

Proof. Since X is projective, we can embed it into some projective space $\mathbb{P}_k^N$ as a closed subset. Consider the following commutative diagram

$$\begin{tikzcd} \Gamma_\varphi \arrow[r, hook] \arrow[d, "\sim"] & Y \times X \arrow[r, hook] & Y \times \mathbb{P}_k^N \arrow[d, "p"] \\ X \arrow[rr, "\varphi"] & & Y \end{tikzcd} \tag{31}$$

Replace X by graph which is a closed subvariety of $Y \times \mathbb{P}_k^N$. For all $y_0 \in Y$, $\varphi^{-1}(y_0)$ is a finite set. Hence we can choose a hyperplane H such that $X \cap (\{y_0\} \times H) = \varnothing$. Note that p is closed map, $p((Y \times H) \cap X)$ is a closed subset of Y not containing y_0. There exists nonempty affine open neighbourhood $V \ni y_0$ such that $V \cap p((Y \times H) \cap X) = \varnothing$.

Now we get $U := \varphi^{-1}(V) \subseteq Y \times H^c$, where $H^c \cong \mathbb{A}_k^N$ and there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & Y \times \mathbb{A}_k^N \arrow[d, "p"] \\ V \arrow[r, hook] & Y \end{tikzcd} \tag{32}$$

Note that $U = X \cap (V \times H^c)$ is a closed subvariety of $V \times \mathbb{A}_k^N$, where V is affine, we get $V \times \mathbb{A}_k^N$ and U both affine. Want to show $A(U)$ is finite $A(V)$-module. We already know that $A(U)$ is finitely generated as $A(V)$-algebra. Remains to show that $A(U)$ is integral over $A(V)$.

Denote coordinate of $\mathbb{A}_k^N$ by $x_1, \cdots, x_N$. By Theorem 3.3, φ is universally closed. For all $a \in A(U)$, consider $U \times \mathbb{A}_k^1 \overset{\varphi\times\mathrm{id}}{\to} V \times \mathbb{A}_k^1$. Let $Z \subseteq U \times \mathbb{A}_k^1$ be the subvariety defined by $ax - 1$. Then $\varphi \times \mathrm{id}(Z)$ is closed in $V \times \mathbb{A}_k^1$.

Denote $J = I(\varphi \times \mathrm{id}(Z))$. There is a ring homomorphism $A(V)[x]/J \to A(U)[x]/(ax - 1)$. For all $g \in J$, write $g = \sum_{i=0}^m a_i x^i$, where $a_i \in A(V)$. Then g is mapping to $(ax - 1)(\sum_{i=0}^{m-1} b_i x^i)$ for some $b_i \in A(U)$. Hence $\varphi^*(a_i) = ab_{i-1} - b_i$, where $b_m = 0$.

For $n > m$, let $f(T) = \sum_{i=0}^m \varphi^*(a_i) T^{n-i}$. Clearly, we have that $f(a) = 0$. So it suffices to show that there exists $g \in J$ such that $a_0 = 1$, which is equivalent to

$$\begin{aligned} & (J, x) = A(V)[x] \\ \iff & V(J) \cap V(X) = \varnothing \\ \iff & \varphi \times \mathrm{id}(Z) \cap (V \times \{0\}) = \varnothing \\ \iff & Z \cap (U \times \{0\}) = \varnothing \end{aligned} \tag{33}$$

By definition of Z, we are done! □

5 Smooth Points and Smooth Morphisms

5.1 Zariski tangent space

Definition 5.1. *Let $X \subseteq \mathbb{A}_k^N$ be an affine variety, $p \in X$. Define the Zariski tangent space T_pX of X at p to be the subspace defined by the linear equations $\{\sum_i \frac{\partial F}{\partial x_i}(p) x_i = 0 \big| \forall F \in I(X)\}$.*

Definition 5.2. *Let $X \subseteq \mathbb{A}_k^N$ be an affine variety, $p \in X$. A derivation of X at p is a k-linear map $D : A(X) \to k$ satisfying that for all $f, g \in A(X)$, $D(fg) = D(f)g(p) + f(p)D(g)$.*

For $X = \mathbb{A}_k^n$ with $A(X) = k[x_1, \cdots, x_n]$, a derivation D of X at p is determined by $D(x_i)$ in the sense that $D = \sum_i D(x_i)\frac{\partial f}{\partial x_i}(p)$. Hence each $a \in T_pX$ correspondends to a derivation with $D(x_i) = a_i$.

For general affine variety X with $A(X) = k[x_1, \cdots, x_n]/I(X)$, a derivation D of X at p is exactly a derivation of $\mathbb{A}_k^n$ at p which kills all $F \in I(X)$, whose corresponding point is contained in T_pX.

We can also identify T_pX with $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, where $\mathfrak{m}_p$ is the maximal ideal of $A(X)$ corresponding to p. Given derivation $D : A(X) \to k$ at p, since $D(fg) = D(f)g(p) + f(p)D(g)$, we have that $D\big|_{\mathfrak{m}_p^2} = 0$ and $D\big|_{\mathfrak{m}_p}$ gives a k-linear function $\theta : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$.

Conversely, given $\theta : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$. Let $D_\theta : A(X) \to k$ be the map defined by $f \mapsto \theta(f - f(p))$. Easy to D_θ satisfies Leibniz's Law and hence is a derivation.

Definition 5.3. *Let X be a variety, $p \in X$. Take an affine open neighbourhood U of p. Define $T_pX := T_pU = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$.*

Remark 5.1. *In fact, for $\mathcal{O}_{X,p} := A(X)_{\mathfrak{m}_p}$, if we denote $\widetilde{\mathfrak{m}_p} = \mathfrak{m}_p\mathcal{O}_{X,p}$, then we have a canonical isomorphism $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \widetilde{\mathfrak{m}_p}/\widetilde{\mathfrak{m}_p}^2$. Hence the definition is independent to our choice of affine open neighbourhood.*

Example 5.1. *Let X, Y be two varieties, $(x, y) \in X \times Y$. Then $T_{(x,y)}(X \times Y) = T_xX \oplus T_yY$.*

5.2 Smooth points and singular points

Proposition 5.1. *Let X be a variety. Then the function $X \to \mathbb{N}$ sending x to $\dim_k T_xX$ is upper semicontinuous.*

Reason 5.1. *By taking affine cover, we may assume X is affine. Then $X \subseteq \mathbb{A}_k^N$ for some N. For all $x \in X$, $T_xX = \{(a_1, \cdots, a_n)\big| \sum_i a_i \frac{\partial F}{\partial x_i}(x) = 0, \forall F \in I(X)\}$. Assume $I(X) = (F_1, \cdots, F_n)$. Then*

$$\begin{aligned} &X_r = \{x \in X \big| \dim_k \ker(\frac{\partial F_i}{x_j}(x)) \geq r\} \\ \iff& \operatorname{rank}(\frac{\partial F_i}{\partial x_j}(x)) \leq N - r \\ \iff& \textit{for all } (N - r + 1)\textit{-minor, determinant vanishes} \end{aligned} \tag{34}$$

Proposition 5.2. *Let X be an irreducible variety. Then for all $x \in X$, $\dim_k T_xX \geq \dim X$ and there exists open dense subset U of X such that for all $x \in U$, $\dim_k T_xX = \dim X$.*

Proof. When $X \subseteq \mathbb{A}_k^{n+1}$ is a hypersurface defined by an irreducible polynomial $0 \neq F$, then $\dim X = n$. For all $p \in X$, T_pX is defined by a linear polynomial $\sum_i \frac{\partial F}{\partial x_i}(p)x_i$. Thus $\dim_k T_pX = n$ or $n + 1$.

By previous proposition, X_{n+1} is a closed subset of X. If $X_{n+1} = X$, then $\frac{\partial F}{\partial x_i}$ vanish at all $x \in X$ so that $\frac{\partial F}{\partial x_i} = 0$. Since F is irreducible, we get $\operatorname{Char} k = p$ and $F = G(x_1^p, \cdots, x_{n+1}^p)$

for some polynomial G. As k is algebraically closed, there exists G_1 such that $G_1^p = F$, contradicting to the assumption that F is irreducible. Hence $X_{n+1} \subsetneq X$ and $U = X_{n+1}^c$ is our desired open subset.

In general, we may assume that X is affine and $n = \dim X = \operatorname{trdeg}(K(X)/k)$. By Proposition 2.4, there exists some hypersurface $X' \subseteq \mathbb{A}_k^{n+1}$ birational to X. Then there exists open dense subset $U \subseteq X$ and $U' \subseteq X'$ such that $U \xrightarrow{\sim} U'$.

Suppose that X_n^c is nonempty. Then since X is irreducible, $X_n^c \cap U$ is also irreducible, contradicting to ${X'_n}^c = \varnothing$. Hence for all $x \in X$, $\dim_k T_x X \geq n$. And by discussion above, there exists open dense subset $U'' \subseteq X'$ such that for all $x' \in U''$, $\dim_k T_{x'} X' = n$. As X' is irreducible, $U' \cap U''$ is nonempty whose inverse image in X is our desired open dense subset. □

Definition 5.4. *Let X be a variety. We say that $x \in X$ is a smooth point if $\dim_k T_x X = \dim_X x$. Otherwise, we would say that $x \in X$ is a singular point.*

Corollary 5.1. *Let X be a variety. If X is smooth at x, then $\dim_k T_x X = \dim X$. Moreover, if X is irreducible, then the converse also holds.*

Reason 5.2. *By Proposition 5.2, we immediately get that $\dim X \leq \dim_k T_x X = \dim_X x \leq \dim X$. When X is irreducible, by Corollary 4.3, we get $\dim_X x = \dim X$ for all x.*

Remark 5.2. *Combining Proposition 5.3 and Corollary 5.1, we get that for any irreducible variety X, there is always an open dense subset of X consists of smooth points in X.*

Example 5.2. *(1)Let $X \subseteq \mathbb{A}_k^3$ be the subvariety defined by (xy, yz). Then $\operatorname{Sing} X = \{0\}$.*
(2)Let X, Y be two varieties. Then $\operatorname{Sing}(X \times Y) = (\operatorname{Sing} X \times Y) \cup (X \times \operatorname{Sing} Y)$.
(3)Let $X \subseteq \mathbb{A}_k^{n+1}$ be the subvariety defined by F. Then $\operatorname{Sing} X$ is the subvariety defined by $(F, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_{n+1}})$. In particular, if $X = \cup V(F_i)$ where F_i are irreducible, then for all $i \neq j$, $V(F_i) \cap V(F_j) \subseteq \operatorname{Sing} X$.
(4)Assume that $X \subseteq \mathbb{A}_k^n$ is an irreducible closed subvariety with $I(X) = (F_1, \cdots, F_r)$. Then $p \in X$ is a smooth point if and only if $(\frac{\partial F_i}{\partial x_j}(p))$ has corank $\dim X$.
(5)Assume that $X \subseteq \mathbb{P}_k^n$ is a hypersurface defined by homogeneous $F(x_0, \cdots, x_{n+1})$. Then $\operatorname{Sing} X$ is defined by $(\frac{\partial F}{\partial x_0}, \cdots, \frac{\partial F}{\partial x_n})$ since $\sum_i x_i \frac{\partial F}{\partial x_i} = dF$ where d is the degree of F.

Lemma 5.1. *Let X be a variety. Then the set of singular points in X is closed, denoted by $\operatorname{Sing} X$.*

Reason 5.3. *By taking affine cover, we may assume X is affine. Hence by Jacobian criterion, we are done.*

Lemma 5.2. *Let X be a variety. Assume V and U are two closed subvariety of X. Then $(V \cup U)^{\mathrm{sm}} \subseteq V^{\mathrm{sm}} \cup U^{\mathrm{sm}}$.*

Proof. We may assume that X is affine. For all $x \in (V \cup U)^{\mathrm{sm}}$, we have that $\dim_{V \cup U} x = \max\{\dim_V x, \dim_U x\}$. Note that defining equations of $T_x(V \cup U)$ are

$$\{\sum \frac{\partial F}{\partial x_j}(x) x_j \big| F \in I(V) \cap I(U)\} \tag{35}$$

Hence $\max\{\dim_V x, \dim_U x\} \leq \max\{\dim_k T_xV, \dim_k T_xU\} \leq \dim_k T_x(V \cup U) = \dim_{V\cup U} x$. Thus all the equalities hold so that $x \in V^{\mathrm{sm}} \cup U^{\mathrm{sm}}$ and $(V \cup U)^{\mathrm{sm}} \subseteq V^{\mathrm{sm}} \cup U^{\mathrm{sm}}$. □

Example 5.3. *Let $X = \mathbb{A}_k^2$ be the affine space. Then x-axis and y-axis are both smooth, while the union of two axis is singular at the origin. Hence the equality in Lemma 5.2 does not necessarily hold.*

5.3 Regular rings

Lemma 5.3. *Let A be a noetherian local ring with residue field k, $\mathfrak{m}_A$ the maximal ideal. Then we always have that $\dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) \geq \dim A$.*

Reason 5.4. *By Krull's Principal Ideal Theorem.*

Definition 5.5. *Let A be a noetherian local ring with residue field k, $\mathfrak{m}_A$ the maximal ideal. We say that A is a regular local ring if $\dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = \dim A$.*

Lemma 5.4. *Let A be a noetherian integral domain. Then A is UFD if and only if all prime ideals of height 1 are principal ideals.*

Theorem 5.1. *A regular local ring is a UFD.*

Proof. Here is a sketch of the proof

- Let R be an integral domain. Then R is a UFD if and only if for all $f, g \neq 0$, $(f) : (g) := \{h \in R \big| gh \in (f)\}$ is a principal ideal.
- Let R be a regular local of dimension n. Then $\widehat{R} \cong k[[x_1, \cdots, x_n]]$.
- R is UFD if and only if $\widehat{R}$ is UFD. □

Corollary 5.2. *Let X be a variety. If X is smooth at x, then $\mathcal{O}_{X,x}$ is a UFD.*

Example 5.4. *Let $R = k[x_1, \cdots, x_n]/(x_1^2 + \cdots + x_n^2)$ and $\operatorname{Char} k \neq 2$.*

- *when $n = 2$, as $x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$, R is not an integral domain.*
- *when $n = 3$, $x_1^2 + x_2^2 + x_3^2$ is irreducible in R, but R is not a UFD as $x_3^2 = -(x_1 + ix_2)(x_1 - ix_2)$.*
- *when $n = 4$, $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is irreducible in R, but R is not a UFD as $(x_1 + ix_2)(x_1 - ix_2) = -(x_3 + ix_4)(x_3 - ix_4)$.*
- *when $n \geq 5$, R is a UFD.*

Lemma 5.5 (***Nagata's Lemma***)**.** *Let R be an integral domain. Assume (x) is a prime ideal of R. Then if $R[\frac{1}{x}]$ is a UFD, so is R.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R of height 1. If $\mathfrak{p} = (x)$, then $\mathfrak{p}$ is principal. Otherwise, $\mathfrak{p}R[\frac{1}{x}]$ is a prime ideal of $R[\frac{1}{x}]$ of height 1. As $R[\frac{1}{x}]$ is a UFD, by Lemma 5.4 there exists some $y \in R[\frac{1}{x}]$ such that $\mathfrak{p}R[\frac{1}{x}] = (y)$.

Assume that $y = y_0 x^n$ where $y_0 \in R \setminus (x)$. Note that $\mathfrak{p}R[\frac{1}{x}] \cap R = \mathfrak{p}$, we get that $\mathfrak{p} = yR[\frac{1}{x}] \cap R = y_0 R[\frac{1}{x}] \cap R = (y_0)$ also principal. Again 5.4, we conclude that R is UFD. $\square$

With Lemma 5.5, we give a proof of the $n \geq 5$ case in Example 5.4.

Lemma 5.6. *Let $R = k[x_1, \cdots, x_n]/(x_1^2 + \cdots + x_n^2)$ and $\operatorname{Char} k \neq 2$. If $n \geq 5$, then R is a UFD.*

Proof. Denote $x = x_{n-1} + ix_n$ and $y = x_{n-1} - ix_n$. As $x_{n-1}^2 + x_n^2 = xy$, we can rewrite R as $k[x_1, \cdots, x_{n-2}, x, y]/(x_1^2 + \cdots + x_{n-2}^2 + xy)$. Since $R/(x) \cong k[x_1, \cdots, x_{n-2}, x, y]/(x_1^2 + \cdots + x_{n-2}^2, x) \cong k[x_1, \cdots, x_{n-2}, y]/(x_1^2 + \cdots + x_{n-2}^2)$ is integral domain, (x) is a prime ideal of R. Taking localization at x, we get $R[\frac{1}{x}] \cong k[x_1, \cdots, x_{n-2}, x]$ is a UFD. By Nagata's Lemma, R is also a UFD. $\square$

Theorem 5.1 is too hard to give a complete proof here. But we can prove for a much more easier but useful result.

Lemma 5.7. *A regular local ring is an integral domain.*

Proof. Induct on dimension of R. When $\dim R = 0$, we immediately get that R is a field and hence an integral domain. Assume $\dim R > 0$ so that $\mathfrak{m}^2 \subsetneq \mathfrak{m}$ otherwise by Nakayama's Lemma, $\mathfrak{m} = 0$. Since noetherian local ring has finitely many minimal prime ideals, we can take some $x \in \mathfrak{m}^2/\mathfrak{m}$ and $x \notin \mathfrak{p}$ for all minimal prime ideal $\mathfrak{p}$.

Then $R/(x)$ with maximal ideal $\mathfrak{m}/(x)$ would be a regular local ring of dimension $\dim R - 1$. By induction, $R/(x)$ is an integral domain so that (x) is prime ideal of R. Hence (x) should contain some minimal prime ideal $\mathfrak{q}$. For all $y \in \mathfrak{q}$, $y = ax$ for some $a \in R$. Note that $x \notin \mathfrak{q}$, we get $a \in \mathfrak{q}$ so that $\mathfrak{q} \subseteq (x)\mathfrak{q} \subseteq \mathfrak{m}\mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q}$ is zero ideal. Conclude that R is an integral domain. $\square$

Remark 5.3. *If X is smooth at x, then x cannot be contianed in two distinct irreducible components since $\mathcal{O}_{X,x}$ is a regular local ring, which by the previous lemma is an integral domain with unique minimal prime ideal (0).*

Proposition 5.3. *Let X be an irreducible variety of dimension n. Assume that $Y \subseteq X$ be a closed subvariety of pure dimension $n - 1$. If $y \in Y$ is a smooth point in X, then there exists an open affine neighbourhood U of y in X and regular function $f \in A(U)$ such that $I(U \cap Y) = (f)$ in U and f is irreducible in $\mathcal{O}_{X,y}$.*

Proof. Assume X is affine and Y is irreducible. Then Y is defined by a prime ideal $\mathfrak{p} \subseteq A(X)$ and $\mathfrak{p}\mathcal{O}_{X,y}$ is still a prime ideal of $\mathcal{O}_{X,y}$ of height 1. By Corollary 5.2, $\mathcal{O}_{X,y}$ is a UFD. Hence $\mathfrak{p}\mathcal{O}_{X,y}$ is generated by some irreducible element $\frac{f}{1}$.

Consider $(\frac{f}{1})\mathcal{O}_{X,y} \cap A(X)$ which is an ideal of $A(X)$. Assume $f_1, \cdots, f_r$ generate the ideal. Then there exist $\frac{h_i}{g_i} \in \mathcal{O}_{X,y}$ such that $\frac{f_i}{1} = \frac{h_i}{g_i} \cdot \frac{f}{1}$. Take $g = \prod_i g_i$ and we have that $A(X)_g \cap (\frac{f}{1})\mathcal{O}_{X,y} = (\frac{f}{1})A(X)_g$. Let $U = D(g)$ and we are done. $\square$

Example 5.5. *Let $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^4$. Easy to verify that X is irreducible of dimension 3. Consider $I = (x_1, x_3)$. Then $V(I) \subseteq X$ is isomorphic to $\mathbb{A}_k^2$ of dimension 2. But around origin o, $I\mathcal{O}_{X,o}$ is not principal. Hence smoothness is necessary in the previous proposition.*

5.4 Zariski's Main Theorem

Theorem 5.2. *Let $\varphi : X \to Y$ be a birational morphism between irreducible varieties of dimension n. Assume that Y is smooth (or factorial). Then there exists open dense subset $U \subseteq Y$ such that*

- *$\varphi^{-1}(U) \xrightarrow{\sim} U$*
- *If $X \setminus \varphi^{-1}(U)$ is not empty, write $X \setminus \varphi^{-1}(U) = \cup_i E_i$ as union of irreducible components. Then $\dim E_i = n - 1$ and $\dim \overline{\varphi(E_i)} \leq n - 2$.*

Proof. For $y \in Y$, take an affine open neighbourhood U_y. For $x \in X$ mapping to y, take an affine open neighbourhood $U_x \subseteq \varphi^{-1}(U_y)$. Then φ induces map

$$\begin{tikzcd} A(U_y) \arrow[r, "\varphi^*"] \arrow[d, hook] & A(U_x) \arrow[d, hook] \\ \mathcal{O}_{Y,y} \arrow[r, "\varphi^*"] \arrow[d, hook] & \mathcal{O}_{X,x} \arrow[d, hook] \\ K(Y) \arrow[r, "\varphi^*", "\sim"'] & K(X) \end{tikzcd} \tag{36}$$

hence all the φ^* are injections.

For the case that $\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective, assume that $t_1, \cdots, t_m$ generate $A(U_x)$ over k. Then there exists $\frac{a_i}{b_i} \in \mathcal{O}_{Y,y}$ such that $\varphi^*(\frac{a_i}{b_i}) = t_i$, where $a_i, b_i \in A(U_y)$ and $b_i(y) \neq 0$. Take $b = \prod_i b_i$. Then $\varphi^* : A(U_y)_b \to A(U_x)_{\varphi^*(b)}$ is bijective and hence $D(\varphi^*(b))$ is isomorphic to $D(b)$.

For the case that φ^* is not surjective, for $t \in \widetilde{\mathfrak{m}_x}$ which is not in $\operatorname{im} \varphi^*$, there exists $a, b \in \widetilde{\mathfrak{m}_y}$ such that $\varphi^*(\frac{a}{b}) = t$. Since Y is smooth, $\mathcal{O}_{Y,y}$ is UFD. We may assume that a, b are coprime. After shrinking U_y, we may assume that $a, b \in A(U_y)$.

Take closed subvarieties $E_1 = V(a)$ and $E_2 = V(b)$ both of pure dimension $n - 1$. They have no common irreducible component through y. Shrinking U_y again, we may assume E_1 and E_2 have no common irreducible component. Then $\dim(E_1 \cap E_2) \leq n - 2$.

Consider $E = V(\varphi^*(b))$ of dimension $n - 1$. Note that $\varphi^*(a) = t\varphi^*(b)$, we get that $\varphi(E) \subseteq E_1 \cap E_2$. Hence $\dim \overline{\varphi(E)} \leq \dim(E_1 \cap E_2) \leq n - 2$, done! □

Remark 5.4. *Through this proof, it is clear that when apply Zariski's Main Theorem, we can take U to be the unique maximal one.*

Example 5.6. *Let $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^4$. Then $\mathcal{O}_{X,0}$ is not UFD. Consider $\varphi : \mathbb{A}_k^3 \to X$ sending (y_1, y_2, y_3) to $(y_1, y_2y_3, y_2, y_1y_3)$. Note the field extension $\varphi^* : K(X) \to$*

$K(y_1, y_2, y_3)$ is an isomorphism and hence φ is birational morphism. Take $U \subseteq X$ defined by $\{(x_1, x_2, x_3, x_4)\big|(x_1, x_3) \neq 0\}$, then $\varphi^{-1}(U) = \{(y_1, y_2, y_3)\big|(y_1, y_2) \neq 0\}$ and $\dim(\mathbb{A}^3_k \setminus \varphi^{-1}(U)) = 1$. Thus Zariski's Main Theorem fails if without factoriality.

Corollary 5.3. *Let $\varphi : X \to Y$ be a quasi-finite birational morphism between irreducible varieties. Assume that Y is smooth. Then φ is an open immersion. In particular, X is also smooth.*

Proof. By Zariski's Main Theorem, there exists an open dense subset $U \subseteq Y$ such that $\varphi^{-1}(U) \xrightarrow{\sim} U$. If $X \neq \varphi^{-1}(U)$, then for all $y \in \operatorname{im}\varphi \setminus U$, since φ is quasi-finite, $\varphi^{-1}(y) \subseteq X \setminus \varphi^{-1}(U)$ is union of some irreducible components of $X \setminus \varphi^{-1}(U)$. Hence all irreducible components E_i of $X \setminus \varphi^{-1}(U)$ would be of dimension 0. By Zariski's Main Theorem, $\dim \overline{\varphi(E_i)} \leq -1$ which is ridiculous, contradiction. Thus $X = \varphi^{-1}(U)$ and hence φ is an open immersion. □

Corollary 5.4. *Let X be a smooth irreducible variety and $\varphi : X \dashrightarrow \mathbb{P}^n_k$ be a rational map. Then φ is defined over $U \subseteq X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$.*

Proof. Let U be the definition of domain of φ and $\Gamma_{\varphi|_U}$ be the graph. Consider its closure $\overline{\Gamma_{\varphi|_U}}$ in $X \times \mathbb{P}^n_k$ and projection $p : \overline{\Gamma_{\varphi|_U}} \to X$. Then p is a birational morphism. Note that continuous map preserves irreducibility, $\overline{\Gamma_{\varphi|_U}}$ is irreducible.

By Zariski's Main Theorem, there exists open dense subset $V \subseteq X$ such that $p^{-1}(V) \xrightarrow{\sim} V$ and $\dim \overline{p(\overline{\Gamma_{\varphi|_U}} \setminus p^{-1}(V))} \leq \dim X - 2$. Hence $\operatorname{codim}_X(X \setminus V) \geq \operatorname{codim} \overline{p(\overline{\Gamma_{\varphi|_U}} \setminus p^{-1}(V))} \geq 2$. While p^{-1} is defined over V, we get $V \xrightarrow{p^{-1}} \overline{\Gamma_{\varphi|_U}} \to \mathbb{P}^n_k$ is a morphism coinciding with $\varphi\big|_U$ on $U \cap V$. Hence by definition, $V \subseteq U$ and $\operatorname{codim}(X \setminus U) \geq \operatorname{codim}(X \setminus V) \geq 2$. □

Corollary 5.5. *Let C be an irreducible smooth curve. Assume that $\varphi : C \dashrightarrow \mathbb{P}^n_k$ is a rational map. Then φ is a morphism.*

Reason 5.5. *By Corollary 5.4, φ is defined over U with $\operatorname{codim}(X \setminus U) \geq 2$. While $\dim C = 1$, we get $X \setminus U = \varnothing$ so that φ is a morphism.*

5.5 Smooth morphisms

Let $\varphi : X \to Y$ be a morphism between varieties, $x \in X$, $y = \varphi(x)$. We may assume that X and Y are affine. Then there is an induced homomorphism $\varphi^* : A(Y) \to A(X)$ such that $\varphi^{*-1}(\mathfrak{m}_x) = \mathfrak{m}_y$. Recall that $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and $T_y Y = (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee$. There is also an induced tangent map $T_x\varphi : T_x X \to T_y Y$.

Example 5.7. *Consider $\varphi : \mathbb{A}^m_k \to \mathbb{A}^n_k$ sending $(x_1, \cdots, x_m)$ to $(f_1, \cdots, f_n)$, where polynomials $f_1, \cdots, f_n \in k[x_1, \cdots, x_m]$. For all $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{A}^m_k$, the tangent map $T_\alpha\varphi$ is defined by $(\frac{\partial f_i}{\partial x_j}(\alpha))$.*

Lemma 5.8. *Given morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$, then $T_x(\psi \circ \varphi) = T_{\varphi(x)}\psi \circ T_x\varphi$ for all $x \in X$.*

Lemma 5.9. *Let $\varphi : X \to Y$ be a morphism, $x \in X$. Assume that X' is the connected component of X_x containing x. Then $T_x X' \subseteq \ker T_x\varphi$.*

Proof. As definition of Zariski tangent space is local, we may assume X and Y are affine. Now there is an induced homomorphism $\varphi^* : A(Y) \to A(X)$. Denote $y = \varphi(x)$. Since $A(X') = A(X)/I$ for some radical ideal containing $\varphi^*(\mathfrak{m}_y)$, we get the composition $\mathfrak{m}_y \xrightarrow{\varphi^*} \mathfrak{m}_x \twoheadrightarrow \mathfrak{m}_x/I$ is zero map. Hence by definition, $T_xX' \subseteq \ker T_x\varphi$. □

Remark 5.5. *It is not necessary to take the connected component here. In fact, $T_xX_x \subseteq \ker T_x\varphi$.*

Proposition 5.4. *Let $\varphi : X \to Y$ be a morphism. Then the function $X \to \mathbb{N}$ sending x to $\dim_k(\ker T_x\varphi)$ is upper semicontinuous i.e. for all $r \in \mathbb{N}$, $X(r) := \{x \in X \big| \dim_k(\ker T_x\varphi) \geq r\}$ is closed.*

Proof. We may assume that $X \subseteq \mathbb{A}_k^n$ is a subvariety defined by $F_1, \cdots, F_s$. And φ is restriction of a polynomial map $\mathbb{A}_k^n \to \mathbb{A}_k^m$ sending $(x_1, \cdots, x_n)$ to $(G_1, \cdots, G_m)$. Note that for all $\alpha \in X$, $T_\alpha X$ is the variety defined by $\{\sum_j \frac{\partial F_i}{\partial x_j}(\alpha)x_j\}_i$ and $T_\alpha\varphi$ is defined by $(\frac{\partial G_i}{\partial x_j}(\alpha))$, we get that $\alpha \in X(r)$ if and only if rank of the following matrix is less than or equal to $n - r$

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\alpha) & \cdots & \frac{\partial F_1}{\partial x_n}(\alpha) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_s}{\partial x_1}(\alpha) & \cdots & \frac{\partial F_s}{\partial x_n}(\alpha) \\ \frac{\partial G_1}{\partial x_1}(\alpha) & \cdots & \frac{\partial G_1}{\partial x_n}(\alpha) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial x_1}(\alpha) & \cdots & \frac{\partial G_m}{\partial x_n}(\alpha) \end{pmatrix} \tag{37}$$

which is a closed condition so that $X(r)$ is closed. □

Example 5.8. *Consider $\varphi : \mathbb{A}_k^3 \to \mathbb{A}_k^2$ sending (x, y, z) to $(z, x^2z + y)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{A}_k^3$, $T_\alpha\varphi : k^3 \to k^2$ is defined by*

$$\begin{pmatrix} 0 & 0 & 1 \\ 2\alpha_1\alpha_3 & 2\alpha_2 & \alpha_1^2 \end{pmatrix} \tag{38}$$

which is of rank at least 1. Hence $X(2) = \{\alpha \in \mathbb{A}_k^3 \big| \dim_k(\ker T_\alpha\varphi) \geq 2\} = \{(0, 0, \alpha_3) \big| \alpha_3 \in k\} \cup \{(\alpha_1, 0, 0) \big| \alpha_1 \in k\}$. As $\varphi(X(2)) = \{(z, 0) \big| z \in k\}$, for all $(s, t) \in \mathbb{A}_k^2$ such that $t \neq 0$, by Jacobian criterion, we have that $\varphi^{-1}((s, t))$ is smooth.

Definition 5.6. *Let X, Y be smooth irreducible varieties. We say a morphism $\varphi : X \to Y$ is smooth at $x \in X$ if $T_x\varphi$ is surjective. If φ is smooth at all points in X, then we say φ is smooth.*

Remark 5.6. *By Proposition 5.4, for morphism φ between smooth varieties, the set of points where φ is not smooth is a closed subset of X.*

Lemma 5.10. *Let X be an irreducible variety, Z irreducible subvariety of X of codimension r. Assume X and Z are both smooth at $x \in Z$. Then there exists affine open neighbourhood $U \ni x$ and a regular sequence $f_1, \cdots, f_r$ in $A(U)$ such that $Z \cap U$ is an irreducible component of $V(f_1, \cdots, f_r) \subseteq U$.*

Proof. We may assume that X is affine. Then Z is defined by some prime ideal $\mathfrak{p}$ of $A(X)$ of height r. As X and Z are both smooth at x, we get that $A(X)_{\mathfrak{m}_x}$ and $(A(X)/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}}$ are both regular local ring. Since $A(X)$ is integral, by Theorem 4.2, $\dim A(X)/\mathfrak{p} = \dim A(X) - r$.

Assume $\dim A(X) = n$. Denote $\overline{\mathfrak{m}_x} = \mathfrak{m}_x/\mathfrak{p}$. Take a basis $a_1, \cdots, a_{n-r}$ of $\overline{\mathfrak{m}_x}/\overline{\mathfrak{m}_x}^2 \cong \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{p})$. Then we can extend it to a basis $a_1, \cdots, a_{n-r}, b_1, \cdots, b_r$ of $\mathfrak{m}_x/\mathfrak{m}_x^2$. Then for all i, there exist $\frac{\alpha_{ij}}{\beta_{ij}} \in A(X)_{\mathfrak{m}_x}$ and $c_i \in \mathfrak{m}_x^2$ such that $f_i := \frac{b_i}{1} + \sum_j \frac{\alpha_{ij}}{\beta_{ij}}\frac{a_j}{1} - \frac{c}{1}$ is contained in $\mathfrak{p}$.

Now we get $f_i \in \mathfrak{p}$ which is clearly a regular sequence. Taking appropriate s, we get $f_i \in U := D(s)$. By Krull's Principal Ideal Theorem, minimal prime ideals in $V(f_1, \cdots, f_r)$ would have height r. Note that $Z \cap U$ also has codimension r and $Z \cap U \subseteq V(f_1, \cdots, f_r)$. We conclude that $Z \cap U$ is an irreducible component of $V(f_1, \cdots, f_r)$. □

Proposition 5.5. *Let X and Y be smooth irreducible varieties, $\varphi : X \to Y$ morphism. Then (1)All nonempty fiber of φ has pure dimension $\dim X - \dim Y$ and φ is dominant.*
(2)Assume that $Z \subseteq Y$ is a smooth closed subvariety, then $\varphi^{-1}(Z)$ is also smooth.

Proof. (1)For all $x \in X$, take X' to be the connected component of X_x containing x. Then by Proposition 5.2 and Theorem 4.5, $\dim_k T_x X' \geq \dim X' \geq \dim X - \dim \overline{\varphi(X)} \geq \dim X - \dim Y$. On the other hand, as φ is smooth, $T_x\varphi$ is surjective. Hence $\dim_k T_x X' \leq \dim_k(\ker T_x\varphi) = \dim_k T_x X - \dim_k T_{\varphi(x)} Y$. Since X and Y are both smooth, by Corollary 5.1, we have that $\dim_k T_x X - \dim_k T_{\varphi(x)} Y = \dim X - \dim Y$. Thus all equalities hold and $\dim \overline{\varphi(X)} = \dim Y$ so that φ is dominant.

(2)We may assume that $Z \subseteq Y$ is smooth, irreducible and of codimension r. For all $z \in \varphi^{-1}(Z)$, $T_z\varphi(T_z\varphi^{-1}(Z)) \subseteq T_{\varphi(z)}Z$. Hence

$$\begin{aligned}\dim_k T_z\varphi^{-1}(Z) &\leq \dim_k T_{\varphi(z)}Z + \dim_k(\ker T_z\varphi) \\ &= \dim Z + \dim X - \dim Y \\ &= \dim X - r\end{aligned} \tag{39}$$

On the other hand, for all $y \in Z$, by Lemma 5.10, there exist affine open neighbourhood $U \ni y$ and a regular sequence $f_1, \cdots, f_r$ in $A(U)$ such that $Z \cap U$ is an irreducible component of $V(f_1, \cdots, f_r)$. As continuous map sending irreducible sets to irreducible sets, $\varphi^{-1}(Z \cap U)$ would be union of some irreducible components of $\varphi^{-1}(V(f_1, \cdots, f_r)) = V(f_1 \circ \varphi, \cdots, f_r \circ \varphi_r)$. By Krull's Principal Ideal Theorem, those irreducible components should be of codimension at most r in $\varphi^{-1}(U)$.

Note that for each irreducible component Z_0 of $\varphi^{-1}(Z)$ with $z \in Z_0$ mapping to y, $Z_0 \cap \varphi^{-1}(U)$ is an irreducible component of $\varphi^{-1}(Z \cap U)$. Hence $\operatorname{codim} Z_0 = \operatorname{codim} Z_0 \cap \varphi^{-1}(U) \leq r$ in X. Since X is irreducible, by Theorem 4.2, $\dim_k T_z\varphi^{-1}(Z) \geq \dim \varphi^{-1}(Z) \geq \dim X - r$. Thus $\dim_k T_z\varphi^{-1}(Z) = \dim \varphi^{-1}(Z) = \dim X - r$. As X is irreducible, $\varphi^{-1}(Z)$ is smooth. □

Remark 5.7. *For $\varphi : X \to Y$ between smooth projective varieties, it is very rare that φ is smooth, except for some trivial case such as projection $Y \times Z \to Y$.*

Proposition 5.6 (***Generic Smoothness***)**.** *Assume* $\operatorname{Char} k = 0$*. Let* $\varphi : X \to Y$ *be a dominant morphism between irreducible varieties. Then there exist open dense subsets* $V \subseteq Y$ *and* $U \subseteq X$ *such that* $\varphi\big|_U : U \to V$ *is a smooth morphism.*

Proof. We can reduce the problem to affine case. Assume that $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$. Moreover, by replacing X by graph of φ, φ is restriction of the projection $\mathbb{A}_k^{n+m} \to \mathbb{A}_k^m$. By induction, we consider the case that $n = 1$ with projection $\mathbb{A}_k^{m+1} \to \mathbb{A}_k^m$ sending $(x_1, \cdots, x_{m+1})$ to $(x_1, \cdots, x_m)$.

When x_{m+1} is transcendental over $K(Y)$, $X \cong Y \times \mathbb{A}_k^1$. Since φ is the projection, φ is smooth. When x_{m+1} is algebraic over $K(Y)$. Let $G(T) = a_d(y)T^d + a_{d-1}(y)T^{d-1} + \cdots + a_0(y)$ be the minimal polynomial of x_{m+1} over $K(Y)$, where $a_i \in K(Y)$. Taking $V = D(a_d)$, we may assume that leading coefficient of $G(T)$ is 1.

By definition, $G(x_{m+1}) \in I(X)$. In addition, since φ is the projection, by Euclidean division, we get $I(X) = (I(Y), G(x_{m+1}))$ so that $\dim X = \dim Y$. By Proposition 5.2, there is always an open dense subset V of Y consisting of smooth points in Y.

For all $\alpha = (\beta, t)$ in $\varphi^{-1}(V)$, we get that $T_\alpha X$ is defined by equations $S \cup \{\sum_j \frac{\partial G}{\partial x_j}(\alpha) x_j\}$, where S is the set of defining equations of $T_\beta Y$. Take $U = D(\frac{\partial G}{\partial x_{m+1}}) \cap \varphi^{-1}(V)$. As G is the minimal polynomial of x_{m+1} over $K(Y)$, $\frac{\partial G}{\partial x_{m+1}}$ would not vanish on X so that U is nonempty.

Hence for $\alpha \in U$, $\dim_k T_\alpha X = \dim_k T_\beta Y$. Thus $\dim X = \dim Y = \dim_k T_\beta Y = \dim_k T_\alpha X$ so that X is smooth at α. In addition, for all $\alpha \in U$, the following matrix has rank $m + 1$

$$\begin{pmatrix} M & 0 \\ \cdots & \frac{\partial G}{\partial x_{m+1}}(\alpha) \\ I_m & 0 \end{pmatrix} \tag{40}$$

where M are the defining equations in S. Hence $\ker T_\alpha \varphi = 0$. As $\dim_k T_\alpha X = \dim_k T_\beta Y$, $T_\alpha \varphi$ is surjective for all $\alpha \in U$. Conclude that $\varphi\big|_U : U \to V$ is a smooth morphism. $\square$

Example 5.9. *When* $\operatorname{Char} k = p$*, generic smoothness would fail. For instance, the tangent map of* $\varphi : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ *sending* x *to* x^p *is zero map everywhere.*

Theorem 5.3 (***Generic Smoothness***)**.** *Assume* $\operatorname{Char} k = p$*. Let* $\varphi : X \to Y$ *be a dominant morphism between irreducible varieties. Then there exists smooth open dense subset* $V \subseteq Y$ *such that* $\varphi^{-1}(V) \cap X^{\mathrm{sm}} \xrightarrow{\varphi} V$ *is a smooth morphism, where* $\varphi^{-1}(V) \cap X^{\mathrm{sm}}$ *is the schemetic intersection i.e. the subvariety with ideal sheaf generated by union of the original two.*

Proof. We may assume that X and Y are smooth. Let $Z = \{x \in X \big| \varphi \text{ is not smooth at } x\}$ which by Proposition 5.4 is closed in X. If $Z = \varnothing$, then take $V = Y$ and we are done.

For all irreducible component Y' of $\overline{\varphi(Z)}$, $\varphi^{-1}(Y')$ is union of some irreducible components of Z. by irreducibility of Y', we can pick some irreducible component X' of Z such that $\overline{\varphi(X')} = Y'$ and hence $\varphi\big|_{X'}$ is dominant. By Proposition 5.6, there exist smooth open dense subsets $U' \subseteq X'$ and $V' \subseteq Y'$ such that $\varphi\big|_{U'} : U' \to V'$ is a smooth morphism.

Hence for $x \in U'$, $\dim Y' = \dim_k T_{\varphi(x)} V' = \dim_k T_x \varphi\big|_{U'}(T_x U') \leq \dim_k T_x \varphi(T_x X) < \dim_k T_{\varphi(y)} Y = \dim Y$. Thus $\overline{\varphi(Z)} \subsetneq Y$. Take $V = Y \setminus \overline{\varphi(Z)}$. Want to show that $\varphi^{-1}(V) \xrightarrow{\varphi} V$

is a smooth morphism. In fact, for all $x \in \varphi^{-1}(V)$, the tangent map is just $T_x\varphi$ which is surjective as $\varphi^{-1}(V) \cap Z = \varnothing$. $\square$

Example 5.10. *Assume* $\operatorname{Char} k = 0$. *Consider* $\varphi : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ *sending* (x, y) *to* $x^2 - y^p$. *For all* $b \in \mathbb{A}_k^1$, $(0, \beta) \in \varphi^{-1}(b)$, *where* $\beta^p = -b$, *is a singular point in* $\varphi^{-1}(b)$. *Hence there does not exist smooth open dense subset* $V \subseteq Y$ *such that* $\varphi^{-1}(V) \to V$ *is smooth.*

5.6 Bertini's Theorem

Theorem 5.4 (***Bertini's Theorem***)**.** *Assume* $\operatorname{Char} k = 0$. *Let* X *be a smooth variety,* $\varphi : X \to \mathbb{P}_k^n$ *morphism. Then for a general hyperplane* H *of* $\mathbb{P}_k^n$ *i.e. for* H *in some nonempty open subset of* ${\mathbb{P}_k^n}^\vee$, $\varphi^{-1}(H)$ *is smooth.*

Proof. Since X is smooth, X is disjoint union of irreducible components. We may assume X is irreducible of dimension at least 1. Then consider the incident variety $I_X := \{(x, [H]) \in X \times {\mathbb{P}_k^n}^\vee | \varphi(x) \in H\}$ and morphism $\varphi \times \mathrm{id} : X \times {\mathbb{P}_k^n}^\vee \to \mathbb{P}_k^n \times {\mathbb{P}_k^n}^\vee$. Clearly, I_X is preimage of the universal hyperplane $I = \{(\alpha, [H]) \in \mathbb{P}_k^n \times {\mathbb{P}_k^n}^\vee | \alpha \in H\}$.

For all $x \in X$, preimage of x under projection $I_X \xrightarrow{p_1} X$ is isomorphic to $\mathbb{P}_k^{n-1}$. Hence by Proposition 4.6, I_X is irreducible and $\dim I_X = \dim X + n - 1$. For all $P = (x, [H]) \in I_X$, want to show that $\dim_k T_P I_X = \dim X + n - 1$.

Denote coordinate of $\mathbb{P}_k^n$ by $y_1, \cdots, y_n$ and coordinate of ${\mathbb{P}_k^n}^\vee$ by $z_1, \cdots, z_n$. By coordinate transformation, We may assume that $\varphi(x) = [1, 0, \cdots, 0]$ and $[H] = [0, \cdots, 0, 1]$. By Proposition 2.1, there exists affine open neighbourhood V of x such that $\varphi\big|_V$ is defined by polynomial $f_1, \cdots, f_n \in A(U)$.

Thus $I_X \cap (V \times D_+(z_n))$ is defined by $z_{0/n} + z_{1/n} f_1 + \cdots + z_{n-1/n} f_{n-1} + f_n$ in $V \times D_+(z_n)$, whose partial derivation with respect to $z_{0/n}$ is constantly 1. Hence $T_P I_X \subseteq T_P(V \times D_+(z_n))$ and $\dim_k T_P I_X \le \dim_k T_P(V \times D_+(z_n)) - 1 = \dim X + n - 1 = \dim I_X$. We get that I_X is smooth.

Now apply generic smoothness to $I_X \xrightarrow{p_2} {\mathbb{P}_k^n}^\vee$, there exists open dense subset $U \subseteq {\mathbb{P}_k^n}^\vee$ such that $I_X \cap p_2^{-1}(U) \xrightarrow{p_2} U$ is a smooth morphism. And for all $[H] \in U$, by Proposition 5.5, the fiber $p_2^{-1}([H]) = \varphi^{-1}(H) \times \{[H]\} \cong \varphi^{-1}(H)$ is smooth since $[H]$ is smooth as a one-point set. $\square$

Theorem 5.5 (***Bertini's Theorem***)**.** *Let* $X \hookrightarrow \mathbb{P}_k^n$ *be an irreducible smooth subvariety. Then for a general hyperplane* H *of* $\mathbb{P}_k^n$, H *does not contain* X *and* $X \cap H$ *is smooth.*

Proof. Consider $S_X := \{(x, [H]) \in X \times {\mathbb{P}_k^n}^\vee | X \subseteq H \text{ or } x \in \operatorname{Sing}(X \cap H)\}$. Suffices to show that S is closed and of dimension $< n$. Denote coordinate of $\mathbb{P}_k^n$ by $y_1, \cdots, y_n$ and coordinate of ${\mathbb{P}_k^n}^\vee$ by $z_1, \cdots, z_n$.

We may assume that X is irreducible. Consider $S_X \xrightarrow{p_1} X$. For $x \in X$ and $(x, [H]) \in p_1^{-1}(x)$, by coordinate transformation assume that $x = [1, 0, \cdots, 0]$. Then we can replace X by $X \cap D_+(y_0)$ and $[H] \in V(z_0)$.

When $X \subseteq H$, the defining equation F of H should be contained in $I(X)$. When H does not contain whole X, $\dim X \cap H = \dim X - 1 = \dim_k T_x X - 1$. Since $X \cap H$ is singular at x,

$\dim_k T_x(X \cap H) > \dim X \cap H$. Note that the defining equations of $T_x(X \cap H)$ are

$$S \cup \{\sum \frac{\partial F}{\partial y_{j/0}}(x) y_{j/0}\} \tag{41}$$

where S are the defining equations of $T_x X$. Hence $\dim_k T_x X \geq \dim_k T_x(X \cap H) > \dim_k T_x X - 1$. Thus $F = \sum \frac{\partial F}{\partial y_{j/0}}(x) y_{j/0}$ is always contained in the linear subspace spanned by S, denoted by $\langle S \rangle$. In fact, $\dim_k \langle S \rangle = n - \dim_k T_x X = n - \dim X$,

Now $\langle S \rangle$ is a subvariety of $\mathbb{P}_k^{n\vee}$ of dimension $n - \dim X$. Assume $G_1, \cdots, G_{n-\dim X}$ generate $I(X)$. Hence $(x, [H]) \in p_1^{-1}(x)$ if and only if $F \in V(z_0)$ and the rank of the following matrix is equal to the rank of the submatrix of the first $n - \dim X$ rows

$$\begin{pmatrix} \frac{\partial G_1}{\partial y_{1/0}}(x) & \cdots & \frac{\partial G_1}{\partial y_{n/0}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{n-\dim X}}{\partial y_{1/0}}(x) & \cdots & \frac{\partial G_{n-\dim X}}{\partial y_{n/0}}(x) \\ \frac{\partial F}{\partial y_{1/0}}(x) & \cdots & \frac{\partial F}{\partial y_{n/0}}(x) \end{pmatrix} \tag{42}$$

which is a closed condition, so that S_X is closed. Thus $p_1^{-1}(x)$ is a projective subspace of $\mathbb{P}_k^{n\vee}$ of codimension $\dim X + 1$. Hence by Proposition 4.6, S_X is irreducible and $\dim S_X = n - 1$. Conclude that $\overline{p_2(S_X)} \neq \mathbb{P}_k^{n\vee}$ and for all $[H] \in \mathbb{P}_k^{n\vee} \setminus \overline{p_2(S_X)}$, $X \cap H$ is smooth. □

Theorem 5.6 (***Bertini's Theorem***)**.** *Let X be an irreducible variety, $\varphi : X \to \mathbb{P}_k^n$ a morphism and $\dim \overline{\varphi(X)} \geq 2$. Then for a general hyperplane H of $\mathbb{P}_k^n$, $\varphi^{-1}(H)$ is irreducible.*

6 Some Interesting Varieties

6.1 Grassmannian

Given an n-dimensional k-linear space V, define Grassmannian variety to be $\mathrm{Gr}(m, V) := \{W \subseteq V \big| \dim W = m\}$, or denoted by $\mathrm{Gr}(m, n)$ when there is no confusion. There are two trivial cases

- when $m = 1$, $\mathrm{Gr}(1, n) \cong \mathbb{P}_k^{n-1}$
- when $m = n - 1$, $\mathrm{Gr}(n-1, n) \cong \mathbb{P}_k^{n-1\vee}$

Just as the trivial cases, $\mathrm{Gr}(m, n)$ and $\mathrm{Gr}(n-m, n)$ are dual spaces for all m. In the following, we are going to show that $\mathrm{Gr}(m, n)$ are really a variety.

Consider m-wedge, there is a natural map

$$\begin{aligned} \mathcal{P} : \mathrm{Gr}(m, n) &\longrightarrow \mathrm{Gr}(1, \binom{n}{m}) \\ W &\longmapsto \wedge^m W \end{aligned} \tag{43}$$

called Plucher embedding. Observe that for $w \in \wedge^m V$ and $e \neq 0$ in V, $e \wedge w = 0$ if and only if there exists $w_1 \in \wedge^{m-1} V$ such that $w = e \wedge w_1$. Then we see that $\mathcal{P}$ is injective. Indeed $W \subseteq V$ is spanned by all the vectors e such that $e \wedge \mathcal{P}(W) = 0$.

We say that $w \in \wedge^m W$ is totally decomposable if $w \in \operatorname{im} \mathcal{P}$, namely there exist $e_1, \cdots, e_m$ linearly independent such that $w = e_1 \wedge \cdots \wedge e_m$.

Lemma 6.1. *Then w is totally decomposable if and only if the linear map $\varphi(w) : V \to \wedge^{m+1} V$ sending e to $e \wedge w$ has rank $\leq n - m$ if and only if $\dim_k(\ker \varphi(w)) \geq m$.*

Proof. "$\Rightarrow$": Assume w is totally decomposable and $w = e_1 \wedge \cdots \wedge e_m$, where $e_1, \cdots, e_m$ are linearly independent. Then $\langle e_1, \cdots, e_m \rangle \subseteq \ker \varphi(w)$ so that $\dim_k(\ker \varphi(w)) \geq m$.

"$\Leftarrow$": Assume that $\dim_k(\ker \varphi(w)) \geq m$. Take sequence $e_1, \cdots, e_m$ linearly independent in $\ker \varphi(w)$. Then $e_1 \wedge w = 0$ and there exists $w_1 \in \wedge^{m-1} V$ such that $w = e_1 \wedge w_1$. Hence $e_2 \wedge e_1 \wedge w_1 = e_2 \wedge w = 0$.

Note that e_1 does not appear in $e_2 \wedge w_1$, we get $e_2 \wedge w_1 = 0$. By induction, we get $w = e_1 \wedge \cdots \wedge e_n$ so that w is totally decomposable. $\square$

Remark 6.1. *Through the proof, it is clear that the inequality in the statement can be replaced by equality.*

Consider matrix representation $M(w)$ of $\varphi(w)$. Then $\mathcal{P}(\mathrm{Gr}(m, n))$ is defined by all $(n - m + 1)$-minors of $M(w)$ and hence closed. In fact, $\mathrm{Gr}(m, n)$ is intersection of the quadric surfaces cut out by Plucher relations.

Theorem 6.1. $\mathrm{Gr}(m, n)$ *is an irreducible and smooth variety of dimension* $(n - m)m$.

Proof. Consider the incident variety $I = \{([W], [H]) \in \mathrm{Gr}(m, n) \times \mathrm{Gr}(1, n) \big| H \subseteq W\}$ and projections $p_1 : I \to \mathrm{Gr}(m, n)$ and $p_2 : I \to \mathrm{Gr}(1, n) \cong \mathbb{P}_k^{n-1}$. Then $p_1^{-1}([W]) \cong \mathrm{Gr}(1, W) \cong \mathbb{P}_k^{m-1}$. And $p_2^{-1}([H]) \cong \mathrm{Gr}(m - 1, V/H)$.

Hence by induction, applying Proposition 4.6, we get that I is irreducible of dimension $\dim \mathrm{Gr}(m - 1, n - 1) + (n - 1) = (n - m)(m - 1) + (n - 1)$. Note that p_1 is surjective, $\mathrm{Gr}(m, n) = p_1(I)$ is also irreducible. Again by Proposition 4.6, $\dim \mathrm{Gr}(m, n) + (m - 1) = (n - m)(m - 1) + (n - 1)$. Thus $\dim \mathrm{Gr}(m, n) = (n - m)m$.

Consider group action $\mathrm{GL}_m(k) \circlearrowleft \mathrm{Gr}(m, n)$. Clearly, this is transitive. Fix $W_0 \in \mathrm{Gr}(m, n)$, define map $\mathrm{GL}_m(k) \to \mathrm{Gr}(m, n)$ sending M to MW_0, which is in fact a morphism between varieties.

Combining Proposition 5.3 and Corollary 5.1, there is an open dense subset $U \subseteq \mathrm{Gr}(m, n)$ consisting of smooth points. Since $\mathrm{GL}_m(k)$ acts transitively, conclude that $\mathrm{Gr}(m, n)$ is smooth everywhere. $\square$

Example 6.1. *Consider Plucher embedding $\mathcal{P} : \mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}_k^5$. Take basis $e_1, \cdots, e_4$ of V. Then $e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4$ is a basis of $\mathbb{P}_k^5$. And $w = \sum x_{ij} e_i \wedge e_j$ is totally decomposable if and only if $w \wedge w = 0$ if and only if $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$. In fact, for all n, $\mathcal{P} : \mathrm{Gr}(2, n) \hookrightarrow \mathbb{P}_k^{\binom{n}{2}-1}$ is always defined by $w \wedge w = 0$.*

Definition 6.1. *A variety X birational to projective space $\mathbb{P}_k^n$ for some n is called a rational variety. In particular, for $n = 1$, we say X is a rational curve.*

Example 6.2. *Assume* $\mathrm{Char}\, k = 0$ *Then all quadric hypersurface is rational. Let X be a smooth cubic hypersurface*

- *when* $n = 2$, X *is not rational*
- *when* $n = 3$, X *is rational*
- *when* $n = 4$, X *is not rational*
- *when* $n = 5$, *when* X *is very good,* X *is rational*

Theorem 6.2. $\mathrm{Gr}(m, n)$ *is a rational variety i.e. birational to a projective space.*

Proof. We can explicitly write down an affine open subset of $\mathrm{Gr}(m, n)$, which is isomorphic to $\mathbb{A}_k^{m(n-m)}$. Consider Plucher embedding $\mathcal{P} : \mathrm{Gr}(m, n) \hookrightarrow \mathrm{Gr}(1, n)$. Take basis $e_1, \cdots, e_n$ of V and $\Gamma := \langle e_{m+1}, \cdots, e_n \rangle$. Then $w_\Gamma = e_{m+1} \wedge \cdots \wedge e_n$ is the image of Γ under Plucher embedding.

Consider the natural map $\varphi : \wedge^m V \to \wedge^n V$ sending w to $w \wedge w_\Gamma$. Want to show that $\mathcal{P}(\mathrm{Gr}(m, n)) \setminus \ker \varphi$ is isomorphic to $\mathbb{A}_k^{m(n-m)}$. For all $w \in \mathcal{P}(\mathrm{Gr}(m, n))$, as w is totally decomposable, there exist linearly independent $f_1, \cdots, f_m$ such that $\mathcal{P}^{-1}(w)$ is spanned by $f_1, \cdots, f_m$. And there is a matrix $M(w) \in \mathrm{Mat}_{m\times n}(k)$ such that

$$\begin{pmatrix} f_1 \\ \cdots \\ f_m \end{pmatrix} = M(w) \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix} \tag{44}$$

Then $w \notin \ker \varphi$ if and only if the determinant of the first m columns does not vanish. By definition, we may assume that the first m columns is just I_m. Then $\mathcal{P}(\mathrm{Gr}(m, n)) \setminus \ker \varphi$ is parameterized by entries of the last $n - m$ columns. Conclude that $\mathcal{P}(\mathrm{Gr}(m, n)) \setminus \ker \varphi \cong \mathbb{A}_k^{m(n-m)}$ which gives a birational map $\mathrm{Gr}(m, n) \dashrightarrow \mathbb{P}_k^{m(n-m)}$. ☐

Example 6.3. *(1)Fix closed subvariety* $X \subseteq \mathbb{P}V$, *define* $F_m(X) = \{[W] \in \mathrm{Gr}(m, n) \big| \mathbb{P}W \subseteq X\}$, *called Fano varieties of dimension* m. *Then* $F_m(X)$ *is naturally a closed subvariety of* $\mathrm{Gr}(m, n)$.

(2)Let X, Y *be subvarieties of* $\mathbb{P}_k^n$. *Then we call* $\mathcal{J}(X, Y) = \overline{\cup_{p\in X, q\neq p\in Y} L_{p,q}}$ *the joint variety of* X *and* Y, *where* $L_{p,q}$ *is the line connecting* p *and* q. *In particular, when* $Y = X$, *we call* $\mathcal{J}(X, X)$ *the second variety of* X.

7 Divisors

7.1 Line bundles

Definition 7.1. *Let* X *be a variety. A line bundle* L *over* X *is a morphism* $P : L \to X$ *such that*

- $\exists$ *open covering* $\{U_i\}$ *of* X *with commutative diagrams*

$$\begin{tikzcd} P^{-1}(U_i) \arrow[r, "\psi_i", "\sim"'] \arrow[dr, "P"'] & U_i \times \mathbb{A}_k^1 \arrow[d, "projection"] \\ & U_i \end{tikzcd}$$

- *for all i, j, the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}_k^1 \to (U_i \cap U_j) \times \mathbb{A}_k^1$ sending (x, t) to $(x, g_{ij}(x)t)$ is linear i.e. g_{ij} is an invertible regular function on $U_i \cap U_j$.*

Usually, we call $\{g_{ij}\}$ transition functions.

Definition 7.2. *Let $P : L \to X$ and $P' : L' \to X$ be two lines bundles over X. We say P and P' (or L and L' when there is no confusion) are isomorphic if there is a commutative diagram*

$$\begin{tikzcd} L \arrow[r, "\phi", "\sim"'] \arrow[d, "P"'] & L' \arrow[dl, "P'"] \\ X & \end{tikzcd} \tag{45}$$

and $\psi_j' \circ \phi \circ \psi_i^{-1}$ is linear.

Definition 7.3. *Let $P : L \to X$ be a line bundle over X. A (resp. rational) section of L is a regular (resp. rational) map $s : X \to L$ such that $P \circ s = id_X$. Denote the set of all sections of L on X by $\Gamma(X, L)$.*

Remark 7.1. *For all line bundle L, there is always a zero section $0_L : X \to L$ glued up by local morphisms $x \mapsto (x, 0)$. Hence $\Gamma(X, L)$ is nonempty.*

Lemma 7.1. *$\Gamma(X, L)$ is a k-linear space.*

Lemma 7.2. *Let $\varphi : X \to Y$ be a morphism between varieties, $P : L \to Y$ line bundle over Y. Set $\varphi^* L = X \times_Y L$. Then the projection $\varphi^* L \to X$ is a line bundle over X. Moreover, $\Gamma(\varphi) : \Gamma(Y, L) \to \Gamma(X, \varphi^* L)$ is a k-linear map.*

Proof. Since $P : L \to Y$ is a line bundle over Y, there is an open covering $\{V_i\}$ of Y with commutative diagrams

$$\begin{tikzcd} P^{-1}(V_i) \arrow[r, "\psi_i", "\sim"'] \arrow[dr, "P"'] & V_i \times \mathbb{A}_k^1 \arrow[d, "projection"] \\ & V_i \end{tikzcd} \tag{46}$$

Take $U_i = \varphi^{-1}(V_i)$, then there are commutative diagrams

$$\begin{tikzcd} & U_i \times \mathbb{A}_k^{-1} \arrow[rr] \arrow[ddl] & & V_i \times \mathbb{A}_k^{-1} \arrow[ddl] \\ U_i \times P^{-1}(V_i) \arrow[ur, "\phi_i", "\sim"'] \arrow[rr] \arrow[d] & & P^{-1}(V_i) \arrow[ur, "\psi_i", "\sim"'] \arrow[d] & \\ U_i \arrow[rr, "\varphi"] & & V_i & \end{tikzcd} \tag{47}$$

such that all blocks are pull back. As $\psi_i \circ \psi_j^{-1}$ sending (y, t) to $(y, g_{ij}(y)t)$ is linear, $\phi_i \circ \phi_j^{-1}$ sending (x, t) to $(x, g_{ij}(\varphi(x))t)$ is also linear. Hence the projection $\varphi^* L \to X$ is a line bundle over X. $\square$

Example 7.1. *We call the projection $X \times \mathbb{A}_k^1 \to X$ the trivial line bundle. Then $P : L \to X$ is isomorphic to the trivial line bundle over X if and only if there exists section $s \in \Gamma(X, L)$ such that s does not meet the 0-section everywhere i.e. $s(x) \neq (x, 0)$ for all $x \in X$.*

Given such a line bundle, note that we have canonical section sending x to (x, a) locally for all $a \in k$, such a section does not meet 0-section. Conversely, given a section not meeting 0-section, we can construct an isomorphism $X \times \mathbb{A}_k^1 \to L$ sending (x, t) to $t \cdot s(x)$. In fact, $\Gamma(X, X \times \mathbb{A}_k^1) = A(X)$.

Example 7.2. *Let V be a vector space over k of dimension $n+1$, $\mathbb{P}V$ parameterizing lines in V through origin. Set $L = \{([l], v) \in \mathbb{P}V \times V \big| v \in l\}$. Denote coordinate of $\mathbb{P}V$ by $x_0, \cdots, x_n$. Then for projection $p : L \to \mathbb{P}V$, there is a canonical isomorphism $p^{-1}(D_+(x_i)) \tilde{\to} D_+(x_i) \times \mathbb{A}_k^1$ defined by $([l], v) \mapsto ([l], v_i)$, where v_i is the ith coordinate of v. In fact, for example, for all $[l] = [1, \alpha_1, \cdots, \alpha_n] \in D_+(x_0)$ and $([l], v) \in p^{-1}([l])$, v is of the form $(v_0, \alpha_1 v_0, \cdots, \alpha_n v_0)$. This line bundle L with transition functions $g_{ij} = \frac{x_i}{x_j}$ is called the tautological line bundle over $\mathbb{P}_k^n$, usually denoted by $\mathcal{O}_{\mathbb{P}_k^n}(-1)$.*

To introduce more interesting properties about line bundles, here we give an equivalent definition of line bundle first.

Definition 7.4. *A line bundle is an equivalence class of a collection of data*

- *an open covering $\{U_i\}$ of X*
- *transition functions $g_{ij} : U_i \cap U_j \to k^*$ satisfying the cocycle condition*

under the equivalent relation that $(\{U_i\}, g_{ij}) \sim (\{V_{i'}\}, g'_{i'j'})$ if there exist regular functions $s_{ii'} : U_i \cap V_{i'} \to k^*$ such that $g'_{i'j'} s_{ii'} = g_{ij} s_{jj'}$ for all i, j, i', j'.

Remark 7.2. *More precisely, such an equivalence class would one-to-one correspond to an isomorphism class of our original definition instead of just a morphism.*

Given a line bundle $L = (\{U_i\}, g_{ij})$, then $L^{-1} := (\{U_i\}, g_{ij}^{-1})$ is a well defined line bundle. Given two line bundles L_1 and L_2, by taking refinement of open covering, we may write $L_1 = (\{U_i\}, g_{ij})$ and $L_2 = (\{U_i\}, h_{ij})$. Then $L_1 \otimes L_2 := (\{U_i\}, g_{ij} h_{ij})$ is a well defined line bundle.

Denote the set of isomorphism classes of line bundles over X by $\mathrm{Pic}(X)$. With discussion above, we successfully give a group structure on $\mathrm{Pic}(X)$, called the Picard group of X.

Definition 7.5. *Let $L = (\{U_i\}, g_{ij})$ be a line bundle over X. A (resp. rational) section of L on X is a set of regular (resp. rational) functions $\{s_i : U_i \to \mathbb{A}_k^1\}$ such that $s_i = g_{ij} s_j$ on $U_i \cap U_j$.*

Example 7.3. *For projective space $\mathbb{P}_k^n$, denote $\mathcal{O}_{\mathbb{P}_k^n}(1) = \mathcal{O}_{\mathbb{P}_k^n}(-1)^{-1}$ and $\mathcal{O}_{\mathbb{P}_k^n}(d) = \mathcal{O}_{\mathbb{P}_k^n}(1)^{\oplus d}$, then $\mathcal{O}_{\mathbb{P}_k^n}(1) = (\{D_+(x_i)\}, \frac{x_j}{x_i})$. Consider section $\{s_i\}$ given by $\frac{x_0}{x_i}$, corresponding to x_0. We get that $S_1 = \oplus_i k x_i \subseteq \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$.*

Proposition 7.1. *Let $S = k[x_0, \cdots, x_n]$. In fact, for all $d \geq 0$, $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d)) = S_d$. Moreover, for all negative d, $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d)) = \{0\}$.*

Proof. Obviously, for all homogeneous polynomial f of degree d, as $\frac{f}{x_i^d} = \frac{x_j^d}{x_i^d} \cdot \frac{f}{x_j^d}$, we get $\{\frac{f}{x_i^d}\}$ is a section of $\mathcal{O}_{\mathbb{P}_k^n}(d)$. Hence $S_d \subseteq \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$. On the other hand, given a section $\{s_i\}$.

Write $s_i = f_i(x_{0/i}, \cdots, x_{n/i})$, then $f_i = \frac{x_j^d}{x_i^d} f_j$. Arguing about degree, we get that $f_i = \frac{g}{x_i^d}$ for some homogeneous polynomial $g \in k[x_0, \cdots, x_n]$ so that $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d)) = S_d$. In particular, for d negative, same argument tells us that the only possible g is zero. $\square$

Remark 7.3. *There is another way to explain why there is only zero section when d is negative. Say s is a section of $\mathcal{O}_{\mathbb{P}_k^n}(-r)$. Note that $\mathcal{O}_{\mathbb{P}_k^n}(r) \otimes \mathcal{O}_{\mathbb{P}_k^n}(-r)$ is just the isomorphism class of trivial bundle, $x_t s$ is section of the trivial bundle for all t. While sections of trivial bundle are just $A(\mathbb{P}_k^n) = S_0$, as $x_t s$ would vanish on $V(x_t)$, $x_t s$ is the zero section for all t. Conclude that s is zero section.*

Proposition 7.2. *Let $L = (\{U_r\}, g_{rl})$ be a line bundle over irreducible variety X. Assume $s_0, \cdots, s_n \in \Gamma(X, L)$ are not all zero sections, where $s_i = \{s_{ir}\}$. Define a rational map $\phi_L : X \dashrightarrow \mathbb{P}_k^n$ by gluing up $U_r \dashrightarrow \mathbb{P}_k^n$ sending x to $[s_{0r}(x), \cdots, s_{nr}(x)]$. If for all $x \in X$, $s_{ir}(x) \neq 0$ for some i, r, then ϕ_L is a well defined morphism. And in this case, $L = \phi_L^* \mathcal{O}_{\mathbb{P}_k^1}(1)$.*

Proof. Clearly, ϕ_L is a morphism, want to show that $(\{\phi_L^{-1}(D_+(x_i))\}, \frac{x_j}{x_i} \circ \phi_L) \sim (\{U_r\}, g_{rl})$. In fact, for all $x \in U_r \cap \phi_L^{-1}(D_+(x_i))$, by definition, $s_{ir}(x) \neq 0$. Hence s_{ir} is an invertible regular function on $U_r \cap \phi_L^{-1}(D_+(x_i))$. And $g_{rl} s_{jl} = s_{jr} = \frac{s_{jr}}{s_{ir}} \cdot s_{ir} = (\frac{x_j}{x_i} \circ \phi_L) \cdot s_{ir}$, done! $\square$

Remark 7.4. *For reducible X, this construction of rational map also makes sense if $U_r \cap (\cup_i D(s_{ir}))$ is dense in U_r for all r.*

Example 7.4. *Recall that for projective spaces, we have Veronese map and Segre embedding. Consider the dth Veronese map $v_d : \mathbb{P}_k^n \to \mathbb{P}_k^{\binom{n+d-1}{d}}$. In fact, v_d is given by sections in $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ that are monomial of degree d with coefficient 1.*

For Segre embedding $\mathbb{P}_k^n \times \mathbb{P}_k^m \hookrightarrow \mathbb{P}_k^{nm+n+m}$, similarly, it is given by sections in $\Gamma(\mathbb{P}_k^n \times \mathbb{P}_k^m, \mathcal{O}_{\mathbb{P}_k^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}_k^m}(1))$ that are $x_i y_j$, where $\mathcal{O}_{\mathbb{P}_k^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}_k^m}(1) := p^* \mathcal{O}_{\mathbb{P}_k^n}(1) \otimes q^* \mathcal{O}_{\mathbb{P}_k^m}(1)$ with p, q projections.

7.2 Weil divisors

Definition 7.6. *Let X be an irreducible variety. A Weil divisor D on X is a finite sum $\sum_i a_i Y_i$, where $a_i \in \mathbb{Z}$ and $Y_i \subseteq X_i$ are irreducible subvarieties of codimension 1. We say D is effective if $a_i \geq 0$ for all i.*

Assume $\operatorname{codim} \operatorname{Sing}(X) \geq 2$. Then for all regular function $f : X \to \mathbb{A}_k^1$, we can define vanishing order $v_Y(f)$ of f along an irreducible subvariety $Y \subseteq X$ of codimension 1.

Since $\operatorname{codim} \operatorname{Sing}(X) \geq 2$, a general point $y \in Y$ is a smooth point of X. Then by Proposition 5.3, there exists an affine open neighbourhood U of y such that $Y \cap U$ is defined by a single equation g. As $\mathcal{O}_{X,y}$ is a UFD, we can write $f = hg^n$ in $\mathcal{O}_{X,y}$, where $g \nmid h$. And $v_Y(f) := n$ is our desired vanishing order.

Lemma 7.3. *The definition of $v_Y(f)$ is independent of choice of y.*

Proof. For two smooth points y_1 and y_2, assume $f = h_i g_i^{n_i}$ in $\mathcal{O}_{X,y_i}$. We may assume that $n_1 \leq n_2$. As X is irreducible, we can take nonempty affine subset $U \subseteq X$ such that $f = h_i g_i^{n_i}$ in $A(U)$. By definition above, we get that $I(U \cap Y) = (g_i)$. Hence $g_1 = ag_2$ and $g_2 = bg_1$ for some $a, b \in A(U)$. As $A(U)$ is an integral domain, we get that $ab = 1$. Then $h_1 g_1^{n_1} = h_1 a^{n_1} g^{n_1}$ and $h_1 = h_2 b^{n_1} g_2^{n_2 - n_1}$. Note that $g_1 \nmid h_1$, we get that $g_2 \nmid h_1$ so that $n_1 = n_2$. □

Remark 7.5. *Similarly, for rational function $f \in K(X)$, we can also define vanishing order $v_Y(f)$.*

Lemma 7.4. *Let X be an irreducible variety. Then for all rational function $f \in K(X)^*$, $v_Y(f) = 0$ except for finitely many irreducible subvarieties Y of codimension 1.*

Proof. Assume $v_Y(f) \neq 0$. For $y \in Y$ smooth, there exists an affine open neighbourhood U such that $f = \frac{f_1}{f_2}$ with $f_1, f_2 \in A(U)$. Then $v_Y(f_1) \neq v_Y(f_2)$. Suffices to show the result holds for regular function.

Now f is a regular function. As $v_Y(f) \neq 0$, by definition, this is equivalent to $g \mid f$ so that $Y \cap U = V(g) \subseteq V(f)$ in U. Take closure in X, then we get $Y \subseteq V(f)$. Note that $V(f)$ has finitely many irreducible components, we are done. □

Under the assumption $\operatorname{codim} \operatorname{Sing}(X) \geq 2$, with lemmas above, for all rational function f, we can define a Weil divisor $\operatorname{div}(f) = \sum_Y v_Y(f) Y$. All these Weil divisors form an abelian subgroup of $Weil(X)$, called the group of principal Weil divisors on X.

Example 7.5. *(1)Let $S = V(xy - z^2) \subseteq \mathbb{A}_k^3$. Then $\operatorname{Sing}(S) = \{(0,0,0)\}$ is of codimension 2. As $V(x) = \{(0,a,0) \big| a \in k\} \cong \mathbb{A}_k^1$ is irreducible, $\operatorname{div}(x) = v_D(x) D$. Since D is defined by z in $D(y)$, we get $x = \frac{z^2}{y}$ and $v_D(x) = 2$.*
(2)Let $\mathbb{P}_k^n$ be projective space. Then $\operatorname{div}(\frac{x_i}{x_j}) = V(x_i) - V(x_j)$.

Corollary 7.1. *Let X be an irreducible variety. Then for all rational function $f \in K(X)^*$ such that $\operatorname{div}(f) = nV$ with $n > 0$, we have that f is a regular function and $V(f) = V$. In particular, when $n = 1$, locally the ideal of $A(X)$ corresponding V is generated by f.*

Reason 7.1. *Similar to proof of Lemma 7.4, we know f is a nonzero regular function and V is the only irreducible subvariety of X of codimension 1, which is contained in $V(f)$. By Krull's Principal Theorem, we know that $V(f)$ is of pure codimension 1 so that $V(f) = V$.*

When $n = 1$, take any affine open subset U meeting V, if $g \in I(V \cap U) \subseteq A(U)$, then $V \cap U \subseteq \operatorname{div}(g)$ so that $\operatorname{div}(g) - V \cap U$ is an effective divisor on $V \cap U$. Note that $\operatorname{div}(f)\big|_U$ is just $V \cap U$, we get $\operatorname{div}(\frac{g}{f}) = \operatorname{div}(g) - V \cap U$ is effective. Hence $\frac{g}{f}$ is regular. Conclude that $I(V \cap U) = (f)$.

7.3 Cartier divisors

Definition 7.7. *Let X be an irreducible variety. A Cartier divisor on X is an equivalence class of a collection of data*

- *an open covering $\{U_i\}$ of X.*

- *rational functions $f_i : U_i \dashrightarrow \mathbb{A}_k^1$ not vanishing on each irreducible component of U_i such that $\frac{f_i}{f_j}$ is a regular function on $U_i \cap U_j$ nowhere vanishing*

under the equivalent relation that $(\{U_i\}, f_i) \sim (\{V_j\}, g_j)$ if $\frac{f_i}{g_j}$ is a regular function on $U_i \cap V_j$ nowhere vanishing.

Just as we did for line bundles, it is natural to give a group structure on the set of Cartier divisors, denoted by $Catier(X)$. We say a Cartier divisor is effective if it can be represented by $(\{U_i\}, f_i)$ with f_i all regular functions. We say a Cartier divisor is principal if it can be represented by (X, f) with $f \in K(X)$. And there is also an abelian group consisting of principal divisors.

Under the assumption $\operatorname{codim} \operatorname{Sing}(X) \geq 2$, there is a natural group homomorphism $\pi : Catier(X) \to Weil(X)$ sending $(\{U_i\}, f_i)$ to $\sum v_Y(D)Y$, where $v_Y(D) := v_{U_i \cap Y}(f_i)$ for Y meeting U_i. Clearly, this is well defined. In particular, $\pi((X, f)) = \operatorname{div}(f)$.

Remark 7.6. *In general, π is neither injective nor surjective.*

- *when X is smooth, π is isomorphic.*
- *when X is normal, π is injective.*

Theorem 7.1. *Let X be an irreducible variety. Then $Catier(X)/Pr(X) \xrightarrow{\sim} \operatorname{Pic}(X)$.*

Proof. Let $D = (\{U_i\}, f_i)$ be a Cartier divisor. Set $g_{ij} = \frac{f_i}{f_j}$. Note that $g_{ij}g_{jk} = \frac{f_i}{f_j} \cdot \frac{f_j}{f_k} = \frac{f_i}{f_k}$, $\mathcal{L}(D) = (\{U_i\}, g_{ij})$ is a line bundle, called the associated line bundle of D. And we can define map $\mathcal{L}(\cdot) : Cartier(X) \to \operatorname{Pic}(X)$ by sending D to $\mathcal{L}(D)$.

Given a line bundle $L = (\{U_i\}, g_{ij})$, fix $i_0 \in I$, then for all $i \in I$, $g_{ii_0} : U_i \cap U_{i_0} \to \mathbb{A}_k^1$ is a regular map. As X is irreducible, $U_i \cap U_{i_0}$ is dense in U_i and hence g_{ii_0} is also a rational function on U_i. Since $\frac{g_{ii_0}}{g_{ji_0}} = g_{ij}$ is a regular function on $U_i \cap U_j$ vanishing nowhere, $D = (\{U_i\}, g_{ii_0})$ is a Cartier divisor and $\mathcal{L}(D) = L$. Thus $\mathcal{L}(\cdot)$ is surjective.

Finally, given $D = (\{U_i\}, f_i)$ with $\mathcal{L}(D)$ trivial. There is a section $s = \{s_i\}$ of $\mathcal{L}(D)$ not meeting the zero section. As s_i is a regular function on U_i vanishing nowhere and $\frac{f_i}{s_i} = \frac{f_j}{s_j}$ for all i, j, $D = (\{U_i\}, f_i) = (\{U_i\}, \frac{f_i}{s_i}) = (X, F)$ for some $F \in K(X)$, which is principal.

Conversely, for any principal divisor $D = (X, f)$, $\mathcal{L}(D) = (X, 1)$ is just the trivial line bundle. Conclude that $\ker \varphi = Pr(X)$ and $Catier(X)/Pr(X) \xrightarrow{\sim} \operatorname{Pic}(X)$. $\square$

Remark 7.7. *When X is smooth in codimension 1 i.e.* $\operatorname{codim} \operatorname{Sing}(X) \geq 2$*, denote $Cl(X) = Weil(X)/Pr(X)$, called the class group of X.*

Lemma 7.5. *Let X be an irreducible variety, $D = (\{U_i\}, f_i)$ a Cartier divisor, $\mathcal{L}(D)$ the associated line bundle of X. Then $\Gamma(X, \mathcal{L}(D)) \xrightarrow{\sim} \{f \in K(X) \big| f = 0$ or $\operatorname{div}(f) + D$ effective$\}$.*

Proof. For all nonzero section $s = \{s_i\}$ of $\mathcal{L}(D)$, as $\frac{s_i}{f_i} = \frac{s_j}{f_j}$ is a rational function on X, denoted by F, $\operatorname{div}(F) + D = (\{U_i\}, s_i)$ is an effective Cartier divisor. Then we can define a map $\varphi : \Gamma(X, \mathcal{L}(X)) \to \{f \in K(X) \big| f = 0 \text{ or } \operatorname{div}(f) + D \text{ effective}\}$ by sending s to F. Since s nonzero, $F = \varphi(s)$ also nonzero so that φ is injective.

Assume $D' := \mathrm{div}(F) + D$ is effective, say $D' = (\{U_i\}, s_i)$. Then $s_i = Ff_i$ is a regular function on U_i and $s_i = Ff_i = Ff_jg_{ij} = Fs_j$. Hence $s = \{s_j\}$ is a nonzero section of $\mathcal{L}(D)$ and $\varphi(s) = F$. Conclude that φ is bijective. □

Example 7.6. *(1)Let $X = \mathbb{A}_k^n$ be n-dimensional affine space. As all irreducible hypersurfaces of X are of the form $V(F)$ for some irreducible polynomial F, we get all divisors on X are principal. Hence $\mathrm{Pic}(X) = Cl(X) = 0$. Similarly, for smooth variety X with $A(X)$ UFD, $\mathrm{Pic}(X) = Cl(X) = 0$.*
(2)Let $X = \mathbb{P}_k^n$ be n-dimensional projective space. As irreducible hypersurface H of X is of the form $V(F)$ for some irreducible homogeneous polynomial F of degree d, $\frac{F}{x_0^d}$ is a rational function on X and $\mathrm{div}(\frac{F}{x_0^d}) = H - dH_0$ where $H_0 = V(x_0)$. Hence $\mathrm{Pic}(X) = Cl(X)$ can be generated by H_0. As $\mathcal{O}_{\mathbb{P}_k^n}(m)$ are nontrivial for all $m \neq 0$, conclude that $\mathrm{Pic}(X) = Cl(X) \cong \mathbb{Z}$.
(3)Let $H \subseteq \mathbb{P}_k^n$ be a hypersurface of degree d, defined by F. Set $U = \mathbb{P}_k^n \setminus H = D_+(F)$ which is affine. Note that any irreducible subvariety of U of codimension 1 is the restriction of some irreducible hypersurface of $\mathbb{P}_k^n$, there is a natural surjection $\pi : \mathrm{Pic}(\mathbb{P}_k^n) \twoheadrightarrow \mathrm{Pic}(U)$.

And for regular function $\frac{x_0^d}{F}$ on U, we get that $\mathrm{div}(\frac{x_0^d}{F}) = d(H_0 \cap U)$ is principal. Hence $\langle dH_0 \rangle \leq \ker \pi$. In fact, the equality holds and $\mathrm{Pic}(U) \cong \mathbb{Z}/d\mathbb{Z}$. Assume that $\mathcal{O}_{\mathbb{P}_k^n} \in \ker \pi$, then $d'(H_0 \cap U)$ is an effective principal Cartier divisor. Since $A(U) = \{\frac{f}{F^m} \big| m \in \mathbb{N}$ and $f \in k[x_0, \cdots, x_n]$ homogeneous of degreemd$\}$, we can write $d'(H_0 \cap U) = \mathrm{div}(\frac{f}{F^m})$ so that H_0 is the only irreducible component of $V(f)$ and hence $f = x_0^{md}$. Conclude that $\ker \pi = \langle dH_0 \rangle$.
(4)As irreducible hypersurface H of $\mathbb{P}_k^n \times \mathbb{P}_k^m$ are defined by some bihomogenous polynomial F of bihomogenous (s, t), we get $\frac{F}{x_0^s y_0^t}$ is a rational function on $\mathbb{P}_k^n \times \mathbb{P}_k^m$ and $\mathrm{div}(\frac{F}{x_0^s y_0^t}) = H - sV_+(x_0) - tV_+(y_0)$ is principal. Conclude that $\mathrm{Pic}(\mathbb{P}_k^n \times \mathbb{P}_k^m) \cong \mathbb{Z} \times \mathbb{Z}$.

Remark 7.8. *In general, for smooth projective varieties X and Y, we do not have that $\mathrm{Pic}(X \times Y) = \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$. And the following proposition would give us an example.*

Lemma 7.6. *Let X be a smooth projective irreducible curve. Then there exists affine open covering $\{U_i\}$ of X with local coordinate t_i.*

Proof. Firstly we cover X with $D_+(x_i)$. Assume that X is defined by irreducible homogeneous polynomial f in $\mathbb{P}_k^2$. Then $X \cap D_+(x_i) = V(f(x_{0/i}, x_{1/i}, x_{2/i}))$. We only need argue for the case that $i = 0$.

Now $X \cap D_+(x_i)$ can be covered by $D(\frac{\partial f}{\partial x_{1/0}})$ and $D(\frac{\partial f}{\partial x_{2/0}})$. Since $x_{2/0} \cdot \frac{\partial f}{\partial x_{2/0}} - \deg(f) \cdot f \in (x_{1/0})$, we get that $x_{1/0}$ is just a local coordinate of $X \cap D(\frac{\partial f}{\partial x_{2/0}})$, done. □

Proposition 7.3. *Let X be a smooth projective irreducible curve not isomorphic to projective line $\mathbb{P}_k^1$. Then $\Delta_X \subseteq X \times X$ is an irreducible subvariety of codimension 1 and the associated line bundle $\mathcal{L}(\Delta_X)$ does not come from $\mathrm{Pic}(X) \times \mathrm{Pic}(X)$.*

Proof. Since $X \subseteq \mathbb{P}_k^2$ is a smooth projective curve not isomorphic to projective line, we may assume that $X = V(f)$ for some irreducible homogeneous polynomial f of degree greater than 1. Denote coordinate of $\mathbb{P}_k^2 \times \mathbb{P}_k^2$ by $x_0, x_1, x_2, y_0, y_1, y_2$. Then $X \times X$ is defined by $\{f(x_0, x_1, x_2), f(y_0, y_1, y_2)\}$.

Note that Δ_X is isomorphic to X and is defined by $\{f(x_0,x_1,x_2), f(y_0,y_1,y_2), x_0y_1 - x_1y_0, x_0y_2 - x_2y_0, x_1y_2 - x_2y_1\}$, we get that δ_X is an irreducible subvariety of $X \times X$ of codimension 1. To better interpret $\mathcal{L}(\Delta_X)$ by transition functions, consider $\Delta_X \hookrightarrow \cup_i(U_i \times U_i) \hookrightarrow X \times X$, where U_i cover X with local coordinate t_i given by previous lemma. It suffices to show that $\mathcal{L}(\Delta_X)$ on $\cup_i(U_i \times U_i)$ does not come from $\mathrm{Pic}(X) \times \mathrm{Pic}(X)$.

Suppose that $\mathcal{L}(\Delta_X) = L_1 \boxtimes L_2$. Then transition functions of $\mathcal{L}(\Delta_X)$ should be of the form $\alpha_{ij}(x)\beta_{ij}(y)$. while $\mathcal{L}(\Delta_X) = (\{U_i \times U_i'\}, \frac{t_i - t_i'}{t_j - t_j'})$, contradiciton! Hence $\mathcal{L}(\Delta_X)$ on $X \times X$ does not come from $\mathrm{Pic}(X) \times \mathrm{Pic}(X)$. $\square$

Let X be a smooth projective irreducible curve. Then any divisor D on X is a finite sum $\sum_i n_i p_i$ with p_i points in X. Set $\deg D = \sum_i n_i$.

Proposition 7.4. *If D is principal, then $\deg D = 0$.*

Reason 7.2. *For $D = \mathrm{div}(f)$ principal, f would give a rational map $X \overset{f}{\dashrightarrow} \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$, denoted by φ. By Corollary 5.5, φ is in fact a morphism. Consider rational map $\frac{x_0}{x_1}$ on $\mathbb{P}^1_k$. Then $\varphi^*(\frac{x_0}{x_1}) = \frac{x_0}{x_1} \circ \varphi = f$ and hence $\mathrm{div}(f) = \mathrm{div}(\varphi^*(\frac{x_0}{x_1})) = \varphi^*[1,0] - \varphi^*[0,1]$. Thus $\deg D = 0$.*

Corollary 7.2. *There is a well defined map $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ sending $\mathcal{O}_X(D)$ to $\deg D$.*

We call kernel of the deg map the Jacobian variety of X, denoted by $J(X)$.

Proposition 7.5. *$\deg$ is injective if and only if $X \cong \mathbb{P}^1_k$.*

Proof. The “ $\Leftarrow$” arrow have been discussed in Example 7.6. We only need prove for the other arrow. For all $x, y \in X$, the divisor $D = [x] - [y]$ has degree 0 and by assumption, $\mathcal{O}_X(D)$ is trivial. Hence by Theorem 7.1, $D = \mathrm{div}(f)$ for some rational function $f \in K(X)$.

Similar to proof of Proposition 7.4, f gives a rational map $X \overset{f}{\dashrightarrow} \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$, denoted by φ, which by Corollary 5.5 is a morphism. In addition, as $\mathrm{div}(f) = \mathrm{div}(\varphi^*(\frac{x_0}{x_1}))$, we have that $x = \varphi^*[1,0]$ and $y = \varphi^*[0,1]$. Conclude that X is of degree 1 and hence isomorphic to $\mathbb{P}^1_k$. $\square$

Example 7.7. *Let $X \subseteq \mathbb{P}^2_k$ be the subvariety defined by $x_1^2x_2 - x_0^3 + x_0x_2^2$, $p_0 = [0,1,0] \in X$. Set $H = V(x_2)$. Then $H \cap X$ is exactly p_0. As $\varphi^*H = 3p_0$, for any irreducible hypersurface H' of $\mathbb{P}^2_k$, we have $\varphi^*H' \sim 3p_0$ and hence φ^*H' is an effective divisor of degree 3. In particular, if P, Q, R are 3 distinct points contained in a line L, then $\varphi^*L = P + Q + R \sim 3p_0$.*

Claim that there is a bijection $X \to J(X)$ sending x to $x - p_0$. For surjectivity, assume $D = \sum_i n_i p_i \in J(X)$. As $\deg D = 0$, we can rewrite $D = \sum_i n_i(p_i - p_0)$. For all distinct p and q, taking the line L connecting p and q, we have that $\varphi^*L = p + q + s \sim 3p_0$ for some $s \in X$.

Then for all i with n_i negative, there exists some $q_i \in X$ such that $p_i + p_0 + q_i \sim 3p_0$ so that $p_i - p_0 \sim p_0 - q_i$. Thus we may assume that $n_i > 0$ for all i. While for different i and j, there is also $q \in X$ such that $p_i + p_j + q \sim 3p_0$. Repeatedly running this process, we get $D \sim n(p_1 - p_0)$ with $n > 0$.

When $n \neq 1$, since X is smooth at p_1, the Zariski tangent space $T_{p_1}X$ is one-dimensional. Hence we get $p_2 \in X$ such that $\varphi^*T_{p_1}X = 2p_1 + p_2 \sim 3p_0$ so that $n(p_1 - p_0) \sim (n-2)(p_1 -$

$p_0) + (p_0 - p_2)$. Thus by this process, we can finally decrease n to 1 so that $X \to J(X)$ is surjective.

For injectivity, if $p_1 - p_0 \sim p_2 - p_0$, then $p_1 - p_2$ is principal. Similar to proof of Proposition 7.5, we would see that $\deg X = 1$ contradicting to our assumption that $\deg X = 3$. Conclude that the map is bijective.