Algebraic Geometry

Contents

1 Schemes

In the whole note, all rings mentioned are commutative rings with identity. Also, some trivial proofs are omitted.

1.1 Spectrum

Definition 1.1. *Let A be a ring. The spectrum of A is the set of all prime ideals of A, denoted by* $\operatorname{Spec} A$.

Definition 1.2. *Let A be a ring, $I \subseteq A$ ideal. Define set $V(I) = \{\mathfrak{p} \in \operatorname{Spec} A \big| I \subseteq \mathfrak{p}\}$ and $D(I) = \operatorname{Spec} A \setminus V(I)$. In particular, for each $f \in A$, define $V(f) = V((f))$ and $D(f) = D((f))$.*

Remark 1.1. *Obviously, if $i \subseteq I$, then $V(I) \supseteq V(J)$.*

Proposition 1.1 (***Zariski Topology***)**.** *Let A be a ring. Then the above definitions satisfy that*

- $\{V(I) \big| I \subseteq A \text{ ideal}\}$ *form the closed subsets of a topology on* $\operatorname{Spec} A$*, called the Zariski topology.*
- $\{D(f) \big| f \in A\}$ *form a topological basis, called standard open subsets.*

Remark 1.2. *Should think $V(f)$ means $f = 0$ while $D(f)$ means $f \neq 0$.*

Definition 1.3. *Let A be a ring, $I \subseteq A$ ideal. Define the radical of I to be the set $\{x \in A \big| \exists n \in \mathbb{N}_+, x^n \in I\}$, denoted by $rad(I)$.*

Lemma 1.1. *Let A be a ring, $I \subseteq A$ ideal. Then $V(I) = V(rad(I))$.*

Reason 1.1. *Immediately comes from result of commutative algebra that $rad(I) = \cap_{\mathfrak{p} \supseteq I} \mathfrak{p}$.*

Proposition 1.2. *Let A be a ring, $I, J \subseteq A$ ideals. Then $V(I) \subseteq V(J)$ if and only if $rad(I) \supseteq rad(J)$.*

Proposition 1.3. *Let A, B be rings, $\varphi : A \longrightarrow B$ ring homomorphism. Then induced map $\varphi^* : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A \quad \mathfrak{P} \longmapsto \varphi^{-1}(\mathfrak{P})$ is continuous. In particular, the preimage of standard open subset $D(f)$ is just $D(\varphi(f))$ and so is $V(f)$.*

Example 1.1. *(1)Let A be a ring, $I \subseteq A$ ideal. Assume $\varphi : A \longrightarrow A/I$ is the natural homomorphism. Then image of φ^* is $V(I) \subseteq \operatorname{Spec} A$. Get $\operatorname{Spec}(A/I) \cong V(I)$*
(2)Let A be a ring, $f \neq 0 \in A$. Assume A_f is the localization and $\varphi : A \longrightarrow A_f \quad a \longmapsto \frac{a}{1}$. Then image of φ^* is $D(f) \subseteq \operatorname{Spec} A$. Get $\operatorname{Spec}(A_f) \cong D(f)$.

Definition 1.4. *Let X be a topological space. We say that X is quasi-compact if any open covering admits a finite subcovering.*

Remark 1.3. *In the topology, we would call this property to be compact not quasi-compact. While, in the history, French mathematicians ask compact topological space also to be Hausdorff. To differ this definition, we say quasi-compact instead of compact.*

Proposition 1.4. *Let A be a ring. Then* $\operatorname{Spec} A$ *is quasi-compact as a topological space.*

Reason 1.2. *The finite subcovering comes from the fact that any element in an ideal can be expressed as a finite sum.*

Since $D(f) \cong \operatorname{Spec}(A_f)$ as topological spaces, we immediately get the following corollary.

Corollary 1.1. *Let A be a ring, $f \in A$. Then $D(f)$ is quasi-compact as a topological space.*

1.2 Sheaves

Definition 1.5. *Let X be a topological space and $\mathfrak{C}$ be a category. Assume $op(X)$ is the category of open subsets of X with morphisms are inclusions. A presheaf on X valued on $\mathcal{C}$ (or say of $\mathfrak{C}$) is a contravariant functor $\mathcal{F} : op(X) \longrightarrow \mathfrak{C}$.*

Remark 1.4. *Though we can consider this general definition of presheaf, we would only consider presheaves at least valued on Ab, where Ab is the category of abelian groups. In addition, we need these presheaves map $\varnothing$ to $\{0\}$.*

Definition 1.6. *Let X be a topological space, $\mathcal{F}$ presheaf of abelian groups on X. For $V \subseteq V$, we call the correspondence homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ a restriction map, denoted by $\cdot\rho_{UV}$. Elements in $\mathcal{F}(U)$ is called a section of $\mathcal{F}$ on U. In particular, if $U = X$, we call elements of $\mathcal{F}(X)$ global sections, otherwise local sections.*

Remark 1.5. *For convenience, for $s \in \mathcal{F}(U)$, we would write $s\big|_V$ to denote image of s instead of $\rho_{UV}(s)$. Also, sometimes $\mathcal{F}(U)$ can be rewritten as $\Gamma(U, \mathcal{F})$.*

Definition 1.7. *Let X be a topological space, $\mathcal{F}$ presheaf of abelian groups on X. $\mathcal{F}$ is a sheaf on X if it satisfies that*
(1)For each open subset U with open covering $\{U_i\}_{i\in I}$ and $s \in \mathcal{F}(U)$, if $s\big|_{U_i} = 0$ for all i, then $s = 0$.
(2)For each open subset U with open covering $\{U_i\}_{i\in I}$, if there exists $s_i \in \mathcal{F}(U_i)$ for all i, satisfying $s_i\big|_{U_i\cap U_j} = s_j\big|_{U_i\cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$ for all i.

Remark 1.6. *The two conditions in fact give a exact sequence for each open subset U with open covering $\{U_i\}_{i\in I}$.*

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

Example 1.2. *(1)Let X be a topological space. Then $U \longmapsto \{$continuous functions $U \longrightarrow \mathbb{R}\}$ give a sheaf on X.*
(2)Let $X = \mathbb{R}^n$. Then $U \longmapsto \{C^\infty$-functions $U \longrightarrow \mathbb{R}\}$ gives a sheaf on X.

(3)Let X be a topological space and A be an abelian group. View A as a topological space with discrete topology. Then $U \longmapsto \{$locally constant functions $U \longrightarrow A\}$ gives a sheaf on X, which is called the constant sheaf of A on X.

Definition 1.8. *Let X be a topological space, $\mathcal{F}$ (pre)sheaf of abelian groups on X. For $x \in X$, define the stalk of $\mathcal{F}$ at x to be $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$. More concretely, an element of $\mathcal{F}_x$ is represented by a pair $(U, s) = s_x$ where U is an open neighbourhood of x and $s \in \mathcal{F}(U)$, called a germ. Two germs (U, s) and (V, t) represent the same element if there exists open neighbourhood W of x such that $W \subseteq U \cap V$ and $s\big|_W = t\big|_W$.*

Proposition 1.5. *Let X be a topological space, $\mathcal{F}$ sheaf of abelian groups on X. Suppose U is an open subset of X and $s, t \in \mathcal{F}(U)$. Then $s = t$ if and only if $s_x = t_x$ for all $x \in U$.*

Proof. " $\Rightarrow$": Obviously.

" $\Leftarrow$": Since for all $x \in U$, we have $s_x = t_x$, there exists open neighbourhood W_x of x such that $W_x \subseteq U$ and $s\big|_{W_x} = t\big|_{W_x}$. Note that $\{W_x\}_{x \in U}$ form an open covering of U. By property of sheaf, we get $s = t$. □

Definition 1.9. *Let X be a topological space, $\mathcal{F}, \mathcal{G}$ (pre)sheaves of abelian groups on X. A morphism φ from $\mathcal{F}$ to $\mathcal{G}$ is a family of group homomorphisms $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ such that for $V \subseteq U$, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

Remark 1.7. *By property of direct limit, morphism φ gives homomorphism $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ for all $x \in X$.*

Definition 1.10. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of (pre)sheaves of abelian groups on X. We say φ is isomorphic if there exists morphism $\psi : \mathcal{G} \longrightarrow \mathcal{F}$ such that for each open subsets U, $\psi(U)\varphi(U) = \mathrm{id}_{\mathcal{F}(U)}$ and $\varphi(U)\psi(U) = \mathrm{id}_{\mathcal{G}(U)}$.*

Proposition 1.6. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of abelian groups on X. Then φ is isomorphic if and only if φ_x is isomorphic for all $x \in X$.*

Proof. "$\Rightarrow$": For all $(U, t) \in \mathcal{F}_x$, since $\varphi(V)$ is isomorphic, there exists $s \in \mathcal{F}(U)$ such that $\varphi(V)(s) = t$. Thus $\varphi_x((U, s)) = (U, t)$, get φ_x is surjective. For all $(U, s) \in \ker(\varphi_x)$, $(U, \varphi(U)(s)) = 0$ in $\mathcal{G}_x$. Then there exists open neighbourhood V of x such that $V \subseteq U$ and $\varphi(U)(s)\big|_V = 0$. By commutative diagram, $\varphi(U)(s)\big|_V = \varphi(V)(s\big|_V)$. Since $\varphi(V)$ is isomorphic, get $s\big|_V = 0$ and so $(U, s) = 0$ in $\mathcal{F}_x$. Get φ_x is isomorphic.

"$\Leftarrow$": For each open subset U, want to show that $\varphi(U)$ is isomorphic. For $s \in \ker \varphi(U)$, $\varphi(U)(s) = 0$. Then for all $x \in U$, $(U, \varphi(U)(s)) = 0$ in $\mathcal{G}_x$. As $(U, \varphi(U)(s)) = \varphi_x((U, s))$ and φ_x is isomorphic, get $(U, s) = 0$ in $\mathcal{F}_x$. Thus there exists open neighbourhood W_x of x such that $W_x \subseteq U$ and $s\big|_{W_x} = 0$. By property of sheaf, get $s = 0$ and so $\varphi(U)$ is injective.

For all $t \in \mathcal{G}(U)$, (U,t) is an element of $\mathcal{G}_x$ for all $x \in U$. Then there exists $(V_x, s(x)) \in \mathcal{F}_x$ such that $\varphi_x((V_x, s(x))) = (U,t)$ and $\varphi(V_x)(s(x)) = t\big|_{V_x}$. Note that $\varphi(V_x \cap V_y)(s(x)\big|_{V_x\cap V_y}) = (t\big|_{V_x})\big|_{V_x \cap V_y} = (t\big|_{V_y})\big|_{V_x\cap V_y} = \varphi(V_x \cap V_y)(s(y)\big|_{V_x\cap V_y})$. Since $\varphi(V_x \cap V_y)$ is injective, get $s(x)\big|_{V_x\cap V_y} = s(y)\big|_{V_x\cap V_y}$. By property of sheaf, there exists $s \in \mathcal{F}(U)$ such that $s\big|_{V_x} = s(x)$. So $\varphi(U)(s)\big|_{V_x} = \varphi(V_x)(s(x)) = t\big|_{V_x}$. By property of sheaf, get $\varphi(U)(s) = t$ and so $\varphi(U)$ is isomorphic. $\square$

Definition 1.11. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of presheaves of abelian groups on X. We have several notions*

- *Define the kernel of φ to be* $\ker(\varphi) : op(X) \longrightarrow Ab \quad U \longmapsto \ker(\varphi(U))$.
- *Define the image of φ to be $im(\varphi) : op(X) \longrightarrow Ab \quad U \longmapsto im(\varphi(U))$.*
- *Define the cokernel of φ to be $coker(\varphi) : op(X) \longrightarrow Ab \quad U \longmapsto coker(\varphi(U))$.*

Remark 1.8. *The definition of cokernel in fact induce the definition of quotient presheaf.*

Definition 1.12 (***Sheafification***)**.** *Let X be a topological space, $\mathcal{F}$ presheaf of abelian groups on X. Define a sheaf $\mathcal{F}^+$ of abelian groups on X by $U \longmapsto \mathcal{F}^+(U)$, where $\mathcal{F}^+(U) = \{(s_x)_{x\in U} \big|$ for all $x \in U$, $\exists$ open $V_x \ni x$ and $t(x) \in \mathcal{F}(V_x)$ such that $t(x)_y = s_y$, $\forall y \in V_x\}$ is a subset of $\prod_{x\in U} \mathcal{F}_x$.*

Proposition 1.7. *Let X be a topological space, $\mathcal{F}$ presheaf of abelian groups on X. Show that there is a natural morphism of presheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{F}^+$ satisfies that*

- *φ_x is identity homomorphism.*
- *If $\mathcal{F}$ is sheaf, then φ is isomorphism.*
- *For any morphism of presheaves $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there exists unique morphism $\phi : \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & \mathcal{G} \\
& \mathcal{F}^+ \arrow[ur, "\phi"'] &
\end{tikzcd} \tag{1}$$

Reason 1.3. *The construction of φ is just $\varphi(U) : s \longmapsto (s_x)_{x\in U}$.*

Definition 1.13. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of abelian groups on X.*
(1)Define the kernel of φ to be $\ker(\varphi) : op(X) \longrightarrow Ab \quad U \longmapsto \ker(\varphi(U))$.
(2)Define the image of φ to be sheafification of $op(X) \longrightarrow Ab \quad U \longmapsto im(\varphi(U))$, denoted by $im(\varphi)$.
(3)Define the cokernel of φ to be sheafification of $op(X) \longrightarrow Ab \quad U \longmapsto coker(\varphi(U))$, denoted by $coker(\varphi)$.

Remark 1.9. *The definition of cokernel in fact induce the definition of quotient sheaf.*

Definition 1.14. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of abelian groups on X.*
(1)We say that φ is injective if $\ker(\varphi) = 0$.
(2)We say that φ is surjective if $im(\varphi) = \mathcal{G}$

Proposition 1.8. *Let X be a topological space, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of abelian groups on X. Then*
(1)φ is injective if and only if $\varphi(U)$ is injective for all open subset U.
(2)φ is injective if and only if φ_x is injective for all $x \in X$.
(3)φ is surjective if and only if φ_x is surjective for all $x \in X$.
(4)φ is isomorphic if and only if φ is injective and surjective.

Remark 1.10. *The proof is similar to proof of Proposition 1.6. In addition, Hartshorne exercise II 1.2 gives a more explicit sketch.*

Definition 1.15 (***Direct Image and Inverse Image***)**.** *Let $f : X \longrightarrow Y$ be continuous map of topological spaces, $\mathcal{F}$ sheaf of abelian groups on X, $\mathcal{G}$ sheaf of abelian groups on Y.*
(1)Define dierct image to be the sheaf $f_*\mathcal{F} : V \longmapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$.
(2)Define inverse image to be the sheafification of $U \longmapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, denoted by $f^{-1}\mathcal{G}$.

Remark 1.11. *In general, we don't have $(f_*\mathcal{F})_{f(x)} \neq \mathcal{F}_x$, while there is a canonical homomorphism $(f_*\mathcal{F})_{f(x)} \longrightarrow \mathcal{F}_x$. On the other hand, we always have $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(X)}$.*

Example 1.3. *Let $i : Z \hookrightarrow X$ be an inclusion of topological spaces, $\mathcal{F}$ sheaf of abelian groups on X. We often write $i^{-1}\mathcal{F}$ as $\mathcal{F}\big|_Z$.*

1.3 Varieties

Example 1.4 (***Affine Varieties***)**.** *Let k be an algebraically closed field, $V \subseteq k^n$ irreducible closed subset. Suppose $\mathfrak{p}$ is the corresponding prime ideal of $\mathfrak{p}$ in $\operatorname{Spec}(k[x_1, \cdots, x_n])$. Define affine coordinate ring of V to be $k[V] = k[x_1, \cdots, x_n]/\mathfrak{p}$. Then $k[V]$ is a finitely generated k-algebra and an integral domain. Define sheaf $\mathcal{O}_V$ of abelian groups on V to be $U \longmapsto \{$regular functions $U \longrightarrow k\}$, where the definition of regular function can be seen in section 3 of Chapter I. We call such a pair $(V, \mathcal{O}_V)$ affine variety.*

Proposition 1.9. *Let k be an algebraically closed field, $V \subseteq k^n$ irreducible closed subset. Then*
(1)$\mathcal{O}_V(V) = k[V]$.
(2)For all $f \in k[V]$, $\mathcal{O}_V(D(f)) = k[V]_f$.
(3)For all $x \in V$, $\mathcal{O}_{V,x} = k[V][f^{-1}\big|f(x) \neq 0]$ is a local ring with maximal ideal $\mathfrak{m}_x$.
(4)For all $x \in V$, $\mathcal{O}_{V,x}/\mathfrak{m}_x = k$.

Example 1.5 (***Gluing Affine Varieties***)**.** *Let k be an algebraically closed field. A abstract prevariety over k is a pair $(X, \mathcal{O}_X)$, where X is an irreducible topological space and $\mathcal{O}_X$ is a sheaf of abelian groups on X, satisfying that there exists finite open covering $\{U_i\}$ of X such that*

$(U_i, \mathcal{O}\big|_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ where $(V_i, \mathcal{O}_{V_i})$ is an affine variety. The isomorphism is a pair $(f_i, f_i^\sharp)$, where $f_i : U_I \longrightarrow V_i$ is a homeomorphism of topological spaces and $f_i^\sharp : \mathcal{O}_{V_i} \longrightarrow f_*\mathcal{O}_X\big|_{U_i}$ is an isomorphism of sheaves.

Example 1.6 (***Affine Line with Two Origins***)**.** *Let k be an algebraically closed field. Assume U_1, U_2 are two affine line $\mathbb{A}_k^1$. Glue them on $U_1 \setminus \{O_1\}$ and $U_2 \setminus \{O_2\}$ and get an abstract variety X. We call X affine line with two origins. Obviously, O_1 and O_2 have no disjoint open neighbourhoods. Thus we say X is some kind of not separated, which is the case we want to avoid.*

Example 1.7 (***Product of Prevarieties***)**.** *Let k be an algebraically closed field, X abstract prevariety over k. Suppose that $\{U_i\}_{i\in I}$ are affine open subsets of X. Consider $X \times X$ as a prevariety, glued by $\{U_i \times U_j\}_{i,j\in I}$, where $U_i \times U_j$ is still isomorphic to affine variety refer to Hartshorne exercise I 3.15.*

Definition 1.16. *Let k be an algebraically closed field, X abstract prevariety over k. X is an abstract variety if $\delta = \{(x,x)\big|x \in X\}$ is closed in $X \times X$.*

1.4 The structure sheaf

From now on, we would consider sheaves of rings. The above propositions about sheaves of abelian groups are still right for sheaves of rings. Here we don't need to repeat the proofs and just acknowledge them. Similar to the theory of abstract, we will define something basic like affine variety first.

Definition 1.17. *Let A be a ring, $X = \operatorname{Spec} A$ spectrum. Define the structure sheaf $\mathcal{O}_X$ of rings on X to be $U \longrightarrow \mathcal{O}_X(U)$, where $\mathcal{O}_X(U) = \{(a_\mathfrak{p})_{\mathfrak{p}\in U}\big|\forall \mathfrak{p} \in U, \exists f(\mathfrak{p}) \in (A \setminus \mathfrak{p})$ and $b(\mathfrak{p}) \in A_{f(\mathfrak{p})}$ such that $D(f(\mathfrak{p})) \subseteq U$ and $b(\mathfrak{p})_\mathfrak{q} = a_\mathfrak{q}$, $\forall \mathfrak{q} \in D(f(\mathfrak{p}))\}$ is a subset of $\prod_{\mathfrak{p}\in U} A_\mathfrak{p}$.*

Theorem 1.1. *Let A be a ring, $\mathcal{O}_{\operatorname{Spec} A}$ structure sheaf. Then for all $f \in A$ and $g \in rad(f)$, we have an isomorphism $A_f \xrightarrow{\sim} \mathcal{O}_X(D(f))$ and the following diagram commutes*

$$\begin{array}{ccc} A_f & \longrightarrow & \mathcal{O}_X(D(f)) \\ \downarrow & & \downarrow \\ A_g & \longrightarrow & \mathcal{O}_X(D(g)) \end{array}$$

Reason 1.4. *Construct the isomorphism as $\frac{a}{f^n} \longmapsto (\frac{a}{f^n})_{\mathfrak{p}\in D(f)}$. The harder part is to show this map is surjective, which need some commutative algebra to prove. For all $(a_\mathfrak{p})_{\mathfrak{p}\in D(f)}$, $\{D(f(\mathfrak{p}))\}$ form an open covering of $D(f)$. By Corollary 1.1, $D(f)$ is quasi-compact and so there exists finitely many $\mathfrak{p}_i \in D(f)$ such that $D(f(\mathfrak{p}_i))$ cover $D(f)$.*

For convenience, rewrite $f(\mathfrak{p}_i)$ as f_i and assume that $b(\mathfrak{p}_i) = \frac{a_i}{f_i^{n_i}}$. Then $\frac{a_i}{f_i^{n_i}}$ and $\frac{a_j}{f_j^{n_j}}$ coincide on $A_\mathfrak{q}$ for all i, j and $\mathfrak{q} \in D(f_i f_j)$. Consider $\Delta_{ij} = a_i f_j^{n_j} - a_j f_i^{n_i}$ and annihilator $\operatorname{ann}_A(\Delta_{ij})$. If $f_i f_j \notin rad(\operatorname{ann}_A(\Delta_{ij}))$, take maximal ideal $\mathfrak{m}_{ij} \supseteq \operatorname{ann}_A(\Delta_{ij})$. Then $\mathfrak{m}_{ij} \in D(f_i f_j)$ and so $\frac{a_i}{f_i^{n_i}} = \frac{a_j}{f_j^{n_j}}$ in $A_{\mathfrak{m}_{ij}}$. Thus there exists $s_{ij} \in (A \setminus \mathfrak{m}_{ij})$ such that $s\Delta_(ij) = 0$ and so $s \in \operatorname{ann}_A(\Delta_{ij})$. But $\operatorname{ann}_A(\Delta_{ij}) \subseteq \mathfrak{m}_{ij}$, get $s \in \mathfrak{m}_{ij}$, contradiction!

Thus $f_if_j \in rad(\mathrm{ann}_A(\Delta_{ij}))$ and there exists $k_{ij} \in \mathbb{M}$ such that $(f_if_j)^{k_{ij}}\Delta_{ij} = 0$. Note that there are finitely many i, we can take a supremum k And so $(f_if_j)^k\Delta_{ij} = 0$ for all i, j. Let $a_i' = a_if_i^k$ and $f_i' = f_i^{n_i+k}$. Can replace $\frac{a_i}{f_i^{n_i}}$ by $\frac{a_i'}{f_i'}$. Then $a_i'f_j' - a_j'f_i' = 0$. On the other hand, $D(f) \subseteq \cup_i D(f_i) = \cup_i D(f_i')$ gives that there exists $n \in \mathbb{N}$ such that $f^n = \sum_i b_if_i'$, where $b_i \in A$. Consider $\frac{\sum_i b_ia_i'}{f^n}$. Then $f^na_j' = \sum_i b_if_i'a_j' = \sum_i b_ia_i'f_j' = (\sum_i b_ia_i')f_j'$ and so $\frac{\sum_i b_ia_i'}{f^n}$ and $\frac{a_i'}{f_i'}$ coincide on $A_\mathfrak{q}$ for all $\mathfrak{q} \in D(f_i') = D(f_i)$. Get $\frac{\sum_i b_ia_i'}{f^n} \longmapsto (a_\mathfrak{p})_{\mathfrak{p}\in D(f)}$ and so homomorphism is surjective.

Corollary 1.2. *Let A be a ring, $\mathcal{O}_{\mathrm{Spec}\,A}$ the structure sheaf. Then $\mathcal{O}_{\mathrm{Spec}\,A}(\mathrm{Spec}\,A) = A$.*

Remark 1.12. *With the structure sheaf, now we can distinguish* $\mathrm{Spec}\,k$ *and* $\mathrm{Spec}\,k'$ *for different fields k and k'.*

Proposition 1.10. *Let A be a ring, $\mathcal{O}_{\mathrm{Spec}\,A}$ the structure sheaf. Then for all $\mathfrak{p} \in \mathrm{Spec}\,A$, we have an isomorphism $A_\mathfrak{p} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Spec}\,A,\mathfrak{p}}$.*

Reason 1.5. *By definition of stalk and property of direct limit, we immediately get the isomorphism.*

1.5 Schemes

Definition 1.18 (***Ringed Spaces***)**.** *Let X be a topological space, $\mathcal{O}_X$ sheaf of rings on X. Then pair $(X, \mathcal{O}_X)$ is called a ringed space.*

Definition 1.19 (***Locally Ringed Spaces***)**.** *Let $(X, \mathcal{O}_X)$ be a ringed space. We say $(X, \mathcal{O}_X)$ is a locally ringed space if for all $x \in X$, $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$ and residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.*

Remark 1.13. *With these definitions we can evaluate sections at point of underlying topological space. For open subset $U \subseteq X$, $x \in U$ and $s \in \mathcal{O}_X(U)$, define $f(x)$ to be the image of f_x in residue field $k(x)$.*

Example 1.8. *Let A be a ring. Then $(Spec(A), \mathcal{O}_{Spec(A)})$ is a locally ringed space. For all $\mathfrak{p} \in \mathrm{Spec}\,A$, $k(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p})$. If moreover A is an integral domain and $\eta = (0)$ is the generic point, then $k(\eta) = \mathrm{Frac}(A)$.*

Definition 1.20. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where $f : X \longrightarrow Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y.*

Definition 1.21. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two locally ringed spaces. A morphism of locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(f, f^\sharp)$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ satisfying that for all $x \in X$ and $y = f(x)$, the first row of the following commutative diagram is a local homomorphism i.e. $(f_x^\sharp)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$*

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[rr, "f_x^\sharp"] \arrow[dr] & & \mathcal{O}_{X.x} \\
& (f_*\mathcal{O}_X)_y \arrow[ur] &
\end{tikzcd}$$

Remark 1.14. *Since $f_x^\sharp$ is a local homomorphism, it induces a field extension $k(f(x)) \hookrightarrow k(x)$.*

Definition 1.22. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of (locally) ringed spaces. We say that $(f, f^\sharp)$ is isomorphic if there exists morphism $(g, g^\sharp) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ such that*
(1)$gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$
(2)$(gf)^\sharp$ is isomorphism of sheaves of rings on X and $(fg)^\sharp$ is isomorphism of sheaves of rings on Y.

Proposition 1.11. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of (locally) ringed spaces. Then $(f, f^\sharp)$ is isomorphic if and only if f is homeomorphism and $f_x^\sharp$ is isomorphic for all $x \in X$.*

Proof. "$\Rightarrow$": Obviously.

"$\Leftarrow$": Since f is homeomorphism, assume the inverse of f is g. Want construct $(g, g^\sharp)$ with $g^\sharp(U) : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f(U))$. We will prove that $f^\sharp(V)$ is isomorphic for all open subset V of Y and take $g^\sharp(U) = (f^\sharp(f(U)))^{-1}$. For $t \in \ker(f^\sharp(V))$, $f^\sharp(V)(t) = 0$. Then $f_x^\sharp((V, t)) = (f^{-1}(V), 0) = 0$ in $\mathcal{O}_{X,x}$ for all $x \in f^{-1}(V)$. As $f_x^\sharp$ is isomorphic, we get $(V, t) = 0$ in $\mathcal{O}_{Y,f(x)}$. Thus there exists open neighbourhood V_x of $f(x)$ such that $t\big|_{V_x} = 0$. Note that f is homomorphism inducing that $\{V_x\}_{x \in f^{-1}(x)}$ is an open covering of V. By property of sheaf get $t = 0$ and so $f^\sharp(v)$ is injective.

For each open subset U of X and $s \in \mathcal{O}_X(U)$, consider $(U, s) \in \mathcal{O}_{X,x}$ for some $x \in U$. Since $f_x^\sharp$ is isomorphic for all $x \in X$, there exists open neighbourhood V_x of $f(x)$ and $t(x) \in \mathcal{O}_Y(V_x)$ such that $f_x^\sharp((V_x, t(x))) = (U, s)$. In fact, by choose appropriate V_x, we can have $f^\sharp(V_x)(t(x)) = s\big|_{V_x}$. By injectivity of $f^\sharp(V_x \cap V_{x'})$, get $t(x)\big|_{V_x \cap V_{x'}} = t(x')\big|_{V_x \cap V_{x'}}$. Then by property of sheaf, there exists $t \in \mathcal{O}_Y(V)$ such that $t\big|_{V_x} = t(x)$. So $f^\sharp(t)\big|_{V_x} = f^\sharp(V_x)(t(x)) = s\big|_{V_x}$. By property of sheaf, get $f^\sharp(t) = s$ and so $f^\sharp(f(U))$ is surjective. Thus $f^\sharp(V)$ is isomorphic for all open subset V of Y. □

Definition 1.23 (***Affine Schemes***)**.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that $(X, \mathcal{O}_X)$ is an affine scheme if there exists a ring A such that $(X, \mathcal{O}_X) \cong (A, \mathcal{O}_{Spec(A)})$ as locally ringed spaces.*

Remark 1.15. *An affine variety over algebraically closed field k is an affine scheme isomorphic to $(Spec(k[V]), \mathcal{O}_{Spec(k[V])})$, where $k[V]$ is affine coordinate ring of some irreducible closed subset V of k^n.*

Theorem 1.2. *Let $\varphi : A \longrightarrow B$ be a ring homomorphism. Then we can associate it with a natural morphism $(f, f^\sharp) : (Spec(B), \mathcal{O}_{Spec(B)}) \longrightarrow (Spec(A), \mathcal{O}_{Spec(A)})$ of locally ringed spaces. Moreover, there is a one-to-one correspondence*

$$\mathrm{Hom}(A, B) \longleftrightarrow Mor((Spec(B), \mathcal{O}_{Spec(B)}), (Spec(A), \mathcal{O}_{Spec(A)}))$$

Proof. For convenience, let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}\,A$. Take f to be the induced map of φ. By Proposition 1.3, we know that f is continuous and in particular, $f^{-1}(D(a)) = D(\varphi(a))$ for all $a \in A$. Note that φ induce a homomorphism $A_a \longrightarrow B_{\varphi(a)}$, while $\mathcal{O}_X(D(\varphi(a))) = B_{\varphi(a)}$ and $\mathcal{O}_Y(D(a)) = A_a$. Thus we can define $f^\sharp(D(a)) : \mathcal{O}_Y(D(a)) \longrightarrow \mathcal{O}_Y(D(\varphi(a))) = f_*\mathcal{O}_X(D(a))$. By gluing, get $f^\sharp : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$. For all $\mathfrak{p} \in X$ and $\varphi^{-1}(\mathfrak{p})$, it is obvious that $f_{\mathfrak{p}^\sharp}$ is local. Thus $(f, f^\sharp)$ is a morphism of locally ringed spaces.

If φ and φ' give the same morphism $(f, f^\sharp)$, consider $f^\sharp(Y)$. By above process, $f^\sharp(Y)$ is just φ and so $\varphi = \varphi'$. For any morphism of locally ringed space $(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, likewise consider the homomorphism of global sections φ. Firstly, want to show that f is the induced map of f. Note that for all $\mathfrak{p} \in X$, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^\sharp_{\mathfrak{p}}"] & B_{\mathfrak{p}} \end{tikzcd}$$

Since $f_{\mathfrak{p}^\sharp}$ is local homomorphism and $f^\sharp_{\mathfrak{p}}(\frac{\varphi^{-1}(\mathfrak{p})}{1}) \subseteq \frac{\mathfrak{p}}{1} \subseteq \mathfrak{m}_{\mathfrak{p}}$, get $\varphi^{-1}(\mathfrak{p}) \subseteq f(\mathfrak{p})$. On the other hand, for all $a \in f(\mathfrak{p})$, we have $\frac{\varphi(a)}{1} \in \mathfrak{p}$ and so $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

It suffices to show that φ just induce $(f, f^\sharp)$ and we only need to check about standard open sets. Note that for all $a \in A$, $\mathfrak{p} \in f^{-1}(D(a))$ if and only if $f(\mathfrak{p}) \in D(a)$ if and only if $\varphi^{-1}(\mathfrak{p}) \in D(a)$ if and only if $\mathfrak{p} \in D(\varphi(a))$, get $f^{-1}(D(a)) = D(\varphi(a))$. In addition, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_a \arrow[r, "f^\sharp(D(a))"] & B_{\varphi(a)} \end{tikzcd}$$

For $\frac{a'}{a^n} \in A_a$, we have

$$\begin{aligned} f^\sharp(D(a))(\frac{a'}{a^n}) &= f^\sharp(D(a))(\frac{a'}{1}) f^\sharp(D(a))(\frac{1}{a^n}) \\ &= \frac{\varphi(a')}{1} f^\sharp(D(a))(\frac{a}{1})^{-n} \\ &= \frac{\varphi(a')}{1}(\frac{\varphi(a)}{1})^{-n} \\ &= \frac{\varphi(a')}{\varphi(a)^n} \end{aligned}$$

Thus $(f, f^\sharp)$ is induced by φ. □

Definition 1.24 (***Schemes***)**.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that $(X, \mathcal{O}_X)$ is a scheme if there exists an open covering $\{U_i\}_{i \in I}$ of X with $(U_i, \mathcal{O}_X\big|_{U_i})$ affine scheme. A morphism of schemes is just a morphism of locally ringed spaces.*

Remark 1.16. *By property of standard open subsets, we can easily see that affine open subsets of a scheme $(X, \mathcal{O}_X)$ form a topological basis.*

Proposition 1.12. *Let $(X, \mathcal{O}_X)$ be an affine scheme. Assume that $X = \mathrm{Spec}\,A$ and $U = \mathrm{Spec}(B) \subseteq X$ is an affine open subset. Then for all $a \in A$, $U \cap D(a) = D(b)$ where b is the image of a under the restriction map.*

Proof. Note that for all $x \in U$, x corresponds to a prime ideal $\mathfrak{p}_x \in \operatorname{Spec} A$ and a prime ideal $\mathfrak{P}_x \in \operatorname{Spec}(B)$. Thus they have the same stalk $\mathcal{O}_{X,x}$, inducing the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}_x} & \xrightarrow{\sim} & B_{\mathfrak{P}_x} \end{array}$$

Then $x \in D(b)$ if and only if $b \in (B_{\mathfrak{P}_x})^\times$ if and only if $a \in (A_{\mathfrak{p}_x})^\times$ if and only if $x \in D(a)$. Get $U \cap D(a) = D(b)$. □

1.6 Projective Schemes

Definition 1.25. *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. An ideal I of S is said to be homogeneous if it is generated by homogeneous elements, or equivalently $I = \oplus_{d \geq 0}(I \cap S_d)$.*

Definition 1.26. *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Define $\operatorname{Proj}(S)$ to be the set of all homogeneous prime ideals of S not containing S_+, where $S_+ = \oplus_{d > 0} S_d$.*

Our goal is to define scheme structure on $\operatorname{Proj}(S)$. So, similar to definition of projective variety in classic algebraic geometry, we will define Zariski topology on $\operatorname{Proj}(S)$.

Definition 1.27. *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring, $I \subseteq S$ homogeneous ideal. Define $V_+(I) = \{\mathfrak{p} \in \operatorname{Proj}(S) \big| I \subseteq \mathfrak{p}\}$ and $D_+(I) = \operatorname{Proj}(S) \setminus V_+(I)$. In particular, $V_+(S_+) = \varnothing$ and $V_+(0) = \operatorname{Proj}(S)$.*

Proposition 1.13 (***Zariski Topology***)**.** *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Then the above definitions satisfy:*
(1)$\{V_+(I) \big| I \subseteq S$ homogeneous ideal$\}$ form the closed subsets of a topology on $\operatorname{Proj}(S)$ called the Zariski topology.
(2)$\{D_+(f) \big| f \in S\}$ form a basis of the topology, called standard open subsets.

Remark 1.17. *The proof has nothing more than Proposition 1.1, but it is necessary to check ideals are homogeneous. In addition, one can check that the Zariski topology on $\operatorname{Proj}(S)$ is in fact the induced topology of Zariski topology on $\operatorname{Spec} A$.*

Lemma 1.2. *Let S be a graded ring, $I, J \subseteq S$ homogeneous ideals. Then $V_+(I) = V_+(J)$ if and only if $J \cap S_+ \subseteq rad(I)$.*

To give a appropriate sheaf of rings on $\operatorname{Proj}(S)$, we need the following lemma

Lemma 1.3. *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring, $f \in S$ homogeneous element of degree $r > 0$. Then*
(1)$u_f : D_+(f) \longrightarrow \operatorname{Spec}(S_f) \quad \mathfrak{p} \longmapsto \mathfrak{p}S_f \cap S_{(f)}$ is bijective, where $S_f \subseteq S_f$ is the subset of all homogeneous elements of degree 0.
(2)Suppose there exists homogeneous element $s \in S$ of degree > 0 with $D_+(g) \subseteq D_+(f)$. Then there exists ring homomorphism $S_{(f)} \longrightarrow S_{(g)}$.

Proof. (1): For $\mathfrak{q} \in \mathrm{Spec}(S_{(f)})$, we have that $\mathfrak{q}S_f$ is a homogeneous ideal of S_f and so is $rad(\mathfrak{q}S_f)$. Let $\mathfrak{p} \in D_+(f)$ be the preimage of $rad(\mathfrak{q}S_f)$ in S (In fact, $\mathfrak{p} = rad(\mathfrak{q}S_f) \cap S$). Want to show that $rad(\mathfrak{q}S_f)$ is prime. If $a, b \in S_f$ homogeneous with $ab \in rad(\mathfrak{q}S_f)$, then $a^m b^m \in \mathfrak{q}S_f$ for some $m \in \mathbb{N}_+$. Get $(\frac{a^r}{f^{\deg(a)}})^m(\frac{b^r}{f^{\deg(b)}})^m \in \mathfrak{q}S_f \cap S_{(f)} = \mathfrak{q}$. As $\mathfrak{q}$ is prime, $a^r \in \mathfrak{q}S_f$ or $b^r \in \mathfrak{q}S_f$. Thus $a \in rad(\mathfrak{q}S_f)$ or $b \in rad(\mathfrak{q}S_f)$ and so $rad(\mathfrak{q}S_f)$ is prime. Then $\mathfrak{p}$ is prime and $u_f(\mathfrak{p}) = \mathfrak{q}$.

For $\mathfrak{p}, \mathfrak{p}' \in D_+(f)$, we claim that $\mathfrak{p}' \subseteq \mathfrak{p}$ if and only if $u_f(\mathfrak{p}') \subseteq \mathfrak{p}$. From the above process, we can easily see that $\mathfrak{p} = rad(u_f(\mathfrak{p})S_f) \cap S$. By the same process, we can prove the claim. And the claim immediately gives the injectivity of u_f.

(2): If $D_+(g) \subseteq D_+(f)$, then $g^m = bf$ for some $b \in S$ and $m \in \mathbb{N}_+$. Note that since f,g are both homogeneous, get b is also homogeneous. Then $\frac{a}{f^n} \longmapsto \frac{ab}{g^{mn}}$ gives the homomorphism. $\square$

Theorem 1.3. *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring, $f \in S$ homogeneous element of degree $r > 0$. Then*
(1)$u_f : D_+(f) \longrightarrow \mathrm{Spec}(S_f)$ as above is a homeomorphism.
(2)If $f, g \in S$ are homogeneous elements of degree > 0 with $D_+(g) \subseteq D_+(f)$, then the following diagram commutes

$$\begin{array}{ccc} D_+(g) & \xrightarrow{u_g} & \mathrm{Spec}(S_{(g)}) \\ \downarrow & & \downarrow \\ D_+(f) & \xrightarrow{u_f} & \mathrm{Spec}(S_{(f)}) \end{array}$$

(3)Via the homeomorphism u_f, we equip $D_+(f)$ with a structure sheaf $\mathcal{O}_{D_+(f)}$. And $\{\mathcal{O}_{D_+(f)}\}$ glue to a structure sheaf $\mathcal{O}_{Proj(S)}$ on $\mathrm{Proj}(S)$. In particular, $(Proj(S), \mathcal{O}_{Proj(S)})$ is a scheme.
(4)For all $\mathfrak{p} \in \mathrm{Proj}(S)$, $\mathcal{O}_{Proj(S),\mathfrak{p}} = S_{(\mathfrak{p})}$.

Proof. (1): By Lemma 1.3, we have already known that u_f is bijective. As the Zariski topology of $\mathrm{Proj}(S)$ is the induced topology of $\mathrm{Spec}(S)$. We can view u_f as the restriction to $D_+(f)$ of $\mathrm{Spec}(S_f) \longrightarrow \mathrm{Spec}(S_{(f)})$. Thus u_f is continuous. On the other hand, want to show that u_f is closed. This comes from a similar claim: for $\mathfrak{p} \in D_+(f)$ and homogeneous ideal $I \subseteq \mathfrak{p}$, we have $I \subseteq \mathfrak{p}$ if and only if $(IS_f \cap S_{(f)}) \subseteq (\mathfrak{p}S_f \cap S_{(f)})$. The proof is similar to the proof of Lemma 1.3. Thus with the claim, get u_f is homogeneous.

(2): Obvious

(3): Since $D_+(f)_{f \in S}$ is a topological basis stable under finite intersection, we only need to show their structure sheaves coincide on the intersection set, which comes from (2).

(4): Same argument as the affine case. $\square$

Now with well defined affine and projective scheme, we can restate definition of affine and projective abstract variety.

Definition 1.28 (***Affine Abstract Varieties***)**.** *Let k be an algebraically closed field. An affine abstract variety over k is a scheme of the form $\mathrm{Spec}(k[x_0, \cdots, x_n]/\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}(k[x_0, \cdots, x_n])$.*

Definition 1.29 (***Projective Abstract Varieties***)**.** *Let k be an algebraically closed field. A projective abstract variety over k is a scheme of the form* $\mathrm{Proj}(k[x_0, \cdots, x_n]/\mathfrak{p})$ *for some* $\mathfrak{p} \in \mathrm{Proj}(k[x_0, \cdots, x_n])$*.*

2 Morphisms of Schemes

Recall a morphism of schemes $(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces. From now on, for convenience, we would just write $f : X \longrightarrow Y$ for a morphism of schemes from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$.

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose that $U \subseteq X$ is an open subset. Then f induce a morphism $f\big|_U : U \longrightarrow Y$, where U is scheme $(U, \mathcal{O}_X\big|_U)$.*

Proof. Obviously, the continuous map of underlying topological spaces can be taken as $f : U \longrightarrow Y$ sending x to $f(x)$. For each $V \subseteq Y$ open subset, define $f\big|_U^\sharp(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X\big|_U(f\big|_U^{-1}(V)) = \mathcal{O}_X(f^{-1}(V) \cap U)$ by sending t to $f^\sharp(V)(t)\big|_{f^{-1}(V)\cap U}$. It suffices to check naturality. Consider the following diagram for open subsets $V' \subseteq V \subseteq Y$,

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{(f|_U)^\sharp(V)} & \mathcal{O}_X\big|_U(f\big|_U^{-1}(V)) \\ \big\downarrow{\scriptstyle \rho_{VV'}} & & \big\downarrow{\scriptstyle \rho_{VV'}} \\ \mathcal{O}_Y(V') & \xrightarrow{(f|_U)^\sharp(V')} & \mathcal{O}_X\big|_U(f\big|_U^{-1}(V')) \end{array}$$

Then for all $t \in \mathcal{O}_Y(V)$

$$\begin{aligned} \rho_{VV'} \circ (f\big|_U)^\sharp(V)(t) &= (f^\sharp(V)(t)\big|_{f^{-1}(V)\cap U})\big|_{f\big|_U^{-1}(V')} \\ &= f^\sharp(V)(t)\big|_{f^{-1}(V')\cap U} \\ &= (f^\sharp(V)(t)\big|_{f^{-1}(V')})\big|_{f^{-1}(V')\cap U} \\ &= (f^\sharp(V')(t\big|_{V'}))\big|_{f^{-1}(V')\cap U} \\ &= (f\big|_U)^\sharp(V')(t\big|_{V'}) \\ &= (f\big|_U)^\sharp(V') \circ \rho_{VV'}(t) \end{aligned}$$

Get the diagram commutes and done! □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose that $V \subseteq Y$ is an open subset and $f(X) \subseteq V$. Then there exists unique morphism $g : X \longrightarrow V$ such that $f = i \circ g$, where $i : V \hookrightarrow Y$ is the inclusion.*

Reason 2.1. *The part of construction is similar to proof of Lemma 2.1. For uniqueness, if there are another $\widetilde{g}$ satisfies the conditions, firstly we can easily see that $g = \widetilde{g}$ in the underlying topological space. While each ring homomorphism in $g^\sharp$ and $\widetilde{g}^\sharp$ is a ring homomorphism in $f^\sharp$, get $g = \widetilde{g}$ as morphisms of schemes.*

Lemma 2.3 (***Gluing Lemma***)**.** *Let X, Y be two schemes. Assume that $\{U_i\}_{i\in I}$ is an open covering of X. Then giving a morphism $f : X \longrightarrow Y$ is equivalent to giving a family of morphisms $f_i : U_i \longrightarrow Y_{i\in I}$ such that $f_i\big|_{U_i\cap U_j} = f_j\big|_{U_i\cap U_j}$.*

Proof. "$\Rightarrow$": Obvious.

"$\Leftarrow$": We need to construct a morphism $f : X \longrightarrow Y$ at first. It is obvious that we can get a continuous map f of underlying topological spaces be gluing f_i, satisfying that $f\big|_{U_i} = f_i$. For each $V \subseteq Y$ open subset and $t \in \mathcal{O}_Y(V)$, we have images $f_i^\sharp(V)(t)_{i\in I}$. Note that $f_i^\sharp(t) \in \mathcal{O}_X(f^{-1}(V) \cap U_i)$, while $f^{-1}(V) \cap U_i$ form an open covering of $f^{-1}(V)$ and $f_i^\sharp(t)$ coincide on $f^{-1}(U)\cap U_i\cap U_j$ for all $i, j \in I$. By property of sheaf, there exists $s \in \mathcal{O}_X(f^{-1}(V))$ such that $s\big|_{f^{-1}\cap U_i} = f_i^\sharp(t)$. Let $f^\sharp(V)$ map t to s. By property of sheaf, it is easty to check naturality and that $f^\sharp(V)$ is a ring homomorphism. Get a morphism f of schemes from X to Y and $f\big|_{U_i} = f_i$. □

Lemma 2.4. *Let X be a scheme, $x \in X$. Then there exists a morphism of schemes* $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$.

Reason 2.2. *For affine case, assume $X = \mathrm{Spec}\, A$. For all $x \in X$, it corresponds to a prime ideal $\mathfrak{p}$ of A and there is a natural homomorphism $A \longrightarrow A_\mathfrak{p}$, where $A_\mathfrak{p}$ is just $\mathcal{O}_{X,x}$, inducing a morphism* $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$.

For general case, we can take an affine open neighbourhood U of x, which gives a morphism $\mathrm{Spec}(\mathcal{O}_{\mathcal{X},\S}) = (\mathcal{O}_X\big|_U)_x \longrightarrow U$. *Composing with $i : U \hookrightarrow X$, get* $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$. *To show that the morphism is independent of choose of U. Consider two affine open neighbourhoods $V \subseteq U$ of x and there is a commutative diagram*

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[rr, "\rho_{UV}"] \arrow[dr] & & \mathcal{O}_X(V) \arrow[dl] \\ & \mathcal{O}_{X,x} & \end{tikzcd}$$

which shows that $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow U$ *is composition of* $\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow V$ *and* $V \hookrightarrow U$.

Remark 2.1. *Since there is a natural morphism* $\mathrm{Spec}(k(x)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,x})$, *we also get* $\mathrm{Spec}(k(x)) \longrightarrow X$.

2.1 S-schemes

Definition 2.1. *Let S be a schemes. An S-scheme (or scheme over S) is a scheme X equipped with a morphism of schemes $X \longrightarrow S$. A morphism of S-schemes is a morphism of schemes $X \longrightarrow Y$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Define $Mor_S(X, Y)$ to be the set of all morphism of S-schemes from X to Y. In particular, if $S = \mathrm{Spec}\, A$ is an affine scheme, we write A-scheme for short.

Definition 2.2. *Let A be a ring, B an A-algebra, X an A-scheme. Define the set of B-points of X to be $X(B) = Mor_{Spec(A)}(Spec(B), X)$.*

Example 2.1. *(1)Let k be a field, l/K field extension, X a k-scheme. Then an l-point of X*

$$\begin{tikzcd} \mathrm{Spec}(l) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}\, k & \end{tikzcd} \longleftrightarrow \begin{tikzcd} l & & \mathcal{O}_{X,x} \arrow[ll] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \textit{for some } x \in X$$

where the first row of the second diagram is a local homomorphism. In particular, $X(k) \longleftrightarrow \{x \in X \big| k(x) = k\}$.
(2)Let $X = \mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[t])$. There are 2 $\mathbb{C}$-points of X where image is $(t^2+1) \in X$ with residue field isomorphic to $\mathbb{C}$. The fact comes from another fact that the following commutative diagram has two choices

$$\begin{tikzcd} \mathbb{C} & & \mathbb{R}[t]/(t^2+1) \arrow[ll] \\ & \mathbb{R} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(3)Let k be a field, X a k-scheme. Assume that $k[\varepsilon] = k[t]/(t^2)$ and $\mathrm{Spec}(k[\varepsilon]) = \{(\bar{t})\}$. Then a $k[\varepsilon]$-point of X

$$\begin{tikzcd} \mathrm{Spec}\, k[\varepsilon] \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}\, k & \end{tikzcd} \longleftrightarrow \begin{tikzcd} k[\varepsilon] & & \mathcal{O}_{X,x} \arrow[ll] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \textit{for some } x \in X$$

where the first row of the second diagram is a local homomorphism. Moreover, here $k(x) = k$.

Definition 2.3 (***Tangent Spaces***)**.** *Let X be a scheme. Assume that $x \in X$ with $k(x) = k$ for some field k. Define the tangent space of X at x to be $T_x(X) = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$, which is the dual space of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a k-vector space.*

Remark 2.2. *With this language of tangent space, we can conclude example 2.1 (3) that $X(k[\varepsilon]) = \{(x, v) \big| x \in X \text{ such that } k(x) = k, v \in T_x(X)\}$.*

2.2 Open Immersions and Closed Immersions

Definition 2.4. *Let X be a scheme. An open subscheme of X is an open subset $U \subseteq X$ equipped with sheaf $\mathcal{O}_X\big|_U$.*

Definition 2.5 (***Open Immersions***)**.** *Let X, Y be two schemes. An open immersion from X to Y is a morphism of schemes $f : X \longrightarrow Y$ inducing an isomorphism between X and some open subset of Y.*

Definition 2.6 (***Closed Immersions***)**.** *Let X, Y be two schemes. A closed immersion from X to Y is a morphism of schemes $f : X \longrightarrow Y$ such that*
(1)f on underlying topological spaces is a homeomorphism between X and some closed subset of Y.
(2)$f^\sharp$ is surjective.

Remark 2.3. *A closed subscheme of Y is a scheme X with a closed immersion of $f : X \longrightarrow Y$. And we would identify two closed subschemes $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y$ if there is an isomorphism $g : X \longrightarrow X'$ such that $f = f' \circ g$.*

Lemma 2.5. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose that f on underlying spaces is a homeomorphism between X and some closed subset $V \subseteq Y$. Then f is closed immersion if and only if $f_x^\sharp$ is surjective for all $x \in X$.*

Proof. "$\Rightarrow$": For all $x \in X$ and $(U, s) \in \mathcal{O}_{X,x}$, we have $f(U)$ is open in V. Thus there exists $W \subseteq Y$ open such that $W \cap V = f(U)$ and so $f^{-1}(W) = U$. Consider $(W, s) \in (f_*\mathcal{O}_X)_{f(x)}$. Since f is closed immersion, get there exists open neighbourhood $W' \subseteq W$ of $f(x)$ and $t \in \mathcal{O}_Y(W')$ such that $f^\sharp(W')(t) = s\big|_{f^{-1}(W')}$. Note that $x \in f^{-1}(W')$ and $(U, s) = (f^{-1}(W'), s\big|_{f^{-1}(W')}) = (f^{-1}(W'), f^\sharp(W')(t)) = f_x^\sharp((W', t))$ in $\mathcal{O}_{X,x}$. Get $f_x^\sharp$ is surjective.

"$\Leftarrow$": For all $y \in Y$, want to show that $f_y^\sharp : \mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is surjective. Note that if $y \notin V$, then $(f_*\mathcal{O}_X)_y = 0$, we only need to prove for $y \in V$. For all $(W, s) \in (f_*\mathcal{O}_X)_y$ where $s \in \mathcal{O}_X(f^{-1}(W))$, consider $(f^W, s) \in \mathcal{O}_{X,x}$ for all $x \in f^{-1}(W)$. As $f_x^\sharp$ is surjective, there exists open neighbourhood W_x of $f(x)$ and $t(x) \in \mathcal{O}_Y(W_x)$ such that $f^\sharp(W_x)(t(x)) = s\big|_{f^{-1}(W_x)}$. Since $y \in V$, there exists $x \in f^{-1}(W)$ such that $y = f(x)$. Now take $(W_x, t(x)) \in \mathcal{O}_{Y,y}$ and get $f_y^\sharp((W_x, t(x))) = (W_x, s\big|_{f^{-1}(W_x)}) = (W, s)$ in $(f_*\mathcal{O}_X)_y$. Thus $f^\sharp$ is surjective. □

Definition 2.7. *Let X be a scheme. For $f \in \mathcal{O}_(X)(X)$, define X_f to be the set of all $x \in X$ such that $f(x) \neq 0$.*

Lemma 2.6. *Let X be a scheme. Then for all $f \in \mathcal{O}_X(X)$ and $x \in X$, we have*
(1)X_f is open in X.
(2)Suppose that X can be covered by finitely many affine open subsets $\{U_i\}_{i \in I}$ satisfying that each $U_i \cap U_j$ is also covered by finitely many affine open subsets. Then $\rho_{XX_f} : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(X_f)$ induces $\mathcal{O}_X(X)_f \xrightarrow{\sim} \mathcal{O}_X(X_f)$.
(3)Let $f : X \longrightarrow Y$ be a morphism of schemes. Then for all $g \in \mathcal{O}_Y(Y)$, $f^{-1}(Y_g) = X_h$, where h is the image of g in $\mathcal{O}_X(X)$.

Proof. (1): For all $x \in X_f$, since $f(x) \neq 0$, we have $f_x \in \mathcal{O}_{X,x}^\times$. Assume that (U, s) is the inverse of f_x. Then $(U, \rho_{XU}(f)s) = 1$ in $\mathcal{O}_{X,x}$. Thus there exists an open neighbourhood $U' \subseteq U$ of x such that $(\rho_{XU}(f)s)\big|_{U'} = 1$ and so $\rho_{XU'}(f) \in \mathcal{O}_X(U')^\times$. Get $U' \subseteq X_f$ and so X_f is open.

(2): For all $x \in X$, as process in (1), there exists open neighbourhood $U_x \subseteq X_f$ of x and $s(x) \in \mathcal{O}_X(U_x)$ such that $f\big|_{U_x} s(x) = 1$. Note that $U_{x x \in X_f}$ form an open covering of X_f and $s(x)\big|_{U_x \cap U_y} = (f\big|_{U_x \cap U_y})^{-1} = s(y)\big|_{U_x \cap U_y}$. By property of sheaf, there exists $g \in \mathcal{O}_X(X_f)$ such that $g\big|_{U_x} = s(x)$. Define $\varphi : \frac{s}{f^n} \longmapsto s\big|_{X_f} g^n$. It is easy to check that φ is a well defined ring homeomorphism.

Want to show that φ is isomorphic. As X is covered by $U_{i i \in I}$, X_f is covered by $V_{i i \in I}$, where $V_i = X_f \cap U_i = D(f\big|_{U_i})$. By property of sheaf and flatness of localization, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(X)_f \arrow[r] \arrow[d, "\varphi"] & \prod_{i\in I} \mathcal{O}_X(U_i)_f \arrow[r] \arrow[d, "\sim"] & \prod_{i,j\in I} \mathcal{O}_X(U_i \cap U_j)_f \arrow[d, "\psi"] \\
0 \arrow[r] & \mathcal{O}_X(X_f) \arrow[r] & \prod_{i\in I} \mathcal{O}_X(V_i) \arrow[r] & \prod_{i,j\in I} \mathcal{O}_X(V_i \cap V_j)
\end{tikzcd}$$

By commutativity of diagram, it is easy to show that φ is injective. While for ψ, we can view $U_1 \cap U_j$ as another scheme with finite affine open covering, inducing that ψ is injective. By 5 Lemma in category theory, we get φ is surjective and thus isomorphic.

(3)Consider the following commutative diagram for all $x \in X$,

$$\begin{tikzcd}
\mathcal{O}_Y(Y) \arrow[r, "f^\sharp(Y)"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\
\mathcal{O}_{Y,f(x)} \arrow[r, "f^\sharp_x"] & \mathcal{O}_{X,x}
\end{tikzcd}$$

As the last row is a local homomorphism, we get the result. □

Proposition 2.1 (***Criterion of Affineness***)**.** *Let X be a scheme. Suppose that there are finitely many elements $f_1, \cdots, f_r \in \mathcal{O}_X$ such that $(f_1, \cdots, f_r) = (1) = \mathcal{O}_X(X)$ and each open set X_{f_i} is affine. Then X is affine.*

Proof. Set $A = \mathcal{O}_X(X)$ and $A_(i) = \mathcal{O}_X(X_{f_i})$. Since $(f_1, \cdots, f_n) = \mathcal{O}_X(X)$, $\{X_{f_i}\}_{i\in I}$ cover X. By Lemma 2.6, there is an isomorphism $A_{f_i} \xrightarrow{\sim} A_i$. Thus there are morphisms $u_i : X_{f_i} = \operatorname{Spec}(A_{f_i}) \longrightarrow \operatorname{Spec} A$. Check if $u_i\big|_{X_{f_i}\cap X_{f_j}} = u_j\big|_{X_{f_i}\cap X_{f_j}}$. Note that $X_{f_i} \cap X_{f_j} = D(f_i\big|_{X_{f_j}})$ in X_{f_j} and $X_{f_i} \cap X_{f_j} = D(f_j\big|_{X_{f_i}})$ in X_{f_i}. Get $u_i\big|_{X_{f_i}\cap X_{f_j}} = (Spec(A[\frac{1}{f_i}, \frac{1}{f_j}]) \longrightarrow \operatorname{Spec} A) = u_j\big|_{X_{f_i}\cap X_{f_j}}$. By Gluing Lemma, there exists morphism $u : X \longrightarrow \operatorname{Spec} A$. It is easy to see that for all $x \in X$, u_x is isomorphic and so is u. □

Lemma 2.7. *Let $\varphi : A \longrightarrow B$ be a ring homomorphism. If φ is injective, then the associated morphism $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ has dense image. Moreover, $f^\sharp : \mathcal{O}_{Spec(A)} \longrightarrow f_*\mathcal{O}_{Spec(B)}$ is also injective.*

Proof. Note for all $a \in A$, $f(D(a)) = D(\varphi(a))$. To show f has dense image, it suffices to show $D(\varphi(a)) \neq \varnothing$ if $D(a) \neq \varnothing$, which immediately comes from φ is injective. As $f^[\sharp]$ is injective if and only if $f^\sharp(D(a))$ is injective for all standard open subset $D(a) \subseteq \operatorname{Spec} A$. While $f^\sharp(D(a)) : A_a \longrightarrow B_{\varphi(a)}$ is obviously injective, get $f^\sharp$ is injective. □

Theorem 2.1. *Let $X = \operatorname{Spec} A$ be an affine scheme, $i : Z \hookrightarrow X$ closed immersion. Then $Z \cong \operatorname{Spec}(A/I)$ for some ideal $I \subseteq A$.*

Proof. First show that Z is affine. Cover Z by affine open subsets ${V_j}_{j\in J}$. Since i is a closed immersion, there exists ${U_j}_{j\in J}$ such that $i^{-1}(U_j) = V_j$. Take standard open covering $\{U_{jk} = D(f_{jk})\}$ of U_j, then Z is covered by $i^{-1}(U_{jk})$. Note that there is a commutative diagram for all $x \in Z$,

$$\begin{tikzcd} A \arrow[r, "i^\sharp(X)"] \arrow[d] & \mathcal{O}_Z(Z) \arrow[d] \\ \mathcal{O}_{X,i(x)} \arrow[r, "i_x^\sharp"] & \mathcal{O}_{Z,x} \end{tikzcd}$$

Assume $h_{jk} = i^\sharp(X)(f_{jk})$. As $i_x^\sharp$ is a local homomorphism and surjective, $x \in Z_{h_{jk}}$ if and only if $h_{jk}(x) \in \mathcal{O}_{Z,x}^\times$ if and only if $f_{jk}(i(x)) \in \mathcal{O}_{X,i(x)}^\times$ if and only if $i(x) \in D(f_{jk})$. Get $i^{-1}(U_{jk}) = Z_{h_{jk}}$. While $i^{-1}(U_{jk}) \subseteq i^{-1}(U_j)$ and $i^{-1}(U_j) = V_j$ is affine, get $i^{-1}(U_{jk}) = Z_{h_{jk}} \cap V_j$ is affine. Since $X = i(Z) \cup (X \setminus i(Z)) = (\cup_{j,k} U_{jk}) \cup (\cup_r D(f_r))$ and X is quasi-compact, there exists finitely many j,k such that U_l cover X and $i^{-1}(D(f_l))$ is affine open subset of Z or empty set. Thus we get a finite affine open covering $\{Z_{h_l}\}$ of Z. As $(h_l|l) \supseteq i^\sharp(X)((f_l|l))$ and $1 \in (f_l|l)$, get $(h_l|l) = \mathcal{O}_Z(Z)$. By Proposition 2.1, Z is affine.

Now we can assume $Z = \operatorname{Spec}(B)$. Then $i : Z \longrightarrow X$ associates with a ring homomorphisms $\varphi : A \longrightarrow B$. It suffices to prove that φ is surjective. Consider injective homomorphism $\widetilde{\varphi} : A/\ker(\varphi) \hookrightarrow B$ and take $X' = \operatorname{Spec}(A/\ker(\varphi))$. Get a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "g"'] & & X \\ & X' \arrow[ur, "i'"'] & \end{tikzcd}$$

Want to show g is isomorphic. As other two morphisms are both injective, i is closed immersion and i' is closed, get g is injective and closed. By Lemma 2.7, get g has dense image and $g^\sharp$ is injective. Thus g on underlying topological spaces is homeomorphism. While for all $x \in Z$, $i_x^\sharp$ and ${i'}^\sharp_{g(x)}$ are both surjective, get $g_x^\sharp$ is surjective. By Proposition 1.11, g is isomorphic. $\square$

2.3 Finite Type and Finite Morphisms

Definition 2.8 (***Local on the Base***)**.** *Let P be a property of morphisms of schemes. We say that P is local on the base if for any morphism $f : X \longrightarrow Y$ and affine open covering $\{V_i\}_{i\in I}$ of Y, the following conditions are equivalent*
(1)f satisfies P
(2)each induced morphism $f_i : f^{-1}(V_i) \longrightarrow V_i$ satisfies P.

Lemma 2.8. *Let P be a property of morphisms of schemes local on the base. Then for any morphism $f : X \longrightarrow Y$, the following conditions are equivalent*
(1)f satisfies P
(2)For any affine open covering $\{V_i\}_{i\in I}$, induced morphisms f_i satisfies P
(3)There exists an affine open covering $\{V_i\}_{i\in I}$ such that induced morphisms f_i satisfies P

Remark 2.4. *Open, closed immersion are local on the base.*

Definition 2.9 (***Quasi-compact***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is quasi-compact if for all affine open subset $V \in Y$, the preimage of V in X is quasi-compact.*

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose Y admits an affine open covering $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subseteq Y$ be an affine open subset. Take standard open covering $\{V_{ik}\}$ of $V_i \cap V$, where $V_{ik} = D(f_{ik})$ in V_i for some $f_{ik} \in \mathcal{O}_Y(V_i)$. Since all such V_{ik} cover V and V is quasi-compact, there is finite open subcovering $\{V_l\}$. As $f^{-1}(V) = \cup_l f^{-1}(V_l)$, it suffices to show that $f^{-1}(V_l)$ is quasi-compact. While $f^{-1}(V_i)$ is quasi-compact, there is finite affine open covering $\{U_{ij}\}$. Since $V_l \subseteq V_i$ for some $i \in I$, $f^{-1}(V_l)$ is covered by $U_{ijl} = U_{ij} \cap f^{-1}(V_l)$. Note that $f^{-1}(V_l) = X_{h_l}$ by Lemma 2.6, U_{ijl} is affine. Thus $f^{-1}(V_l)$ is quasi-compact and so is $f^{-1}(V)$. Get f is quasi-compact. □

Definition 2.10 (***Locally of Finite Type***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is locally of finite type if for all affine open subset $V = \operatorname{Spec} A \subseteq Y$, there exists affine open covering $\{U_i = \operatorname{Spec}(B_i)\}$ of $f^{-1}(V)$ such that B_i are all A-algebra of finite type. Or equivalently, for all affine open subset $U = \operatorname{Spec}(B) \subseteq f^{-1}(V)$, B is an A-algebra of finite type.*

Lemma 2.9. *Let A be a ring, B an A-algebra. Then*
(1)Suppose that $\operatorname{Spec}(B)$ covered by $\{D(b_i)\}_{i\in I}$, where $b_i \in B$., satisfying that each B_{b_i} is an A-algebra of finite type. Then B is also an A-algebra of finite type.
(2)If $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ is an open immersion, then B is an A-algebra of finite type.

Proof. (1): Since $\operatorname{Spec}(B)$ is quasi-compact, we can assume that $\{D(b_i)\}_{i\in I}$ is finite. As B_{b_i} is A-algebra of finite type, assume that $B_{b_i} = A[\frac{a_{ij}}{b_i^{n_{ij}}}]$, where $a_{ij} \in B$. Consider the A-subalgebra C of B generated by $\{b_i\}$ and $\{a_{ij}\}$. Then $B_{b_i} \subseteq C_{b_i} \subseteq B_{b_i}$, get $C_{b_i} = B_{b_i}$. Since $D(b_i)$ cover B, we have $\sum_{i\in I} b_i' b_i = 1_B$ where $b_i' \in B$. Consider the A-subalgebra D of B generated by C and $\{b_i'\}$. Obviously, D is of finite type over A. While for all $b \in B$, since $\frac{b}{1} \in B_{b_i}$, there exists $k_i \in \mathbb{N}$ such that $b_i^{k_i} b = b_i^{k_i} c_i$ for some $c_i \in C$. By finiteness of i, we can take a supremum k of k_i. Thus $b = b(\sum_i b_i' b_i)^{\sharp(I)k} = b\sum_i b_i^k d_{ik} = \sum_i b_i^k c_i d_{ik} \in D$. Get $B = D$ and so B is of finite type over A.

(2): View $\operatorname{Spec}(B)$ is an open subset of $\operatorname{Spec} A$. Cover $\operatorname{Spec}(B)$ by finite standard open subsets $\{D(a_i) = D(b_i)\}$. Since $B_{b_i} = \mathcal{O}_{Spec(B)}(D(b_i)) = \mathcal{O}_{Spec(A)}(D(a_i)) = A_{a_i}$, get B_{b_i} is of finite type over A_{a_i} and so is of finite type over A. BY (1), get B is of finite type over A. □

Proposition 2.3. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose Y admits an affine open covering $\{V_i\}_{i\in I}$ with each $f^{-1}(V_i)$ covered by affine open subsets $\{U_{ij}\}$ such that $\mathcal{O}_X(U_{ij})$ is an $\mathcal{O}_Y(V_i)$-algebra of finite type via $\rho_{f^{-1}(V_i U_{ij})} \circ f^\sharp(V_i)$. Then f is locally of finite type.*

Proof. Let $V = \operatorname{Spec} A \subseteq Y$ be an affine open subset. Similar to proof of Proposition 2.2, we can take standard open covering V_{ik} of $V \cap V_i$, where $V_{ik} = D(g_{ik})$ in V_i. Each $f^{-1}(V_i)$ can be covered by affine open subsets $\{U_{ij}\}$. Consider $U_{ijk} = U_{ij} \cap f^{-1}(V_{ik})$. As $f^{-1}(V_{ik}) = X_{h_{ik}}$, get U_{ijk} is affine. Also since V_{ik} is localization of V_i, get $\mathcal{O}_X(U_{ijk})$ is of finite type over $\mathcal{O}_Y(V_{ik})$. Note $V_{ik} \hookrightarrow V$ is an open immersion, by Lemma 2.9, get $\mathcal{O}_Y(V_{ik})$ is of finite type over $\mathcal{O}_Y(V)$. Thus we can cover $f^{-1}(V)$ with affine open subsets $\{U_{ijk}\}$ such that $\mathcal{O}_X(U_{ijk})$ is of finite type over $\mathcal{O}_Y(V)$.

Let $U \subseteq f^{-1}(V)$ be affine open subset. For all $x \in U$, there exists standard open subset $D(b(x)) \subseteq U$ of x such that $D(b_i) \subseteq U_{ijk}$ for some U_{ijk}. Thus $D(b(x))$ form an open covering of U. Note for each $D(b(x))$, $\mathcal{O}_X(D(b(x)))$ is localization of $\mathcal{O}_X(U_{ijk})$ and so is of finite type over $\mathcal{O}_Y(V)$. By Lemma 2.9, get $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_Y(V)$. Thus f is locally of finite type. □

Definition 2.11 (***of Finite Type***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is of finite type if f is quasi-compact and locally of finite type.*

Remark 2.5. *With Proposition 2.2 and Proposition 2.3, of finite type is local on the base.*

Example 2.2. *(1)Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of affine schemes. Then f is of finite type if and only if the associated ring homomorphisms $\varphi : A \longrightarrow B$ is of finite type.*
(2)Let k be an algebraically closed field k. Affine or projective abstract varieties over k is of finite type over $\operatorname{Spec} k$.
(3)Let k be an algebraically closed field k. k-morphisms between affine or projective abstract varieties over k is locally of finite type.
(4)An open immersion is always locally of finite type but not quasi-compact in general. For example, we can take $Y = \operatorname{Spec}(k[x_1, x_2, \cdots])$, $X = Y \setminus \{(0)\}$ and $i : X \hookrightarrow Y$, then $i^{-1}(Y) = X$ is not quasi-compact.

Proposition 2.4. *(1)The composition of two morphisms of finite type (resp. locally of finite type, quasi-compact) is still of finite type (resp. locally of finite type, quasi-compact).*
(2)Let $f : X \longleftarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms of schemes. If $g \circ f$ is locally of finite type, then f is locally of finite type.

Proof. (1): Trivial.

(2): Take affine open covering $\{W_i\}$ of Z. Then $g^{-1}(W_i)$ form an open covering of Y and for each W_i, $f^{-1}g^{-1}(W_i)$ has an affine open covering $\{U_{ij}\}$ such that $\mathcal{O}_X(U_{ij})$ is of finite type over $\mathcal{O}_Z(W_j)$. For all affine open subset $V \subseteq Y$, can cover $V \cap g^{-1}(W_i)$ with standard open subsets $\{V_{ik}\}$, where $V_{ik} = D(g_{ik})$ in $g^{-1}(W_i)$. Consider $U_{ijk} = U_{ij} \cap f^{-1}(V_{ik})$ which is affine. Thus $f^{-1}(V)$ is covered by U_{ijk}.

For all affine open subset $U \subseteq f^{-1}(V)$ and $x \in U$, can take standard open neighbourhood $D(b(x)) \subseteq U$ of x such that $D(b(x)) \subseteq U_{ijk}$ for some U_{ijk}. While $\mathcal{O}_X(D(b(x)))$ is localization of $\mathcal{O}_X(U_{ijk})$ and so is of finite type over $\mathcal{O}_X(U_{ijk})$, consider if $\mathcal{O}_X(U_{ijk})$ is of finite type over $\mathcal{O}_Y(V)$. Note that there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(U_{ijk}) & & \mathcal{O}_Z(V_i) \arrow[ll] \arrow[dl] \\ & \mathcal{O}_Y(V_{ik}) \arrow[ul] & \end{tikzcd}$$

Get $\mathcal{O}_X(U_{ijk})$ is of finite type over $\mathcal{O}_Y(V_{ik})$. In addition, $\mathcal{O}_Y(V_{ik})$ is of finite type over $\mathcal{O}_Y(V)$. Get $\mathcal{O}_X(D(b(x)))$ is of finite type over $\mathcal{O}_Y(V)$. By Lemma 2.9, get $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_Y(V)$. □

Definition 2.12 (***Affine***)**.** *Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of schemes. We say that f is affine if for all affine open subset $V \subseteq Y$, $f^{-1}(V)$ is affine.*

Definition 2.13 (***Finite***)**.** *Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of schemes. We say that f is finite if for all affine open subset $V \subseteq Y$, $f^{-1}(V)$ is affine and $\mathcal{O}_X(f^{-1}(V))$ is finite over $\mathcal{O}_Y(V)$.*

Proposition 2.5. *Composition of two finite (resp. affine) morphisms is still finite (resp. affine).*

Lemma 2.10. *Let A be a ring, B an A-algebra via ring homomorphism $\varphi : A \longrightarrow B$. Suppose that $\operatorname{Spec} A$ is covered by $\{D(a_i)\}_{i\in I}$ such that each $B_{\varphi(a_i)}$ is finite over A_{a_i}. Then B is finite over A.*

Proof. Since $\operatorname{Spec} A$ is quasi-compact, we can assume that $\{D(a_i)\}_{i\in I}$ is finite. Let $b_i = \varphi(a_i)$. As B_{b_i} is finite A_{a_i}-algebra, assume that $B_{b_i} = \sum_j A_{ai} \frac{b_{ij}}{b_i^{n_{ij}}}$, where $b_{ij} \in B$. Consider the A-submodule C of B generated by $\{b_{ij}\}$. Then $B_{b_i} \subseteq C_{b_i} \subseteq B_{b_i}$, get $C_{b_i} = B_{b_i}$. Since $D(a_i)$ cover B, we have $\sum_{i\in I} a_i' a_i = 1_A$ where $a_i' \in A$. While for all $b \in B$, since $\frac{b}{1} \in B_{b_i}$, there exists $k_i \in \mathbb{N}$ such that $b_i^{k_i} b = b_i^{k_i} c_i$ for some $c_i \in C$. By finiteness of i, we can take a supremum k of k_i. Thus

$$\begin{aligned} b &= (\sum_i a_i' a_i)^{\sharp(I)k} b \\ &= \sum_i a_{ik} a_i^k b \\ &= \sum_i a_{ik} b_i^k b \\ &= \sum_i a_{ik} b_i^k c_i \\ &= \sum_i a_{ik} a^k c_i \in C \end{aligned}$$

Get $B = C$ and so B is finite over A. □

Proposition 2.6. *Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of schemes. Suppose that Y admits an affine open covering $\{V_i\}$ such that each $U_i = f^{-1}(V_i)$ is affine. Then*
(1)f is affine
(2)If moreover each $\mathcal{O}_X(U_i)$ is finite over $\mathcal{O}_Y(V_i)$, then f is finite.

Proof. (1): Let $V \subseteq Y$ be an affine open subset. For all $y \in V \cap V_i$, take standard open subset $D(a(y)) \subseteq V$ such that $y \in D(a(y)) \subseteq V \cap V_i$. Now $D(a(y))$ is an affine subset of V_i. Take standard open subset $D(b(y)) \subseteq V_i$ such that $y \in D(b(y))$. By Proposition 1.12, we have that $D(a(y)) \cap D(b(y))$ is a standard open subset of $D(a(y))$. Thus we can take V_{ik} cover V, satisfying that V_{ik} is standard open subset both in V and V_i. Since V is quasi-compact, can assume that $\{V_{ik}\}$ is finite. Thus U can be covered by finitely many $f^{-1}(V_{ik})$. While $f^{-1}(V_{ik}) = f^{-1}(V_{ik}) \cap f^{-1}(V_i)$ is affine, get $f^{-1}(V_{ik})$ is affine subset of U of the form U_h for some $h \in \mathcal{O}_X(U)$. Assume $V_{ik} = D(g_{ik})$ in V, get $f^{-1}(V_{ik}) = U_{f^\sharp(V)(g_{ik})}$. In addition, since $(f^\sharp(V)(g_{ik})\big| i,k) \supseteq f^\sharp(V)((g_{ik}\big| i,k))$ and $1 \in (g_{ik}\big| i,k)$, we have $(f^\sharp(V)(g_{ik})\big| i,k) = \mathcal{O}_X(U)$. Thus by criterion of affineness, get U is affine.

(2): By assumption, $\mathcal{O}_X(f^{-1}(V_i))$ is finite over $\mathcal{O}_Y(V_i)$. Then their localizations still satisfy that $\mathcal{O}_X(f^{-1}(V_{ik}))$ is finite over $\mathcal{O}_Y(V_{ik})$. By Lemma 2.10, $\mathcal{O}_X(U)$ is finite over $\mathcal{O}_Y(V)$. Thus f is finite. $\square$

Example 2.3. *(1)Closed immersion is finite.*
(2)Open immersion is not finite in general, as A_f is not finite over A in general.
(3)Open immersion is not affine in general, as $\mathbb{A}^1_k \setminus \{(0)\}$ is not affine.
(4)Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of affine schemes. Then f is finite of and only if the associated ring homomorphism $\varphi : A \longrightarrow B$ makes B finite over A.

2.4 Gluing and Fibre Product

Lemma 2.11 (***Gluing Lemma***)**.** *Let S be a scheme, $\{X_i\}_{i\in I}$ a collection of S-schemes. For each i, give open subschemes $\{X_{ij}\}_{j\in I}$ of X_i and for each i,j, give S-isomorphism $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$, satisfying that*
(1)$f_{ij} = f_{ji}^{-1}$. In particular, f_{ii} is identity.
(2)$f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$.
(3)$(f_{jk} \circ f_{ij})\big|_{X_{ij}\cap X_{ik}} = f_{ik}\big|_{X_{ij}\cap X_{ik}}$.

Then there exists an S-scheme X unique up to isomorphism equipped with S-open immersions $g_i : X_i \longrightarrow X$ such that $X = \cup_{i\in I} g_i(X_i)$ and $(g_j \circ f_{ij})\big|_{X_{ij}} = g_i\big|_{X_{ij}}$. X is called the gluing S-scheme of $\{X_i\}_{i\in I}$ along $\{X_{ij}\}$.

Remark 2.6. *(1)Let $S = \operatorname{Spec}(\mathbb{Z})$. Then this lemma gives a way to glue scheme.*
(2)Gluing $\{X_i\}_{i\in I}$ along $\{X_{ij} = \varnothing\}$ gives $X = \sqcup_{i\in I} X_i$.

Proof. Consider topological space $X = (\sqcup_{i\in I} X_i)/\sim$, where $\sim$ is an equivalence relation and $x \sim y$ if $x \in X_{ij}$, $y \in X_{ji}$ and $y = f_{ij}(x)$ for some i,j. Get open inclusions $g_i : X_i \hookrightarrow X$ and $g_j \circ f_{ij} = g_i\big|_{X_{ij}}$. Need to construct sheaf of rings on X. Assume that $U_i = g_i(U_i)$. Take sheaf $\mathcal{O}_{U_i}$ to be $(g_i)_*\mathcal{O}_{X_i}$. Note that $\mathcal{O}_{U_i}\big|_{U_i\cap U_j} = \mathcal{O}_{U_j}\big|_{U_i\cap U_j}$ and U_i cover X. By gluing lemma for sheaves, we get a sheaf $\mathcal{O}_X$ on X such that $\mathcal{O}_X\big|_{U_i} = \mathcal{O}_{U_i}$. As g_i is isomorphic, take $h_i : U_i \xrightarrow{g_i^{-1}} X_i \longrightarrow S$ and we have that $h_i\big|_{U_i\cap U_j} = h_j\big|_{U_i\cap U_j}$. By gluing lemma of morphisms, get $X \longrightarrow S$ and so X is an S-scheme. Uniqueness up to isomorphism is obvious. $\square$

Definition 2.14 (***Fibre Product***)**.** *Let S be a scheme, X, Y two S-schemes. Define the fiber product of X and Y over S to be a S-scheme $W = X \times_S Y$ equipped with two S-morphisms $p : W \longrightarrow X$ and $q : W \longrightarrow Y$ satisfying the following universal property.*

$$\begin{tikzcd}
\forall Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Proposition 2.7 (***Existence and Uniqueness of Fibre Product***)**.** *Let S be a scheme, X, Y two S-schemes. Then the fiber product of X and Y over S exists and is unique up to isomorphism.*

Proof. Uniqueness up to isomorphism immediately comes from universal property. Here we only talk about the existence. For affine case, if $X = \operatorname{Spec} A$, $Y = \operatorname{Spec}(B)$ and $S = Spec(C)$, then we can take $X \times_S Y = \operatorname{Spec}(A \otimes_C B)$. With universal property of tensor product, we can easily check $\operatorname{Spec}(A \otimes_C B)$ is fiber product.

If $X \times_S Y$ exists, then obviously $Y \times_S X$ exists and $U \times_S Y = p^{-1}(U)$ exists for all open subset $U \subseteq X$. Consider the case that Y,S are still affine while X abtrary. Cover X by affine open subsets U_i. Then $U_i \times_S Y$ exists and so does $(U_i \cap U_j) \times_S Y$. Taking $X_i = U_i \times_S Y$ and $X_{ij} = (U_i \cap U_j) \times_S Y$, we get isomorphisms $f_{ij} : X_{ij} \longrightarrow X_{ji}$. Bu gluing lemma of schemes, there exists gluing scheme W. It is easy to check that W satisfies the conditions for fiber product.

Consider the case that S is affine while X,Y abtrary. Cover Y by affine open subsets V_i. Then $X \times_S V_i$ exists. Similar to the above process, get $X \times_S Y$ exists.

For general case, cover S by affine open subsets W_i. Assume $f : X \longrightarrow S$ and $g : Y \longrightarrow S$. Then $f^{-1}(W_i) \times_{W_i} g^{-1}(W_i)$ exists for all i. Thus $f^{-1}(W_i \cap W_j) \times_{W_i} g^{-1}(W_i \cap W_j)$ exists for all i,j. Note that $f^{-1}(W_i \cap W_j) \times_{W_i} g^{-1}(W_i \cap W_j)$ and $f^{-1}(W_i \cap W_j) \times_{W_i \cap W_j} g^{-1}(W_i \cap W_j)$ satisfy the same universal property. We can view $f^{-1}(W_i \cap W_j) \times_{W_i \cap W_j} g^{-1}(W_i \cap W_j)$ as open subset of $f^{-1}(W_i) \times_{W_i} g^{-1}(W_i)$. Take $X_i = f^{-1}(W_i) \times_{W_i} g^{-1}(W_i)$ and $X_{ij} = f^{-1}(W_i \cap W_j) \times_{W_i \cap W_j} g^{-1}(W_i \cap W_j)$. There are isomorphisms $X_{ij} \xrightarrow{\sim} X_{ji}$. By gluing lemma of schemes, $X \times_S Y$ exists. □

Proposition 2.8. *Let S be a scheme, X, Y S-schemes, . Then*
(1)$X \times_S S \cong X$
(2)$X \times_S Y \cong Y \times_S X$
(3)$(X \times_S Y) \times_S Z \cong X \times_S (Y \times_S Z)$.
(4)If moreover Z is a Y-scheme, then $(X \times_S Y) \times_Y Z \cong X \times_S Z$.

Example 2.4. *Let X be an affine or projective abstract variety over algebraically closed field k, l/k field extension. Then $X \times_k l$ is affine or projective abstract "variety" over l and $(X \times_k l)(l) = X(l)$.*

Definition 2.15. *Let $f : X \longrightarrow Y$ be a morphism of schemes. For $y \in Y$ with residue field $k(y)$, define the fiber of f over y to be the $k(y)$-scheme $X_y = X \times_Y k(y)$.*

Proposition 2.9. *Let $f : X \longrightarrow Y$ be a morphism of schemes, $y \in Y$. Then projection $p : X_y \longrightarrow X$ induces a homeomorphism between X_y and $f^{-1}(y)$.*

Proof. Let $V \subseteq Y$ be an open subset. Then $X_y = X \times_Y k(y) \cong (X \times_Y V) \times_V k(y)$. Note that $X \times_Y V = f^{-1}(V)$, get $X_y \cong f^{-1}(V)_y$. Thus by taking affine open neighbourhood of y, we can assume that $Y = \operatorname{Spec} A$.

Let $U \subseteq X$ be an open subset. Then $p^{-1}(U) \cong U \times_Y k(y) = U_y$ so that we can also assume $X = \operatorname{Spec}(B)$. Since f now is a morphism of affine schemes, we can take its associated ring homomorphisms $\varphi : A \longrightarrow B$ and y corresponds to a prime ideal $\mathfrak{p}$ of A. Then p corresponds to the following commutative diagram

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & B \otimes_A \mathrm{Frac}(A/\mathfrak{p}) \\ & B \otimes_A A_{\mathfrak{p}} \arrow[ur] & \end{tikzcd}$$

For all $\mathfrak{q} \in im(p)$, we have that $\mathfrak{q} \supseteq \varphi(\mathfrak{p})B$ and $\mathfrak{q} \cap \varphi(A \setminus \mathfrak{p}) = \varnothing$. While for all $\mathfrak{q} \in \{\mathfrak{q} \in \mathrm{Spec}(B) \big| \mathfrak{q} \supseteq \varphi(\mathfrak{p})B, \mathfrak{q} \cap \varphi(A \setminus \mathfrak{p}) = \varnothing\} = f^{-1}(\mathfrak{p})$, consider subset $\mathfrak{P} = \{b \otimes \frac{\overline{a'}}{\overline{a}} \big| b \in \mathfrak{q}, \overline{a}, \overline{a'} \in A/\mathfrak{p}\} \subseteq B \otimes_A A/\mathfrak{p}$. With homomorphisms $B \otimes_a \mathrm{Frac}(A/\mathfrak{p}) \longrightarrow \mathrm{Frac}(B/\mathfrak{q})$, it is easy to see that $\mathfrak{P}$ is a prime ideal of $B \otimes_a \mathrm{Frac}(A/\mathfrak{p})$ and $p(\mathfrak{P}) = \mathfrak{q}$. Thus $im(p) = f^{-1}(\mathfrak{p})$.

Want to show that $\widetilde{p} : X_y \longrightarrow im(p)$ is an open map. It suffices to prove that $p(D(b \otimes 1))$ is open in $im(p)$ for all $b \in B$. Obviously, we have $p(D(b \otimes 1)) \subseteq D(b) \cap im(p)$. While for all $\mathfrak{q} \in D(b) \cap im(p)$, there exists $\mathfrak{P} \in X_y$ such that $p(\mathfrak{P}) = \mathfrak{q}$. If $b \times 1 \in \mathfrak{P}$, then $b \in p(\mathfrak{P}) = \mathfrak{q}$, contradiction. Thus $\mathfrak{P} \in D(b \otimes 1)$. Thus p induces a homeomorphism between X_f and $f^{-1}(\mathfrak{p})$. $\square$

Definition 2.16 (***Base Change***)**.** *Let S be a scheme, X an S-scheme. If S' is an S-scheme, we call $X \times_S S' \longrightarrow S \times_S S' \cong S'$ the base change of X in the morphism $S' \longrightarrow S$.*

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

A property of morphisms P is called stable under base change if for all $X \longrightarrow Y$ satisfying P and Y' Y-scheme, $X \times_Y Y' \longrightarrow Y'$ still satisfies P.

Proposition 2.10. *The following propositions of morphisms are stable under base change.*
(1)open immersion
(2)closed immersion
(3)locally of finite type
(4)quasi-compact
(5)of finite type
(6)affine
(7)finite

Corollary 2.1. *Let S be a scheme, X, Y two S-schemes of finite type (resp. locally of finite type, quasi-compact, affine, finite). Then so is $X \times_S Y$ over S.*

Reason 2.3. *These properties are all stable under composition and base change.*

Corollary 2.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type. Then for all $y \in Y$, X_y is finite type over $k(y)$.*

3 Global Properties of Schemes

Definition 3.1. *Let X be a topological space. We say that X is noetherian if all decreasing sequence of closed subsets stabilizes. Or equivalently, all increasing sequence of open subsets* *stabilizes.*

Remark 3.1. *If A is a noetherian ring, then* $\operatorname{Spec} A$ *is noetherian. But if* $\operatorname{Spec} A$ *is noetherian, A need not be noetherian in general. For example, we can take $A = k[x_1, x_2, \cdots]/(x_1, x_2^2, \cdots)$.*

Proposition 3.1. *Let X be a topological space. Then*
(1)If X is noetherian, then all open or closed subsets of X is also noetherian.
(2)X is noetherian if and only if all open subsets of X are quasi-compact.
(3)If X is covered by finitely many $\{X_i\}$ which are noetherian, then X is noetherian.

Definition 3.2. *Let X be a topological space. We say that X is irreducible if X is nonempty and X cannot be covered by two proper closed subsets. Or equivalently, X is nonempty and any 2 nonempty open subsets intersect.*

Proposition 3.2. *Let X be a topological space. Then*
(1)If X is irreducible, then every nonempty open subset of X is also irreducible.
(2)Let $Y \subseteq X$ be a topological subspace. If Y is irreducible, then closure $\overline{Y}$ of Y in X is also irreducible.

Theorem 3.1. *Let X be a noetherian topological space. Then every nonempty closed subset Y of X can be uniquely written as a finite union $Y = \cup_i Y_i$, where Y_i is irreducible closed subset and $Y_i \not\supseteq Y_j$ for all $i \neq j$. The decomposition of Y is called the irreducible components of Y.*

Proof. Existence: suppose there exists nonempty closed subset Y of X with no such finite decomposition. As X is noetherian, we can take a minimal Y. Obviously, Y is not irreducible. Thus $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are proper closed subsets. As Y_1 and Y_2 are nonempty, get Y_1 and Y_2 can be written as such finite decompositions, inducing that Y can be written as such finite decomposition, contradiction. Thus all nonempty closed subsets have such finite decomposition.

Uniqueness: Assume $Y = \cup_i Y_i$ and $Y = \cup_j Z_j$ are two decompositions. Then $Z_j \subseteq Y = \cup_i Y_i$ and so $Z_j = \cup_i (Z_j \cap Y_i)$. Since Z_j is irreducible, there exists i such that $Z_j \subseteq Y_i$. While for Y_i there also exists j' such that $Y_i \subseteq Z_{j'}$, inducing $Z_j \subseteq Z_{j'}$. By conditions for decomposition, get $j = j'$ and $Y_i = Z_j$. Thus we give a bijection between $\{Y_i\}$ and $\{Z_j\}$ so that $Y = \cup_i Y_i$ and $Y = \cup_j Z_j$ are the same decomposition. □

Remark 3.2. *Without noetherian condition, we can use the Axiom of Choice to prove weak edition of this theorem i.e. decomposition without finiteness.*

3.1 Noetherian Schemes

Definition 3.3 (***Locally Noetherian Schemes***)**.** *Let X be a scheme. We say that X is locally noetherian if for all affine open subset $U = \operatorname{Spec} A$ of X, A is noetherian.*

Lemma 3.1. *Let X be a locally noetherian scheme. Then the minimal irreducible closed subsets of X are closed points.*

Proof. Assume V is one minimal irreducible closed subset of X. Then for all $x \in V$, $\overline{\{x\}} = V$. If there are two points in V, suppose they are x and y. Take a noetherian affine open neighbourhood $U = \operatorname{Spec} A$ of x. Then since $\overline{\{y\}} = V$, we have that $y \in X$. While minimal irreducible closed subsets in an affine scheme are the maximal ideals. Get $x = y$, thus V is one-point set. $\square$

Definition 3.4 (***Noetherian Schemes***)**.** *Let X be a scheme. We say that X is noetherian if X can be covered by finitely many affine open subset $U = \operatorname{Spec} A$ of X with A is noetherian.*

Remark 3.3. *If X is noetherian scheme, then the underlying topological space of X is noetherian. While if the underlying topological space of X is noetherian, X is not noetherian in general. Example can be taken the same as Remark 3.1.*

Proposition 3.3. *Let X be a noetherian scheme, $U \subseteq X$ open subset. Then U is noetherian as scheme.*

Proposition 3.4. *Let X be a noetherian scheme. If $i : U \hookrightarrow X$ is an open immersion, then i is of finite type.*

Reason 3.1. *By noetherian, we know i is quasi-compact. By open immersion, we know is locally of finite type.*

Proposition 3.5. *Let S be a scheme, X, Y two S-schemes. If X is noetherian and of finite type over S, then all S-morphism $f : X \longrightarrow Y$ are of finite type.*

Reason 3.2. *By Proposition 2.4, we know that f is locally of finite type. Since X is noetherian, get f is also quasi-compact.*

Remark 3.4. *All affine or projective abstract varieties over algebraically closed field k is noetherian. Thus k-morphism between affine or projective abstract varieties is of finite type.*

Theorem 3.2. *Let X be a scheme. Suppose that X admits an affine open covering $\{U_i = Sepc(A_i)\}$ with each A_i noetherian. Then X is locally noetherian.*

Proof. Let U be an affine open subset of X. Similarly, we can cover U by U_{ik} such that $U_{ik} \subseteq U$ is of the form $D(f_{ik})$ and $U_{ik} \subseteq U_i$ is of the form $D(g_{ik})$. Note that localization of noetherian ring is noetherian, get $\mathcal{O}_X(U_{ik})$ is noetherian. Remains to show that if $\operatorname{Spec} A$ is covered by $\{D(f_i) = \operatorname{Spec}(A_{f_i})\}$ with each A_{f_i} noetherian, then A is noetherian.

Since $\operatorname{Spec} A$ is quasi-compact, can assume $\{D(f_i) = \operatorname{Spec}(A_{f_i})\}$ finite. Consider $\varphi_i : A \longrightarrow A_{f_i}$ localization. For any increasing sequence of ideals of A, $I_1 \subseteq I_2 \subseteq \cdots$, by localization, we get increasing sequence of ideals of A_{f_i}. Since A_{f_i} is noetherian and there are only finitely many i, there exists n such that $\forall k \geq n$, $\varphi_i(I_k)A_{f_i} = \varphi_i(I_{k+1})A_{f_i}$. Thus it suffices to show that $I_k = \cap_i(\varphi_i(I_k)A_{f_i} \cap A)$. For $s \in \cap_i(\varphi_i(I_k)A_{f_i} \cap A)$, assume that $s = \frac{a_i}{f_i^{n_i}}$ in A_{f_i} where $a_i \in I_k$ Similar to proof of Theorem 1.1, we can take a_i' and f_i' such that $\frac{a_i'}{f_i'} = s$, $a_i' f_j' = a_j' f_i'$ and $\sum_i b_i f_i' = 1$ where $b_i \in A$. Consider element $\frac{\sum_i b_i a_i'}{1}$. Then as $a_j' = \sum_i b_i f_i' a_j' = \sum_i b_i a_i' f_j'$, $\frac{\sum_i b_i a_i'}{1} = s$ in A_{f_j} for all j. Thus $s = \frac{\sum_i b_i a_i'}{1}$ and so $s \in I_k$. Get $I_k = \cap_i(\varphi_i(I_k)A_{f_i} \cap A)$. $\square$

3.2 Reduced and Integral Schemes

Definition 3.5 (***Reduced Schemes***)**.** *Let X be a scheme. A scheme X is called reduced if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ are reduced.*

Remark 3.5. *In an affine scheme, viewing each ring element as a function, then function vanishing at all prime ideals is equivalent to be nilpotent. Thus, geometrically, reducedness means that function is determined by its values at each points.*

Proposition 3.6. *Let X be a scheme. Then X is reduced if and only if for all open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced.*

Proof. "$\Rightarrow$": If $s \in \mathcal{O}_X(U)$ satisfies that there exists $n \in \mathbb{N}_+$ such that $s^n = 0$, then for all $x \in U$, $(s_x)^n = 0$ in $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is reduced, get $s_x = 0$. Thus there exists open neighbourhood U_x of x such that $s\big|_{U_x} = 0$. As U_x cover U, by property of sheaf, $s = 0$. Thus $\mathcal{O}_X(U)$ is reduced.

"$\Leftarrow$": For all $x \in X$, if $(U, s) \in \mathcal{O}_{X,x}$ satisfies that there exists $n \in \mathbb{N}_+$ such that $(U, s)^n = 0$, then there exists open neighbourhood V of x such that $(s\big|_V)^n = 0$. Since $\mathcal{O}_X(V)$ is reduced, get $s\big|_V = 0$. As $(U, s) = (V, s\big|_V)$, $\mathcal{O}_{X,x}$ is reduced. □

Example 3.1. *(1)Let A be a ring. Then* $\operatorname{Spec} A$ *is reduced if and only if A is reduced.*
(2)Let A be a ring. Consider $A_{red} = A/rad(0)$. There is a closed immersion $\operatorname{Spec}(A_{red}) \longrightarrow \operatorname{Spec} A$. *Reduced scheme can also be defined for abtrary scheme, see in Hartshorne exercise II 2.3.*
(3)Let A be a ring. Then reduced closed subscheme of $\operatorname{Spec} A$ *is given by some radical ideal $rad(I)$ of A. For general case, if V is a closed subset of scheme X, then there exists unique structure of reduced closed subscheme on V gluing by reduced closed subscheme $V \cap V_i$ of affine open subset V_i. Details can be seen in Hartshorne Chapter II example 3.2.6, p86.*

Definition 3.6 (***Integral Schemes***)**.** *Let X be a scheme. We say that X is integral if X is irreducible and reduced.*

Proposition 3.7. *Let X be a scheme. If X is noetherian and connected, then X is integral if and only if for all $x \in X$, $\mathcal{O}_{X,x}$ is integral domain.*

Proof. "$\Rightarrow$": For all $x \in X$, if $(U, s), (V, t) \in \mathcal{O}_{X,x}$ satisfy that $(U, s)(V, t) = 0$, then there exists affine open neighbourhood W of x such that $s\big|_W t\big|_W = 0$. Since X is irreducible, by Proposition 3.2, W is irreducible. While $D(s\big|_W) \cap D(t\big|_W) = D(s\big|_W t\big|_W) = \varnothing$, get $D(s\big|_W) = \varnothing$ or $D(t\big|_W) = \varnothing$. Thus (U, s) is nilpotent or (V, t) is nilpotent. Since $\mathcal{O}_{X,x}$ is reduced, get $\mathcal{O}_{X,x}$ is integral domain.

"$\Leftarrow$": Since $\mathcal{O}_{X,x}$ is integral domain, $\mathcal{O}_{X,x}$ is reduced and so X is reduced. Want to show that X is irreducible. Suppose that X is not irreducible. Since X is noetherian, by Theorem 3.1, X has finite decomposition of irreducible components. Assume $X = \cup_i X_i$ and $X \neq X_1$. As X is connected, $X_1 \cap (\cup_{i \neq 1} X_i) \neq \varnothing$. Without loss of generality, can assume that $X_1 \cap X_2 \neq \varnothing$ and take $x \in X_1 \cap X_2$. Consider an affine open neighbourhood $U = \operatorname{Spec} A$ of X. Take generic points $\eta_1 \in X_1$ and $\eta_2 \in X_2$ (By Proposition 3.10). Then $\eta_1, \eta_2 \in U$ and correspond to different prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively. While x also corresponds to a prime ideal $\mathfrak{p}$, we have $\mathfrak{p}_1 \subseteq \mathfrak{p}$ and $\mathfrak{p}_2 \subseteq \mathfrak{p}$, inducing that there are two minimal prime ideals in $A_{\mathfrak{p}}$ contradicting to that $A_{\mathfrak{p}}$ is integral domain. $\square$

Proposition 3.8. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then*
(1)Closed subset $V(I) \subseteq X$ is irreducible if and only if $rad(I)$ is prime. In particular, X is irreducible if and only if X has a unique minimal prime ideal.
(2)X is integral if and only if A is integral domain.
(3)If A is noetherian, then we have that irreducible components of X are $\{V(\mathfrak{p}_i)\}$, where $\mathfrak{p}_i$ are minimal prime ideals of A.

Proposition 3.9. *Let X be a nonempty scheme. Then X is integral if and only if for all nonempty open subset U of X, $\mathcal{O}_X(U)$ is integral domain.*

Reason 3.3. *Proof of "$\Rightarrow$" is similar to proof of Proposition 3.7. For "$\Leftarrow$", it suffices to show that all affine open subsets are irreducible, which comes from Proposition 3.8.*

Example 3.2. *Let A be an integral domain. Then $\mathbb{A}^n_A$ and $\mathbb{P}^n_A$ are integral.*

Proposition 3.10. *Let X be an irreducible scheme. Then X contains a unique point η such that $\overline{\{\eta\}} = X$ called the generic point of X. If moreover X is integral, then $\mathcal{O}_{X,\eta}$ is a field called the function field of X.*

Remark 3.6. *If X is integral, then for all affine open neighbourhood $U = \operatorname{Spec} A$ of η, we have that $\mathcal{O}_{X,\eta} = \operatorname{Frac}(A)$.*

Proof. For affine case, obviously, $\eta = (0)$. For abtrary X, if η exists, then for all affine open subset $U = \operatorname{Spec} A$ of X, we have $\eta \in U$. Thus η is also generic point of U, inducing the uniqueness of generic point.

On the other hand, take $\eta = \eta_U$ for some fixed affine open subset U of X. For all open subset V of X, since X is irreducible, $U \cap V$ is nonempty. Thus $U \cap V$ is a nonempty open subset of U and so $\eta \in U \cap V \subseteq V$. Get $\overline{\{\eta\}} = X$ and η is the wanted generic point. $\square$

3.3 Geometrically Global Properties

Now let X be a k-scheme for some field k, we are interested in $X_{\overline{k}} = X \times_k \overline{k}$.

Definition 3.7. *Let X be a k-scheme for some field k. We say that X is geometrically integral (resp. geometrically irreducible, geometrically reduced) if $X_{\overline{k}}$ is integral (resp. irreducible, reduced).*

Remark 3.7. *(1)These properties are dependent on the base field k. For instance, $\operatorname{Spec}(\mathcal{C})$ is geometrically irreducible over $\mathbb{C}$ but not over $\mathbb{R}$.*
(2)Geometrically irreducible induces irreducible but the oppsite is false. For $\operatorname{Spec}(\mathbb{C})$ over $\mathbb{R}$ is irreducible but not geometrically irreducible.
(3)Geometrically reduced induces reduced but the oppsite is false. For instance, take $X = \operatorname{Spec}(\mathbb{R}[x]/(x^2+1))$ over $\mathbb{R}$, then X is reduced but not geometrically reduced.

(4)Geometrically integral induces but the oppsite is false. Can take the same example as (2) or (3). In addition, if take $k = \mathbb{F}_p(t)$ *and* $X = \mathrm{Spec}(k[x]/(x^p - t)) = k(t^{\frac{1}{p}})$*, then this is an example that is integral but neither geometrically irreducible nor geometrically reduced.*

Proposition 3.11. *Let* k *be a perfect field,* X *integral* k*-scheme,* K *function field of* X*. Then the following conditions are equivalent*
(1)X *is geometrically integral.*
(2)The ring $K \otimes_k \overline{k}$ *is an integral scheme.*
(3)k *is algebraically closed in* K*.*
(4)For all finite field extension l/k*,* $X \times_k l$ *is integral.*
(5)For all algebraic field extension l/k*,* $X \times_k l$ *is integral.*
(6)For all field extension l/k*,* $X \times_k l$ *is integral.*

3.4 Dimension Theory

Theorem 3.3 (***Cohen-Seidenberg, Going-up Theorem***)**.** *Let* $A \hookrightarrow B$ *be an injective ring homomorphism. Suppose that* B *is integral over* A*. Then*
(1)The associated map of spectrums $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}\, A$ *is surjective.*
(2)$\dim(A) = \dim(B)$*.*

Remark 3.8. *Precisely, Going-up Theorem is saying that if* $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ *are two prime ideals of* A *and there exists* $\mathfrak{P}_1 \in \mathrm{Spec}(B)$ *over* $\mathfrak{p}_1$*, then there also exists* $\mathfrak{P}_2 \in \mathrm{Spec}(B)$ *over* $\mathfrak{p}_2$ *such that* $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$*.*

Theorem 3.4 (***Noetherian Normalization Theorem***)**.** *Let* A *be an algebra of finite type over a field* k*. Then*
(1)There exists $y_1, \cdots, y_r \in A$ *algebraically independent over* k *such that* A *is integral over* $k[y_1, \cdots, y_r]$*.*
(2)If A *is integral domain, then* $\dim(A)$ *equals to the transcandence degree of* $\mathrm{Frac}(A)$ *over* k*, which by (1) is* r*. In addition, for all prime ideal* $\mathfrak{p} \in \mathrm{Spec}\, A$*, we have that* $\dim(A) = \dim(A/\mathfrak{p}) + ht(\mathfrak{p})$*.*

Definition 3.8. *Let* X *be a nonempty topological space. Define the dimension of* X *is the supremum of* n *for which there exists* $V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n$ *sequence of irreducible closed subsets of* X*.*

Proposition 3.12. *Let* $X = \mathrm{Spec}\, A$ *be an affine scheme. Then we have* $\dim(X) = \dim(A)$*.*

Reason 3.4. *Comes from the fact irreducible closed subset one-to-one corresponds to prime ideal of* A*.*

Proposition 3.13. *Let* X *be a topological space. Then*
(1) For all subset $Y \subseteq X$ *with induced topology, then* $\dim(Y) \leq \dim(X)$*.*
(2) Suppose that X *is irreducible and finite-dimensional. If* $Y \subseteq X$ *is a closed subset and* $\dim(X) = \dim(Y)$*, then* $X = Y$*.*

(3)If X is noetherian as topological space, then $\dim(X) = \max_{irreducible\ components}\{\dim(V)\}$.
(4)If $\{U_i\}$ is an open covering of X, then $\dim(X) = \sup_i\{\dim(U_i)\}$.

Example 3.3. *(1)Let X be an irreducible scheme. If $\dim(X) = 0$, then underlying topological space of X is a one-point set.*
(2) Let X be a noetherian scheme. If $\dim(X) = 0$, then irreducible components of X are all one-point sets and disjoint.
(3)Let A be a noetherian ring. Then $\dim(\mathbb{A}^n_A) = \dim(\mathbb{P}^n_A) = \dim(A) + n$.
(4)$\dim(X) = \dim(X_{red})$.
(5)Let X be a scheme, $U \subseteq X$ dense open subset. It can happen that $\dim(U) = \dim(X)$ even when X is noetherian integral scheme. For instance, take $X = \operatorname{Spec} A$ for some discrete valuation ring A.

Definition 3.9. *Let X be a topological space, $Y \subseteq X$ irreducible closed. Define the codimension $codim_X(Y)$ to be the supremum of n for which there exists $V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n = Y$ sequence of irreducible closed subsets of X.*

Remark 3.9. *Obviously,* $\dim(X) \geq \dim(Y) + codim_X(Y)$.

Theorem 3.5. *Let X be an integral scheme of finite type over field k, K function field of X. Then*
(1)$\dim(X) = tr\dim(K/k) < \infty$
(2)For all nonempty open subset U of X, $\dim(X) = \dim(U)$.
(3)For all closed point $x \in X$, $\dim(X) = \dim(\mathcal{O}_{X,x})$.

Proof. (1): By Proposition 3.13, $\dim(X)$ equals to the supremum of dimension of affine open subsets. While for all affine open subset $U = \operatorname{Spec} A$ of X, we have that A is of finite type over k, $\dim(U) = \dim(A)$ and $\operatorname{Frac}(A) = K$. By Noetherian Normalization Theorem, get $\dim(U) = tr\dim(K/k)$ and so $\dim(X) = tr\dim(K/k)$.

(2): Since U is also integral scheme with function field K of finite type over k, by (1), $\dim(U) = \dim(X)$.

(3): For all closed point $x \in X$, take affine open neighbourhood $U = \operatorname{Spec} A$ of x. Then x corresponds to a prime ideal $\mathfrak{p}_x$ of A and $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$. While x is also closed point in U, get $\mathfrak{p}_x$ is maximal ideal. Thus by Noetherian Normalization Theorem, $\dim(\mathcal{O}_{X,x}) = \dim(A) = \dim(U) = \dim(X)$. □

Corollary 3.1. *Let X be a scheme of finite type over field k. Then $\dim(X) < \infty$. If moreover X is irreducible, then $\dim(X) = \dim(U)$ for all nonempty open subset U of X.*

Proof. Since X is of finite type over k, get X is noetherian and so X has finite decomposition of irreducible components $\{X_i\}$. Then $\dim(X) = \max_i\{\dim(X_i)\}$. While $\dim(X_i) = \dim((X_i)_{red})$ where $(X_i)_{red}$ is integral scheme of finite type over k. By Proposition 3.5, get $\dim(X)$ finite. □

Definition 3.10 (***Pure Schemes***)**.** *Let X be a noetherian scheme. We say that X is pure (or say equidimensional) if all irreducible components of X have the same finite dimension.*

Proposition 3.14. *Let X be a scheme of finite type over field k. Suppose X is pure. Then (1)For all nonempty open subset U of X, $\dim(U) = \dim(X)$.*
(2)For all irreducible closed subset $Y \subseteq X$, $\dim(X) = \dim(Y) + codim_X(Y)$.

Reason 3.5. *The proof is similar to proof of Theorem 3.5 and Corollary 3.1.*

Theorem 3.6. *Let X be a scheme of finite type over field k. Then the set of all closed points of X is dense in X.*

Proof. By taking reduced scheme of X, can assume that X is reduced. If suffices to show that there exists some closed point in U for all affine open subset $U = \operatorname{Spec} A$of X. Take $x \in U$ such that x corresponds to maximal ideal $\mathfrak{m}_x \subseteq A$. Want to show that x is a closed point in X. Since X is of finite type over k, get X is noetherian and so that X has finite decomposition of irreducible components. Suppose that $x \in X_i$ where X_i is some irreducible component. Only need to show that x is closed point in X_i.

Note that X_i with reduced closed subscheme structure is integral scheme of finite type over k. Replace X by X_i. If there is another point $y \in \overline{\{x\}}$, then for all open neighbourhood V of y, $x \in V$. Take $V = \operatorname{Spec}(B)$ to be affine and assume that x,y correspond to prime ideals $\mathfrak{q}_x$ and $\mathfrak{q}_y$ respectively. Then it is easy to see that $\mathfrak{q}_x \subseteq \mathfrak{q}_y$. Thus $\dim(X) = \dim(U) = \dim(A_{\mathfrak{m}_x}) = \dim(B_{\mathfrak{q}_x}) < \dim(B_{\mathfrak{q}_y}) \leq \dim(B) = \dim(V) = \dim(X)$, contradiction! Get x is a closed point in X. So it is obviously that the set of all closed points of X is dense in X. □

Corollary 3.2. *Let X be a scheme of finite type over field k. Assume Z is an irreducible component of X. Then there exists closed point $x \in Z$ such that x is not in any other irreducible component of X.*

Proof. Since X is of finite type over k, X is noetherian. Thus X has finitely many irreducible components. Assume other irreducible components are $Z_1, \cdots, Z_n$ and $\Sigma_1 \subseteq Z_1, \cdots, \Sigma_n \subseteq Z_n$ are subsets of closed points respectively. If all closed points of Z are in some other irreducible component, then the subset of all closed points in X is $\cup_{i=1}^n \Sigma_i$. By Theorem 3.6, we have that $X = \overline{\cup_{i=1}^n \Sigma_i} \subseteq \overline{\cup_{i=1}^n Z_i} = \cup_{i=1}^n Z_i$, contradiction! □

Theorem 3.7. *Let $f : X \longrightarrow Y$ be a finite and surjective morphism of schemes. Then $\dim(X) = \dim(Y)$.*

Proof. Since f is affine, can cover both X and Y by affine open subsets. Thus the question can be reduced to the affine case. Assume that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec} A$ with A,B reduced. Note that B is finite over A and so B is integral over A. By Cohen-Seidenberg, $\dim(X) = \dim(A) = \dim(B) = \dim(Y)$. □

Definition 3.11. *Let $f : X \longrightarrow Y$ be morphism of integral schemes. We say that f is dominant if $f(X)$ is dense in Y. Or equivalently, $f(X)$ contains the generic point of Y. Or equivalently, $f(\eta_X) = \eta_y$.*

Theorem 3.8. *Let X, Y be two integral schemes of finite type over field k, $f : X \longrightarrow Y$ dominant k-morphism, K function field of X, F function field of Y. Then*

(1)The generic fiber X_{η_Y} is an integral F-scheme whose function field is K.
(2)$\dim(X_{\eta_Y}) = \dim(X) - \dim(Y)$.

Remark 3.10. *(1): Take affine open covering $\{V_i = \mathrm{Spec}(A_i)\}$ of Y with $\mathrm{Frac}(A_i) = F$. Then we can coevr X by affine open subsets $U_{ij} = \mathrm{Spec}(B_{ij}) \subseteq f^{-1}(V_i)$. Note that $X_{\eta_Y} = \cup_{i,j}(U_{ij} \times_{V_i} F) = \cup_{i,j} \mathrm{Spec}(B_{ij} \otimes_{A_i} F)$. It suffices to show that $B_{ij} \otimes_{A_i} F$ is integral domain with fraction field K. As f is dominant, $f(\eta_X) = \eta_Y$. Consider the associated ring homomorphism $\varphi_{ij} : A_i \longrightarrow B_{ij}$. Then $\ker(\varphi_{ij}) = \{0\}$. Since F is a localization of A_i, $B_{ij} \otimes_{A_i} F$ is a localization of B_{ij} so that $B_{ij} \otimes_{A_i} F$ is integral domain with fraction field $K = \mathrm{Frac}(B_{ij})$.*

(2): By Theorem 3.5, $\dim(X_{\eta_Y}) = tr\dim(K/F) = tr\dim(K/k) - tr\dim(F/k) = \dim(X) = \dim(Y)$.

Remark 3.11. *Fibre over generic point is integral doesn't induce other fibers are integral.*

4 Global Properties of Morphisms

4.1 Separated Morphisms

Definition 4.1. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Define the diagonal morphism associated to f to be $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ induced by $(\mathrm{id}_X, \mathrm{id}_X)$ with the following commutative diagram.*

$$\begin{tikzcd}
X \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[dr, "\Delta_{X/Y}"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Remark 4.1. *For $x \in X \times_Y X$, if images of x under two projections are same, we don't have that $x \in \Delta_{X/Y}(X)$. For example, take $X = \mathrm{Spec}(\mathbb{C})$ and $Y = \mathrm{Spec}(\mathbb{R})$. Then X has only one point but $X \times_Y Y$ has two different points.*

In addition, if take $W = \cup U \times_V U$, where $V \subseteq Y$ and $U \subseteq f^{-1}(V)$ are both affine open and U consist of an open covering of X, then $\Delta(X) \subseteq W$ is closed in each $U \times_V U$ so that it is closed in W.

Definition 4.2 (***Separated Morphisms***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is separated if the diagonal morphism associated to f is a closed immersion. In particular, we say that a S-scheme X is separated over S if $X \longrightarrow S$ is separated.*

Proposition 4.1. *Let $f : X \longrightarrow Y$ be a morphism of affine schemes. Then f is separated.*

Proof. Assume that $X = \mathrm{Spec}(B)$ and $\mathrm{Spec}\, A$. Then $X \times_Y X = \mathrm{Spec}(B \otimes_A B)$ and diagonal morphism associates to a ring homomorphism $\varphi : B \otimes_A B \longrightarrow B$. Since φ is obviously surjective, get $B \cong (B \otimes_A B)/I$ for some ideal I and so $\mathrm{Spec}(B) = \mathrm{Spec}((B \otimes_A B)/I)$ is a closed subscheme of $X \times_Y X$. Thus $\Delta_{X/Y}$ is a closed immersion. $\square$

With this definition, we can state definition of abstract variety over general field k.

Definition 4.3 (***Abstract Varieties***)**.** *Let k be a field. An abstract variety over k is a separated and integral scheme of finite type over k.*

Proposition 4.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Then f is separated if and only if $\Delta_{X/Y}(X)$ is closed in $X \times_Y X$.*

Proof. "$\Rightarrow$": Obvious.

"$\Leftarrow$": There is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "\Delta_{X/Y}"'] & & X \\ & X \times_Y X \arrow[ur, "p"'] & \end{tikzcd}$$

It is easy to see that $\Delta_{X/Y}$ induces a homeomorphism between X and $\Delta_{X/Y}(X)$ in this case. Just need to show that $\Delta^\sharp_{X/Y}$ is surjective.

By Lemma 2.5, if suffices to prove that $\Delta^\sharp_{X/Y,x}$ is surjective for all $x \in X$. Take affine open neighbourhood V of $f(x)$ and affine open neighbourhood $U \subseteq f^{-1}(V)$ of x. Then $\Delta^\sharp_{X/Y,x} = \Delta^\sharp_{U/V,x}$. Note that $\Delta_{U/V}$ is a closed immersion, done! $\square$

Proposition 4.3. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Suppose that Y admits an open covering $\{Y_i\}$ such that $f^{-1}(Y_i) \longrightarrow Y_i$ is separated. Then f is separated.*

Proof. Let $X_i = f^{-1}(Y_i)$. Then $X \times_Y X$ is covered by $X_i \times_{Y_i} X_i$ with each $X_i \longrightarrow Y_i$ separated. Thus $\Delta_{X/Y}(X) \cap (X_i \times_{Y_i} X_i) = \Delta_{X_i/Y_i}(X_i)$ is closed in $X_i \times_{Y_i} X_i$. Get $\Delta_{X/Y}(X)$ is closed in $X \times_Y X$. By Proposition 4.2, f is separated. $\square$

Proposition 4.4 (***Criterion of Separated***)**.** *Let $S = \operatorname{Spec} A$ be an affine scheme, X an S-scheme. The following conditions are equivalent*

(1)X is separated over S.

(2)For all U, V affine open subsets of X, $U \cap V$ is still affine and $\varphi_{UV} : \mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$ is surjective.

(3)There exists affine open covering $\{U_i\}$ of X such that $U_i \cap U_j$ is still affine and $\varphi_{U_iU_j} : \mathcal{O}_X(U_i) \otimes_A \mathcal{O}_X(U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j)$ is surjective

Proof. Prove as (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): For all U,V affine subsets of X, $U \cap V = \Delta^{-1}_{X/S}(U \times_S V)$. Since $\Delta_{X/S}$ is closed immersion, get $\Delta_{X/S}\big|_{U\cap V} : U \cap V \longrightarrow U \times_S V$ still closed immersion. As $U \times_X V$ is affine, $U \cap V$ is affine and $\varphi_{UV} : \mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$ is surjective.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Similarly, it suffices to show that $\Delta_{X/S}(X)$ is closed in $X \times_S X$. $\square$

Proposition 4.5. *(1)Open or closed immersions are separated.*

(2)The composition of two separated morphisms is separated.

(3)Separated is stable under base change.

(4)Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms of schemes. If $g \circ f$ is separated, then f is separated. In particular, k-morphisms between abstract k-varieties are separated.

Proof. Here only prove (4). Consider $h : X \times_Y X \longrightarrow X \times_Z X$ with the commutative diagram.

$$\begin{tikzcd}
X \times_Y X \arrow[dr, "h"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& X \times_Z X \arrow[r, "q_2"] \arrow[d, "q_1"] & X \arrow[d, "g\circ f"] \\
& X \arrow[r, "g\circ f"] & Z
\end{tikzcd}$$

Want to show $\Delta_{X/Y}(X) = h^{-1}(\Delta_{X/Z})$ which is closed in $X \times_Y X$. Obviously, $\Delta_{X/Y}(X) \subseteq h^{-1}(\Delta_{X/Z})$. For the other side, if for $s, t \in X \times_Y X$ satisfy that $h(s) = h(t) \in \Delta_{X/Z}(X)$, we need $s = t$. Assume that $h(s) = h(t) = \Delta_{X/Z}(x)$. Take affine open neighbourhoods U, V, W of $x, f(x), g(f(x))$ respectively such that $U \subseteq f^{-1}(V)$ and $V \subseteq g^{-1}(W)$. Then $h\big|_{U\times_V U}$ is injective. It is easy to see that s and t are both in $U \times_V U$. Thus $s = t$. □

Proposition 4.6. *Let S be a scheme, X reduced S-scheme, Y separated S-scheme. Suppose that $f, g : X \longrightarrow Y$ are two S-morphisms coincider on a dense open subset $U \subseteq X$. Then $f = g$.*

Proof. Consider $h : X \longrightarrow Y \times_S Y$ induced by (f, g). We have that $\Delta_{Y/S} \circ f$ coincide with h on U. Thus $h(U) = \Delta_{Y/S} \circ f(U) \subseteq \Delta_{Y/S}(Y)$ so that $U \subseteq h^{-1}(\Delta_{Y/S}(Y))$. Note that $h^{-1}(\Delta_{Y/S}(Y))$ is closed in X. While U is dense in X, get $h^{-1}(\Delta_{Y/S}(Y)) = X$ and so $h(X) \subseteq \Delta_{Y/S}(Y)$.

For all $x \in X$, we have that $p_1 \circ h(x) = f(x)$ and $p_2 \circ h(x) = g(x)$. While $p_1 \circ \Delta_{Y/S} = \mathrm{id}_Y = p_2 \circ \Delta_{Y/S}$, get $f(x) = g(x)$ for all $x \in X$. Now it suffices to show that $f^\sharp(V) = g^\sharp(V)$ for all affine open subset V of Y. Thus can assume that $Y = \operatorname{Spec} A$. Also, by taking affine open covering of X, can assume $X = \operatorname{Spec}(B)$. Then $f, g : X \longrightarrow Y$ associate to $\varphi, \psi : A \longrightarrow B$. For all $a \in A$, take $b = \varphi(a) - \psi(a) \in B = \mathcal{O}_X(X)$. Then $b\big|_U = 0$ so that $U \subseteq V(b)$. Note that $V(b)$ is closed, get $V(b) = \operatorname{Spec}(B)$ so that $b \in rad(B) = 0$. Thus $\varphi = \psi$ and so $f = g$. □

4.2 Proper Morphisms

Definition 4.4 (***Universally Closed Morphisms***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is universally closed if for any base change $Y' \longrightarrow Y$, $X \times_Y Y' \longrightarrow Y'$ is closed.*

Definition 4.5 (***Proper Morphisms***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is proper if f is separated, of finite type and universally closed. In particular, an S-scheme X is proper over S if $X \longrightarrow S$ is proper.*

Example 4.1. *(1)Closed immersion is always proper.*
(2)Open immersion is not proper in general, since it is not quasi-compact in general.
(3)Let k be an algebraically closed field. Then $\mathbb{A}^1_k \longrightarrow k$ is not proper, since $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$

is not closed. The image of $\operatorname{Spec}(k[x,y]/(xy-1))$ *is* $\mathbb{A}_k^1 \setminus \{(x)\}$. *Will see that projective abstract k-varieties are proper over k.*

Proposition 4.7. *Let* $f : X \longrightarrow Y$ *be a morphism of schemes. Suppose that* Y *admits an open covering* $\{Y_i\}$ *such that* $f^{-1}(Y_i) \longrightarrow Y_i$ *is proper. Then f is proper.*

Proof. We already know of finite type and separated are local on the base. It suffices to show that universally closed is local on the base. Let $X_i = f^{-1}(Y_i)$. For all base change $g : Y' \longrightarrow Y$, consider the following commutative diagram.

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{\widetilde{f}} & Y' \\ \downarrow{\scriptstyle \widetilde{g}} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then $X \times_Y Y'$ can be covered by $X_i \times_{Y_i} g^{-1}(Y_i)$ which is a base change of $X_i \longrightarrow Y_i$. Thus $X_i \times_{Y_i} g^{-1}(Y_i) \longrightarrow g^{-1}(Y_i)$ is closed. For all closed subset V of $X \times_Y Y'$, $V \cap X_i \times_{Y_i} g^{-1}(Y_i)$ is closed in $X_i \times_{Y_i} g^{-1}(Y_i)$ so that $\widetilde{f}(V) \cap g^{-1}(Y_i)$ is closed in $g^{-1}(Y_i)$. As $g^{-1}(Y_i)$ cover Y', get $\widetilde{f}(V)$ is closed in Y'. Thus $X \longrightarrow Y$ is universally closed. □

Theorem 4.1. *Let* $f : X \longrightarrow Y$ *be a finite morphism of schemes. Then f is proper.*

Proof. Obviously, f is of finite type. Since proper is local on the base, can assume that $Y = \operatorname{Spec} A$. While f is affine, get $X = f^{-1}(Y)$ affine. Assume that $X = \operatorname{Spec}(B)$. Now f is a morphism of affine schemes, automatically being separated.

Consider the associated ring homomorphism $\varphi : A \longrightarrow B$. φ factors $A \longrightarrow A/\ker(\varphi) \longrightarrow B$. Can assume that φ is injective. As f is finite, B is finite over A. By Cohen-Seidenberg, f is closed. Similar discussion show that f is universally closed. □

Proposition 4.8. *(1)The composition of two proper morphisms is proper.*
(2)Proper is stable under base change.
(3)Let $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *be two morphisms of schemes. If* $g \circ f$ *is proper and g is separated, then f is proper. In particular, k-morphisms between abstract k-varieties are proper.*

Theorem 4.2 (***Cancellation Theorem***)**.** *Let P be a property of morphisms that is stable under composition and base change. Assume that* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *are two morphisms of schemes. If* $g \circ f$ *satisfies P and g is separated, then f satisfies P.*

Reason 4.1. *Comes from the following commutative diagram.*

$$\begin{tikzcd} X \cong X \times_Y Y \arrow[rr] \arrow[d] & & X \times_Z Y \arrow[rr] \arrow[d] \arrow[dr] & & Z \times_Z Y \cong Y \arrow[d] \\ Y \arrow[rr, "\Delta_{Y/Z}"] & & Y \times_Z Y & X \arrow[r, "g\circ f"] & Z \end{tikzcd}$$

Lemma 4.1. *Let* $\varphi : A \longrightarrow B$ *be injective ring homomorphism. If the associated map of spectrum* $f : \operatorname{Spec} A \longrightarrow \operatorname{Spec}(B)$ *is closed, then* $\varphi^{-1}(B^\times) = A^\times$.

Proof. By Lemma 2.7, the image of f is dense in $\operatorname{Spec} A$. While f is closed, get f is surjective. Thus for all $b \in B^\times$ and $a \in \varphi^{-1}(b)$, if $a \notin A^\times$, then there exists maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$ such that $a \in \mathfrak{m}$. Since f is surjective, there exists prime ideal $\mathfrak{P} \in \operatorname{Spec}(B)$ such that $\varphi^{-1}(\mathfrak{P}) = \mathfrak{m}$. Thus $b = \varphi(a) \in \mathfrak{P}$, contradiction. Get $\varphi^{-1}(B^\times) = A^\times$. □

Lemma 4.2. *Let $\varphi : A \longrightarrow B$ be injective ring homomorphism. If the associated map of spectrum $f : \operatorname{Spec}(A[t]) \longrightarrow \operatorname{Spec}(B[t])$ is closed, then B is integral over A.*

Proof. Consider ideal $I = (bt - 1) \subseteq B[t]$ and $J = I \cap A[t] \subseteq A[t]$. Since φ is injective, the induced homomorphism $A[t] \longrightarrow B[t]$ is also injective and so is $\varphi' : A[t]/J \longrightarrow B[t]/I$. And $\operatorname{Spec}(B[t]/I) \longrightarrow \operatorname{Spec}(A[t]/J)$ is closed. By Lemma 4.1, we have that ${\varphi'}^{-1}((B[t]/I)^\times) = (A[t]/J)^\times$. Note that $\overline{bt} = \overline{1}$, get $\overline{t} \in (B[t]/I)^\times$. As $\widetilde{\overline{t}} = \overline{t}$, $\overline{t} \in (A[t]/J)^\times$. Thus there exists polynomial $a_n t^n + \cdots + a_1 t + a_0 \in A[t]$ such that $\overline{t a_n t^n + \cdots + a_1 t + a_0} = \overline{1}$ in $A[t]/J$ so that in $B[t]/I$. Multiplied by $\overline{b^{n+1}}$, get $\overline{a_n + \cdots + a_1 b^{n-1} + a_0 b^n} = \overline{b^{n+1}}$. Thus $b^{n+1} - a_0 b^n - \cdots - a_{n-1} b - a_n \in I = (bt - 1)$. Since $I \cap B = \{0\}$, get $b^{n+1} - a_0 b^n - \cdots - a_{n-1} b - a_n = 0$ and so b is integral over A. In conclusion, B is integral over A. □

Proposition 4.9. *Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a proper morphism of affine schemes. Then B is finite over A-module.*

Proof. Consider the associated ring homomorphism $\varphi : A \longrightarrow B$ and $\ker(\varphi)$. There is a commutative diagram.

$$\begin{tikzcd} \operatorname{Spec}(B) \arrow[rr, "f"] \arrow[dr, "g"'] & & \operatorname{Spec} A \\ & \operatorname{Spec}(A/\ker(\varphi)) \arrow[ur, "h"'] & \end{tikzcd}$$

As h is separated, by Proposition 4.8, g is proper. Note that $A/\ker(\varphi)$ is finite over A, can assume that φ is injective.

Obviously, B is of finite type over A. It suffices to show that B is integral over A. Note that $B[t] \cong A[t] \otimes_A B$, get $\operatorname{Spec}(B[t]) \longrightarrow \operatorname{Spec}(A[t])$ is given by base change $\operatorname{Spec}(A[t]) \longrightarrow \operatorname{Spec} A$ on f. As f is universally closed, $\operatorname{Spec}(B[t]) \longrightarrow \operatorname{Spec}(A[t])$ is closed. Thus by Lemma 4.2, B is integral over A. □

4.3 Projective Morphisms

Definition 4.6. *Let Y be a scheme. Define projective n-space over Y to be $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$, where $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x_0, \cdots, x_n])$.*

Definition 4.7 (***Projective Morphisms***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is projective if it factors through for some projective n-space over Y,*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where i is a closed immersion and p is the projection.

Remark 4.2. *This definition is same to the definition in Hartshorne, which is more general than the definition in EGA. For instance, finite induce EGA's projective but not Hartshorne's projective.*

Lemma 4.3. *Let A be a ring, B an A-algebra, S a graded A-algebra. Then $S \otimes_A B$ is a graded B-algebra graded by $(S \otimes_A B)_d = S_d \otimes B$. In addition, we have that* $\mathrm{Proj}(S \otimes_A B) = \mathrm{Proj}(S) \times_A B$.

Proof. It is easy to check $S \otimes_A B$ is a graded B-algebra. There is a graded homomorphism $\varphi : S \longrightarrow S \otimes_A B \quad s \longmapsto s \otimes 1_B$ and $\varphi(S_+)(S \otimes_A B) = (S \otimes_A B)_+$. Get A-morphism $g : \mathrm{Proj}(S \otimes_A B) \longrightarrow \mathrm{Proj}(S)$. On the other hand, $B \longrightarrow S \otimes_A B$ gives A-morphism $\pi : \mathrm{Proj}(S \otimes_A B) \longrightarrow \mathrm{Spec}(B)$. By universal property of $\mathrm{Proj}(S) \times_A B$, there is an A-morphism $h : \mathrm{Proj}(S \otimes_A B) \longrightarrow \mathrm{Proj}(S) \times_A B$ induced by (g, π). And we have that $h^{-1}(D_+(f) \times_A B) = g^{-1}(D_+(f)) = D_+(\varphi(f))$. $D_+(\varphi(f)) \longrightarrow D_+(f) \times_A B$ associates to ring homomorphism $\psi : S_{(f)} \otimes_A B \longrightarrow (S \otimes_A B)_{(\varphi(f))}$. It suffices to show ψ is isomorphic. While $S_{(f)} \otimes_A B$ is the subset of elements of degree 0 in $S_f \otimes_A B$, $(S \otimes_A B)_{(\varphi(f))}$ is the subset of elements of degree 0 in $(S \otimes_A B)_{\varphi(f)}$ and $S_f \otimes_A B \cong (S \otimes_A B)_{\varphi(f)}$, get ψ isomorphic. □

Theorem 4.3. *Let $f : X \longrightarrow Y$ be a projective morphism of schemes. Then f is proper.*

Proof. Since proper is stable under composition and closed immersion is proper, we only need to show that $\mathbb{P}^n_Y \longrightarrow Y$ is proper. In addition, as proper is stable under base change, it suffices to prove $p : \mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is proper. Note that $\mathbb{P}^n_{\mathbb{Z}} = \cup_{i=0}^n D_+(x_i)$. It is easy to use the affine open covering to show p is of finite type and separated.

To show p is universally closed, need to prove $\mathbb{P}^n_S \longrightarrow S$ is closed. While closed is local on the base, can assume $S = \mathrm{Spec}\, A$ so that by Lemma 4.3, $\mathbb{P}^n_S = \mathrm{Proj}(A[x_0, \cdots, x_n])$. It is easy to show that $\mathrm{Spec}(A[x_0, \cdots, x_n]_{(\frac{1}{x_i})}) \longrightarrow \mathrm{Spec}\, A$ is proper for all $0 \leq i \leq n$. Thus f is proper. □

Theorem 4.4 (***Valuation Criterion for Separated and Proper***)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes with X noetherian. Then*
(1)f is separated if and only if for all valuation ring R with $Rrad(R) = K$ and commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists at more one morphism $\mathrm{Spec}(R) \longrightarrow X$ *making the diagram commutative.*
(2)If moreover f is of finite type, then f is proper if and only if for all valuation ring R with $Rrad(R) = K$ and commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

there exists unique morphism $\mathrm{Spec}(R) \longrightarrow X$ *making the diagram commutative.*

Proof. See in the Hartshorne Chapter II Theorem 4.3, p97 and Theorem 4.7, p101. □

5 Local Properties of Schemes and Morphisms

5.1 Normal Schemes

Definition 5.1. *Let X be a scheme. We say that X is normal at $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is integrally closed domain. In particular, we say that X is normal if X is normal at x for all $x \in X$.*

Example 5.1. *Let A be an integral domain. Then $\operatorname{Spec} A$ is normal if and only if A is integrally closed.*

Theorem 5.1 (***Algebraic Hartogs***)**.** *Let A be a noetherian integrally closed domain, $K = \operatorname{Frac}(A)$. Then $A = \cap_{\mathfrak{p} \in \operatorname{Spec} A, ht(\mathfrak{p}) \leq 1} A_{\mathfrak{p}} \subseteq K$.*

Remark 5.1. *In algebraic geometry, this fact can be translated into that functions defined in codim 1 are defined everywhere.*

Corollary 5.1. *Let X be a noetherian normal scheme. Assume that $Z \subseteq X$ is a closed subset and all irreducible components of Z are of codimension ≥ 2. Then the restriction $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(X \setminus Z)$ is isomorphic.*

Proof. Since X is integral, both $\mathcal{O}_X(X)$ and $\mathcal{O}_X(X \setminus Z)$ are subrings of the function field K of X. Suffice to prove the surjectivity for the case that Z is irreducible. For all affine open subset $U \subseteq X$ meeting Z, we have $codim_X(Z) = codim_U(Z \cap U)$. By gluing lemma, can assume that $X = \operatorname{Spec} A$ so that $Z = V(\mathfrak{q})$ for some $\mathfrak{q} \in X$.

For all $\mathfrak{p} \in X$ with $ht(\mathfrak{p}) \leq 1$, we have that $\mathfrak{p} \in X \setminus Z$. Now we can view $\mathcal{O}_X(X \setminus Z)$ as a subring of $A_{\mathfrak{p}}$. Thus by Theorem 5.1 $A \subseteq \mathcal{O}_X(X \setminus Z) \subseteq \cap_{\mathfrak{p} \in \operatorname{Spec} A, ht(\mathfrak{p}) \leq 1} A_{\mathfrak{p}} = A$, done! □

Definition 5.2. *Let X be a scheme, $x \in X$. We say that x is of codimension r if codimension of the irreducible closed subset $\overline{\{x\}}$ is r.*

Proposition 5.1. *Let X be a scheme, $x \in X$. Then x is of codimension r if and only if $\dim(\mathcal{O}_{X,x}) = r$.*

Reason 5.1. *Take affine open neighbourhood U of x.*

Lemma 5.1. *Let S be a scheme, X, Y two S-schemes with Y locally of finite type. Assume that either X integral or S locally noetherian. For all $x \in X$ and S-morphism $f_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow Y$, f_x extends an open subset of X containing x.*

Proof. Can assume X, Y and S are affine. Set $X = Spec(B), Y = \operatorname{Spec} A$ and $S = \operatorname{Spec}(C)$. Want all C-homomorphism $\varphi : A \longrightarrow B_{\mathfrak{p}}$ factors as

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr] & & B_{\mathfrak{p}} \\
& B_b \arrow[ur] &
\end{tikzcd}$$

for some $b \notin \mathfrak{p}$.

If B is integral domain, choose generators of A as C-algebra $\{a_1, \cdots, a_r\}$. Then we can take $b \notin \mathfrak{p}$ such that $\varphi(a_i) \in B_b$ for all i, done!

If C is noetherian, write $A = C[t_1, \cdots, t_n]/I$, where $I = (P_1, \cdots, P_m)$ is finitely generated. Now since $P_j(\overline{t_1}, \cdots, \overline{t_n}) = 0$, $P_j(\varphi(\overline{t_1}), \cdots, \varphi(\overline{t_n})) = 0$ in $B_{\mathfrak{p}}$. By finiteness of j, can take $b \notin \mathfrak{p}$ such that $bP_j(\varphi(\overline{t_1}), \cdots, \varphi(\overline{t_n})) = 0$ in B, done! □

Theorem 5.2. *Let S be a noetherian scheme, X, Y two S-schemes of finite type. Assume that Y is proper over S and X is normal with function field K. For all nonempty open subset U of X and S-morphism $f : U \longrightarrow Y$ (resp. $f_{\eta_X} : \operatorname{Spec} K \longrightarrow Y$), there exists open subset $V \subseteq U$ (resp. nonempty open subset V) of X, containing all points of codimension 1 of X, such that f extends uniquely to V. In particular, if $\dim(X) = 1$, then f extends to whole X.*

Proof. Uniqueness: For open subset case, if there are two extension to V, denote by f_1 and f_2. Then f_1 and f_2 coincide on U which is a dense open subset of V. Note that X is reduced and Y is separated over S, by Proposition 4.6, $f_1 = f_2$.

Existence: For each case, there exists S-morphism $f_{\eta_X} : \operatorname{Spec} K \longrightarrow Y$. For all point $x \in X$ of codimension 1, $\mathcal{O}_{X,x}$ is 1-dimensional, noetherian and integrally closed local domain so that is a discrete valuation ring. By valuation criterion of properness, f_{η_X} extends uniquely to S-morphism $f_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow Y$. Thus by Lemma 5.1, f_x extends to $g_x : U_x \longrightarrow Y$ for some open neighbourhood U_x of x. Note that $f(\eta_X) = g_x(\eta_X)$. Take affine affine open covering $f(\eta_X) \in \operatorname{Spec} A \subseteq Y$ and affine open covering $\operatorname{Spec}(B) \subseteq f^{-1}(Spec(A)) \cap g^{-1}(Spec(A))$. Then f and g_X coincide on $\operatorname{Spec}(B)$ induced by the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] & B \arrow[r, hook] & K \end{tikzcd}$$

Get f and g_x coincide on $U \cap U_x$. For $x' \in X$ another point of codimension 1, similar argument gives f, g_x and $g_{x'}$ coincide on the intersections of U, U_x and $U_{x'}$. By gluing, f can extends to $U \cup (\cup_{\text{x of codimension 1}} U_x)$. □

Definition 5.3. *Let X, Y be two abstract k-varieties. We say that X and Y are k-birational if their function fields are k-isomorphic.*

Corollary 5.2. *Let X, Y be two normal and complete curves over k. If X, Y are k-birational, then X, Y are k-isomorphic.*

Definition 5.4 (***Normalization***)**.** *Let $X = \operatorname{Spec} A$ be an affine scheme with A integral domain. Consider $\widetilde{X} = \operatorname{Spec}(\widetilde{A})$, where $\widetilde{A}$ is the integral closure of A in $\operatorname{Frac}(A)$. We say that $\widetilde{X}$ is the normalization of X with the $\pi : \widetilde{X} \longrightarrow X$ and universal property: for all dominant morphism $f : Y \longrightarrow X$ with Y normal, there exists morphism $\widetilde{f} : Y \longrightarrow \widetilde{X}$ such that $f = \pi \circ \widetilde{f}$. By gluing, we can also define the normalization of abtrary integral scheme X over its function field with the same universal property.*

Example 5.2. *(1)Let $X = \operatorname{Spec}(\mathbb{Z}[\sqrt{-3}])$ not normal. Then $\widetilde{X} = \operatorname{Spec}(\mathbb{Z}[\frac{-1+\sqrt{-3}}{2}])$.*
(2)Let $X = \operatorname{Spec}(k[x, y]/(y^2 - x^3))$ not normal. Then $\widetilde{X} = \operatorname{Spec}(k[t])$. Morphism $\pi : \widetilde{X} \longrightarrow X$

is given by $x \longmapsto t^2$ *and* $y \longmapsto t^3$*.*
(3)Let $X = \mathrm{Spec}(k[x,y]/(y^2 - x^2(x+1)))$ *not normal. Then* $\widetilde{X} = \mathrm{Spec}(k[t])$*. Morphism* $\pi : \widetilde{X} \longrightarrow X$ *is given by* $x \longmapsto t^2 - 1$ *and* $y \longmapsto t(t^2-1)$*.*

Remark 5.2. *(1)Let* A *be an integral domain,* $K = \mathrm{Frac}(A)$*,* L/K *field extension. Since we can define the integral closure* $\widetilde{A}_L$ *of* A *in* L*, we can also define the normalization of* $Sepc(A)$ *in* L *to be* $\mathrm{Spec}(\widetilde{A}_L)$*.*

Proposition 5.2. *Let* X *be an integral scheme,* K *function field of* X*. Then the natural morphism* $\pi : \widetilde{X} \longrightarrow X$ *is affine and surjective. If moreover* X *is of finite type over* k *for some field* k*,* π *is finite.*

Proof. The question is local. Can assume that $X = \mathrm{Spec}\, A$ is affine. Then $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$ and $K = \mathrm{Frac}(A)$. Obviously $\pi : \mathrm{Spec}(\widetilde{A}) \longrightarrow \mathrm{Spec}\, A$ is affine. And since $\widetilde{A}$ is integral over A, by Cohen-Seidenberg Theorem, get π is surjective.

If moreover A is of finite type over k, by noetherian normalization Theorem, get there exists $x_1, \cdots, x_r \in A$ algebraically independent over k such that A is integral over $k[x_1, \cdots, x_r]$. Thus K is a finite extension over $k(x_1, \cdots, x_r)$ and $\widetilde{A}$ is the integral closure of $k[x_1, \cdots, x_r]$ in K. The proof of the fact that $\widetilde{A}$ is of finite type over $k[x_1, \cdots, x_r]$ is similar to the proof in *Algebraic Nunber Theory*, Milne, p35. Thus $\widetilde{A}$ is finite over $k[x_1, \cdots, x_n]$ and so that $\widetilde{A}$ is finite over A. □

5.2 Regular Schemes

Definition 5.5. *Let* X *be a locally noetherian scheme. We say that* X *is regular at* $x \in X$ *if the local ring* $\mathcal{O}_{X,x}$ *is regular. If not, then* x *is called a singular point of* X*. In particular, we say that* X *is regular if* X *is regular at all points.*

Example 5.3. *(1)Let* X *be a normal and locally noetherian scheme. Since discrete valuation ring is regular local ring, all points of codimension 1 in* X *are nonsingular points.*
(2)Let $X = \mathrm{Spec}(k[x,y,z]/(z^2 - xy))$*. Then* X *is normal and noetherian but not regular at* $(\overline{x}, \overline{y}, \overline{z})$*.*

Proposition 5.3. *Let* X *be a locally noetherian scheme. If* X *is regular at closed points, then* X *is regular.*

Proof. For all $x \in X$, $\overline{\{x\}}$ is irreducible. Consider the minimal irreducible closed subset $V \subseteq \{x\}$. By Lemma 3.1, V must be a one-point set. Assume that $V = \{y\}$. Take an affine open neighbourhood U of y, then $x \in U$. As $y \in \overline{\{x\}}$, get $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$. Since X is regular at y, X is also regular at x. □

Remark 5.3. *Let* X *be a locally noetherian scheme,* $x \in X$ *closed point. Recall that* $T_{X,x} = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$*. Thus* X *is regular at* x *if and only if* $\dim_{k(x)}(T_{X,x}) = \dim(\mathcal{O}_{X,x})$*.*

Lemma 5.2. *Let* $V = k^n$ *be a vector space over field* k*,* $y \in V$*,* V^* *the dual space. Take* $J \subseteq k[x_1, \cdots, x_n]$ *to be the maximal ideal corresponding to* y*. Define map* D *to be*

$D : k[x_1, \cdots, x_n] \longrightarrow V^* \quad P \longmapsto ((t_1, \cdots, t_n \mapsto \sum_j \frac{\partial P}{\partial x_j}(y)t_j))$. Then $D\big|_J$ induces an isomorphism $J/J^2 \xrightarrow{\sim} V^*$.

Proof. Assume $y = (y_1, \cdots, y_n)$. Foa all $P \in J$, $P(y) = 0$. Consider Taylor Expansion of P at y is $P = P(y_1, \cdots, y_n) + \sum_j \frac{\partial P}{\partial x_j}(y)(x_j - y_j) + Q$. Note that $P \in J$ and $Q \in J^2$, get $\overline{P} = \overline{\sum_j \frac{\partial P}{\partial x_j}(y)(x_j - y_j)}$ in J/J^2. Thus for all $P \in J^2$, get $\sum_j \frac{\partial P}{\partial x_j}(y)(x_j - y_j) \in J^2$. It is easy to see that $\frac{\partial P}{\partial x_j}(y) = 0$ for all j. Thus $D(P)$ is zero map so that D induces an isomorphism $J/J^2 \xrightarrow{\sim} V^*$. □

Theorem 5.3 (Jacobson Criterion). *Let k be a field and $X = \mathrm{Spec}(k[x_1, \cdots, x_n]/I)$ be an affine k-variety where $I = (P_1, \cdots, P_r)$, $x \in X$ closed point with $k(x) = k$. Then X is regular at x if and only if the Jacobson matrix $J_x = (\frac{\partial P_i}{\partial x_j}(x))_{i,j}$ is of rank $n - \dim(X)$.*

Remark 5.4. *More generally, for closed point $x \in X$ with residue field $k(x)$. If J_x is of rank $n - \dim(X)$, then X is regular at x. On the other hand, only when $k(x)$ is separable over k can we get J_x is of rank $n - \dim(X)$ if X is regular at x.*

Proof. Let $f : X \longrightarrow \mathbb{A}_k^n$ be the closed immersion, $y = f(x)$. Set $J \subseteq k[x_1, \cdots, x_n]$ be the corresponding maximal ideal to $y \in Y$. Note that $k \subseteq k(y) \subseteq k(x) = k$, get $k(y) = k$. Thus $k[x_1, \cdots, x_n]/J \cong k$. Assume the images of x_j under the natural homomorphism $k[x_1, \cdots, x_n] \longrightarrow k[x_1, \cdots, x_n]/J$ are y_j respectively. Then by the Lemma 5.1, get $J/J^2 \cong V^n$, where $V = k^n$. As the maximal corresponding to x is J/I, $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong (J/I)/(J/I)^2 \cong J/(J^2 \cap I)$. Consider the following exact sequence

$$0 \longrightarrow I/(J^2 \cap I) \longrightarrow I/J^2 \longrightarrow J/(J^2 \cap I) \longrightarrow 0$$

Thus

$$\begin{aligned} dim_k T_{X,x} &= \dim_{k_x}(\mathfrak{m}/\mathfrak{m}^2) \\ &= \dim_{kx}(J/J^2) - \dim_{kx}(I/(J^2 \cap I)) \\ &= n - rank(J_x) \end{aligned}$$

Thus X is regular at x if and only if the Jacobson matrix $J_x = (\frac{\partial P_i}{\partial x_j}(x))_{i,j}$ is of rank $n - \dim(X)$. □

Example 5.4. *Let k be an algebraically field and $X = \mathrm{Spec}(k[x_1, \cdots, x_m])/(P)$ be an affine k-variety where P is irreducible. Now $\dim(X) = n - 1$. And X is regular at closed point $x \in X$ if and only if at least one $\frac{\partial P}{\partial x_j}(x) \neq 0$.*

5.3 Flat Morphisms

Definition 5.6. *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is flat at $x \in X$ if the induced homomorphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is flat. In particular, we say that f is flat if it is flat at all points.*

Remark 5.5. *(1)Open immersions are flat.*
(2)Flatness is stable under composition and base change.

Proposition 5.4. *Let $f : X \longrightarrow Y$ be a finite morphism of schemes with Y integral and locally noetherian. Then f is flat if and only if for all $y \in Y$, the fiber X_y is of constant dimension.*

Proposition 5.5. *Let $f : X \longrightarrow Y$ be a finite and surjective morphism between regular schemes. Then f is flat.*

Remark 5.6. *Surjectivity of f and regularity of Y are both necessary, while regularity of X can be reduced to Cohen-Macaulay condition.*

Theorem 5.4 (***Going-down Theorem for Flat Condition***)**.** *Let $\varphi : A \longrightarrow B$ be a flat ring homomorphism. If $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq A$ are prime ideals such that $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ and $\mathfrak{P}_1 \subseteq B$ is prime ideal over $\mathfrak{p}_1$, then there exists $\mathfrak{P}_2 \subseteq B$ prime ideal over $\mathfrak{p}_2$.*

Theorem 5.5. *Let $f : X \longrightarrow Y$ be flat morphism of finite type between noetherian schemes. Then f is open.*

Reason 5.2. *Comes from Chavalley's Theorem and Going-down Theorem. Proof of Chavalley's Theorem can be seen in the Midterm24.pdf.*

Corollary 5.3. *Let $f : X \longrightarrow Y$ be flat morphism of finite type between noetherian schemes. If Y is irreducible, then all nonempty open subset $U \subseteq X$ dominates Y i.e. $f(U)$ is dense in Y. Moreover, all irreducible components of X dominate Y.*

Lemma 5.3. *Let A be a reduced noetherian ring, $a \in A$ zero-divisor. Then a is in a minimal prime ideal of A.*

Proof. Since a is a zero-divisor, the natural homomorphism $A \longrightarrow A_a$ is not injective with nontrivial kernel I. While A is reduced, get $V(I) \neq \operatorname{Spec} A$. Note that $D(a) \subseteq V(I)$, $\operatorname{Spec} A$ contains an irreducible component whose generic point η is not in $D(a)$. While η corresponds to a minimal prime ideal $\mathfrak{p}$ of A, get $a \in \mathfrak{p}$. □

Theorem 5.6. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Assume that X is noetherian and reduced and Y is noetherian, normal and one-dimensional. Suppose that all irreducible components of X dominate Y. Then f is flat.*

Proof. For generic point η_Y, $\mathcal{O}_{Y,\eta_Y}$ is a field, so there is nothing to prove. For other $x \in X$ which not maps to η_Y, since Y is one-dimensional, $y = f(x)$ is a closed point. Then $\mathcal{O}_{Y,y}$ is a discrete valuation ring. Take uniformizer ω of $\mathcal{O}_{Y,y}$ and assume $f_x^\sharp(\omega) = t$. If t is in a minimal prime ideal $\mathfrak{p}$ of $\mathcal{O}_{X,x}$, consider the generic point η of some irreducible component containing x corresponds to $\mathfrak{p}$. As irreducible components of X dominate Y, $f(\eta) = \eta_Y$. Then there is commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[r, "f_x^\sharp"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\
\operatorname{Frac}(\mathcal{O}_{Y,y}) \arrow[r, "f_\eta^\sharp"] & \mathcal{O}_{X,\eta}
\end{tikzcd}$$

while ω is not zero in $\mathrm{Frac}(\mathcal{O}_{Y,y})$, its image in $\mathcal{O}_{X,\eta}$ should be invertible, contradicting to $t \in \mathfrak{p}$. Thus t is not in some minimal prime ideal of A. By Lemma 5.3, get t is not zero-divisor. Thus $\mathcal{O}_{X,x}$ is torsion-free as $\mathcal{O}_{Y,y}$-module. While $\mathcal{O}_{Y,y}$ is principal ideal domain, get $\mathcal{O}_{X,x}$ free so that is flat. $\square$

Theorem 5.7. *Let A be a noetherian ring, $a \in A$ not invertible. Then each minimal prime ideal of A containing a has height ≤ 1 and "=" holds if a is not a zero-divisor.*

Remark 5.7. *Geometrically, for $X = \operatorname{Spec} A$ affine scheme and $a \in A$ not invertible, each irreducible component of $V(a)$ has codimension ≤ 1 and "=" holds if a is not a zeo-divisor.*

Corollary 5.4. *Let A be a noetherian local ring, $\mathfrak{m} \subseteq A$ maximal ideal, $a \in \mathfrak{m}$. Then* $\dim(A/(a)) \geq \dim(A) - 1$ *and "=" holds if a is not a zero-divisor.*

Theorem 5.8. *Let $f : X \longrightarrow Y$ be a flat morphism between locally noetherian schemes, $x \in X$, $y = f(x)$. Then* $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$.

Proof. Can assume $Y = \operatorname{Spec} A$ affine. Consider base change $\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y$. Get $X \times_Y \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y,y}$ is flat. Thus by replacing Y by $\mathcal{O}_{Y,y}$, can assume that A is noetherian local ring and y corresponds to the maximal ideal $\mathfrak{m}$ of A.

Induct on dimension of A. If $\dim(A) = 0$, then A is field and Y is one-point set. Now $X_y \cong X$ so that $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$.

Suppose that statement is ok for $\dim(A) = d$. When $\dim(A) = d+1$, consider base change $Y_{red} \longrightarrow Y$. Can assume that A is reduced. Let $t \in A$ not invertible and not a zero-divisor, which means $t \in \mathfrak{m}$ and t is not in some minimal prime ideal. t exists because $\cup_{\mathfrak{p}\text{ minimal prime ideal}} \mathfrak{p} \subsetneq \mathfrak{m}$. Now $A \longrightarrow \mathcal{O}_{X,x}$ is flat. Denote $\mathcal{O}_{X,x}$ by B. Since $A \xrightarrow{\cdot t} A$ is injective, $B \xrightarrow{\cdot t} B$ is also injective. Thus the image of t in B is not zeo-divisor. By Corollary 5.4, $\dim(A/tA) = \dim(A) - 1$ and $\dim(B/tB) = \dim(B) - 1$.

Set $Y' = \mathrm{Spec}(A/tA) \subseteq Y$ and $X' = X \times_Y Y'$, then $X' \longrightarrow Y'$ is flat. While $t \in \mathfrak{m}$, get $y \in Y'$ and $X'_y = X_y$ By induction, $\dim(\mathcal{O}_{X'_y,x}) = \mathcal{O}_{X',x} - \dim(\mathcal{O}_{Y',y})$ so that $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$. $\square$

Remark 5.8 (***Partial converse of Theorem "Miracle Flatness"***)**.** *Let $f : X \longrightarrow Y$ be a morphism between regular schemes. If for all $x \in X$ and $y = f(x)$, we have that* $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$, *then f is flat.*

Lemma 5.4. *Let $f : X \longrightarrow Y$ be a morphism of schemes, $x \in X$, $y = f(x)$. Then the local ring of X_y at x is $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$.*

Proof. Since we are discussing about local rings, we can assume that $X = \mathrm{Spec}(B)$ and $Y = \operatorname{Spec} A$. Then f corresponds a ring homomorphism $\varphi : A \longrightarrow B$. Assume that x corresponds to $\mathfrak{q} \in \mathrm{Spec}(B)$ and y corresponds to $\mathfrak{p} \in \mathrm{Spec}\, A$. Then

$$\begin{aligned}\mathcal{O}_{X_y,x} &= (B \otimes_A k(\mathfrak{p}))_{\mathfrak{q}\otimes_A k(\mathfrak{p})} \\ &\cong B_{\mathfrak{q}} \otimes_A k(\mathfrak{p}) \\ &\cong B_{\mathfrak{q}} \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \\ &\cong B_{\mathfrak{q}} \otimes_A A_{\mathfrak{p}} \otimes_A A/\mathfrak{p} \\ &\cong B_{\mathfrak{q}} \otimes_A A/\mathfrak{p} \\ &\cong B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} \\ &\cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}\end{aligned}$$

□

Theorem 5.9. *Let $f : X \longrightarrow Y$ be a flat k-morphism between schemes of finite type over field k. Suppose Y is irreducible and X is pure. Then for all $y \in Y$, the fiber X_y is either empty or pure with* $\dim(X_y) = \dim(X) - \dim(Y)$.

Proof. For $y \in f(X)$, consider $Z \subseteq X_y$ irreducible component. As X_y is of finite type over $k(y)$, by Corollary 3.2, we can choose $x \in Z$ closed point not in any other irreducible component of X_y. Assume $x \in X_0 \subseteq X$ with X_0 irreducible component of X. Then since X is pure,

$$\begin{aligned}\dim(\mathcal{O}_{X,x}) &= \dim(\mathcal{O}_{X_0,x}) \\ &= \dim(X_0) - \dim(\overline{\{x\}}) \\ &= \dim(X) - \dim(\overline{\{x\}})\end{aligned}$$

We also have that $\dim(\mathcal{O}_{Z,x}) = \dim(Z)$ and $\dim(\mathcal{O}_{Y,y}) = \dim(Y) - \dim(\overline{\{y\}})$. By Theorem 5.8, $\dim(\mathcal{O}_{Z,x}) = \dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{\mathcal{Y},\dagger})$. It remains to show that $\dim(\overline{\{x\}}) = \dim(\overline{\{y\}})$.

Note that if we take open neighbourhood $V = \mathrm{Spec}\, A$ of y and y corresponds to $\mathfrak{p}$, then $\dim(\overline{\{y\}}) = \dim(A/\mathfrak{p}) = tr\dim(k(y)/k)$. Similarly, we also have $\dim(\overline{\{x\}}) = tr\dim(k(x)/k)$. While by Lemma 5.4, $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, get the residue field of x in X_y is also $k(x)$. Take affine open neighbourhood $U = \mathrm{Spec}(B)$ of x in X_y. Then x corresponds to a maximal ideal $\mathfrak{m}$ of B and $k(x) = B/\mathfrak{m}$. Thus $B/\mathfrak{m}$ is a field of finite type over $k(y)$. By Zariski's Lemma, $B/\mathfrak{m}$ is finite over $k(y)$ so that it is algebraic over $k(y)$, inducing $tr\dim(k(x)/k) = tr\dim(k(y)/k)$. □

5.4 Etale Morphisms and Smooth Schemes and Morphisms

Definition 5.7. *Let $f : X \longrightarrow Y$ be a morphism of finite type between locally noetherian schemes, $x \in X$, $y = f(x)$. We say that f is unramified at x if*
(1)$\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$.
(2)The field extension $k(x)/k(y)$ is finite and separable.

In particular, we say that f is unramified if it is unramified at all points.

Remark 5.9. *Recall that $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$. Thus if f is unramified at x, then $\mathcal{O}_{X_y,x}$ is a field.*

Proposition 5.6. *Let $f : X \longrightarrow Y$ be a morphism of finite type between locally noetherian schemes. Then f is unramified if and only if for all $y \in Y$, X_y is finite over $k(y)$ and reduced and for all $x \in X_y$, the extension $k(x)/k(y)$ is finite and separable.*

Definition 5.8. *Let $f : X \longrightarrow Y$ be a morphism of finite type between locally noetherian schemes. We say that f is etale at x if f is both flat and unramified at x. In particular, we say that f is etale if it is etale at all points.*

Example 5.5. *(1)*$\operatorname{Spec} K \longrightarrow \operatorname{Spec} k$ *is unramified/etale if and only if the field extension K/k is finite and separable.*
(2)Close immersions are unramified.
(3)Open immersions are etale.

Proposition 5.7. *unramified and etale are both stable under composition and base change.*

Proposition 5.8. *Let $f : X \longrightarrow Y$ be an etale morphism between locally noetherian schemes, $x \in X$, $y = f(x)$. Then*
(1)$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.
(2)The tangent map $T_{X,x} \longrightarrow T_{Y,y} \otimes_{k(y)} k(x)$ is an isomorphism.

Proof. (1): By Theorem5.8, $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$. While f is unramified, get $\mathcal{O}_{X_y,x}$ is a field so that $\dim(\mathcal{O}_{X_y,x}) = 0$. Thus $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

(2): Note that $T_{Y,y} \otimes_{k(y)} k(x) = \operatorname{Hom}_{k(x)}(\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{k(y)} k(x), k(x))$. It suffices to prove $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{k(y)} k(x)$. Firstly, we have that

$$\begin{aligned} \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{k(y)} k(x) &\cong (\mathfrak{m}_y \otimes_{\mathcal{O}_{\mathcal{Y},\dagger}} k(y)) \otimes_{k(y)} k(x) \\ &\cong \mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} k(x) \end{aligned}$$

and

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

Since f is unramified, $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{X,x}$ so that $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathfrak{m}_x$ is surjective. Since f is flat, $\mathfrak{O}_{X,x}$ is flat over $\mathfrak{O}_{Y,y}$ so that $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathfrak{m}_x$ is injective. Thus $\mathbf{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} = \mathfrak{m}_x$ so that $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{k(y)} k(x)$. ∎

Proposition 5.9. *Let X be a k-scheme of finite type.*
(1)Let K/k be a field extension, $x \in X$ closed point with residue field k. Assume $x_K \in X_K = X \times_k K$ is a closed point over x with residue field K. Then X is regular at x if and only if X_K is regular at x_K.
(2)Suppose k is perfect field with algebraic closure $\overline{k}$. Then X is regular if and only if $X_{\overline{k}}$ is regular.

Proof. (1): Can assume that $X = \operatorname{Spec} A$ affine. As X is of finite type over k, A is of the form $k[x_1, \cdots, x_r]/I$. Thus $X_k = \operatorname{Spec}(A \otimes_k K) \cong \operatorname{Spec}(K[x_1, \cdots, x_r])/I$ so that the Jacobson matrixes at $x \in X$ and $x_K \in X_K$ are same. Note that $X_K \longrightarrow X$ is given by $\operatorname{Spec} K \longrightarrow \operatorname{Spec} k$ under base change $X \longrightarrow \operatorname{Spec} k$, $X_K \longrightarrow X$ is flat. Assume x corresponds to maximal ideal $\mathfrak{m}$ of A. Then $\mathfrak{m}_x$ is $\mathfrak{m}A_{\mathfrak{m}}$ and $\mathfrak{m}_{x_K} = \mathfrak{m}A_{\mathfrak{m}} \otimes_k K$. Thus $\mathfrak{m}_{x_K} = \mathfrak{m}_x \mathcal{O}_{X_K,x_K}$, inducing that the local ring of fiber X_{K_x} is a field. By Theorem 5.8, $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X_k,x_K})$. Thus by Jacobson Criterion, get X is regular at x if and only if X_K is regular at x_K.

(2): For $x \in X$ closed point, assume the residue field of x is $k(x)$. Consider $X' = X \times_k k(x)$. Take affine open neighbourhood $U = \operatorname{Spec} A$ of x in X. Then $\operatorname{Spec}(A \otimes_k k(x))$ is an affine open subset of X'. While A is of finite type over k, by Noetherian Normalization Theorem, there exists $X_1, \cdots, x_r \in A$ algebraically independent over k such that A is integral over $k[x_1, \cdots, x_r]$. It is easy to show that $(x_1, \cdots, x_r)$ is a maximal ideal of A so that $(x_1, \cdots, x_r) \otimes_k k(x)$ is a maximal ideal of $A \otimes_k k(x)$. Thus we can take the closed point x' corresponding to $(x_1, \cdots, x_r) \otimes_k k(x)$ with residue field equal to $k(x)$. By (1), it suffices to prove that X is regular at x if and only if X' is regular at x'.

Since $k(x)/k$ is finite and separable, $\operatorname{Spec}(k(x)) \longrightarrow \operatorname{Spec} k$ is etale. As etale is stable under base change, get $X' \longrightarrow X$ is etale. By Proposition 5.8, we have that $\dim_{k(x)}(T_{X',x'}) = \dim_{k(x)}(T_{X,x} \otimes_{k(x)} k(x)) = \dim_{k(x)}(T_{X,x})$ and $\dim(\mathcal{O}_{X',x'}) = \dim(\mathcal{O}_{X,x})$. Thus X is regular at x if and only if X' is regular at x' as Remark 5.3. ☐

Definition 5.9. *Let X be a scheme of finite type over field k, $\overline{k}$ algebraic closure of k. We say that X is smooth (or say nonsingular) over k if $X_{\overline{k}} = X \times_k \overline{k}$ is regular.*

Example 5.6. *Let k be a field. Then $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are smooth over k.*

Remark 5.10. *By Proposition 5.9 (2), it is easy to see that if k is perfect, then smoothness over k is equivalent to regularity over k. In addition, X is smooth over k if and only if for all K/k field extension, $X_k = X \times_k K$ is regular.*

Definition 5.10. *Let $f : X \longrightarrow Y$ be a morphism of finite type between locally noetherian schemes. We say that f is smooth if it is flat and for all $y \in Y$, the fiber X_y is smooth over $k(y)$. In addition, we say that f is smooth of relative dimension n if moreover all nonempty fibers are pure of dimension n.*

Remark 5.11. *Somewhat etale is equivalent to smooth of relative dimension 0.*

Theorem 5.10. *Let $f : X \longrightarrow Y$ be a smooth morphism between locally noetherian schemes. If Y is regular, then X is regular.*

Remark 5.12. *If X is regular, then we also have Y is regular even when f is just flat.*

Proof. For $x \in X$ and $y = f(x)$, set $m = \dim(\mathcal{O}_{X,x})$ and $n = \dim(\mathcal{O}_{Y,y})$. Then by Theorem 5.8, $\mathcal{O}_{X_y,x} = m - n$. Since f is smooth, X_y is smooth over $k(y)$ so that X_y is regular at $x \in X_y$. Thus the maximal ideal of $\mathcal{O}_{X_y,x}$ can be generated by $m - n$ elements $b_{n+1}, \cdots, b_m$. Since $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, can lift b_i to $a_i \in \mathfrak{m}_x$ and complete them with $a_1, \cdots, a_n \in \mathfrak{m}_y$ generators of $\mathfrak{m}_y$. Get m generators $a_1, \cdots, a_m \in \mathfrak{m}_x$. Thus $\mathcal{O}_{X,x}$ is regular. ☐

Corollary 5.5. *Smooth morphisms are stable under composition and base change.*

Reason 5.3. *Theorem 5.10 and remark.*

6 Sheaves of Modules

6.1 Modules over Ringed Space

Definition 6.1. *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{F}$ on X such that for all $U \subseteq X$ open, $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ which is compatible with restriction maps.*

Definition 6.2. *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}_X$-modules. A morphism φ from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of sheaves compatible with the module structure i.e. for all $U \subseteq X$ open, $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.*

Remark 6.1. *Kernel, image and cokernel of a morphism of $\mathcal{O}_X$-modules are still $\mathcal{O}_X$-modules. Can also define submodule, quotient module and direct sum.*

Definition 6.3. *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space. An sheaf ideal on X is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.*

Definition 6.4 (***Tensor Products***)**.** *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}_X$-modules. Define tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheafification of $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.*

Remark 6.2. *The stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ at $x \in X$ is just $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Definition 6.5 (***Free $\mathcal{O}_X$-modules***)**.** *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is free if $\mathcal{F}$ is isomorphic to a direct sum of $\mathcal{O}_X$. In addition, we say that $\mathcal{F}$ is free of rank r if $\mathcal{F} \cong \mathcal{O}_X^r$.*

Definition 6.6 (***Locally Free $\mathcal{O}_X$-modules***)**.** *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is locally free if X can be covered by open subsets $\{U_i\}_{i \in I}$ such that $\mathcal{F}\big|_{U_i}$ is free as $\mathcal{O}_{U_i}$-module. In addition, we say that $\mathcal{F}$ is locally free of rank r if X can be covered by open subsets $\{U_i\}_{i \in I}$ such that $\mathcal{F}\big|_{U_i} \cong \mathcal{O}_{U_i}^r$ as $\mathcal{O}_{U_i}$-module.*

Definition 6.7 (***Invertible $\mathcal{O}_X$-modules***)**.** *Let X be a scheme or $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is invertible if it is locally free of rank 1.*

Remark 6.3. *If take $\mathcal{G} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$.*

Lemma 6.1. *Let $f : X \longrightarrow Y$ be a morphism of schemes or ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module, $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then the direct image $f_*\mathcal{F}$ of $\mathcal{F}$ is an $\mathcal{O}_Y$-module induced by $f^\sharp$. On the other hand, the inverse image $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ of $\mathcal{G}$ is an $\mathcal{O}_X$-module where $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is induced by $f^\sharp$.*

6.2 Quasi-coherent and Coherent Sheaves

Definition 6.8. *Let A be a ring, $X = \operatorname{Spec} A$, $M \in Mod_A$. Define an $\mathcal{O}_X$-module $\widetilde{M}$ on X by the same process as $\mathcal{O}_X$ replacing A by M.*

Remark 6.4. *The definition gives a fully faithful functor from Mod_A to category of $\mathcal{O}_{Spec(A)}$-modules $M \longmapsto \widetilde{M}$.*

Proposition 6.1. *Let A be a ring, $X = \operatorname{Spec} A$, $M \in Mod_A$. Then*
(1)$\widetilde{M}(X) = M$,
(2)$\widetilde{M}(D(f)) = M_f = M \otimes_A A_f$
(3)$\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$.

Proposition 6.2. *Let A be a ring, $X = \operatorname{Spec} A$. Then*
(1)Let M_i be A-modules. Then $\widetilde{\oplus_i M_i} \cong \oplus_i \widetilde{M_i}$.
(2)Let M,N be A-modules. Then $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.
(3)Let L, M, N be A-modules. Then $L \longrightarrow M \longrightarrow N$ is exact if and only if $\widetilde{L} \longrightarrow \widetilde{M} \longrightarrow \widetilde{N}$ is exact.
(4)Let $f : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec} A$ be a morphism of affine schemes corresponding to ring homomorphism $\varphi : A \longrightarrow B$, $M \in Mod_A$, $N \in Mod_B$. Then $f_*\widetilde{N} \cong \widetilde{N}$, where N is viewed as A-module, and $f^*\widetilde{M} \cong \widetilde{M \otimes_A B}$.

Definition 6.9 (***Quasi-coherent Sheaves***)**.** *Let X be a scheme, $\mathcal{F}$ a sheaf of abelian groups on X. We say that $\mathcal{F}$ is quasi-coherent if for all affine open subset $U = \operatorname{Spec} A \subseteq X$, $\mathcal{F}\big|_U \cong \widetilde{M}$ for some A-module M.*

Definition 6.10 (***Coherent Sheaves***)**.** *Let X be a noetherian scheme, $\mathcal{F}$ a sheaf of abelian groups on X. We say that $\mathcal{F}$ is coherent if for all affine open subset $U = \operatorname{Spec} A \subseteq X$, $\mathcal{F}\big|_U \cong \widetilde{M}$ for some finite A-module M.*

Theorem 6.1. *Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. Suppose X admits an affine open subset covering $\{U_i = \operatorname{Spec}(A_i)\}_i$ such that for all i, $\mathcal{F}\big|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i. Then $\mathcal{F}$ is quasi-coherent. If moreover X is noetherian and M_i are finite over A_i, then $\mathcal{F}$ is coherent.*

Proof. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Cover each $U \cap U_i$ with $\{U_{ik}\}$ such that $U_{ik} \subseteq U_i$ of the form $D(g_{ik})$. Then $\mathcal{F}\big|_{U_{ik}} \cong \widetilde{M_{i g_{ik}}}$. Can assume $X = U = \operatorname{Spec} A$ and finitely many affine open subset U_i such that $\mathcal{F}\big|_{U_i} \cong \widetilde{M_i} = \widetilde{\mathcal{F}(U_i)}$.

To prove $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$, it suffices to show that for all $f \in A$, $\mathcal{F}(X)_f = \mathcal{F}(X) \otimes_A A_f \longrightarrow \mathcal{F}(D(f))$ is isomorphic. Set $V_i = U_i \cap D(f) = D(f\big|_{U_i})$. As tensor product commutes with finite direct sum, there is commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \prod_i \mathcal{F}(U_i)_f & \longrightarrow & \prod_{i,j} \mathcal{F}(U_i \cap U_j)_f \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \prod_i \mathcal{F}(V_i) & \longrightarrow & \prod_{i,j} \mathcal{F}(V_i \cap V_j) \end{array}$$

Similar to proof of Lemma 2.6, get $\mathcal{F}$ is quasi-coherent.

Further if A is noetherian, $\mathcal{F} = \widetilde{M}$. Can cover X by finitely many $\{D(f_i)\}$ such that $\mathcal{F}(D(f_i)) = M_{f_i}$ is finite over A_{f_i} Assume $M_{f_i} = \sum_j \frac{m_{ij}}{1} A_{f_i}$. Take $N = \sum_{i,j} m_{i,j} A$. Then N is a finitely generated A-submodule of M and for all i, $N_{f_i} = M_{f_i}$. Thus there is a natural morphism $\widetilde{N} \longrightarrow \widetilde{M}$ and $\widetilde{N}\big|_{D(f_i)} \xrightarrow{\sim} \widetilde{M}\big|_{D(f_i)}$ is isomorphic for all i. As $d(f)_i$ cover X, get $\widetilde{N} \longrightarrow \widetilde{M}$ is isomorphic so that $N = M$ and M is finitely generated. Thus $\mathcal{F}$ is coherent. $\square$

Proposition 6.3. *Let X be a scheme. Then*
(1)Kernel, image and cokernel of morphism of quasi-coherent sheaves are still quasi-coherent.
(2)Direct sum of quasi-coherent sheaves is quasi-coherent.
(3)Tensor product of two quasi-coherent sheaves is quasi-coherent.

Remark 6.5. *If X is moreover noetherian, then (1) and (3) still hold for coherent and (2) is also ok if the direct sum is finite.*

Proposition 6.4. *Let X be an affine scheme. Then for all exact sequence of quasi-coherent sheaves*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

the sequence of global sections

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \longrightarrow 0$$

is exact.

Proposition 6.5. *Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for all $x \in X$, there exists open neighbourhood $U \ni x$ such that there is an exact sequence*

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}\big|_U \longrightarrow 0$$

If moreover X is noetherian, then $\mathcal{F}$ is coherent if and only if for all $x \in X$, there exists open neighbourhood $U \ni x$ such that there is an exact sequence with I,J finite

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}\big|_U \longrightarrow 0$$

Proof. "$\Rightarrow$": For $\mathcal{F}$ quasi-coherent and all $x \in X$, there is affine open neighbourhood $U = \operatorname{Spec} A \ni x$ such that $\mathcal{F}\big|_U \cong \widetilde{M}$ for some A-module M. Note that there is an exact sequence for some index sets I,J

$$A^{\oplus J} \longrightarrow A^{\oplus I} \longrightarrow M \longrightarrow 0$$

By Proposition 6.2 (3), get exact sequence

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}\big|_U \longrightarrow 0$$

"$\Leftarrow$": For all $x \in X$, there exists open neighbourhood $U \ni x$ such that there is an exact sequence

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}\big|_U \longrightarrow 0$$

Can assume that U is affine. By Proposition 6.4, get exact sequence

$$\mathcal{O}_U(U)^{\oplus J} \longrightarrow \mathcal{O}_U(U)^{\oplus I} \longrightarrow \mathcal{F}\big|_U(U) \longrightarrow 0$$

By Proposition 6.2, get exact sequence

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \widetilde{\mathcal{F}\big|_U(U)} \longrightarrow 0$$

Thus $\mathcal{F}\big|_U \cong \widetilde{\mathcal{F}\big|_U(U)}$. By Theorem 6.1, $\mathcal{F}$ is quasi-coherent. For coherent, the same argument would make sense. $\square$

Example 6.1. *(1)Let X be a scheme. Then locally free $\mathcal{O}_X$-modules are quasi-coherent.*
(2)Let X be a noetherian scheme. Then locally free $\mathcal{O}_X$-modules of rank $< \infty$ are coherent.
(3)Let $i : Y \longrightarrow X$ be a closed immersion with X, Y noetherian. Then $i_*\mathcal{O}_Y$ is coherent on X.
(4)Let X be an integral noetherian scheme with function field K. Then constant sheaf K_X is quasi-coherent but not coherent in general.
(5)Let X be a scheme, $U \subseteq X$ open subset with inclusion $j : U \longrightarrow X$, $\mathcal{F}$ sheaf on U. Define $j_!\mathcal{F}$ to be the sheafification of $V \longmapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$. Then stalk $(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$. Now let $X = \operatorname{Spec} A$ be an affine scheme with A integral domain, $U \subsetneq X$ nonempty open subset with inclusion $j : U \longrightarrow X$. Then $j_!\mathcal{O}_U$ is an $\mathcal{O}_X$-module but not quasi-coherent.

6.3 Direct and Inverse Images

Lemma 6.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes, $\mathcal{M}, \mathcal{N}$ two $\mathcal{O}_Y$-modules. Then $f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})$.*

Proof. By definition, it suffices to prove that $f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N} \cong f^{-1}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})$. Firstly, want to show that

$$\varinjlim_{V \supseteq f(U)} \mathcal{M}(V) \otimes_{\varinjlim_{V \supseteq f(U)} \mathcal{O}_Y(V)} \varinjlim_{V \supseteq f(U)} \mathcal{N}(V)$$

is isomorphic to $\varinjlim_{V \supseteq f(U)} (\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(V))$ for all opne subset $U \subseteq X$. Note that there are two ring homomorphisms $\varphi : \sum_i (m_i, V_i) \otimes (n_i, V_i') \longmapsto (\sum_i m_i\big|_V \otimes n_i\big|_V, V)$, where $V = \cap_i (V_i \cap V_i')$, and $\psi : (\sum_i m_i \otimes n_i, V) \longmapsto \sum_i (m_i, V) \otimes (n_i, V)$. Obviously, $\varphi \circ \psi(\sum_i m_i \otimes n_i, V) = \sum_i m_i \otimes n_i, V$ and $\psi \circ \varphi(\sum_i (m_i, V_i) \otimes (n_i, V_i')) = \sum_i (m_i, V_i) \otimes (n_i, V_i')$ so that get the isomorphism.

Now we have ring homomorphisms $\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(V) \longrightarrow (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(V)$ compatible with restriction maps for each open subset $V \supseteq f(U)$. By property of directed limit, there is an induced ring homomorphism $\varinjlim_{V \supseteq f(U)} (\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(V)) \longrightarrow \varinjlim_{V \supseteq f(U)} (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(V)$. For $U' \subseteq U$ inclusion of open subsets of X, it is easy to show the following diagram commutes

$$\begin{array}{ccc}
\varinjlim_{V \supseteq f(U)} (\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(V)) & \longrightarrow & \varinjlim_{V \supseteq f(U)} (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(V) \\
\downarrow & & \downarrow \\
\varinjlim_{V \supseteq f(U')} (\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(V)) & \longrightarrow & \varinjlim_{V \supseteq f(U')} (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(V)
\end{array}$$

By sheafification, get a morphism of $\mathcal{O}_X$-modules $f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N} \longrightarrow f^{-1}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})$. As their stalks are same everywhere, it is an isomorphism. $\square$

Proposition 6.6. *Let $f : X \longrightarrow Y$ be a morphism of schemes, $\mathcal{G}$ a quasi-coherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Can assume $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec} A$. Then $\mathcal{G} = \widetilde{M}$ for some A-module M so that $f^* = \widetilde{M \otimes_A B}$. Get $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module. $\square$

Proposition 6.7. *Let $f : X \longrightarrow Y$ be a morphism between noetherian schemes, $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a coherent $\mathcal{O}_X$-module.*

Proof. Can assume $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec} A$. Then $\mathcal{G} = \widetilde{M}$ for some finitely generated A-module M so that $f^* = \widetilde{M \otimes_A B}$. Since M is finitely generated A-module, get $M \otimes_A B$ finitely generated as B-module. Get $f^*\mathcal{G}$ is a coherent $\mathcal{O}_X$-module. □

Remark 6.6. *For direct image, if N is finitely generated B-module, we don't have that N is finitely generated as A-module in general. Thus the similar property for direct image is much more complicated.*

Theorem 6.2. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Assume that either X is noetherian or f is quasi-compact and separated. Then*
(1)If $\mathcal{F}$ is quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent.
(2)If $\mathcal{F}$ is coherent on X with X, Y noetherian and f is finite, then $f_*\mathcal{F}$ is coherent.

Proof. For general case, can assume that $Y = \operatorname{Spec} A$. Let $g \in A$ with image $h \in B = \mathcal{O}_X(X)$. Thus $f^{-1}(D(g)) = X_h$ so that $(f_*\mathcal{F})(D(g)) = \mathcal{F}(X_h)$. It suffices to show that $\mathcal{F}(X_h) = f_*\mathcal{F}(Y)_g = \mathcal{F}(X) \otimes_A A_g$. Note that $B \otimes_A A_g \cong B_h$, we have $\mathcal{F}(X) \otimes_A A_g \cong \mathcal{F}(X)_h$. Cover X be affine open subsets $\{U_i\}$. As either X is noetherian or f is quasi-compact and separated, the covering is finite. There is a commutative diagram with exact rows,

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}(X)_h & \longrightarrow & \prod_i \mathcal{F}(U_i)_h & \longrightarrow & \prod_{i,j} \mathcal{F}(U_i \cap U_j)_h \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathcal{F}(X_h) & \longrightarrow & \prod_i \mathcal{F}(X_h \cap U_i) & \longrightarrow & \prod_{i,j} \mathcal{F}(X_h \cap U_i \cap U_j)
\end{array}$$

On the one hand, when X is noetherian, $U_i \cap U_j$ can still be covered by finitely many affine open subsets. On the other hand, when f is quasi-compact and separated, $U_i \cap U_j$ is affine. Thus similar to proof of Lemma 2.6, get $\mathcal{F}(X_h) \cong \mathcal{F}(X)_h$ so that $\mathcal{F}$ is quasi-coherent.

For finite case, can assume $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec} A$ with B finite over A. Since $\mathcal{F}$ is coherent, get $\mathcal{F} = \widetilde{N}$ for some finitely generated B-module. By Proposition 6.2, get $f_*\mathcal{F} = f_*\widetilde{N} = \widetilde{N \text{ as } A\text{-module}}$. While N is finitely generated and B is finite over A, get N is also finitely generated as A-module. Thus $f_*\mathcal{F}$ is coherent. □

Example 6.2 (***Ideal Sheaves***)**.** *Let $i : Y \longrightarrow X$ be a closed immersion. Define the ideal sheaf of Y to be $I_Y = \ker(\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y)$. Then I_Y is quasi-coherent and coherent if X, Y are noetherian. Conversely, all quasi-coherent ideal sheaves $\mathcal{F}$ on X are of the form I_Y for a uniquely determined closed subscheme Y. Precisely, we have that $Y = \{x \in X \big| (\mathcal{O}_X/\mathcal{F})_x \neq 0\}$.*

In particular, if $X = \operatorname{Spec} A$, there are some one-to-one correspondences

$$\begin{array}{ccc}
\{\textit{ideals of } A\} & \longleftrightarrow & \{\textit{closed subschemes of } X\} \\
\nwarrow\!\!\searrow & & \swarrow\!\!\nearrow \\
& \{\textit{quasi-coherent ideal sheaves on } X\} &
\end{array}$$

6.4 Quasi-coherent Sheaves on Projective Schemes

Assume that S is a graded ring. Let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a graded C-module. Define $\widetilde{M}$ on $X = \mathrm{Proj}(S)$ by the same process on $\mathcal{O}$ replacing S by M. In particular, $\widetilde{M}\big|_{D_+(f)} = \widetilde{M_{(f)}}$ and $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$ for $\mathfrak{p} \in X$. Then $\widetilde{M}$ is quasi-coherent on X and coherent if X is noetherian and M is finitely generated. In addition, we should note that $\widetilde{M}$ doesn't determine M. For instance, if $M = \oplus_{d\geq 0} M_d$ and $N = \oplus_{d\geq d_0} M_d$ for some $d_0 > 0$, then $\widetilde{M} = \widetilde{N}$.

Definition 6.11 (***Twisting***)**.** *Let S be a graded ring and $X = \mathrm{Proj}(S)$, $\mathcal{F}$ an $\mathcal{O}_X$-module. Set $\mathcal{O}_X(n) = \widetilde{S(n)}$ where $S(n)$ is the graded C-module given by $S(n)_d = S_(n+d)$. Also set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.*

Proposition 6.8. *Let S be a graded ring and $X = \mathrm{Proj}(S)$. Assume that S is generated by S_1 as S_0-algebra. Then $\mathcal{O}_X$ is an invertible sheaf. In addition, For all M graded S-module, we have $\widetilde{M(n)} = \widetilde{M}(n)$. In particular, $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$.*

Proof. Since S is generated by S_1 as S_0-algebra, $\{D_+(f)\}_{f\in S_1}$ cover X. Suffice to show for all $f \in S_1$, $S(n)_{(f)}$ is free of rank 1 over $S_{(f)}$. Note that there is a canonical isomorphism $S(n)_{(f)} \longrightarrow S_{(f)} \quad s \longmapsto f^{-n}s$, done! □

Lemma 6.3 (***qcqs Lemma***)**.** *Let X be a scheme, $\mathcal{F}$ quasi-coherent sheaf on X, $\mathcal{L}$ invertible sheaf on X, $s \in \mathcal{L}(X)$. Assume that X is either noetherian or quasi-compact and separated over* $\mathrm{Spec}(\mathbb{Z})$. *Then*
(1)Let $f \in \mathcal{F}(X)$. If $f\big|_{X_s} = 0$, then there exists $n > 0$ such that $f \otimes s^n = 0$ in $(\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{L}^n)(X)$, where $X_s = \{x \in X \big| s_x\mathcal{O}_{X,x} = \mathcal{L}_x\}$ is open and $\mathcal{L}^n$ denotes $\mathcal{L}^{\otimes n}$.
(2)Let $g \in \mathcal{F}(X_s)$. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, $g \otimes (s^n\big|_{X_s})$ extends to a section in $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)(X)$.

Proof. Cover X by finitely many affine open subsets $\{U_i\}$ such that $\mathcal{L}\big|_{U_i}$ is free and generated by $e_i \in \mathcal{L}(U_i)$. Then $s_i = s\big|_{U_i} = h_ie_i$ for some $h_i \in \mathcal{O}_X(U_i)$. Get $X_s \cap U_i = D(h_i) \subseteq U_i$.

(1): Note that $\mathcal{F}\big|_{U_i} \cong \widetilde{\mathcal{F}(U_i)}$. Since $f\big|_{D(h_i)}$, there exists $n > 0$ such that for all i, $f_i = f\big|_{U_i}$ satisfies that $h_i^n f_i = 0$ in $\mathcal{F}(U_i)$. Thus $(f \otimes s)\big|_{U_i} = f_i \otimes s_i^n = h_i^n f_i \otimes e_i^n = 0$. As U_i cover X, get $f \otimes s^n = 0$.

(2): Set $g_i = g\big|_{D(h_i)}$, Then there exists $m > 0$ such that for all i, there exists $g_i' \in \mathcal{F}(U_i)$ such that $h_i^m g_i = g_i'$. Set $t_i = g_i' \otimes e_i^m \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m)(U_i)$. Note that t_i and t_j coincide on $X_s \cap U_i \cap U_j$. Thus $(t_i\big|_{U_i\cap U_j} - t_j\big|_{U_i\cap U_j})\big|_{X_s\cap U_i\cap U_j}$. As $U_i \cap U_j$ is either noetherian or affine, by (1), there exists $p > 0$ such that $(t_i\big|_{U_i\cap U_j} - t_j\big|_{U_i\cap U_j}) \otimes s\big|_{U_i\cap U_j}^p = 0$ in $((\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{L}^m)\big|_{U_i\cap U_j} \otimes_{\mathcal{O}_{U_i\cap U_j}} (\mathcal{L}\big|_{U_i\cap U_j})^p)(U_i \cap U_j)$. While $(\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{L}^m)\big|_{U_i\cap U_j} \otimes_{\mathcal{O}_{U_i\cap U_j}} (\mathcal{L}\big|_{U_i\cap U_j})^p \cong (\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{L}^{m+p})\big|_{U_i\cap U_j}$, get $t_i \otimes s_i^p$ and $t_j \otimes s_j^p$ coincide on $U_i \cap U_j$. Thus we can glue them to obtain $t \in (\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{L}^{m+p})(X)$ such that $t\big|_{X_s} = g \otimes (s\big|_{X_s}^{m+s})$. Take $n_0 = m + p$, done! □

Definition 6.12. *Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is generated by global sections (or say globally generated) if there exists a family of global sections $\{s_i\}$ such that for all $x \in X$, $\{(s_i)_x\}$ generate $\mathcal{F}_x$.*

Remark 6.7. *Note that $\mathcal{F}$ is generated by global sections if and only if $\mathcal{F}$ is a quotient of a free $\mathcal{O}_X$-module.*

Example 6.3. *(1)Let $X = \mathbb{P}_k^n$ be projective k-scheme. Then $\mathcal{O}_X(-1)$ is not generated by global sections since there is no global sections at all.*
(2)Let $X = \operatorname{Spec} A$ be an affine scheme. Then all quasi-coherent sheaves on X are generated by global sections.

Definition 6.13. *Let A be a ring and X be a projective scheme over A with commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & \mathbb{P}_A^d \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

Set $\mathcal{O}_X(n) = i^*\mathcal{O}_{\mathbb{P}_A^d}(n)$. For $\mathcal{F}$ an $\mathcal{O}_X$-module, also set $\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Lemma 6.4 (***Projection Formula for Affine Case***). *Let $f : X \longrightarrow Y$ be an affine morphism, $\mathcal{F}$ a quasi-coherent sheaf on X, $\mathcal{G}$ a quasi-coherent sheaf on Y. Then $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$.*

Proof. Can assume that $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{N}$ and $\widetilde{M}$. Then $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong \widetilde{P}$, where $P = N \otimes_B (M \otimes_A B)$ as A-module. On the other hand, $f_*\mathcal{F} \otimes \mathcal{G} \cong \widetilde{Q}$, where $Q = N \otimes_A M$ as A-module. As $P \cong Q$, done! □

Theorem 6.3 (***Serre's Theorem***)**.** *Let A be a noetherian ring and X be a projective scheme over A with commutative diagram,*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & \mathbb{P}_A^d \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

Assume that $\mathcal{F}$ is a coherent sheaf on X. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F}(n)$ is generated by finitely many global sections.

Proof. Since i is a closed immersion, i is finite so that $i_*\mathcal{F}$ is coherent. By Lemma 6.4, $i_*(\mathcal{F}(n)) \cong (i_*\mathcal{F})(n)$. Thus global sections of $i_*(\mathcal{F}(n))$ are equal to global sections of $(i_*\mathcal{F})(n)$ and stalks of $i_*(\mathcal{F}(n))$ are equal to stalks of $(i_*\mathcal{F})(n)$. It suffices to prove Theorem for $X = \mathbb{P}_A^d = \operatorname{Proj}(A[x_0, \cdots, x_d])$.

Cover X by $\{U_i = D_+(x_i)\}$. Each $\mathcal{F}\big|_{U_i}$ is generated by finitely many sections $g_{ij} \in \mathcal{F}(U_i)$. By Lemma 6.3, there exists $n_0 > 0$ such that $g_{ij} \otimes x_i^n$ extends to a global section for all i,j. Thus $\mathcal{F}(n)$ is generated by these extensions. □

Corollary 6.1. *Let A be a noetherian ring and X be a projective scheme over A with commutative diagram,*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & \mathbb{P}_A^d \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

Assume that $\mathcal{F}$ is a coherent sheaf on X. Then there exists $m \in \mathbb{Z}$ and $r < \infty$ such that $\mathcal{F}$ is a quotient of $\mathcal{O}_X(m)^{\oplus r}$.

Reason 6.1. *Since $\mathcal{F}(n)$ is generated bv global sections for large enough n, $\mathcal{F}(n)$ is quotient of $\mathcal{O}_X^{\oplus I}$ with finite index set. Tensor by $\mathcal{O}_X(-n)$, done!*

Definition 6.14. *Let S be a graded ring generated by finitely many elements of S_1 as S_0-algebra and $X = \mathrm{Proj}(S)$, $\mathcal{F}$ an $\mathcal{O}_X$-module. Define a graded C-module $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{F}(n))$.*

Proposition 6.9. *Let A be a ring, $S = A[x_0, \cdots, x_d]$ with $d > 0$ and $X = \mathrm{Proj}(S) = \mathbb{P}^d_A$. Then $\Gamma_*(\mathcal{O}_X) = S$.*

Proof. Want to show $\Gamma(X, \mathcal{O}_X(n)) = \begin{cases} S_n & n \geq 0 \\ 0 & n < 0 \end{cases}$. Let $B = A[x_0, \cdots, x_d, x_0^{-1}, \cdots, x_d^{-1}]$. Then any global section of $\mathcal{O}_X(n)$ gives an element $f \in B$ which is in $x_0^n\mathcal{O}_X(D_+(x_0))$ by restriction. Thus f is of the form $\frac{P}{x_0^m}$ where P is a homogeneous polynomial. But there is also a restriction in $x_1^n\mathcal{O}_X(D_+(x_1))$, get f is homogeneous polynomial of degree n if $n \geq 0$ and $f = 0$ if $n < 0$. Conversely, such an f is obviously in $\Gamma(X, \mathcal{O}_X(n))$. □

Theorem 6.4. *Let S be a graded ring generated by finitely many elements of S_1 as S_0-algebra and $X = \mathrm{Proj}(S)$. Then for all quasi-coherent sheaf $\mathcal{F}$ on X, we have $\mathcal{F} \cong \widetilde{\Gamma_*(\mathcal{F})}$.*

Proof. Let $s \in S_1$. Set $\Gamma_*(\mathcal{F}) = M$ and $U = D_+(s)$. Want to show that there is a canonical isomorphism $\varphi_s : M_{(s)} \xrightarrow{\sim} \mathcal{F}(U)$. Assume $s^{-n}t \in M_{(s)}$ with $t \in \Gamma(X, \mathcal{F}(n))$. Note that $\mathcal{F}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n) \cong \mathcal{F}$, get $t\big|_U \otimes s^{-n} \in \Gamma(U, \mathcal{F})$. Then define $\varphi : s^{-n}t \longmapsto t\big|_U \otimes s^{-n}$. By Lemma 6.3, for each element $g \in \mathcal{F}(U)$, there exists $m > 0$ such that $g \otimes s^m$ extends to $t \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m))(X) \subseteq M$. Thus $\varphi_s(\frac{t}{s^m}) = t\big|_U \otimes s^{-m} = g \otimes s^m \otimes s^{-m} = g$ so that φ_s is surjective. On the other hand, if $s^{-n}t \in \ker(\varphi_s)$, then $t\big|_U \otimes s^{-n} = 0$ in $\mathcal{F}(U)$. Note that $t\big|_U \otimes s^{-n} \otimes s^n = t\big|_U$, by Lemma 6.3, get there exists $m > 0$ such that $t \otimes s^m = 0$ in $\mathcal{F}(m)(X)$ so that $s^{-n}t = 0$ in $M_{(s)}$. Thus φ_s is isomorphic. By gluing lemma of morphism, get $\widetilde{M} \xrightarrow{\sim} \mathcal{F}$ isomorphic. □

Theorem 6.5. *Let A be a ring, $S = A[x_0, \cdots, x_d]$ with $d > 0$ and $X = \mathrm{Proj}(S) = \mathbb{P}^d_A$. Then any closed subscheme Z of X is of the form $\mathrm{Proj}(S/I)$ for uniquely determined homogeneous ideal I.*

Proof. Can assume that $d > 0$ since when $d = 0$, $\mathrm{Proj}(X) = D_+(x_0)$ is just the affine case. Take $\mathcal{I}$ to be the ideal sheaf of Z. Then $\mathcal{I}$ is quasi-coherent By Theorem 6.4, $\mathcal{I} = \widetilde{I}$ where $I = \Gamma_*(\mathcal{I})$. Since I is a homogeneous submodule of $\Gamma_*(\mathcal{O}_X)$, by Theorem 6.3, I is a homogeneous ideal of S. Restrict to standard open subsets and by gluing lemma, we get $(Z, \mathcal{O}_Z) \cong (Proj(S/I), \mathcal{O}_{Proj(S/I)})$. □

6.5 Ample and Very Ample Invertible Sheaves

Proposition 6.10. *Let A be a ring, X an A-scheme, $Y = \mathrm{Proj}(A[x_0, \cdots, x_d]) = \mathbb{P}^d_A$. Then (1)Let $f : X \longrightarrow Y$ be an A-morphism. Then $f^*\mathcal{O}_Y(1)$ is an invertible sheaf on X, generated*

by $d+1$ global sections.
(2)Conversely, if $\mathcal{L}$ is an invertible sheaf on X, generated by global sections $s_0, \cdots, s_d$, then there exists unique A-morphism $f : X \longrightarrow Y$ such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ and f^*x_i is identified with s_i.

Proof. (1): $\mathcal{O}_Y(1)$ is generated by $d+1$ global sections, inducing global sections $s_1, \cdots, s_d$ of $f^*\mathcal{O}_Y(1)$. While for all $x \in X$ and $y = f(x)$, then $(f^*\mathcal{O}_Y(1))_x = \mathcal{O}_Y(1)_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. Thus $s_{0x}, \cdots, s_{dx}$ generate $(f^*\mathcal{O}_Y)_x$.

(2): As assumption, X is covered by $\{X_{s_i}\}$. For all i, want to define $f_i : X_{s_i} \longrightarrow D_+(x_i)$. By Hartshorne exercise II 2.4, it is equivalent to give a ring homomorphism between global sections $A[\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_d}{x_i}] \longrightarrow \mathcal{O}_{X_{s_i}}$, which can be defined by mapping $\frac{x_j}{x_i}$ to $\frac{s_j}{s_i}$, where $\frac{s_j}{s_i}$ is the unique element $a \in \mathcal{O}_X(X_{s_i})$ such that $s_j\big|_{X_{s_i}} = as_i\big|_{X_{s_i}}$. It is easy to check that f_i satisfy conditions for gluing lemma. Get $f : X \longrightarrow Y$. □

Remark 6.8. *Intuitively, f should be $x \longmapsto (s_0(x) : \cdots : s_d(x))$. In addition, f is a closed immersion if and only if for all i, X_{s_i} is affine and the A-algebra is generated by $\{s_j/s_i\}$.*

Definition 6.15 (***Immersions***). *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is an immersion if f factors as $X \overset{\text{open immersion}}{\longrightarrow} Z \overset{\text{closed immersion}}{\longrightarrow} Y$.*

Proposition 6.11. *There are lots of properties of immersions.*
(1)Closed immersions are immersions.
(2)Composition of immersions are immersions.
(3)Immersions are stable under base change.

Remark 6.9. *Since (1),(2),(3) are in fact the properties (a),(b),(c) in Hartshorne exercise ii 4.8, by the exercise, we immediately get that (d),(e),(f) also holds for immersion.*

Definition 6.16 (***Quasi-projective***). *Let A be a ring, X an A-scheme. We say that X is quasi-projective over A if $X \longrightarrow \operatorname{Spec} A$ factors as*

$$\begin{tikzcd} X \arrow[r, "immersion"] \arrow[dr] & \mathbb{P}^d_A \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

In Midterm24.pdf, there is also a definition of immersion which is a little different from here and we would rename it.

Definition 6.17 (***Locally Closed Immersions***). *Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is a locally closed immersion if f factors as $X \overset{\text{closed immersion}}{\longrightarrow} Z \overset{\text{open immersion}}{\longrightarrow} Y$.*

Proposition 6.12. *Let $f : X \longrightarrow Y$ be a morphism of schemes. If f is a immersion, then f is a locally closed immersion. On the other hand, if f is a locally closed immersion and suppose that either X is noetherian or f is quasi-compact, then f is an immersion.*

Definition 6.18 (***Very Ample Invertible Sheaves***)**.** *Let A be a ring, X an A-scheme, $\mathcal{L}$ an invertible sheaf on X. We say that $\mathcal{L}$ is very ample relative to A if there exists A-immersion $f : X \longrightarrow \mathbb{P}^d_A$ such that $\mathcal{L} \cong \mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}^d_A}(1)$.*

Remark 6.10. *$\mathcal{L}$ is very ample if and only if $\mathcal{L}$ can be generated by finitely many global sections such that the associated A-morphism $f : X \longrightarrow \mathbb{P}^d_A$ is an immersion. If moreover $X \longrightarrow \operatorname{Spec} A$ is proper, then f is a closed immersion. Thus X is projective over A if and only if X is proper over A and admits a very ample invertible sheaf.*

Definition 6.19 (***Ample Invertible Sheaves***)**.** *Let X be a noetherian scheme, $\mathcal{L}$ an invertible sheaf on X. We say that $\mathcal{L}$ is ample if for all coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is generated by global sections.*

Remark 6.11. *Serre's Theorem are saying that for $X \longrightarrow \operatorname{Spec} A$ proper with A noetherian, a very ample invertible sheaf $\mathcal{L}$ on X is ample. In fact, can remove proper by reducing to proper case with the following nontrivial fact which is Hartshorne exercise ii 5.15.*

Fact: Let X be a noetherian scheme, $U \subseteq X$ open subset, $\mathcal{F}$ coherent sheaf on U. Then there exists coherent sheaf on X whose restriction to U is just $\mathcal{F}$.

Example 6.4. *(1)Any invertible sheaf on an affine noetherian scheme is ample.*
(2)If $\mathcal{L}$ is ample, then $\mathcal{L}^r$ is ample for all $r > 0$. Conversely, if $\mathcal{L}^r$ is ample for some $r > 0$, then $\mathcal{L}$ is ample.
(3)Let $X = \mathbb{P}^d_A$. Then $\mathcal{O}_X(n)$ is very ample/ample if and only if $n > 0$.

Lemma 6.5. *Let X be a noetherian scheme, $\mathcal{L}$ ample invertible sheaf on X. For all $x \in X$, there exists $n > 0$ and $s \in \mathcal{L}^n(X)$ such that $x \in X_s \subseteq X$ and X_s is an affine open subset.*

Proof. Let $U \subseteq X$ be an affine open neighbourhood of x such that $\mathcal{L}\big|_U$ is free. Our goal is to find an affine open subset inside U of the form X_s containing x. Consider $Z = X \setminus U$ with the reduced scheme structure and the ideal sheaf $\mathcal{I}$ on Z. For all $n > 0$, since $\otimes_{\mathcal{O}_X} \mathcal{L}^n$ is exact functor, $I\mathcal{L}^n \subseteq \mathcal{L}^n$ is a subsheaf. As $\mathcal{L}$ is ample, for n big enough, $\mathcal{I}\mathcal{L}^n$ is generated by global sections. In particular, since $x \notin Z$, there exists global section $s \in \mathcal{I}\mathcal{L}^n(X) \subseteq \mathcal{L}^n(X)$ such that s_x generates $(\mathcal{I}\mathcal{L}^n)_x \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}^n_x \cong \mathcal{L}^n_x$. Thus $x \in X_s$. On the other hand, $X_s \subseteq U$ since if $y \in X_s$, then $(\mathcal{I}\mathcal{L}^n)_u$ contains s_y which generates $\mathcal{L}^n_y$. Thus $X_s = X_s \cap U$ is affine. □

Theorem 6.6. *Let A be a noetherian ring and X be a scheme of finite type over A, $\mathcal{L}$ an ample invertible sheaf on X. Then there exists $r > 0$ such that $\mathcal{L}^r$ is very ample.*

Proof. Since X is noetherian, we can cover X by finitely many $\{X_{s_i}\}$ as in Lemma 6.5. Can also assume $s_i \in \mathcal{L}^n(X)$ for same n and can even assume that $n = 1$ by replacing $\mathcal{L}$ by $\mathcal{L}^n$. Note that X is of finite type over A, $\mathcal{O}_X(X_{s_i})$ is generated as A-algebra by finitely many $\{g_{ij}\}$. By Lemma 6.3, there exists $r > 0$ such that $g_{ij} \otimes s_i\big|^r_{X_{s_i}}$ extends to a global section $s_{ij} \in (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^r)(X) = \mathcal{L}^r(X)$ for all i,j. Now $\{s_i^r\}$ generate $\mathcal{L}^r$ since $\{X_{s_i} = X_{s_i^r}\}$ cover X. Thus $\{s_i^r, s_{ij}\}$ also generate $\mathcal{L}^r$.

By Proposition 6.10, there is a morphism $f : X \longrightarrow \operatorname{Proj}(A[x_i, x_{ij}]) = \mathbb{P}$ such that $f^*\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{L}^n$. Now for all i, $\mathcal{O}_{\mathbb{P}}(D_+(x_i)) \longrightarrow \mathcal{O}_X(X_{s_i}) \quad x_{ij}/x_i \longmapsto g_{ij}$ is surjective, inducing by $g_{ij} \otimes s\big|_{X_{s_i}}^{r}$ extends to s_{ij}. Then f is a closed immersion from X to $\cup_i D_+(x_i) \subseteq \mathbb{P}$, since X_{s_i} and $D_+(x_i)$ are both affine. Get f is locally closed immersion. As X is Noetherian, by Proposition 6.12, get f is immersion. Thus $\mathcal{L}^r$ is very ample by remark 6.9. □

Corollary 6.2. *Let A be a noetherian ring and X be a scheme of finite type over A, $\mathcal{L}$ an ample invertible sheaf on X. Then X is quasi-projective (resp. projective) over A if and only if X admits an ample invertible sheaf (resp. X admits an ample invertible sheaf and X is proper).*

Remark 6.12. *Let A be a noetherian ring and X be a scheme of finite type over A, $\mathcal{L}$ an ample invertible sheaf on X. If X admits an ample invertible sheaf, then X is separated.*

7 Sheaf Cohomology

7.1 Cohomology Group

There are lots of abelian categories in Algebraic Geometry.

Example 7.1. *(1)Ab is the category of abelian groups.*
(2)Mod_A is the category of A-modules.
(3)$Ab(X)$ is the category of sheaves of abelian groups on a topological space X.
(4)$Mod(\mathcal{O}_X)$ is the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.
(5)$Qcoh(X)$ is the category of quasi-coherent sheaves on a scheme X.
(6)$Coh(X)$ is the category of coherent sheaves on a noetherian scheme X.

There are some special functors in Algebraic Geometry.

Example 7.2. *(1)$\mathrm{Hom}_{\mathcal{C}}(A, \cdot) : \mathcal{C} \longrightarrow Ab$ is left exact covariant functor.*
(2)$\mathrm{Hom}_{\mathcal{C}}(\cdot, A) : \mathcal{C} \longrightarrow Ab$ is left exact contravariant functor.
(3)$\Gamma(X, \cdot) : Ab(X) \longrightarrow Ab$ is left exact covariant functor.
(4)$M \otimes_A \cdot : Mod_A \longrightarrow Mod_A$ is right exact covariant functor. In particular, $M \otimes_A \cdot$ is exact if and only if M is flat A-module.

Definition 7.1. *Let $\mathcal{C}$ be a category with enough injective objects, $\mathcal{F}$ a left exact covariant functor, $J \in \mathcal{C}$. We say that J is acyclic for the functor $\mathcal{F}$ (or say $\mathcal{F}$-acyclic) if $R^i\mathcal{F}(J) = 0$ for all $i > 0$.*

Proposition 7.1. *Let $\mathcal{C}$ be a category with enough injective objects, $\mathcal{F}$ a left exact covariant functor, $A \in \mathcal{C}$. Assume that there is an $\mathcal{F}$-acyclic resolution J^* of A i.e. $0 \longrightarrow A \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots$ exact with each J_i $\mathcal{F}$-acyclic. Then $R^i\mathcal{F}(A) \cong H^i(\mathcal{F}(J^*))$.*

Proposition 7.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then $Mod(\mathcal{O}_X)$ has enough injective objects. In particular, $Ab(X)$ has enough injective objects if viewing X as $(X, \mathbb{Z}_X)$ ringed space, where $\mathbb{Z}_X$ is the constant sheaf.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For all $x \in X$, can embed $\mathcal{F}_x$ in an injective $\mathcal{O}_{X,x}$-module I_x since Mod_A has enough injective objects for all ring A. View I_x as a sheaf on $\{x\}$. Set $\mathcal{I} =$

$\prod_{x\in X} j_*I_X$ where $j:\{x\}\longrightarrow X$ is inclusion. For all $\mathcal{G}$ $\mathcal{O}_X$-module, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) = \prod_{x\in X}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_*I_x) = \prod_{x\in X}\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. With $\mathcal{F}_x \hookrightarrow I_x$, get a monomorphism $\mathcal{F}\longrightarrow \mathcal{I}$ with $\mathcal{I}$ injective. □

Definition 7.2 (***Sheaf Cohomology***)**.** *Let X be a topological space, $\Gamma(X,\cdot): Ab(X)\longrightarrow Ab$. Define the cohomology functor $H^i(X,\cdot) = R^i\Gamma(X,\cdot)$. For $\mathcal{F}\in Ab(X)$, we call $H^i(X,\mathcal{F})$ the ith cohomology group of $\mathcal{F}$.*

Definition 7.3. *Let X be a topological space, $\mathcal{F}\in Ab(X)$. We say that $\mathcal{F}$ is flasque if for all $V\subseteq U$ inclusion of open subsets, the restriction map $\mathcal{F}(U)\longrightarrow\mathcal{F}(V)$ is surjective.*

Proposition 7.3. *(1)Direct products of flasque sheaves are flasque.*
(2)Direct image of flasque sheaves are flasque.
(3)quotient of flasque sheaves by flasque sheaves are flasque.
(4)If $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}\longrightarrow\mathcal{H}\longrightarrow 0$ is exact with $\mathcal{F}$ flasque, then $0\longrightarrow\mathcal{F}(U)\longrightarrow\mathcal{G}(U)\longrightarrow\mathcal{H}(U)\longrightarrow 0$ is exact for all open subset $U\subseteq X$.

Proposition 7.4. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then every injective $\mathcal{O}_X$-module is flasque.*

Proof. Consider $U\subseteq X$ is an inclusion of open subsets. Set $\mathcal{G}_U = j_!\mathcal{O}_U$. Let $\mathcal{F}$ be an injective $\mathcal{O}_X$-module, $V\subseteq U$ open subset. Then the natural morphism $\mathcal{G}_V\longrightarrow\mathcal{G}_U$ is monic so that $0\longrightarrow\mathcal{G}_V\longrightarrow\mathcal{G}_U$ is exact. Since $\mathcal{F}$ is injective, get $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}_U,\mathcal{F})\longrightarrow\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}_V,\mathcal{F})\longrightarrow 0$ exact. Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}_U,\mathcal{F})\big|_U = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U,\mathcal{F}\big|_U) = \mathcal{F}(U)$. Thus $\mathcal{F}(U)\longrightarrow\mathcal{F}(V)\longrightarrow 0$ is exact so that $\mathcal{F}$ is flasque. □

Corollary 7.1. *Let X be a topological space. Then every injective object in $Ab(X)$ is flasque by viewing X as ringed space $(X,\mathbb{Z}_X)$ where $\mathbb{Z}_X$ is the constant sheaf.*

Proposition 7.5. *Let X be a topological space. Then every flasque sheaf on X is acyclic for the functor $\Gamma(X,\cdot)$.*

Proof. Let $\mathcal{F}$ be a flasque sheaf on X. Consider exact sequence $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{I}\longrightarrow\mathcal{G}\longrightarrow 0$ where $\mathcal{I}$ is an injective object in $Ab(X)$. By Corollary 7.1, $\mathcal{I}$ is flasque. Thus $\mathcal{G}$ is flasque since it is quotient of flasque sheaf by flasque sheaf. Take the long exact sequence $0\longrightarrow\mathcal{F}(X)\longrightarrow\mathcal{I}(X)\longrightarrow\mathcal{G}(X)\longrightarrow H^1(X,\mathcal{F})\longrightarrow H^1(X,\mathcal{I})\longrightarrow\cdots$. While $\mathcal{F}$ is flasque, we have $0\longrightarrow\mathcal{F}(X)\longrightarrow\mathcal{I}(X)\longrightarrow\mathcal{G}(X)\longrightarrow 0$ is exact. Also, as $\mathcal{I}$ is injective, $H^i(X,\mathcal{I})=0$ for all $i>0$. Thus $H^i(X,\mathcal{F})=0$ for all $i>0$ so that $\mathcal{F}$ is acyclic for the functor $\Gamma(X,\cdot)$. □

Corollary 7.2. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the right derived functor of $\Gamma(X,\cdot): Mod(\mathcal{O}_X)\longrightarrow Ab$ coincide with $H^i(X,\cdot)$.*

Reason 7.1. *Take injective resolution in $Mod(\mathcal{O}_X)$. By Proposition 7.4 and Proposition 7.5, it is a $\Gamma(X,\cdot)$-acyclic resolution. Thus by Proposition 7.1, the two right derived functors coincide.*

Corollary 7.3. *Let X be a topological space, $Y\subseteq X$ closed subset, $i: Y\longrightarrow X$ inclusion. Then for all $\mathcal{F}\in Ab(Y)$, we have $H^i(Y,\mathcal{F})\cong H^i(X,j_*\mathcal{F})$.*

Reason 7.2. *Take injective resolution J^* of $\mathcal{F}$ in $Ab(Y)$. Then f_*J^* is a flasque resolution of $f_*\mathcal{F}$.*

Remark 7.1 (***Gabber Theorem***)**.** *Let X be a scheme. Then $Qcoh(X)$ has enough injective objects.*

Remark 7.2. *Let X be a scheme. Suppose that X is either noetherian or quasi-compact and separated. Then the right derived functor of $\Gamma(X,\cdot) : Qcoh(X) \longrightarrow Ab$ coincide with $H^i(X),\cdot$. In fact, this property is equivalent to each injective object is $\Gamma(X,\cdot)$-acyclic.*

Theorem 7.1 (***Grothendieck***)**.** *Let X be a noetherian topological space of dimension $n < \infty$. Then for all $\mathcal{F} \in Ab(X)$, we have that $H^i(X,\mathcal{F}) = 0$ for all $i > n$.*

7.2 Cohomology of Noetherian Affine Schemes

Theorem 7.2 (***Artin-Rees***)**.** *Let A be a noetherian ring, $M \subseteq N$ finitely generated A-modules, $\mathfrak{a} \subseteq A$ ideal. Then for all $n > 0$, there exists $m > n$ such that $\mathfrak{a}M \supseteq M \cap a^m N$.*

Remark 7.3. *This theorem says that the $\mathfrak{a}$-adci topology of M is compatible with the induced topology of $\mathfrak{a}$-adic topology of N on M.*

Lemma 7.1. *Let A be a noetherian ring, I an injective A-module, $\mathfrak{a} \subseteq A$ ideal. Assume $J \subseteq I$ is the A-submodule consist of $x \in I$ such that $\mathfrak{a}^n x = 0$ for some $n > 0$. Then J is injective.*

Proof. It suffices to show that for all $\mathfrak{b} \subseteq A$ ideal and $\varphi : \mathfrak{b} \longrightarrow J$ homomorphism of A-modules, there exists $\psi : A \longrightarrow J$ extending φ satisfies the following diagram commutes

$$\begin{tikzcd} \mathfrak{b} \arrow[r, hook] \arrow[d, "\varphi"] & A \arrow[dl, "\psi"] \\ J & \end{tikzcd}$$

Since A is noetherian, B is finitely generated. Thus there exists $n > 0$ such that $\mathfrak{a}^n\varphi(\mathfrak{b}) = 0$. By Artin-Rees, there exists $m > n$ such that $\mathfrak{a}^n\mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^m$. Thus $\varphi(\mathfrak{b} \cap \mathfrak{a}^m) = 0$ so that φ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^m)$. Consider $\varphi_I : \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^m) \longrightarrow J \hookrightarrow I$. As I is injective, φ_I extends to $\psi_I : A/\mathfrak{a}^m \longrightarrow I$. Note that $im(\psi_I) \subseteq J$ since $\mathfrak{a}^m im(\psi I) = 0$, get ψ. $\square$

Lemma 7.2. *Let A be a noetherian ring, I an injective A-module. Then for all $f \in A$, the localization $\theta : I \longrightarrow I_f$ is surjective.*

Proof. For all $i > 0$, set $\mathfrak{b}_i = \operatorname{ann}_A(f^i)$. Get an increasing sequence of ideals $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$. As A is noetherian, $\{\mathfrak{b}_i\}$ stabilize from $\mathfrak{b}_r$ for some $r < \infty$. For $x \in I_f$, we can write $x = \frac{\theta(y)}{f^n}$ where $y \in I$ and $n > 0$. There is a canonical homomorphism $\varphi : (f^{n+r}) \longrightarrow I \quad f^{n+r} \longmapsto f^r y$. Easy to check that φ is well defined. Since I is injective, φ extends to $\psi : A \longrightarrow I$. Set $z = \psi(1)$, then $f^{n+r}z = f^r y$. Thus $\theta(z) = \frac{z}{1} = \frac{f^{n+r}z}{f^{n+r}} = \frac{f^r y}{f^{n+r}} = \frac{\theta(y)}{f^n} = x$ so that θ is surjective. $\square$

Proposition 7.6. *Let A be a noetherian ring, I an injective A-module. Then $\widetilde{I}$ is flasque on $X = \operatorname{Spec} A$.*

Proof. Set $Y = Supp(\widetilde{I}) = \{x \in X \big| \widetilde{I}_x \neq 0\} \subseteq X$ closed. If $Y = \varnothing$, then $\widetilde{I}$ is flasque. If $Y \neq \varnothing$, since A is noetherian, we can assume for all injective A-module J with support $Supp(\widetilde{J}) \subsetneq Y$, $\widetilde{J}$ is flasque. Want to show $\widetilde{I}(X) \longrightarrow \widetilde{I}(U)$ is surjective for all open subset $U \subseteq X$.

If $Y \cap U = \varnothing$, then $\widetilde{I}(U) = 0$, done! If $Y \cap U \neq \varnothing$, take standard open subset $D(f) \subseteq U$ and $D(f) \cap Y \neq \varnothing$. Set $Z = X \setminus D(f)$. Let $s \in \widetilde{I}(U)$, then $s\big|_{D(f)} \subseteq \widetilde{I}(D(f)) = I_f$. By Lemma 7.2, we can lift $s\big|_{D(f)}$ to $t \in \widetilde{I}(X) = I$. Thus $(s - t\big|_U)\big|_{D(f)} = 0$ so that $(s - t\big|_U) \in \Gamma_{Z \cap U}(U, \widetilde{I}\big|_U)$ i.e. support of it is in Z. Remains to show $\Gamma_Z(X, \widetilde{I}) \longrightarrow \Gamma_{Z\cap U}(U, \widetilde{I}\big|_U)$.

Note that $J = \Gamma_Z(X, \widetilde{I})$ is exactly the submodule consist of $x \in I$ such that $f^n x = 0$ for some $n > 0$. By Lemma 7.1, J is injective A-module with $Supp(\widetilde{J}) \subsetneq Y$. By induction hypothesis, $\widetilde{J}$ is flasque. Thus $\widetilde{J}(X) \longrightarrow \widetilde{J}(U)$ is surjective. While $\widetilde{J}(X) = \Gamma_Z(X, \widetilde{I})$ and $\widetilde{J}(U) = \Gamma_{Z\cap U}(U, \widetilde{I}\big|_U)$, done! □

Theorem 7.3 (***Serre***)**.** *Let A be a noetherian ring and $X = \operatorname{Spec} A$. Then for all $\mathcal{F}$ quasi-coherent sheaf in X, we have that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. For $\mathcal{F}$ quasi-coherent, set $M = \mathcal{F}(X)$. Take injective resolution I^* of M in Mod_A. Thus $0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{I^0} \longrightarrow \cdots$ is exact, which is a $\Gamma(X, \cdot)$-acyclic resolution by Proposition 7.6. As $\widetilde{I^i}$ is $\Gamma(X, \cdot)$-acyclic, can apply $\Gamma(X, \cdot)$ to calculate $H^i(X, \mathcal{F})$. But $\Gamma(X, \cdot)$ gives back to $0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots$ exact, get $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. □

Corollary 7.4. *Let X be a noetherian affine scheme. Then for all exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of $\mathcal{O}_X$-module with $\mathcal{F}$ quasi-coherent, we have that $0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \longrightarrow 0$ is exact.*

Corollary 7.5. *Let X be a noetherian scheme. Suppose that $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of $\mathcal{O}_X$-module. If $\mathcal{F}$ and $\mathcal{H}$ are both quasi-coherent (resp. coherent), then $\mathcal{G}$ is quasi-coherent (resp. coherent).*

Proof. Can assume that $X = \operatorname{Spec} A$ is affine. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathcal{F}(X)} \arrow[r] \arrow[d, "\sim"] & \widetilde{\mathcal{G}(X)} \arrow[r] \arrow[d] & \widetilde{\mathcal{H}(X)} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

Thus by 5 Lemma, $\mathcal{G}$ is quasi-coherent.

If moreover $\mathcal{F}$ and $\mathcal{H}$ are coherent, then $\mathcal{F}(X)$ and $\mathcal{H}(X)$ are finitely generated. Then $\mathcal{G}(X)$ is also finitely generated so that $\mathcal{G}$ is coherent. □

Lemma 7.3. *Let X be a noetherian scheme. Assume that $H^1(X, \mathcal{I}) = 0$ for all $\mathcal{I}$ coherent ideal sheaf. Then we can cover X by affine open subsets of the form X_f for $f \in A = \Gamma(X, \mathcal{O}_X)$.*

Proof. Suffice to show that every closed point $x \in X$ admits an affine open neighbourhood X_f. For $x \in X$ closed point, choose affine open neighbourhood U of x and set $Y = X \setminus U$. Get $0 \longrightarrow \mathcal{I}_{Y \cup \{x\}} \longrightarrow \mathcal{I}_Y \longrightarrow i_* \mathcal{O}_{\{x\}} \longrightarrow 0$ is exact with reduced scheme structures on each set, where $i : \{x\} \longrightarrow X$ is inclusion. By assumption, $\Gamma(X, \mathcal{I}_Y) \longrightarrow \Gamma(X, i_* \mathcal{O}_{\{x\}})$ is surjective. Take $f \in \Gamma(X, \mathcal{I}_Y)$ such that f is mapped to 1. Then $x \in X_f$. On the other hand, $X_f \subseteq U$, get X_f is affine. □

Lemma 7.4. *Let X be a noetherian scheme. Assume that $H^1(X, \mathcal{I}) = 0$ for all $\mathcal{I}$ coherent ideal sheaf. Then for all $e > 0$ and $\mathcal{F} \subseteq \mathcal{O}_X^{\oplus r}$ coherent, we have that $H^1(X, \mathcal{F}) = 0$.*

Proof. Define filtration $\mathcal{F} \supseteq \mathcal{F} \cap \mathcal{O}_X^{\oplus (r-1)} \supseteq \cdots \supseteq \mathcal{F} \cap \mathcal{O}_X \supseteq 0$ such that the associated quotients are coherent ideal sheaves. By long exact sequence and induction, get $H^1(X, \mathcal{F} \cap \mathcal{O}_X^i) = 0$ for all $i > 0$ so that $H^1(X, \mathcal{F}) = 0$. □

Theorem 7.4 (***Serre, Cohomology Criterion of Affineness***)**.** *Let X be a noetherian scheme. Then the following conditions are equivalent:*
(1)X is affine
(2)$H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and $\mathcal{F}$ quasi-coherent.
(3)$H^1(X, \mathcal{I}) = 0$ for all $\mathcal{I}$ coherent ideal sheaf.

Proof. See in the Hartshorne Chapter III p.215-216. □

Remark 7.4. *In fact, noetherian condition is not necessary.*

7.3 Čech Cohomology

Definition 7.4. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. For $i_0, \cdots, i_p \in I$, set $U_{i_0, \cdots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define complex $C^*(\mathcal{I}, \mathcal{F})$ to be*

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0, \cdots, i_p} \mathcal{F}(U_{i_0, \cdots, i_p})$$

with coboundary map

$$d^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$
$$\alpha = (\alpha_{i_0, \cdots, i_p})_{i_0, \cdots, i_p} \longmapsto (\sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \cdots, \widehat{i_k}, \cdots, i_{p+1}}|_{U_{i_0, \cdots, i_{p+1}}})_{i_0, \cdots, i_{p+1}}$$

Remark 7.5. *If we don't assume that I is well-ordered, we need to set $U_{i_0, \cdots, i_p} = 0$ if there are repeated index and $\alpha_{i_0, \cdots, i_p} = (-1)^{sgn(\sigma)} \alpha_{i_{\sigma(0)}, \cdots, i_{\sigma(p)}}$.*

Definition 7.5. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Define $\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^*(\mathcal{U}, \mathcal{F}))$, called the pth Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$.*

Example 7.3. *(1)For any $\mathcal{F}$ and $\mathcal{U}$, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.*
(2)For $\mathcal{U} = \{X\}$, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Lemma 7.5. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Suppose that $\mathcal{U}$ contains $U_i = X$ for some i. Then $0 \longrightarrow \mathcal{F}(X) \longrightarrow C^*(\mathcal{U},\mathcal{F})$ is exact.*

Proof. Obviously, exactness at $\mathcal{F}(X)$ and $C^0(\mathcal{U},\mathcal{F})$ is ok. For $p \geq 1$, without loss of generality, may assume that $i = 0$ which is the smallest element. Define homotopy $k^p : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^{p-1}(\mathcal{U},\mathcal{F}) \quad \alpha \longmapsto (\alpha_{0,i_0,\cdots,i_{p-1}})_{i_0,\cdots,i_{p-1}}$. Check that $d^{p-1} \circ k^p + k^{p+1} \circ d^p = \mathrm{id}_{C^*(\mathcal{U},\mathcal{F})}$. Thus identity is homotopic to zero map, which inducing $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 0$.. □

7.4 Comparison Theorem

Definition 7.6. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Define complex of sheaves on X $\mathcal{C}^*(\mathcal{U},\mathcal{F})$ to be*

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0,\cdots,i_p} f_*(\mathcal{F}|_{U_{i_0,\cdots,i_p}})$$

where f is the inclusion, with coboundary map

$$d^p : \mathcal{C}^p(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F})$$

defined accordingly.

Remark 7.6. *Obviously, $\Gamma(X, \mathcal{C}^p(\mathcal{U},\mathcal{F})) = C^p(\mathcal{U},\mathcal{F})$.*

Proposition 7.7. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Then $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^*(\mathcal{U},\mathcal{F})$ is exact.*

Reason 7.3. *Since exactness of complex of sheaves is equivalent to exactness at all stalks. Set $x \in X$, we can represent elements in stalks by sections on V, which is small enough to be contained in U_j for some j. Apply Lemma 7.5, get exactness of complex of sections on V.*

Proposition 7.8. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ flasque sheaf of abelian groups on X. Then $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 0$.*

Proof. Since flasque is stable under restriction to open subset, direct limit and product, $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ is flasque for all p. Thus $\mathcal{C}^*(\mathcal{U},\mathcal{F})$ is a flasque resolution of $\mathcal{F}$. By Proposition 7.5 and Proposition 7.1, get

$$\check{H}^p(\mathcal{U},\mathcal{F}) = h^p(\Gamma(X, \mathcal{C}^*(\mathcal{U},\mathcal{F}))) = H^p(X,\mathcal{F}) = 0$$

□

Lemma 7.6. *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Then for all $p \geq 0$, there is a natural homomorphism $\check{H}^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(X,\mathcal{F})$.*

Proof. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^*$ be an injective resolution of $\mathcal{F}$ in $Ab(X)$. By Comparison Theorem in Homological Algebra, there exists a morphism of complexes which is unique up to homotopy, inducing the natural homomorphisms we want. □

Theorem 7.5 (***Comparison Theorem***). *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X with well-ordered index set I, $\mathcal{F}$ sheaf of abelian groups on X. Suppose that for all p and all $i_0, \cdots, i_p$, we have that $H^q(U_{i_0,\cdots,i_p}, \mathcal{F}\big|_{U_{i_0,\cdots,i_p}}) = 0$ for all $q > 0$. Then the natural homomorphism $\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ is isomorphic for all p.*

Proof. For $p = 0$, the natural homomorphism is identity map of global sections of $\mathcal{F}$. For $p > 0$, embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. There is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

By assumption, for all $i_0, \cdots, i_p$,

$$0 \longrightarrow \mathcal{F}(U_{i_0,\cdots,i_p}) \longrightarrow \mathcal{I}(U_{i_0,\cdots,i_p}) \longrightarrow \mathcal{G}(U_{i_0,\cdots,i_p}) \longrightarrow 0$$

is exact. Taking products, get

$$0 \longrightarrow C^*(\mathcal{U}, \mathcal{F}) \longrightarrow C^*(\mathcal{U}, \mathcal{I}) \longrightarrow C^*(\mathcal{U}, \mathcal{G}) \longrightarrow 0$$

By Homological Algebra, there is an long exact sequence of cohomological groups

$$\cdots \longrightarrow \check{H}^{p-1}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^{p-1}(\mathcal{U}, \mathcal{I}) \longrightarrow \check{H}^{p-1}(\mathcal{U}, \mathcal{G}) \longrightarrow \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

Since $\mathcal{I}$ is injective, get $\mathcal{I}$ is flasque and $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$. Thus $\check{H}^p(\mathcal{U}, \mathcal{F}) = \check{H}^{p-1}(\mathcal{U}, \mathcal{G})$. By induction, done! □

Corollary 7.6. *Let X be a (noetherian) separated scheme, $\mathcal{U}$ affine open covering of X, $\mathcal{F}$ quasi-coherent sheaf on X. Then the natural homomorphism $\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ is isomorphic for all p.*

Remark 7.7. *This corollary immediately comes from Serre Vanishing Theorem.*

Corollary 7.7. *Let X be a (noetherian) separated scheme covered by $r+1$ affine open subsets. Then for all $\mathcal{F}$ quasi-coherent on X, we have that $H^i(X, \mathcal{F}) = 0$ for all $i > r$. In particular, when $r = 0$, this specializes to Serre Vanishing Theorem.*

7.5 Cohomology of Projective Space

Theorem 7.6. *Let A be a noetherian ring, $S = A[x_0, \cdots, x_r]$, $X = \mathrm{Proj}(S) = \mathbb{P}^r_A$, $r \geq 1$. Then we have that*
(1)$H^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < r$ and all $n \in \mathbb{Z}$
(2)$H^r(X, \mathcal{O}_X(-r-1)) \cong A$ is free A-module of rank 1
(3)the natural pairing $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1))$ is a perfect pairing of finite free A-modules for all $n \in \mathbb{Z}$.

Remark 7.8. *In fact, we have already known that* $H^0(X, \mathcal{O}_X(n)) = \begin{cases} S_n & n \geq 0 \\ 0 & n < 0 \end{cases}$ *and* $H^i(X, \mathcal{O}_X(n)) = 0$ *for all* $i > r$ *and all* $n \in \mathbb{Z}$. *Thus with this theorem, we have figured out all the cohomological groups of* $\mathbb{P}^r_A$. *Especially, all of them are finite free* A*-modules.*

Proof. Consider $\mathcal{F} = \oplus_{n\in\mathbb{Z}}\mathcal{O}_X(n)$ quasi-coherent. Since cohomology commutes with direct sum, $H^i(X, \mathcal{F}) = \oplus_{n\in\mathbb{Z}}H^i(X, \mathcal{F})$. In particular, $H^0(X, \mathcal{F}) = \Gamma_*(\mathcal{O}_\mathcal{X}) \cong S$. Note that X is noetherian, separated and covered by $r + 1$ affine open subsets. By Corollary 7.7, get $\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$ for all p.

For all $i_0, \cdots, i_p$, note that $U_{i_0,\cdots,i_p} = D_+(x_{i_0}, \cdots, x_{i_p})$. Get $\mathcal{F}(U_{i_0,\cdots,i_p}) \cong S_{x_{i_0},\cdots,x_{i_p}}$. Thus $C^*(\mathcal{U}, \mathcal{F})$ is given by

$$0 \longrightarrow \prod_{i_0} S_{x_{i_0}} \longrightarrow \cdots \longrightarrow S_{x_0,\cdots,x_r}.$$

Since $\check{H}^r(\mathcal{U}, \mathcal{F})$ is the cokernel of $d^{r-1} : \prod_k S_{x_0,\cdots,\widehat{x_k},\cdots,x_r} \longrightarrow S_{x_0,\cdots,x_r}$. Note that $S_{x_0,\cdots,x_r}$ is free A-module spanned by $\{x_0^{l_0}..x_r^{l_r}\}$, where $l_0, \cdots, l_r \in \mathbb{Z}$. In addition, $im(d^{r-1})$ is free A-module spanned by $\{x_0^{l_0} \cdots x_r^{l_r}\}$, where $l_0, \cdots, l_r \in \mathbb{Z}$ and at least one $l_i \geq 0$. Get $H^r(\mathcal{U}, \mathcal{F})$ is free A-module spanned by $\{x_0^{l_0} \cdots x_r^{l_r}\}$, where $l_i < 0$. While the only "monomial" of degree $-r - 1$ is $x_0^{-1} \cdots x_r^{-1}$, get $H^r(X, \mathcal{O}_X(-r - 1)) \cong A$.

For (3), we also find $H^r(X, \mathcal{O}_X(-n - r - 1)) = 0$ if $n < 0$. On the other hand, $H^0(X, \mathcal{O}_X(n)) = 0$ if $n < 0$. Thus, (3) is trivially true for $n < 0$. For $n \geq 0$, $H^0(X, \mathcal{O}_X(n))$ has basis $\{x_0^{m_0} \cdots x_r^{m_r}\}$, where $m_i \geq 0$ and $\sum_{i=0}^r m_i = n$. The natural pairing is generated by $(x_0^{m_0} \cdots x_r^{m_r}, x_0^{l_0} \cdots x_r^{l_r}) \longmapsto x_0^{m_0+l_0} \cdots x_r^{m_r+l_r}$, here that $x_i^{m_i+l_i} = 0$ if $m_i + l_i \geq 0$. Then it is a perfect pairing since it is easy to see that dual basis of $x_0^{m_0} \cdots x_r^{m_r}$ is $x_0^{-m_0-1} \cdots x_r^{-m_r-1}$.

For (1), take induction on r. When $r = 1$, there is nothing to prove. Suppose that $r \geq 2$. Consider localization of $C^*(\mathcal{U}, \mathcal{F})$ with respect to x_r. By Theorem 2.1, the localization gives the Čech complex of $\mathcal{F}\big|_{U_r}$ with respect to $\mathcal{U}\big|_{U_r}$. As localization is exact and U_r is affine, $H^i(X, \mathcal{F})_{x_r} = H^i(U_r, \mathcal{F}\big|_{U_r}) = 0$. Suffice to show that for all $0 < i < r$, x_r acts injectively on $H^i(X, \mathcal{F})$. View x_r as a homomorphism of A-modules like below

$$x_r : H^i(X, \mathcal{F}(-1)) \longrightarrow H^i(X, \mathcal{F})$$

here $\mathcal{F}(-1)$ is same sheaf on $\mathcal{F}$ with grading shifted. Consider exact sequence of graded C-modules

$$0 \longrightarrow S(-1) \xrightarrow{x_r} S \longrightarrow S/(x_r) \longrightarrow 0$$

Get exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{x_r} \mathcal{O}_X \longrightarrow j_*\mathcal{O}_H \longrightarrow 0$$

where H is the hyperplane given by $x_r = 0$ and $j : H \hookrightarrow X$ is closed immersion. Take direct sum, get exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \xrightarrow{x_r} \mathcal{F} \longrightarrow \mathcal{F}_H \longrightarrow 0$$

where $\mathcal{F}_H = \oplus_{n\in\mathbb{Z}}j_*(\mathcal{O}_H(n))$. Get long exact sequence

$$\cdots \longrightarrow H^{i-1}(X, \mathcal{F}_H) \longrightarrow H^i(X, \mathcal{F}(-1)) \xrightarrow{x_r} H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})_H \longrightarrow \cdots$$

Now $H \cong \mathbf{P}_A^{r-1}$ and since j is closed immersion, $H^i(X, \mathcal{F}_H) = H^i(H, \oplus_{n\in\mathbb{Z}} O_H(n))$. By induction hypothesis, $H^i(X, \mathcal{F}_H) = 0$ for all $0 < i < r-1$. Thus x_r is injective on $H^i(X, \mathcal{F}(-1))$ for all $1 < i < r$. When $i = 1$, find that $H^0(X, \mathcal{F}) = S \longrightarrow H^0(X, \mathcal{F}_H) = S/(x_r)$ is surjective. Get x_r is injective on $H^i(X, \mathcal{F}(-1))$ for all $0 < i < r$, done! □

Theorem 7.7 (***Serre***)**.** *Let A be a noetherian ring, X projective scheme over A, $\mathcal{F}$ coherent sheaf on X. Then*
(1)for all $i \geq 0$, $H^i(X, \mathcal{F})$ is a finite A-module.
(2)for $j : X \hookrightarrow \mathbb{P}_A^r$ closed immersion, there exists integer n_0 relative to $\mathcal{F}$ such that $H^i(X, \mathcal{F}(n)) = 0$ for all $i > 0$ and $n \geq n_0$.

Proof. Since j is finite, $j_*(\mathcal{F})$ is coherent on $\mathbb{P}_A^r$ and by Projection Formula, $j_*(\mathcal{F}(n)) = (j_*(\mathcal{F}))(n)$. What's more, since j is closed immersion, $H^i(X, \mathcal{F}) = H^i(\mathbb{P}_A^r, j_*(\mathcal{F}))$. Thus we can reduce this problem to $X = \mathbb{P}_A^r$. For $\mathcal{F} = \mathcal{O}_X(m)$, this is just things told by 7.6. Also, for general $\mathcal{F}$, as $\mathbb{P}_A^r$ can be covered by $r+1$ affine open subsets, $H^i(X, \mathcal{F}(n)) = 0$ for all $i > r$ and $n \in \mathbb{Z}$.

For (1), by induction, suppose theorem is correct for $i+1$. By Corollary 6.1, there exists $m \in \mathbb{Z}$ and $r < \infty$ such that $\mathcal{F}$ is quotient of $\mathcal{O}_X(m)^r$, inducing the following exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(m)^r \longrightarrow \mathcal{F} \longrightarrow 0$$

Consider the long exact sequence

$$\cdots \longrightarrow H^i(X, \mathcal{O}_X(m)^r) \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{G}) \longrightarrow \cdots$$

By hypothesis, $H^{i+1}(X, \mathcal{G})$ is finite A-module. As A is noetherian, $H^i(X, \mathcal{F})$ is also finite.

For (2), by above discuss, we only need to check for $0 < i \leq r$. Induct on i, assume for $i+1$ and all coherent sheaf $\mathcal{F}$, there exists $n_{i+1}(\mathcal{F})$ such that $H^{i+1}(X, \mathcal{F}(n)) = 0$ for all $n \geq n_{i+1}(\mathcal{F})$. Take same short exact sequence. Tensor the short exact sequence by $\mathcal{O}_X(n)$. Get exact sequence

$$0 \longrightarrow \mathcal{G}(n) \longrightarrow \mathcal{O}_X(m+n)^r \longrightarrow \mathcal{F}(n) \longrightarrow 0$$

Consider the long exact sequence

$$\cdots \longrightarrow H^i(X, \mathcal{O}_X(m+n)^r) \longrightarrow H^i(X, \mathcal{F}(n)) \longrightarrow H^{i+1}(X, \mathcal{G}(n)) \longrightarrow \cdots$$

By hypothesis, there exists $n_{i+1}(\mathcal{G})$ such that $H^{i+1}(X, \mathcal{G}(n)) = 0$ for all $n \geq n_{i+1}(\mathcal{G})$. By Theorem 7.6, $H^i(X, \mathcal{O}_X(m+n)^r) = 0$ for large enough n. Thus there exists $n_i(\mathcal{F})$ such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_i(\mathcal{F})$. As there are finitely many i, we can take a uniform upper $n_0(\mathcal{F})$ bound for $n_i(\mathcal{F})$. □

Corollary 7.8. *Let $f : X \longrightarrow Y$ be a projective morphism with X, Y noetherian, $\mathcal{F}$ coherent sheaf on X. Then $f_*\mathcal{F}$ is coherent on Y.*

Reason 7.4. *Can assume $Y = \operatorname{Spec} A$. Then $f_*\mathcal{F} \cong H^0(\widetilde{X, \mathcal{F}})$ which is a finite A-module by Theorem 7.7.*

Corollary 7.9. *Let X be a geometrically integral scheme projective over field k. Then $\Gamma(X, \mathcal{O}_X) = k$.*

Reason 7.5. *By Theorem 7.7, $L = \Gamma(X, \mathcal{O}_X)$ is finite over k. While X is integral, get L is integral domain. Thus by Noetherian Normalization Lemma, L/k is finite field extension. Note by Proposition 3.11, geometrically integral if and only if k is algebraically closed in function field $k(X)$. Get $L = k$.*

Corollary 7.10. *Let A be a noetherian ring, X proper over $\operatorname{Spec} A$, $\mathcal{L}$ ample invertible sheaf on X. Then for all $\mathcal{F}$ coherent sheaf on X, there exists n_0 relative to $\mathcal{F}$ such that $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ for all $i > 0$ and $n \geq n_0$.*

Reason 7.6. *By Theorem 6.6, there exists $m > 0$ such that $\mathcal{L}^m$ is very ample. Get closed immersion $i : X \hookrightarrow \mathbb{P}^r_A$ such that $\mathcal{L}^n = \mathcal{O}_X(1)$. Apply Theorem 7.7 to $\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}, \cdots, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{m-1}$.*

Theorem 7.8 (***Serre Cohomological Criteria for Ampleness***)**.** *Let A be a noetherian ring, X proper over $\operatorname{Spec} A$, $\mathcal{L}$ invertible sheaf on X. Then the following conditions are equivalent*
(1)$\mathcal{L}$ is ample
(2)for all $\mathcal{F}$ coherent sheaf on X, there exists n_0 relative to $\mathcal{F}$ such that $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ for all $i > 0$ and $n \geq n_0$.
(3)for all $\mathcal{F}$ coherent sheaf on X, there exists n_0 relative to $\mathcal{F}$ such that $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$.

Proof. Step 1. Since by Corollary 7.10, we have (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) is obvious. Only need to prove (3) $\Rightarrow$ (1). Moreover, suffices to prove that for all $\mathcal{F}$ coherent sheaf on X and closed point $x \in X$, there exists open neighbourhood $U_x \ni x$ and $n_0(x) > 0$ such that for all $n \geq n_0(x)$ and $y \in U_x$, the stalk $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_y$ is generated by global sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$, since X noetherian, it can be covered by only finitely many U_x so that we can take a uniform upper bound of $n_0(x)$.

Step 2. Claim that it suffices to prove that for all $\mathcal{F}$ coherent sheaf on X, closed point $x \in X$ and large enough n, there exists open neighbourhood $U_x \ni x$ depending on n such that for all $y \in U_x$, the stalk $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_y$ is generated by global sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$. Assume we have proved this. Apply it to $\mathcal{F} = \mathcal{O}_X$, there exists large enough n_1 and $V \ni x$ such that for all $y \in V$, $(\mathcal{L}^{n_1})_y$ is generated by global sections. Then apply this to $\mathcal{F}$, there exists large enough n_0 and open neighbourhoods $U_0, U_1, \cdots, U_{n_1-1}$ of x such that for all i and $y \in U_i$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{n+i})_y$ is generated by global sections. Take $U_x = V \cap U_0 \cap \cdots \cap U_{n_1-1}$. For all $n \geq n_0$ and $y \in U_x$, assume $n = n_0 + mn_1 + i$, then $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_y = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{n_0+i})_y \otimes_{\mathcal{O}_{X,y}} (\mathcal{L}^{n_1})_y^{\otimes m}$ is generated by global sections.

Step 3. Let $x \in X$ be a closed point, $\mathcal{I}_x$ the ideal sheaf of $\{x\} \overset{i}{\hookrightarrow} X$, where $\{x\}$ has the same scheme structure as $\operatorname{Spec}(k(x))$ with structure sheaf denoted by $k(x)$ for convenience. We have short exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_x \longrightarrow i_* k(x) \longrightarrow 0$$

Tensor by $\mathcal{F}$, get

$$\mathcal{I}_x\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow 0$$

Tensor by $\mathcal{L}^r$, get

$$\mathcal{I}_x\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n \longrightarrow i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{L}^n \longrightarrow 0$$

Note that, by hypothesis, $H^1(X, \mathcal{I}_X\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ for all $n \geq n_0$. So $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) \longrightarrow \Gamma(X, i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)$ is surjective for all $n \geq n_0$. Note that $i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is only supported at x, we have that $\Gamma(X, i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = (i_*k(x) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_x = k(x) \otimes_{\mathcal{O}_{X,x}} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_x$. Denote $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_x$ by M. As $M/\mathfrak{m}_x M$ is generated by global sections, by Nakayama's Lemma, M is generated by global sections.

As $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is coherent, take an affine open neighbourhood $U = \operatorname{Spec} A$ of x so that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)\big|_U = \widetilde{N}$ for some finitely generated A-module. Assume x corresponds to maximal ideal $\mathfrak{m}$ and N is generated by $m_1, \cdots, m_l$. Since $N_{\mathfrak{m}}$ is generated by global sections, $\frac{m_k}{1} = \frac{\sum_j a_{kj} s_{j,x}}{b_k}$, where $a_{kj} \in A$, $b_k \notin \mathfrak{m}$ and s_j are global sections. Then there exists $c_k \notin \mathfrak{m}$ such that $(b_k m_k - \sum_j a_{kj} s_{j,x}) c_k = 0$. Consider $b = \prod_{k=1}^l b_k c_k$, then $\mathfrak{m} \in D(b)$. Obviously, for each $\mathfrak{p} \in D(b)$, $\frac{m_k}{1}$ can be generated by global sections in $N_{\mathfrak{p}}$. Thus for all $n \geq n_0$, there exists open neighbourhood $U_x \ni x$ depending on n such that for all $y \in U_x$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)_y$ is generated by global sections, done! $\square$

Remark 7.9. *In fact, all finiteness results still hold for X proper over A. Prooves can be seen in EGA or Stacks Project.*

7.6 Higher Direct Images

Definition 7.7. *Let $f : X \longrightarrow Y$ be a morphism of schemes (can be replaced by topological spaces or ringed space). For all $i \geq 0$, define $R^i f_* : Ab(X) \longrightarrow Ab(Y)$ as the right derived functor of $f_* : Ab(X) \longrightarrow Ab(Y)$, which is called the higher direct image of f.*

Remark 7.10. *In fact, for $\mathcal{F} \in Ab(X)$, $R^i f_*\mathcal{F}$ is the sheaf associated with the presheaf $V \subseteq Y \longmapsto H^i(f^{-1}(V), \mathcal{F}\big|_{f^{-1}(V)})$. Details can be seen in Hartshorne Chapter III section 8. In particular, flasque sheaves are acyclic for the functor f_*. It follows that the derived functors of $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ coincide with $R^i f_*$.*

Moreover, if X is noetherian and $Y = \operatorname{Spec} A$ is affine, then for all $\mathcal{F}$ quasi-coherent sheaf on X, we have that $R^i f_*\mathcal{F} \cong \widetilde{H^i(X, \mathcal{F})}$. Details can also be seen in Hartshorne Chapter III section 8.

In particular, for $f : X \longrightarrow Y$ morphism of schemes with X noetherian and $\mathcal{F}$ quasi-coherent sheaf on X, the derived functor $R^i f_*\mathcal{F}$ is quasi-coherent on Y. Also true if $f : X \longrightarrow Y$ is quasi-compact and separated.

Proposition 7.9. *Let $f : X \longrightarrow Y$ be an affine morphism between noetherian schemes. Then for $\mathcal{F}$ quasi-coherent on X, we have*
(1)$R^i f_*\mathcal{F} = 0$ for all $i > 0$
(2)$H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for all i

Proof. For (1), note that $R^i f_*\mathcal{F}$ is the sheaf associated with the presheaf $V \subseteq Y \longmapsto H^i(f^{-1}(V), \mathcal{F}\big|_{f^{-1}(V)})$, consider affine open subset $V \subseteq Y$. Since f is affine, $f^{-1}(V)$ is affine. Note $\mathcal{F}\big|_{f^{-1}(V)}$ is still quasi-coherent on $f^{-1}(V)$, by Theorem 7.4, get $H^i(f^{-1}(V), \mathcal{F}\big|_{f^{-1}(V)}) = 0$ for all $i > 0$. By definition of sheafification, it is easy to see that $R^i f_*\mathcal{F} = 0$.

For (2), the general proof, we need to use property of spectrum sequence. Here, we only gieve a proof assuming Y separated. Take affine open covering $\{V_i\}_{i\in I}$ of Y. Since f is affine, get $\{U_i = f^{-1}(V_i)\}_{i\in I}$ is an affine open covering of X. As we can compute cohomology with Čech complexes, suffices to show they have same complexes. For open covering $\{V_i\}_{i\in I}$ of Y and $f_*\mathcal{F}$, the Čech complex should be

$$0 \longrightarrow f_*(\mathcal{F})(Y) = \mathcal{F}(X) \longrightarrow \prod_{i\in I} f_*\mathcal{F}(V_i) = \prod_{i\in I} \mathcal{F}(U_i) \longrightarrow \cdots$$

Thus $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for all i. □

Proposition 7.10. *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes. Then for each $\mathcal{F}$ coherent sheaf on X, $R^k f_*\mathcal{F}$ is coherent on Y for all $k \geq 0$.*

Proof. Can assume that $Y = \operatorname{Spec} A$ is affine. Want to show that $R^k f_*\mathcal{F} \cong \widetilde{H^k(X, \mathcal{F})}$, which is a finitely generated A-module by Theorem 7.7. Suffice to prove that for all $f \in A$, $H^k(f^{-1}(D(f)), \mathcal{F}\big|_{f^{-1}(D(f))}) = H^k(X, \mathcal{F})_f$. Note that $D(f) = Y_f$, by Lemma 2.6, get $f^{-1}(D(f)) = f^{-1}(Y_f) = X_g$, where g is the image of f under the map $f^\sharp(Y)$. Since f is projective, by Theorem 4.3, f is proper so that f is of finite type and separated. Thus X can be covered by finitely many affine open subsets $U_1, \cdots, U_r$. And $U_i \cap U_j$ is still affine. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \prod_{i=1}^r \mathcal{F}(U_i) & \longrightarrow & \prod_{i=1}^r \prod_{j=1}^r \mathcal{F}(U_i \cap U_j) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X_g) & \longrightarrow & \prod_{i=1}^r \mathcal{F}(X_g \cap U_i) & \longrightarrow & \prod_{i=1}^r \prod_{j=1}^r \mathcal{F}(X_g \cap U_i \cap U_j) & \longrightarrow & \cdots
\end{array}$$

By similar argument in Lemma 2.6, we can show that $\mathcal{F}(X_g) \cong \mathcal{F}(X)_g \cong \mathcal{F}(X)_f$. Moreover, we have $\mathcal{F}(X_g \cap U_{i_1} \cap \cdots \cap U_{i_l}) \cong \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_l})_{g\big|_{U_{i_1}\cap\cdots\cap U_{i_l}}} \cong \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_l})_g \cong \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_l})_f$. Thus for open covering $\{U_i \cap X_g\}_{i\in I}$ of X_g and $\mathcal{F}\big|_{X_g}$, the Čech complex is localization of the Čech complex for open covering $\{U_i\}_i$ of X and $\mathcal{F}$ with respect to f. By exactness of localization, get $H^k(f^{-1}(D(f)), \mathcal{F}\big|_{f^{-1}(D(f))}) = H^k(X, \mathcal{F})_f$. □

Remark 7.11. *Obviously, in the proof, we only need f to proper. Thus this proposition still holds if projective is replaced by proper, as Remark 7.9.*

7.7 Flat Base Change

Definition 7.8 (***Flat Module of Sheaves***)**.** *Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$ module. We say that $\mathcal{F}$ is flat if for all $x \in X$, $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$.*

Remark 7.12. *With this definition, we can restate the definition of flat morphism between schemes. We say $f : X \longrightarrow Y$ is flat if $\mathcal{O}_X$ is a flat $f^{-1}\mathcal{O}_Y$-module. More generally, an $\mathcal{O}_X$-module $\mathcal{F}$ is flat over Y if $\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_Y$-module.*

Example 7.4. *Assume X is noetherian and $\mathcal{F}$ is coherent on X. Then $\mathcal{F}$ is flat if and only if it is locally free of finite rank, since flatness over a local ring is equivalent to free and X is noetherian.*

We are interested in cartesian diagrams of noetherian schemes like below

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

Proposition 7.11. *Further assume $f : X \longrightarrow Y$ separated in the diagram above. Then for all $\mathcal{F}$ quasi-coherent sheaf on X, we have that*
(1)There are natural morphisms which are called base change morphism,

$$u^*(R^i f_*\mathcal{F}) \longrightarrow R^i g_*(v^*\mathcal{F})$$

(2)if moreover $u : Y' \longrightarrow Y$ is flat, then the natural morphisms are isomorphic.

Proof. Since everything is local on Y and Y', can assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Want natural morphisms

$$H^i(X, \mathcal{F}) \otimes_A A' \longrightarrow H^i(X', i^*\mathcal{F})$$

Consider the Čech complexes. Take affine open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X. By construction of fiber product, easy to see $v^{-1}(\mathcal{U}) = \{v^{-1}(U_i)\}_{i\in I}$ is affine open covering of X'. Observe that $C^*(v^{-1}(\mathcal{U}), v^*\mathcal{F}) = C^*(\mathcal{U}, \mathcal{F}) \otimes_A A'$. Get $H^i(X, \mathcal{F}) \otimes_A A' \longrightarrow H^i(X', i^*\mathcal{F})$. By universal property of sheafification, get natural morphism $u^*(R^i f_*\mathcal{F}) \longrightarrow R^i g_*(v^*\mathcal{F})$. Further, if A' is flat over A, then $\otimes_A A'$ is exact so that commutes with cohomology, inducing the isomorphism. Thus the local homomorphisms between stalks are isomorphic, so is the natural morphism between schemes. □

Proposition 7.12 (***Projection Formula, Flat Case***)**.** *Let $f : X \longrightarrow Y$ be a separated morphism between noetherian schemes. Then for all $\mathcal{F}$ quasi-coherent on X and $\mathcal{G}$ quasi-coherent on Y, we have that*
(1)there are natural morphisms

$$(R^i f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \longrightarrow R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$$

(2)if $\mathcal{G}$ is flat over Y, then the natural morphisms are isomorphic.

Remark 7.13. *The proof is same as Proposition 7.11*

8 Divisors

8.1 Weil Divisors

For convenience, we assume that scheme X is noetherian, integral, (separated) and regular in dimension 1. Denote the condition by $(*)$.

Definition 8.1. *Let X be a scheme satisfying condition $(*)$. A prime divisor on X is an intgral closed subscheme of codimension 1. A Weil divisor on X is an element of $Div(X)$, the free abelian group generated by prime divisors on X.*

Definition 8.2. *Let X be a scheme satisfying condition $(*)$, $D \in Div(X)$. D is called effective if coefficients of prime divisors are all non-negative in it. Define support of D to be $Supp(D)(= |D|) := \cap_{n_Y \neq 0} Y$.*

Want to construct an equivalence relation in $Div(X)$. Consider function field $k(X)$ of X. By our assumptions, for all Y prime divisor on X, there is a discrete valuation $v_Y : k(X)^\times \longrightarrow \mathbb{Z}$ on $\mathcal{O}_{X,\eta_Y}$, where η_Y is the generic point of Y.

Definition 8.3. *Let X be a scheme satisfying condition $(*)$, $k(X)$ the function field of X, $f \in k(X)$. Define the divisor of f to be $div(f) = \sum_Y v_Y(f)Y$. Devisors of the form $div(f)$ are called principal divisors, denoted by $Pr(X)$.*

Lemma 8.1. *Let X be a scheme satisfying condition $(*)$, $k(X)$ the function field of X, $f \in k(X)$. Then $v_Y(f) = 0$ for all but finitely many Y.*

Proof. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Since X is noetherian, by Theorem 3.1, $X \setminus U$ is union of finitely many irreducible components which are of codimension ≥ 1 in X. Hence we can reduce to the case $X = \operatorname{Spec} A$. Since $f \in k(X) = \operatorname{Frac}(A)$, assume $f = \frac{a}{b}$ where $a, b \in A$. Take $D(b) \subseteq \operatorname{Spec} A$, then $f \in A_b$. Hence we can assume that $f \in A \setminus \{0\}$. Thus for all Y prime divisor, $f \in \mathcal{O}_{X,\eta_Y}$ so that $v_Y(f) \geq 0$ and $v_Y(f) > 0$ if and only if $\eta_Y \subseteq V(f)$. While again by Theorem 3.1, $V(f)$ can be covered by finitely many irreducible components which are of codimension ≥ 1 in X, done! $\square$

Definition 8.4. *Let X be a scheme satisfying condition $(*)$. Two divisors D_1, D_2 on X are linearly equivalent, denoted by $D_1 \sim D_2$, if $D_1 - D_2$ is principal. Define divisor class group of X to be $Cl(X) := Div(X)/\sim$.*

Proposition 8.1. *Let A be a noetherian integral domain, $X = \operatorname{Spec} A$. Then X is normal and $Cl(X) = 0$ if and only if A is unique factorization domain.*

Proof. "$\Leftarrow$": Since A is unique factorization domain, A is integrally closed so that X is normal. For all Y prime divisor, the generic point η_Y corresponds to a prime ideal $\mathfrak{p}_Y$ of height 1 in $\operatorname{Spec} A$. Since $A_{\mathfrak{p}_Y}$ is a DVR, we can take its uniformizer $\omega_Y = \frac{a_Y}{b_Y}$, where $a_Y, b_Y \in A$. Consider $a_Y \in \mathfrak{p}_Y$. Since A is UFD, it is easy to show that $\mathfrak{p}_Y = (a_Y)$. Since $\mathfrak{p} \in V(a_Y)$ if and only if $\mathfrak{p}_Y \subseteq \mathfrak{p}$, as the proof of Lemma 8.1, get $Y = div(a_Y)$ is principal. Thus $Cl(X) = 0$.

"$\Rightarrow$": By knowledge of commutative algebra, a noetherian integral domain is UFD if and only if all prime ideals of height 1 is principal. For $\mathfrak{p} \in \operatorname{Spec} A$ of height 1 and $Y = V(\mathfrak{p})$, as $Cl(X) = 0$, there exists $f \in k(X)^\times$ such that $div(f) = Y$. Since $v_Y(f) = 1$, get $f \in A_\mathfrak{p}$ and $\mathfrak{p}A_\mathfrak{p} = (f)$. If $\mathfrak{p}'$ is another prime ideal of height 1, then $v_{V(\mathfrak{p}')}(f) = 0$ so that $f \in A_{\mathfrak{p}'}^\times$. By Algebraic Hartogs, $A = \cap_{ht(\mathfrak{p})=1} A_\mathfrak{p}$. Get $f \in A$ so that $f \in A \cap \mathfrak{p}A_\mathfrak{p} = \mathfrak{p}$. Remains to show that $\mathfrak{p} = (f)$. For all $g \in \mathfrak{p} \setminus \{0\}$, we have that $\frac{g}{f} \in A_\mathfrak{p}$ and $V_{V(\mathfrak{p}')}(\frac{g}{f}) \geq 0$ so that $\frac{g}{f} \in A_{\mathfrak{p}'}$. Again with Algebraic Hartogs, get $\frac{g}{f} \in A$. Thus $\mathfrak{p} = (f)$. □

Proposition 8.2. *Let k be a field. Then*
(1)the prime divisors of $\mathbb{A}_k^n$ are of the form $V(f)$ with f irreducible polynomial.
(2)the prime divisors of $\mathbb{P}_k^n$ are of the form $V_+(f)$ with f irreducible homogeneous polynomial.

Proof. For (1), since $\mathbb{A}_k^n = \operatorname{Spec}(k[x_1, \cdots, x_n])$, any prime divisor corresponds a prime ideal of height 1. By knowledge of commutative algebra, a noetherian integral domain is UFD if and only if all prime ideals of height 1 is principal. Thus prime divisors of $\mathbb{A}_k^n$ are of the form $V(f)$ with f irreducible polynomial.

For (2), consider affine open covering $\{D_+(x_i)\}$. For all prime divisor Y, take generic point η of Y. Assume that $\eta \in D_+(x_i)$. Then $Y \cap D_+(x_i)$ is prime divisor in $D_+(x_i)$. With same argument, $Y \cap D_+(x_i)$ is of the form $V(f)$ with f prime in $k[x_0, \cdots, x_n]_{(x_i)}$. Assume that $f = \frac{g}{x_i^k}$, where $x_i \not| g$, g is irreducible and $\deg(g) = k$. Since generic point $V(f)$ is (f) and (f) corresponds to (g) in $\mathbb{P}_k^n$, get $\overline{\{(g)\}} = Y$. Thus $Y = V_+(g)$ with g irreducible homogeneous polynomial. □

Remark 8.1. *With this proposition, we can define degree for prime divisor of $\mathbb{P}_k^n$. For prime divisor $Y = V_+(f)$, define $\deg(Y) = \deg(f)$. Get a group homomorphism $\deg : Div(\mathbb{P}_k^n) \longrightarrow \mathbb{Z}$.*

Proposition 8.3. *Let k be a field and $X = \mathbb{P}_k^n$, $H = V_+(x_0) \subseteq X$ hyperplane. Then*
(1)every divisor D of degreee d is equivalent to dH
(2)for all $f \in k(X)^\times$, where $k(X)$ is the function field, we have that $\deg(div(f)) = 0$
(3)the degree homomorphism $\deg : Div(\mathbb{P}_k^n) \longrightarrow \mathbb{Z}$ descends to isomorphism $\deg : Cl(\mathbb{P}_k^n) \xrightarrow{\sim} \mathbb{Z}$.

Proof. For (3), deg is obviously surjective. It suffices to show that ker(deg) is just principal divisors. Assume we have proved (1) and (2). By (1), $\ker(\deg) \subseteq Pr(X)$. By (2), $Pr(X) \subseteq \ker(\deg)$, done!

For (2), for all $f \in K^\times$, f can be reducedly written as $\frac{g}{h}$, where $g, h \in k[x_0, \cdots, x_n]$ homogeneous polynomials of same degree d. Factorize $g = g_1^{n_1} \cdots g_r^{n_r}$, where g_i is irreducible homogeneous polynomial of degree d_i. By Proposition 8.2, g_i corresponds to a prime divisor $Y_i = V_+(g_i)$ and $\deg(Y_i) = d_i$. We can define a divisor $div(g) = \sum_{i=1}^r n_i Y_i$ whose degree is d. Also, there is a divisor $div(h)$. It is clear that $div(f) = div(g) - div(h)$. Thus $\deg(div(f)) = \deg(div(g)) - \deg(div(h)) = 0$.

For (1), let D be a divisor of degree d. Write $D = D_1 - D_2$ where D_1, D_2 are two effective divisors. Assume $\deg(D_1) = d_1$ and $\deg(D_2) = d_2$. As proof of (2), we can write $D_1 = div(g_1)$

and $D_2 = div(g_2)$, where g_1, g_2 are two homogeneous polynomials. Then $D - dH = div(\frac{g_1}{x_0^d g_2})$ so that D is equivalent to dH. $\square$

Remark 8.2. *Can also show that $Cl(\mathbb{P}_k^1 \times_k \mathbb{P}_k^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. While $\mathbb{P}_k^2 \cong \mathbb{Z}$, get $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$ is not isomorphic to $\mathbb{P}_k^2$. However, it is not true that $Cl(X \times_k X) \cong Cl(X) \oplus Cl(X)$ in general, not even for curves.*

If $U \subseteq X$ is a nonempty open subset, then the restriction map $Cl(X) \longrightarrow Cl(U)$ is surjective, whose kernel consists of equivalence classes of divisors supported on $Z = X \setminus U$. If Z is irreducible closed subset of codimension 1, then kernel is spanned by Z. For example, if $Y \subseteq \mathbb{P}_k^2$ is irreducible curve of degree d, then $Cl(\mathbb{P}_k^2 \setminus Y) \cong \mathbb{Z}/d\mathbb{Z}$.

8.2 Weil Divisors for Curve Case

Definition 8.5 (***Curves***)**.** *Let X be a scheme of finite type over field k. We say that X is a curve over k if it is geometrically integral of dimension 1.*

Definition 8.6. *Let X be a curve which is smooth and proper over field k, $D = \sum_i n_i P_i \in Div(X)$. Define the degree of D to be $\sum_i n_i [k(P_i) : k]$, where $k(P_i)$ is the residue field of generic point of P_i.*

Remark 8.3. *Similar to projective abstract variety case, we will see that* $\deg : Div(X) \longrightarrow \mathbb{Z}$ *can also descend to* $\deg : Cl(X) \longrightarrow \mathbb{Z}$. *However, the homomorphism is not necessarily surjective if k is not algebraically closed.*

8.3 Cartier Divisors

Definition 8.7. *Let X be an arbitrary scheme. For all $U \subseteq X$ open subset, set $K(U)$ to be localization of $\mathcal{O}_X(U)$ with respect to the multiplicative system $S(U)$, where $S(U)$ is the subset of elements which are non zero-divisors in each $\mathcal{O}_{X,x}$ for all $x \in U$. In particular, if $U = \operatorname{Spec} A$ is affine, then $K(U)$ is the localization of A with respect to non zero-divisors.*

Define $\mathcal{K}$ to be the sheafification of $U \longmapsto K(U)$, called the function field of X. It is clear that if X is integral, then $\mathcal{K}$ is the constant sheaf corresponding to $K(X)$. Can also define $\mathcal{K}^* \subseteq \mathcal{K}$ and $\mathcal{O}_X^* \subseteq \mathcal{O}_x$ to be the subsheaves of invertible elements.

Definition 8.8. *Let X be a scheme. A Cartier divisor D on X is a global section of the sheaf (of multiplicative groups) $\mathcal{K}^*/\mathcal{O}_X^*$. Denote $\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ by $CaDiv(X)$.*

$D \in CaDiv(X)$ is called principal if it is in the image of $\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$. Two Cartier divisors are linearly equivalent if their difference is principal. Define $CaCl(X) = CaDiv(X)/im(\Gamma(X, \mathcal{K}^*))$, called the Cartier divisor class group of X.

Remark 8.4. *More concretely, a Cartier divisor D on X is represented by an (affine) open covering $\{U_i\}$ of X and corresponding elements $f_i \in \mathcal{K}^*(U_i)$ such that the restriction of $\frac{f_i}{f_j}$ to $U_i \cap U_j$ lies in $\mathcal{O}^*(U_i \cap U_j)$ for all i, j. $D \in CaDiv(X)$ is called effective if it can be represented by (U_i, f_i) with $f_i \in \mathcal{K}^*(U_i) \cap \mathcal{O}_X(U_i)$*

Lemma 8.2. *Let X be a scheme, $D \in CaDiv(X)$. Define the support of D to the set of points satisfying that the local equation f_i of D is not invertible in the stalk of $\mathcal{O}_X$, denoted by $Supp(D)$ or $|D|$. Then $Supp(D)$ is a closed subset of X.*

Remark 8.5. *Just a corollary of the fact that support of a section is closed.*

Definition 8.9 (***Locally Factorial***)**.** *Let X be a scheme. We say that X is locally factorial if all its stalks are UFD.*

Example 8.1. *Regular schemes are locally factorial.*

Remark 8.6. *Since UFD is not a local property, we cannot cover a locally factorial scheme with affine open subsets $U_i = \mathrm{Spec}(A_i)$ with each A_i UFD, such as curve $y^2 = x^3 + x$. In addition, since UFD is integrally closed, local factorial implies normal.*

Theorem 8.1. *Let X be a scheme which is noetherian, integral, (separated) and locally factorial. Then there is isomorphism $Div(X) \xleftarrow{\sim} CaDiv(X)$. Further, the principal (resp. effective) Weil divisors correspond to principal (resp. effective) Cartier divisors under the isomorphism. In particular, we get the isomorphism $Cl(X) \xleftarrow{\sim} CaCl(X)$.*

Proof. Since X is integral, $\mathcal{K} = K(X)_X$. Let D be a Cartier divisor represented by (U_i, f_i). For all Y prime Weil divisor, set $v_Y(D) = v_Y(f_i)$ for i such that U_i meets Y. This is well defined since $\frac{f_i}{f_j}$ lies in $\mathcal{O}_X^*(U_i \cap U_j)$. Since X is noetherian, there are only finitely many prime divisor Y such that $v_Y(D) \neq 0$. Then $\sum_Y v_Y(D)Y$ is a Weil divisor.

Conversely, let D be a Weil divisor on X. Then for all $x \in X$, D restricts to a Weil divisor D_x on $\mathrm{Spec}(\mathcal{O}_{X,x})$. For all $Y \ni x$ prime divisor, take an affine open neighbourhood $U = \mathrm{Spec}\, A$ of x. Assume that x corresponds to a prime ideal $\mathfrak{p}$. Then $Y \cap U \neq \varnothing$ so that $Y \cap U$ is a irreducible closed subset of codimension 1 of U and of the form $V(\mathfrak{q})$ with $\mathfrak{q} \subseteq \mathfrak{p}$ of height 1. Then we restirct Y to $V(\mathfrak{q}A_{\mathfrak{p}}) \subseteq \mathrm{Spec}(A_{\mathfrak{p}})$. It is clear that the restriction is well defined.

Note that $\mathcal{O}_{X,x}$ is UFD, by Proposition 8.1, get D_x is principal. Thus there exists $f_x \in K(Spec(\mathcal{O}_{X,x})) = \mathrm{Frac}(\mathcal{O}_{X,x}) = K(X)$ such that $D_x = div(f_x)$. Since $f_x \in K(X)$, we can also consider Weil divisor $\widetilde{div(f_x)}$ on X. Then D and $\widetilde{div(f_x)}$ have the same restriction to $\mathrm{Spec}(\mathcal{O}_{X,x})$. Thus they differ by finitely many prime divisors not containing x. Consider the complementary set of union of such prime divisors, we get an open neighbourhood U_x of x such that D and $\widetilde{div(f_x)}$ have the same restriction to U_x. Note that U_x form an open covering of X, want to check that (U_x, f_x) is a Cartier divisor on X.

Take affine open covering $\{V_i = \mathrm{Spec}(B_i)\}$ of $U_x \cap U_{x'}$. Since $\widetilde{div(f_x)}$ coincides with $\widetilde{div(f_{x'})}$ on $U_x \cap U_{x'}$, for all Y prime divisor meeting $U_x \cap U_{x'}$ with generic point y, it is clear that $\frac{f_x}{f_{x'}}$ is invertible in $\mathcal{O}_{X,y}$. Thus for each $\mathfrak{q} \in V_i$ of height 1, $\frac{f_x}{f_{x'}}$ is invertible in $(B_i)_{\mathfrak{q}}$. By Algebraic Hartogs, $B_i = \cap_{\mathfrak{q}} (B_i)_{\mathfrak{q}}$. Get $\frac{f_x}{f_{x'}}$ is invertible in B_i so that $\frac{f_x}{f_{x'}} \in \mathcal{O}_X^*(U_i \cap U_j)$. It is easy to show that the two constructions are inverse to each other and correspondences of principal (resp. effective) divisors. □

Remark 8.7. *If X is only noetherian, integral and normal, then there is still an injection $CaDiv(X) \hookrightarrow Div(X)$ with image equal to the set of Weil divisors which are locally principal.*

When we say a Weil divisor D is locally principal, it means that there is an open covering $\{U_i\}$ such that $D\big|_{U_i}$ is principal divisor on U_i. Similarly, the injection also descends to $Cadiv(X) \hookrightarrow Div(X)$. However, without locally factorial, the map is not necessarily surjective.

8.4 Invertible Sheaves

Recall that invertible sheaf on scheme X is equal to locally free $\mathcal{O}_X$-module of rank 1.

Definition 8.10. *Let X be a scheme. Define the Picard group of X to be the set of isomorphic classes of invertible sheaves on X with multiplicative operation given by tensor product, denoted by $Pic(X)$.*

Example 8.2. *For local ring A, it is easy to show that $Pic(Spec(A))$ is trivial.*

Definition 8.11. *Let $D = (U_i, f_i)$ be a Cartier divisor on scheme X. We can define an invertible sheaf $\mathcal{O}_X(D)$ on X as the $\mathcal{O}_X$-submodule of $\mathcal{K}$ generated by f_i^{-1} on U_i*

Example 8.3. *Let k be a field and $X = \mathbb{P}_k^d$, $D = H$ hyperplane. Then $\mathcal{O}_X(D) \cong \mathcal{O}_X(1)$.*

Proposition 8.4. *Let X be a scheme. Then*
(1)The map $D \longmapsto \mathcal{O}_x(D)$ is a bijection between Cartier divisors and invertible subscheaves if $\mathcal{K}$.
(2)for all Cartier divisors D_1, D_2, $\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)^{-1}$
(3)for all Cartier divisors D_1, D_2, if $D_1 - D_2$ is principal, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-modules.

Proof. (1)Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{K}$. Assume that $\{U_i\}$ is the open covering of X such that $\mathcal{L}\big|_{U_i}$ is free of rank 1 over $\mathcal{O}_{U_i}$. Then for all i, we can take the preimage of 1 under $\mathcal{L}(U_i) \xrightarrow{\sim} \mathcal{O}_X(U_i)$, denoted by g_i. Obviously, $g_i \in \mathcal{K}^*(U_i)$. Set $f_i = g_i^{-1}$, want to show (U_i, f_i) is a Cartier divisor, which is clear since two bases of $\mathcal{L}\big|_{U_i \cap U_j}$ differ by an element of $\mathcal{O}_X^*(U_i \cap U_j)$. Obviously, the construction is inverse of $D \longmapsto \mathcal{O}_X(D)$.

(2)For two Cartier divisors D_1, D_2, when representing them, we can use the same open covering. Assume that $D_1 = (U_i, f_i)$ and $D_2 = (U_i, g_i)$. Then $D_1 - D_2 = (U_i, \frac{f_i}{g_i})$ so that $\mathcal{O}_X(D_1 - D_2)\big|_{U_i}$ is generated by $\frac{f_i}{g_i}$. Thus $\mathcal{O}_X(D_1 - D_2) \subseteq \mathcal{K}$. Denote $\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)^{-1}$ and $\mathcal{O}(D_1 - D_2)$ by $\mathcal{L}$ and $\mathcal{J}$. For all open subset $V \subseteq X$, consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{J}(V) & \longrightarrow & \prod_{i \in I} \mathcal{J}(V \cap U_i) & \longrightarrow & \prod_{i,j \in I} \mathcal{J}(V \cap U_i \cap U_j) \\
& \downarrow & & \downarrow\wr & & \downarrow\wr \\
0 \longrightarrow & \mathcal{L}(V) & \longrightarrow & \prod_{i \in i} \mathcal{L}(V \cap U_i) & \longrightarrow & \prod_{i,j \in I} \mathcal{L}(V \cap U_i \cap U_j)
\end{array}$$

The last two isomorphisms between rows come from the isomorphisms between stalks. Then, by 5 Lemma, get $\mathcal{J} \cong \mathcal{L}$.

(3)Suffice to show that Cartier divisor D is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. If D is principal and defined by $f \in \Gamma(X, \mathcal{K}^*)$, then $\mathcal{O}_X(D)$ globally generated by f^{-1}. Thus $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Conversely, if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, then the image of $1 \in \mathcal{O}_X(X)$ fives the f^{-1}. $\square$

Corollary 8.1. *Let X be a scheme. Then the map $D \longmapsto \mathcal{O}_X(D)$ descends to an injective group homomorphism $CaCl(X) \hookrightarrow Pic(X)$.*

Proposition 8.5. *Let X be an integral scheme. Then $CaCl(X) \xrightarrow{\sim} Pic(X)$ is an isomorphism.*

Proof. Since X is integral, $\mathcal{K} = K(X)_X$ is a constant sheaf. For any invertible sheaf $\mathcal{L}$ on X, set $\mathcal{L}' = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$. Assume $\{U_i\}$ is an open covering of X trivializing $\mathcal{L}$. Thus for all i, $\mathcal{L}'\big|_{U_i} \cong \mathcal{K}\big|_{U_i}$ is a constant sheaf on U_i. As X is irreducible, $\mathcal{L}'\big|_{U_i}$ glue up to a constant sheaf on X. Moreover, we have that $\mathcal{L}' \cong \mathcal{K}$. Note that for all $x \in X$, $\mathcal{K}_x$ is a localization of $\mathcal{O}_{X,x}$ so that $\mathcal{K}_x$ is flat over $\mathcal{O}_{X,x}$. Then $\mathcal{L} \hookrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}$ expresses $\mathcal{L}$ as a subsheaf on $\mathcal{K}$. □

Corollary 8.2. *Let k be a field and $X = \mathbb{P}^d_k$. Then all invertible shaeves on X are isomorphic to $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$.*

Proof. By Proposition 8.5, Theorem 8.1 and Proposition 8.3, get

$$Pic(X) \cong CaCl(X) \cong Cl(X) \cong \mathbb{Z}$$

□

In conclusion, if X is a scheme which is noetherian, integral, (separated) and locally factorial, then there is a commutative diagram

$$\begin{tikzcd} Cl(X) \arrow[dr, "f_3"'] & & CaCl(X) \arrow[ll, "f_1"'] \arrow[dl, "f_2"] \\ & Pic(X) & \end{tikzcd}$$

f_1 is given by

$$D = (U_i, g_i) \longmapsto \sum_Y v_Y(g_i)Y$$

f_2 is given by

$$D = (U_i, g_i) \longmapsto \mathcal{O}_X(D) \subseteq \mathcal{K}$$

f_3 is given by

$$D = \sum_Y n_Y Y \longmapsto (\mathcal{O}_X(D) : U \longmapsto \{g \in \mathcal{K}^* \big| D\big|_U + div(f)\big|_U \text{ effective}\} \cup \{0\}) \subseteq \mathcal{K}$$

8.5 Effective Divisors

Let X be a scheme. Recall that $D = (U_i, f_i)$ is an effective Cartier divisor on X if and only if for all i, $f_i \in \mathcal{K}^*(U_i) \cap \mathcal{O}_X(U_i)$. Thus an effective Cartier divisor one-to-one corresponds to an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ which is locally generated by a non zero-divisor f_i. Moreover, $\mathcal{I} \cong \mathcal{O}_X(-D) = \mathcal{O}_X^{-1}$.

Definition 8.12 (***Zero Divisor of Global Sections***)**.** *Let X be an integral scheme, $\mathcal{L}$ invertible sheaf on X, $s \neq 0 \in \Gamma(X, \mathcal{L})$. For all open subset $U \subseteq X$ trivializing $\mathcal{L}$, there is an isomorphism $\varphi : \mathcal{L}\big|_U \xrightarrow{\sim} \mathcal{O}_U$. Assume that $\{U_i\}$ is an open covering of X trivializing $\mathcal{L}$ with* *isomorphisms φ_i. Define the zero divisor of s to be $Z(s) := (U_i, \varphi_i(U_i)(s))$. In particular, for $\mathcal{L} \cong \mathcal{O}_X$, get zero divisor of a regular function.*

Remark 8.8. *It is clear that $Z(s)$ is a well defined Cartier divisro.*

Proposition 8.6. *Let X be a scheme which is geometrically integral, smooth and projective over field k, D Cartier divisor on X, $\mathcal{L} = \mathcal{O}_X(D)$ invertible sheaf associated with D. Then*
(1)for all $s \neq 0 \in \Gamma(X, \mathcal{L})$, the zero divisor $Z(s)$ is effective and linearly equivalent to D.
(2)All effective divisors linearly equivalent to D are of the form $Z(s)$ for some $s \neq 0 \in \Gamma(X, \mathcal{L})$.
(3)Two sections $s, s' \neq 0 \in \Gamma(X, \mathcal{L})$ have the same zero divisor if and only if they differ by a nonzero constant in $k^\times$.

Proof. (1)Since X is integral, view $\mathcal{L}$ as a subsheaf of $\mathcal{K} = K(X)_X$. Then also since X is integral, s corresponds to an element $f \in \mathcal{K}^*(X) = K(X)^\times$. Assume that $D = (U_i, f_i)$. Then $\mathcal{L}$ is locally generated by f_i^{-1} and the isomorphism φ_i is inducing by multiplication with f. Get $Z(s) = (U_i, f_i f) = D + div(f)$ is effective and linearly equivalent to D.

(2)If D' is effective and $D' = D + div(f)$ for some $f \in \mathcal{K}^*(X) = K(X)^\times$. Then $f_i f \in \mathcal{K}^*(U_i) \cap \mathcal{O}_X(U_i)$ so that $f\big|_{U_i} = \frac{f_i f}{f_i}$ glue up to a global section of $\mathcal{L}$, denoted by s. Same argument, get $Z(s) = D'$.

(3)Assume s, s' correspond to $f, f' \in K(X)^\times$. Then $Z(s) = Z(s')$ if and only if $div(\frac{f}{f'}) = 0$ if and only if $\frac{f}{f'} \in \mathcal{O}_X^*(X)$. As X is geometrically integral and projective over k, by Corollary 7.9, get $\mathcal{O}_X^*(X) = k^\times$, done! $\square$

Definition 8.13. *Let X be a scheme which is geometrically integral, smooth and projective over field k, D Cartier divisor on X. The set of divisors that are effective and linearly equivalent to D is in bijection with $\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}/k^\times$. It is called the (complete) linear system associated with D, denoted by $|D|$.*

8.6 Vector Bundles

Definition 8.14 (***Vector Bundles***)**.** *Let X be a scheme. A vector bundle of rank r on X is a scheme E with a morphism $\pi : E \longrightarrow X$ satisfying that there exists an open covering $\{U_i\}$ of X and commutative diagrams*

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\varphi_i", "\sim"'] \arrow[dr] & & \mathbb{A}^r_{U_i} \arrow[dl] \\
& U_i &
\end{tikzcd}$$

where isomorphism φ_i is called the local trivialization, such that the transition $\varphi_i \circ \varphi_j^{-1}$ is linear over $U_i \cap U_j$ i.e. for all affine open subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, $\varphi_i \circ \varphi_j^{-1}$: $\operatorname{Spec}(A[y_1, \cdots, y_r]) \longrightarrow \operatorname{Spec}(A[x_1, \cdots, x_r])$ is given by an A-linear isomorphism

$$\begin{aligned}
A[x_1, \cdots, x_r] &\longrightarrow A[y_1, \cdots, y_r] \\
x_i &\longmapsto \sum_j a_{ij} y_j
\end{aligned}$$

Equivalently giving transition functions $g_{ij}: U_i \cap U_j \longrightarrow (GL_r(\mathcal{O}_X(U_i \cap U_j)))_{U_i \cap U_j}$ *such that* $g_{ii} = id$, $g_{ij}^{-1} = g_{ji}$ *and* $g_{ik} = g_{jk} \circ g_{ij}$ *on* $U_i \cap U_j \cap U_k$.

Definition 8.15. *Let* X *be a scheme and* $\pi : E \longrightarrow X$ *be a vector bundle on* X. *A section of* E *is a morphism* $s : X \longrightarrow E$ *such that* $\pi \circ s = \mathrm{id}_X$. *If* E *is determined by transitions functions* g_{ij}, *then* s *is determined by morphisms* $s_i : U_i \longrightarrow \mathbb{A}^r_{U_i}$ *such that* $s_i = g_{ij} s_j$.

Now given $\pi : E \longrightarrow X$ vector bundle of rank r, define its sheaf of sections to be

$$\mathcal{E} : U \longmapsto \{\text{sections of } \pi\big|_U : \pi^{-1}(U) \longrightarrow U\} \tag{2}$$

Then $\mathcal{E}$ is locally free sheaf of rank r. To see this, take open subset $U \subseteq X$ with commutative diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i", "\sim"'] \arrow[dr] & & \mathbb{A}^r_{U_i} \arrow[dl] \\ & U_i & \end{tikzcd}$$

Then sections of $\pi\big|_U$ are one-to-one corresponding to sections of $\mathbb{A}^r_U \longrightarrow U$. Note that $\mathbb{A}^r_U$ is the fiber product of U and $\mathbb{A}^r_{\mathbb{Z}}$ over $\mathbb{Z}$. By universal property of fiber product, sections are one-to-one corresponding to a commutative diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & \mathbb{Z} \\ & \mathbb{A}^r_{\mathbb{Z}} \arrow[ur] & \end{tikzcd}$$

which is equivalent to

$$\begin{tikzcd} \mathcal{O}_U(U) & & \mathbb{Z} \arrow[ll] \arrow[dl] \\ & \mathbb{Z}[x_1, \cdots, x_r] \arrow[ul] & \end{tikzcd}$$

Since the diagram is determined by the image of each x_i, it is clear that the rank is r.

Conversely, a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r comes from a vector bundle of rank r unique up to isomorphism. To see this, either use transition functions or directly define $E := \mathrm{Spec}(Sym(\mathcal{E}^\vee))$, where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual space. For all affine open subset $U = \mathrm{Spec}\, A \subseteq X$, take $\mathrm{Spec}(Sym_A(\mathcal{E}^\vee(U)))$. By gluing lemma, get E. In conclusion, get the correspondence between isomorphic classes of locally free $\mathcal{O}_X$-module of rank r and isomorphic classes of vector bundles of rank r. However, this is not an equivalence of categories, as morphism of vector bundles are typically required to be of constant rank.

Remark 8.9. *For field* k *and vector space* V, $Sym_k(V^\vee) = \{$*polynomials on* $V\}$. *Precise definition can be seen in Hartshorne, Chapter II, exercise 5.16.*

Definition 8.16 (***Zero Schemes of Sections of Vector Bundles***)**.** *Let* X *be a scheme and* $\pi : E \longrightarrow X$ *be a vector bundle,* $s : X \longrightarrow E$ *section. Assume that* s *is determined by* $s_i : U_i \longrightarrow \mathbb{A}^r_{U_i}$. *Since* s_i *is equivalent to a ring homomorphism* $A[x_1, \cdots, x_r] \longrightarrow \mathcal{O}_X(U_i)$, s_i *is determined by* $s_i^1, \cdots, s_i^r \in \mathcal{O}_X(U_i)$. *Define the zero scheme of* s *to be a closed subscheme* $Z(s) \subseteq X$ *corresponding to the ideal sheaf generated by* $s_i^1, \cdots, s_i^r$ *on* U_i.

8.7 Riemann-Roch Theorem for Curve Case

Recall that a curve is a scheme of finite type over field k which is geometrically integral of dimension 1. Thus irreducible closed subsets of codimension 1 are just the closed points. Let X be a curve which is smooth and proper over k, $D = \sum_i n_i P_i$ a Weil divisor on X. Then $\deg(D) = \sum_i n_i [k(P_i) : k]$. Set $\ell(D) := \dim_k(\Gamma(X, \mathcal{O}_X(D))) = \dim_k(H^0(X, \mathcal{O}_X(D)))$. Since X is projective over k and $\mathcal{O}_X(D)$ is coherent, get $\ell(D) < \infty$.

Lemma 8.3. *Let X be a curve which is smooth and proper over k, D a Weil divisor on X. If $\ell(D) \neq 0$, then $\deg(D) \geq 0$. If $\ell(D) \neq 0$ and $\deg(D) = 0$, then $D \sim 0$ i.e. $\mathcal{O}_X(D) \cong \mathcal{O}_X$.*

Proof. If $\ell(D) \neq 0$, then $\mathcal{O}_X(D) \neq 0$ and there exists an effective divisor $D' \sim D$ so that $\deg(D) = \deg(D') \geq 0$. Since D' is effective, if moreover $\deg(D) = 0$, then $D' = 0$ so that $D \sim 0$. $\square$

Definition 8.17 (***Canonical Divisors***)**.** *A canonical divisor K_X is any Cartier divisor corresponding to canonical sheaf ω_X. Set $g(X)(= p_g(X)) := \dim_k(\Gamma(X, \omega_X))$ to be the geometric genus of X.*

Theorem 8.2 (***Riemann-Roch Theorem for Curve Case***)**.** *Let X be a curve which smooth and proper over field k, D a Weil divisor on X. Then $\ell(D) - \ell(K_X - D) = \deg(D) + 1 - g(X)$.*

Remark 8.10. *We will prove the theorem assuming $\ell(K_X - D) = \dim_k(H^1(X, \mathcal{O}_X(D)))$, which is a result from Serre duality. In particular, assuming the theorem, for $D = 0$, we have that $g(X) = \ell(K_X) + 1 - \ell(0)$. Since $\ell(0) = \dim_k(\Gamma(X, \mathcal{O}_x)) = \dim_k(k) = 1$, get $g(x) = \ell(K_X)$, called the arithmetirc genus of X.*

For $\mathcal{F}$ coherent sheaf on X, set $\chi(\mathcal{F}) := \sum_i (-1)^i \dim_k(H^i(X, \mathcal{F}))$, called the Euler characteristic of $\mathcal{F}$. By Serre duality, can rewrite Riemann-Roch Theorem as $\chi(\mathcal{O}_X(D)) = \deg(D) + 1 - g(X)$. Note that $1 - g(X) = \chi(\mathcal{O}_X)$, get $\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \deg(D)$.

Proof. As remark, $D = 0$ is ok. Let $P \in X$ be a closed point. Suffice to show that Riemann-Roch is ok for D if and only if Riemann-Roch is ok for $D + P$. View $i : \{P\} \hookrightarrow X$ as a closed subscheme with reduced scheme structure. Then $\mathcal{I}_P \cong \mathcal{O}_X(-P)$. Note that there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_{\{P\}} \longrightarrow 0$$

Tensot with $\mathcal{O}_X(D + P)$, get exact sequence

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow i_* \mathcal{O}_{\{P\}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D + P) \longrightarrow 0$$

Since no prime divisor can restrict to $\mathrm{Spec}(\mathcal{O}_{X,P})$, $\mathcal{O}_X(D + P)$ is generated by 1 on some open neighbourhood of P. Thus $i_* \mathcal{O}_{\{P\}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D + P) \cong i_* \mathcal{O}_{\{P\}}$. Consider the long exact sequence given by cohomology, get $\chi(\mathcal{O}_X(D + P)) = \chi(\mathcal{O}_X(D)) + \chi(i_* \mathcal{O}_{\{P\}})$, where $\chi(i_* \mathcal{O}_{\{P\}}) = \dim_k(H^0(X, i_* \mathcal{O}_{\{P\}})) = \dim_k(H^0(Spec(k(P)), \mathcal{O}_{Spec(k(P))})) = [k(P) : k]$. Thus $\chi(\mathcal{O}_X(D + P)) - \chi(\mathcal{O}_X(D)) = [k(P) : k]$. Since $\deg(D + P) = \deg(D) + [k(P) : k]$, done! $\square$

Example 8.4. *Let X be a curve which smooth and proper over field k, D a Weil divisor on X.*
(1)If $\deg(D) > 0$, then dimension of complete linear system of nD is $\dim_k(|nD|) = n\deg(D) - g(X)$ for n large enough, since $\ell(K_X - nD) = 0$ for n large enough.
(2)Let $D = K_X$, then $\ell(K_X) - 1 = \deg(K_X) + 1 - g(X)$. Get $\deg(K_X) = 2g(X) - 2$.
(3)If $g(X) = 1$, then $\deg(K_X) = 0$ and $\ell(K_X) = g(X) = 1$. Get $K_X \sim 0$.
(4)Let X be an elliptic curve i.e. $g(X) = 1$ and the k-points of X isn't empty, $P_0 \in X(k)$. Take $Pic^0(X) \subseteq Pic(X)$ to be the subgroup corresponding to divisor classes of degree zero. Then $X(k) \longrightarrow Pic^0(X) \quad P \longmapsto \mathcal{O}_X(P - P_0)$ is bijective inducing a group structure over $X(k)$. Injection is obvious. For surjection, let D be a divisor of degree 0, then $\deg(D + P_0) = \deg(P_0) = 1$. By Riemann-Roch, $\ell(D+P_0) - \ell(K_X - D - P_0) = 1$. While $\deg(K_X - D - P_0) < 0$, get $\ell(K_X - D - P_0) = 0$ and $\ell(D + P_0) = 1$. Thus there exists effective divisor $P \sim D + P_0$ so that $P \longmapsto \mathcal{O}_X(D)$.
(5)If $\deg(D) \geq 2g(X) + 1$, then $\mathcal{O}_X(D)$ is very ample. Thus $\mathcal{O}_X(D)$ is ample if and only if $\deg(D) > 0$.

8.8 Higher Dimension Divisor Theory

Let X be a scheme which is geometrically integral, smooth and proper over k. Define $Z^i(X)$ to be the free abelian group spanned by closed and integral subschemes of X of codimension i. Each element of $Z^i(X)$ is of the form $\alpha = \sum_V n_V V$, called an algebraic cycle of codimension i.

Definition 8.18 (***Rational Equivalence***)**.** *$\sim_{rat}$ is the equivalence on $Z^i(X)$ generated by $j_*(div(f))$, where $W \subseteq X$ is closed and integral subscheme of codimension $i-1$, $\widetilde{W}$ is the normalization of W, j is the map $\widetilde{W} \longrightarrow W \hookrightarrow X$ and $f \in k(W) = k(\widetilde{W})$. Set $CH^i(X) := Z^i(X)/\sim_{rat}$ to be the Chow group of algebraic cycles of codimension i*

Want to define intersection between $Z^i(X)$ and $Z^j(X)$ for all i, j. However, $V \cap W$ is not necessarily reduced as a scheme and the intersection is not always proper i.e. codimension of $V \cap W$ might be less than $i + j$. There is an important theorem that can help us to solve these problems.

Theorem 8.3 (***Chow's Moving Lemma***)**.** *For all $\alpha \in Z^i(X)$ and $\beta \in Z^j(X)$, there exists $\alpha' \sim_{rat} \alpha$ such that every component of α' intersects every component of β properly.*

Remark 8.11. *Then we can define $\alpha \cdot \beta = [\alpha' \cap \beta] \in CH^{i+j}(X)$. However, there is another problem about this definition that if it is well defined. Even though the answer is correct, this is a hard problem, but there are other ways to avoid this problem. Fulton and Macpherson established a geometric, refined, moving lemma-free intersection theory. In particular, get well defined*

$$\cdot : CH^i(X) \times CH^j(X) \longrightarrow CH^{i+j}(X) \times (CH^*(X), \cdot)$$

There is also some generalization of Riemann-Roch in higher dimension case.

Theorem 8.4 (***Hirzebruch-Riemann-Roch***)**.** *Let X be a scheme which is geometrically integral, smooth and proper over field k, E vector bundle on X. Then $\chi(E) = \int_X ch(E)+d(X)$.*

Remark 8.12. *$ch(E)$ is the Chern character of E and $d(X)$ is the Todd class of X which is independent of E.*

9 Differentials

9.1 Kähler Differentials in Algebraic Theory

Let X be a scheme, $x \in X$ closed point. Recall that the tangent space of X at x is $T_xX := (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. However, in the world of algebraic geometry, it is easier to study cotangent space $\mathfrak{m}/\mathfrak{m}^2$. Further, not only the tangent space, but also we want to get differentials in the world of algebraic geometry. Firstly, let's see three equivalent definitions of Kähler differentials.

Definition 9.1. *Let A be a ring, B an A-algebra. The module of relative differentials of B over A, denoted by $\Omega_{B/A}$ (or $\Omega^1_{B/A}$), is defined by taking the quotient module of the free B-module spanned by symbols $\{db \big| b \in B\}$ under the following relations*
(1)da, $\forall a \in A$,
(2)$d(b_1 + b_2) - db_1 - db_2$, $\forall b_1, b_2 \in B$,
(3)$d(b_1b_2) - b_1db_2 - b_2db_1$, $\forall b_1b_2 \in B$.
Get an A-linear map $d : B \longrightarrow \Omega_{B/A} \quad b \longmapsto db$.

Remark 9.1. *This is an constructive definition, which directly shows us the structure of module of relative differentials. It is clear that those relation are copied from Differential Geometry.*

Definition 9.2. *Let A be a ring, B an A-algebra, $M \in Mod_B$. An A-derivation of B into M is a map $d : B \longrightarrow M$ such that*
(1)$da = 0$, $\forall a \in A$,
(2)$d(b_1 + b_2) = db_1 + db_2$, $\forall b_1, b_2 \in B$.
(3)$d(b_1b_2) = b_1db_2 + b_2db_1$, $\forall b_1, b_2 \in B$.

Denote the set of A-derivations of B into M by $\mathrm{Der}_A(B, M)$, which is automatically an A-module. Then $d : B \longrightarrow \Omega_{B/A}$ has the universal property that for all $M \in Mod_B$ and A-derivation $d' : B \longrightarrow M$, there exists unique B-module homomorphism $\phi : \Omega_{B/A} \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[d, "d'"] & \Omega_{B/A} \arrow[dl, "\phi"] \\ M & \end{tikzcd}$$

The universal property gives an canonical isomorphism $Hom_B(\Omega_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M)$.

Example 9.1. *(1)Let A be a ring, $B = A[x_1, \cdots, x_n]$. Then $\Omega_{B/A}$ is the free module spanned by $\{dx_i\}$.*
(2)If B is either localization of quotient of A, then $\Omega_{B/A} = 0$. For localization case, assume $B = S^{-1}A$. Then for all $b \in B$, there exists $s \in S$ such that $bs \in A$ so that $s(db) = d(bs) = 0$. Note that $s \in B^\times$, get $db = 0$. For quotient case, obviously.

Consider ring homomorphism $A \longrightarrow B \longrightarrow C$. There are natural homomorphisms $\alpha : \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \quad db \otimes c \longmapsto d(bc)$ and $\beta : \Omega_{C/A} \longrightarrow \Omega_{C/B} \quad dc \longmapsto dc$.

Proposition 9.1. *Let A be a ring, B an A-algebra.*
(1)(base change) Let A' be an A-algebra, $B' := B \otimes_A A'$. Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$ is a canonical isomorphism of B-modules.
(2)(cotangent sequence) Let $A \longrightarrow B \longrightarrow C$ be ring homomorphisms. Then there is an exact sequence

$$\Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \longrightarrow 0$$

(3)(localization) Let S be a multiplicative system of B. Then $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$ is a canonical isomorphism of $S^{-1}B$-modules.
(4)(conormal sequence) Let $I \subseteq B$ be an ideal, $C := B/I$. Then there is an exact sequence

$$I/I^2 \xrightarrow{\sigma} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} = 0$$

where σ maps $\bar{b} \longmapsto db \otimes 1$.

Proof. For (1), $d : B \longrightarrow \Omega_{B/A}$ induces a homomorphism $d' = d \otimes \mathrm{id}_{A'} : B' \longrightarrow \Omega_{B/A} \otimes_A A' = \Omega_{B/A} \otimes_B B'$. It is clear that d' is an A'-derivation. Suffice to show that d' has the universal property. For all $M \in Mod_{B'}$ and A'-derivation $d'' : B' \longrightarrow M$, define map $\phi : \Omega_{B/A} \otimes_A A' \longrightarrow M$ to be $db \otimes a' \longmapsto a'd(b \otimes 1)$. It is easy to check that ϕ is a B'-module homomorphism and $\phi \circ d' = d''$.

For (2), use the property of Hom functor. It suffices to show that for all $M \in Mod_C$, the following sequence is exact

$$0 \longrightarrow \mathrm{Hom}_C(\Omega_{C/B}, M) \longrightarrow \mathrm{Hom}_C(\Omega_{C/A}, M) \longrightarrow \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M)$$

As $\mathrm{Hom}_C(\Omega_{C/B}, M) \cong \mathrm{Der}_B(C, M)$, $\mathrm{Hom}_C(\Omega_{C/A}, M) \cong \mathrm{Der}_A(C, M)$ and $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, \mathrm{Hom}_C(C, M)) \cong \mathrm{Hom}_B(\Omega_{B/A}, M) \cong \mathrm{Der}_A(B, M)$, get sequence

$$\begin{array}{rrr} 0 \longrightarrow \mathrm{Der}_B(C, M) \longrightarrow & \mathrm{Der}_A(C, M) \longrightarrow & \mathrm{Der}_A(B, M) \\ (d : C \to M) \longmapsto & (d : C \to M) \longmapsto & (B \to C \xrightarrow{d} M) \end{array}$$

Obviously, the sequence is exact, done!

For (3), taking $C = S^{-1}B$ in (2), get the surjectivety since $\Omega_{C/B} = 0$. For injectivety, compose with $\Omega_{S^{-1}B/A} \longrightarrow S^{-1}\Omega_{B/A} \quad d\frac{b}{s} \longmapsto \frac{db}{s}$ so that we get the $S^{-1}B$-module endomorphism of $S^{-1}\Omega_{B/A}$. By the universal property of $S^{-1}\Omega_{B/A}$, it should be identity map. Thus the canonical homomorphism is injective.

For (4), recall that $I/I^2 \cong I \otimes_B C$. Similar, suffice to show that for all $M \in Mod_C$, the following sequence is exact

$$0 \longrightarrow \mathrm{Hom}_C(\Omega_{C/A}, M) \longrightarrow \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M) \longrightarrow \mathrm{Hom}_C(I/I^2, M)$$

Since $\mathrm{Hom}_C(\Omega_{C/A}, M) \cong \mathrm{Der}_A(C, M)$, $\mathrm{Hom}_c(\Omega_{B/A} \otimes_B C, M) \cong \mathrm{Der}_A(B, M)$ as in (2) and $\mathrm{Hom}_C(I/I^2, M) \cong \mathrm{Hom}_C(I \otimes_B C, M) \cong \mathrm{Hom}_B(I, \mathrm{Hom}_C(C, M)) \cong \mathrm{Hom}_B(I, M)$, get sequence

$$\begin{array}{rcccc}
0 \longrightarrow \mathrm{Der}_A(C, M) \longrightarrow & \mathrm{Der}_A(B, M) & \longrightarrow & \mathrm{Hom}_B(I, M) \\
(d : C \to M) \longmapsto & (C \to B \xrightarrow{d} M) & & \\
& (d' : B \to M) & \longmapsto & (d'\big|_I : I \longrightarrow M)
\end{array}$$

Obviously, the sequence is exact, done! □

Corollary 9.1. *Let A be a ring, B an algebra of finite type over A. Then $\Omega_{B/A}$ is a finite B-module and for all localization $C := S^{-1}B$ of B, $\Omega_{C/A}$ is a finite C-module.*

Reason 9.1. *Follows from example 9.1 and Proposition 9.1 (3) and (4).*

Definition 9.3. *Let A be a ring, B an A-algebra. Consider the diagonal homomorphism $f : B \otimes_A B \longrightarrow B \quad b \otimes b' \longmapsto bb'$. Assume I is the kernel of f. Then I/I^2 is a B-module by multiplication on the left. Define map $d : B \longrightarrow I/I^2 \quad b \longmapsto I/I^2 \quad b \longmapsto \overline{1 \otimes b - b \otimes 1}$.*

Lemma 9.1. *The map defined above is an A-derivation.*

Proof. For all $a \in A$, $da = \overline{1 \times a - a \otimes 1} = 0$. For all $b_1, b_2 \in B$,

$$\begin{aligned}
d(b_1 + b_2) &= \overline{1 \otimes (b_1 + b_2) - (b_1 + b_2) \otimes 1} \\
&= \overline{1 \otimes b_1 - b_1 \otimes 1} + \overline{1 \otimes b_2 - b_2 \otimes 1} \\
&= db_1 + db_2
\end{aligned}$$

and

$$\begin{aligned}
d(b_1 b_2) - b_1 db_2 - b_2 db_1 &= \overline{1 \otimes b_1 b_2 - b_1 b_2 \otimes 1} - b_1 \overline{1 \otimes b_2 - b_2 \otimes 1} - b_2 \overline{1 \otimes b_1 - b_1 \otimes 1} \\
&= \overline{1 \otimes b_1 b_2 - b_1 b_2 \otimes 1} - \overline{b_1 \otimes b_2 - b_1 b_2 \otimes 1} - \overline{b_2 \otimes b_1 - b_1 b_2 \otimes 1} \\
&= \overline{(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1)} \\
&= 0
\end{aligned}$$

□

Proposition 9.2. *Assumption as above, then $(I/I^2, d) \cong (\Omega_{B/A}, d)$.*

Proof. By universal property, there exists unique $\phi : \Omega_{B/A} \longrightarrow I/I^2 \quad db \longmapsto \overline{1 \otimes b - b \otimes 1}$. As $1 \otimes b - b \otimes 1$ are generators of I, ϕ is surjective. Consider $B \otimes_A B \longrightarrow \Omega_{B/A} \quad b \otimes b' \longmapsto bdb'$ restricting to I. It is easy to check that the image of I^2 is O. Get $\psi : I/I^2 \longrightarrow \Omega_{B/A}$. Suffice to show that $\psi \circ \phi$ is identity. While $\psi \circ \phi(db) = \psi(\overline{1 \otimes b - b \otimes 1}) = db$, done! □

9.2 Sheaf of Differentials

Definition 9.4. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Then there exists quasi-coherent sheaf $\Omega_{X/Y}$ on X such that for all affine open subsets $V \subseteq Y$ and $U \subseteq f^{-1}(V)$, we have that $\Omega_{X/Y}\big|_U \cong \widetilde{\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)}}$ and for all $x \in X$, $\Omega_{X/Y,x} \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$. $\Omega_{X/Y}$ is called the sheaf of relative differentials of X over Y.*

Even though we can get $\Omega_{X/Y}$ by gluing lemma, the following proposition is a much faster way to see the uniqueness of sheaf of relative differentials.

Proposition 9.3. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Consider the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$. Note that by Remark 4.1, $\Delta(X)$ is closed in an open subset $U \subseteq X \times_Y X$. Set $\mathcal{I}$ to be the ideal sheaf corresponding to $\Delta(X)$ in U. Then $\Omega_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$.*

Remark 9.2. *Even though we take the inverse image of $\mathcal{I}/\mathcal{I}^2$ under Δ in the equation, $\mathcal{I}/\mathcal{I}^2$ is automatically viewed as an $\mathcal{O}_X$-module.*

Proof. For all affine open subset $V = \operatorname{Spec} A \subseteq V$ and $U = \operatorname{Spec}(B) \subseteq f^{-1}(V)$, then $\Delta(U) \subseteq W := U \times_V U$. Note that $\mathcal{O}_{X\times_Y X}(W) = B \otimes_A B$ and $\mathcal{I}\big|_W = \widetilde{I}$, where I is the kernel of $B \otimes_A B \longrightarrow B \quad b \otimes b' \longmapsto bb'$. Thus $I/I^2 \subseteq \mathcal{I}/\mathcal{I}^2(W)$. Define map $d : B \longrightarrow \Delta^*(\mathcal{I}/\mathcal{I}^2)(U)$ to be $b \longmapsto (W, \overline{1 \otimes b - b \otimes 1})$. Same argument as proof of Lemma 9.1, get d is an A-derivation. By universal property, there exists unique homomorphism $\phi : \Omega_{B/A} \longrightarrow \Delta^*(\mathcal{I}/\mathcal{I}^2)(U)$, which corresponds to a morphism $\varphi : \Delta^*(\mathcal{I}/\mathcal{I}^2)\big|_U \longrightarrow \widetilde{\Omega_{B/A}}$.

Suffice to show that φ_x is isomorphic for each $x \in U$. Assume x corresponds to prime ideal $\mathfrak{p} \in \operatorname{Spec}(B)$ and $f(x)$ corresponds to prime ideal $\mathfrak{P} \in \operatorname{Spec}(B \otimes_A B)$. Then $(\widetilde{\Omega_{B/A}})_x = \Omega_{B_\mathfrak{p}/A}$ by Proposition 9.1 (3) and $(\Delta^*(\mathcal{I}/\mathcal{I}^2)\big|_U)_x = I_\mathfrak{P}/I_\mathfrak{P}^2 \otimes_{(B\otimes_A B)_\mathfrak{P}} B_\mathfrak{p}$. Note that

$$I_\mathfrak{P}/I_\mathfrak{P}^2 \otimes_{(B\otimes_A B)_\mathfrak{P}} B_\mathfrak{p} \cong I/I^2 \otimes_{B\otimes_A B} (B \otimes_A B)_\mathfrak{P} \otimes_{(B\otimes_A B)_\mathfrak{P}} B_\mathfrak{p} \cong I/I^2 \otimes_{B\otimes_A B} B_\mathfrak{p}$$

it is clear that $\varphi(\frac{b}{s}) = \overline{1 \otimes b - b \otimes 1} \otimes \frac{1}{s}$. As $B \otimes_A B \longrightarrow B$ is surjective, $I/I^2 \otimes_{B\otimes_A B} B_\mathfrak{p} \cong I/I^2 \otimes_B B_p \cong (\Omega_{B/A})_\mathfrak{p}$ as B-modules. Thus φ_x is isomorphic, done! $\square$

Proposition 9.4. *Let $f : X \longrightarrow Y$ be a morphism of finite type with X noetherian. Then $\Omega_{X/Y}$ is coherent on X.*

Reason 9.2. *By Corollary 9.1, for all local affine open subset $V = \operatorname{Spec} A$ and $U = \operatorname{Spec}(B)$, we have that $\Omega_{B/A}$ is a finite B-module, done!*

Example 9.2. *(1)Let Y be a scheme, $X = \mathbb{A}^n_Y$. Then $\Omega_{X/Y} \cong \mathcal{O}_X^n$.*
(2)Let A be a ring, $X = \mathbb{P}^1_A = \operatorname{Proj}(A[x,y])$. Then $\Omega_{X/Y} \cong \mathcal{O}_X(-2)$.

Proposition 9.5. *Let $f : X \longrightarrow Y$ be a morphism of schemes.*
(1)(base change) Let Y' be a Y-scheme, $X' = X \times_Y Y'$ with projection $p : X' \longrightarrow X$. Then $\Omega_{X'/Y'} \cong p^*\Omega_{X/Y}$.
(2)(cotangent sequence) Let $X \xrightarrow{f} Y \longrightarrow Z$ be morphisms of schemes. Then there is an exact sequence

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

(3)(localization) Let $U \subseteq X$ be an open subset. Then $\Omega_{X/Y}\big|_U \cong \Omega_{U/Y}$ and for all $x \in X$, $(\Omega_{X/Y})_X \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$.
(4)(conormal sequence) Let $i \hookrightarrow X$ be a closed immersion with ideal sheaf $\mathcal{I}$. Then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} i^*\Omega_{X/Y} \longrightarrow \Omega_{Z/Y} \longrightarrow 0$$

Proof. For (1), the question is local. Can reduce to the case $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}\, A$ and $Y' = \mathrm{Spec}(C)$. Then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ and $\Omega_{X'/Y'} = \widetilde{\Omega_{B\otimes_A C/C}}$. While by Proposition 9.1 (1), $\Omega_{B\otimes_A C/C} \cong \Omega_{B/A} \otimes_B (B \otimes_A C)$, get $p^*\widetilde{\Omega_{B/A}} \cong \Omega_{B/A} \otimes_B \widetilde{(B \otimes_A C)} \cong \widetilde{\Omega_{B\otimes_A C/C}}$, done!

For (2), (3) and (4), the questions are also local, similarly by Proposition 9.1, it is easy to prove for the reduced cases. $\square$

Remark 9.3. *$\mathcal{I}/\mathcal{I}^2$ is naturally an $\mathcal{O}_Z$-module, called the conormal sheaf of $Z \subseteq X$.*

Example 9.3. *Let k be a field and $Y = \mathrm{Spec}\, k$, X, Z geometrically integral and smooth over k. Will see that $\Omega_X := \Omega_{X/k}$, Ω_Z and $\mathcal{I}/\mathcal{I}^2$ are all locally free and there is an exact sequence*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_X \longrightarrow \Omega_Z \longrightarrow 0$$

Taking dual, get an exact sequence

$$0 \longrightarrow \mathcal{T}_Z := \mathcal{H}om_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z) \longrightarrow i^*T_X \longrightarrow \mathcal{N}_{Z/X} := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z) \longrightarrow 0$$

which is obvious a algebraic version of exact sequence in Differential Geometry. However, when sheaf is not locally free, taking dual would lose some information. That's why considering cotangent space and conormal space is more convenient in algebraic geometry.

Theorem 9.1 (***Euler Sequence***)**.** *Let A be a ring, $X = \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \cdots, x_n])$. Then there is an exact sequence*

$$0 \longrightarrow \Omega_{X/A} \longrightarrow \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proof. Denote $S = A[x_0, \cdots, x_n]$. Define graded homomorphism $\phi : S(-1)^{\oplus n+1} \longrightarrow S$ mapping $(s_0, s_1, \cdots, s_n)$ to $x_0s_0 + x_1s_1 + \cdots + x_ns_n$. Set $M = \ker(\phi)$, then there is an exact sequence

$$0 \longrightarrow M \longrightarrow S(-1)^{\oplus n+1} \longrightarrow S$$

Note that ϕ is surjective in all degree ≥ 1, $\widetilde{\mathrm{im}(\phi)}\widetilde{\oplus_{d\geq 1} S_d} = \mathcal{O}_X$. Taking $\widetilde{\cdot}$, get

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Remains to show that $\widetilde{M} \cong \Omega_{X/A}$. First observe that for all i, M_{x_i} is a free S_{x_i}-module of rank n with basis

$$(0, \cdots, \underset{k}{1}, \cdots, \underset{i}{-\frac{x_k}{x_i}}, \cdots, 0)$$

Thus $\widetilde{M}\big|_{D_+(x_i)}$ is a free $\mathcal{O}_{D_+(x_i)}$-module of rank n with basis

$$\begin{array}{c} (0,\cdots,\underset{k}{\frac{1}{x_i}},\cdots,\underset{i}{-\frac{x_k}{x_i^2}},\cdots,0) \end{array}$$

of degree 0. Define isomorphism $\varphi_i : \Omega_{X/A}\big|_{D_+(x_i)} \xrightarrow{\sim} \widetilde{M}\big|_{D_+(x_i)}$ mapping $d(\frac{x_k}{x_i})$ to

$$\begin{array}{c} (0,\cdots,\underset{k}{\frac{1}{x_i}},\cdots,\underset{i}{-\frac{x_k}{x_i^2}},\cdots,0) \end{array}$$

Suffice to check that φ_i can be glued up, which is clear. $\square$

9.3 Smoothness Revisit

Let k be a field of characteristic p, $l = k[t]/(t^p - a)$, l/k inseparable. Consider the conormal sequence

$$I/I^2 \longrightarrow \Omega_{k[t]/k} \otimes_{k[t]} l \longrightarrow \Omega_{l/k} \longrightarrow 0$$

where $I = (t^p - a)$. Thus $\Omega_{l/k} = l \cdot dt/l \cdot d(t^p - a) = l$. On the other hand, if l/k is finite separable extension, then $\Omega_{l/k} = 0$. There is a fact that if l/k is finite field extension, then $\Omega_{l/k}$ is a finite dimensional l-vector space and $\dim_l(\Omega_{l/k}) \geq \operatorname{trdeg}_k(l)$. And the equality is reached if and only if l/k is separable. Back to schemes, we have the following lemma.

Lemma 9.2. *Let A be a k-algebra of finite type. Assume that $x \in \operatorname{Spec} A$ is a closed point corresponding to a maximal ideal $\mathfrak{m} \subseteq A$ such that $A/\mathfrak{m} = k(x)$ is separable over k. Then the canonical map $\delta : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{A/k} \otimes_A k(x)$ is isomorphic.*

Reason 9.3. *Consider the conormal sequence again and note by the previous fact, $\Omega_{k(x)/k} = 0$ so that δ is surjective. For injectivety, assume $A = k[x_1, \cdots, x_n]/I = B/I$ and $\mathfrak{M} \subseteq B$ is the inverse image of $\mathfrak{m}$. We have the following commutative diagram with exact rows*

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \mathfrak{M}/\mathfrak{M}^2 \arrow[r] \arrow[d, "\delta'"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] \arrow[d] & 0 \\ I \arrow[r] & \Omega_{B/k} \otimes_B k(x) \arrow[r] & \Omega_{A/k} \otimes_A k(x) \arrow[r] & 0 \end{tikzcd}$$

By 5 Lemma, it is reduced to prove δ' is surjective. Then repeat the argument, done!

Let X be a scheme of finite type over k. Recall that X is smooth at $x \in X$ if the points of $X_{\overline{k}}$ above x are regular. X is smooth over k if $X_{\overline{k}}$ is regular. A morphism $f : X \longrightarrow S$ of finite type between locally noetherian schemes is smooth at $x \in X$ if it is flat at x and $X_s \longrightarrow \operatorname{Spec}(k(s))$ is smooth at x, where $s = f(x)$. f is smooth if it is flat and for all $s \in S$, the fiber X_s is smooth over $k(s)$.

Lemma 9.3. *Let X be a scheme of finite type over k which is pure of $\dim(X) = n$. Then for any field extension l/k, we have that X_l is also pure of dimension n.*

Remark 9.4. *Can been seen in Rising Sea.*

Proposition 9.6. *Let X be a scheme of finite type over k which is pure of $\dim(X) = n$. For all $x \in X$, he following conditions are equivalent*
(1)X is smooth at x
(2)$\Omega_{X/k,x}$ is a free $\mathcal{O}_{X,x}$-module of rank n.
(3)X is smooth in an open neighbourhood of x.

Proof. (3) $\Rightarrow$ (1) is obvious.

For (2) $\Rightarrow$ (3), consider an affine open neighbourhood $U = \operatorname{Spec} A$ of x. Then $\Omega_{X/k}\big|_U$ corresponds a finitely generated A-module. By argument about generators, it is clear that there is an open neighbourhood $V \ni x$ such that $\Omega_{X/k}\big|_V$ is free $\mathcal{O}_V$-module of rank n. By base change, get $\Omega_{V_{\overline{k}}/\overline{k}}$ also locally free of rank n. Note that for all closed point $y \in V_{\overline{k}}$, $k(y)/\overline{k}$ is separable. Similar to proof of Lemma 9.2, we have $\mathfrak{m}_y/\mathfrak{m}_y^2 \cong \Omega_{V_{\overline{k}}/\overline{k},y} \otimes_{\mathcal{O}_{V_{\overline{k}},y}} k(y)$. Thus $\dim_{k(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2) = n$. By Lemma 9.3, $\dim(\mathcal{O}_{V_{\overline{k}},y}) = n$. Thus $V_{\overline{k}}$ is regular at closed points so that $V_{\overline{k}}$ is regular. Get V is smooth.

For (1) $\Rightarrow$ (2), let $x' \in X_{\overline{k}}$ be a point above x. Then with regularity at x', there exists open neighbourhood $W \ni x'$ such that W is regular. Can also assume that W is of the form $U_{\overline{k}}$ for some open neighbourhood $U \ni x$. $\square$

Remark 9.5. *For the proof of the* (1) $\Rightarrow$ (2)*, there are several hints.*
(1)firstly, reduce to finite extension l/k. Show that there exists $W' \subseteq X_l$ such that W' is regular.
(2)As $X_l \longrightarrow X$ is flat and of finite type, it is open. Then project W' to X through it.
(3)if one $z' \in X_{\overline{k}}$ above $z \in X$ is regular, then all points above z are regular.
(4)consider closed point $z \in U$ and $z' \in U_{\overline{k}}$ above z. Then by Lemma 9.2, $\dim_{k(z)}(\Omega_{U/k,z} \otimes_k k(z)) = \dim_{k(z')}(\Omega_{U_{\overline{k}},z'} \otimes_{\overline{k}} k(z')) = n$.

Lemma 9.4. *Let X be a noetherian scheme, $\mathcal{F}$ coherent sheaf on X. Define* $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_{X,x}} k(x))$. *Then φ is upper semicontinuous i.e.* $\{x \in X \big| \varphi(x) \geq m\}$ *is closed. If moreover X is reduced and φ is constant, then $\mathcal{F}$ is locally free.*

Proof. Both two statements follow from Nakayama. The proof of the first statement can be find in Hartshorne exercise II 5.8. Here, we only prove the second statement. For all $x \in X$, there exists affine open neighbourhood $U = \operatorname{Spec} A$ of x such that $\mathcal{F}\big|_U = \widetilde{M}$, where M is a finitely generated A-module. Then there is a natural homomorphism $\phi : A^n \twoheadrightarrow M \quad (a_1, \cdots, a_n) \longmapsto \sum a_i m_i$. Suppose that ϕ is not injective. Then there exists $(a_1, \cdots, a_n) \in \ker(\phi)$ with $a_i \neq 0$ for some i. As A is reduced, there exists $\mathfrak{p} \in \operatorname{Spec} A$ such that $a_i \notin \mathfrak{p}$. Thus a_i is invertible in $A_\mathfrak{p}$ so that $M_\mathfrak{p}$ can be generated by less than n elements, contradiction! Thus M is free so that $\mathcal{F}$ is locally free. $\square$

Lemma 9.5 ([***Liu Qing***])**.** *Let $f : X \longrightarrow S$ be a morphism of finite type between locally noetherian schemes. Assume $s \in S$, $x \in X_s$ and set* $n = \dim_{k(X)}(\Omega_{X_s/k(s),x} \otimes_{\mathcal{O}_{X_s,x}} k(x))$. *Then there exist open neighbourhood $U \ni x$ and closed immersion $U \hookrightarrow Z$ with Z smooth over S at x such that Z_s is pure of $\dim(Z_s) = n$ and $\Omega_{Z/S,x}$ is free of rank n over $\mathcal{O}_{Z,x}$.*

Proof. Can assume that X and S are affine. First write X as a closed subscheme of an affine S-scheme Y which is smooth over S at x and Y_s is pure and $\Omega_{Y/S,x}$ is free, since X is of finite type over S. In particular, we can take $Y = \mathbb{A}^r_S$. Assume $\mathcal{I} \subseteq \mathcal{O}_Y$ is the ideal sheaf of X in Y. Consider the conormal sequence for all $x \in X$.

$$\mathcal{I}_x/\mathcal{I}_x^2 \longrightarrow \Omega_{Y/S,x} \otimes_{\mathcal{O}_{Y,x}} \mathcal{O}_{X,x} \longrightarrow \Omega_{X/S,x} \longrightarrow 0$$

Tensor with $k(s)$ over $\mathcal{O}_{S,s}$, we can restrict to the fiber of s.

$$\mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k(s) \longrightarrow \Omega_{Y_s/k(s),x} \otimes_{\mathcal{O}_{Y_s,x}} \mathcal{O}_{X_s,x} \longrightarrow \Omega_{X_s/k(s),x} \longrightarrow 0$$

Agian, tensor with $k(x)$ over $\mathcal{O}_{X_s,x}$

$$\mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k(s) \otimes_{\mathcal{O}_{X_s,x}} k(x) \longrightarrow \Omega_{Y_s/k(s),x} \otimes_{\mathcal{O}_{Y_s,x}} k(x) \longrightarrow \Omega_{X_s/k(s),x} \otimes_{\mathcal{O}_{X_s,x}} k(x) \longrightarrow 0$$

If $\dim(Y) = n$, then we can take $Z = Y$.

Suppose that $\dim(Y) = m > n$. As Y is smooth over S at x, by Proposition 9.6, $\dim_{k(x)}(\Omega_{Y_s/k(s),x} \otimes_{\mathcal{O}_{Y_s,x}} k(x)) = m$. Thus there exists $f \in \mathcal{I}_x$ whose image $d\overline{f}$ can be completed to a basis $\{d\overline{f}, d\overline{f}_2, ..., d\overline{f}_m\}$. Furthermore, by restricting Y, can assume $f, f_i \in \mathcal{O}_Y$. Set $Z = V(f) \subseteq Y$, $X \subseteq Z \subseteq Y$. By Nakayama, $\{df_1, \cdots, df_m\}$ span $\Omega_{Y/S,x}$. As $\Omega_{Y/S}$ is free of rank m, $\{df_1, \cdots, df_m\}$ form a basis of $\Omega_{Y/S,x}$. Now $\Omega_{Z/S,x} = \Omega_{Y/S,x}/(df) \otimes_{\mathcal{O}_{Y,x}} \mathcal{O}_{Z,x}$ is free of rank $m-1$. By base change, $\Omega_{Z_s/k(s),x}$ is free of rank $m-1$. By Proposition 9.6, Z_s is smooth at x. As $f \in \mathcal{O}_{Y_s,x}$ is a non-zero-divisor in an integral domain, irreducible components of Z_s are of dimension $m-1$. Finally by "local criterion for flatness", the Hartshorne Chapter III Lemma 10.5 A, flatness of $\mathcal{O}_{Z,x}$ over $\mathcal{O}_{S,s}$ is ok. Now we can finish by induction. □

Proposition 9.7. *Let $f : X \longrightarrow S$ be a morphism of finite type between locally noetherian schemes. Assume $s \in S$, $x \in X_s$ and X_s is pure of dimension n. If X is smooth over S at x, then $\Omega_{X/S}$ is free of rank n in an open neighbourhood of x.*

Proof. Let $U \hookrightarrow Z$ be as in Lemma 9.5, then $\dim(X_s) = \dim(Z_s) = n$. Thus $X_s = Z_s$ in an open neighbourhood of x. As X is flat over S at x, $X = Z$ in an open neighbourhood of x. Get $\Omega_{X/S,x} = \Omega_{Z/S,x}$ is free of rank n. Then $\Omega_{X/S}$ is free of rank n in an open neighbourhood of x. □

Theorem 9.2. *Let $f : X \longrightarrow S$ be a morphism of finite type between locally noetherian schemes with fibers pure of dimension n. Then f is smooth if and only if f is flat and $\Omega_{X/S}$ is locally free of rank n.*

Reason 9.4. *The theorem is immediately given by Proposition 9.6 and Proposition 9.7.*

Remark 9.6. *Flatness is needed. Otherwise if $X \hookrightarrow Y$ is a closed immersion, then $\Omega_{X/Y} = 0$.*

Corollary 9.2. *Let $f : X \longrightarrow Y$ be a dominant morphism between integral schemes of finite type over k of characteristic 0. Then there exists nonempty open subset $U \subseteq X$ such that $f\big|_U : U \longrightarrow Y$ is smooth.*

Proof. Since characteristic of k is 0, $K(X)$ is separable over $K(Y)$. Thus

$$\dim(\Omega_{X/Y,\eta_X}) = \mathrm{trdeg}_{K(Y)}(K(X)) = \dim(X) - \dim(Y) = \dim(X_{\eta_Y}) = n$$

Get $\Omega_{X/Y}$ is free of rank n on a nonempty open subset of X. Note that f is also flat with fiber pure of dimension n on an nonempty open subset of X. By Theorem 9.2, f is smooth on a nonempty open subset of X. $\square$

Example 9.4. *This example shows that of characteristic* 0 *is needed in the previous corollary. Consider Frobinius map* $\mathbb{A}^1_{\mathbb{F}_p} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p}$ *corresponding to ring homomorphism* $\mathbb{F}_p[t] \longrightarrow \mathbb{F}_p[t] \quad t \longmapsto t^p$*. Then it is nowhere smooth.*

9.4 Some Notions

For simplicity, all schemes in this subsection are assumed geometrically intgral k-schemes of finite type and all morphisms are k-morphisms.

Definition 9.5. *Let X be a smooth k-scheme pure of dimension n. Then by Proposition 9.7, $\Omega_{X/k}$ is locally free of rank n. Set $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_{\mathcal{X}})$ to be the tangent sheaf, which is locally free of rank n. Also, we can define tangent bundle $T_X := \mathrm{Spec}(Sym(\Omega_{X/k}))$, which is a vector bundle of rank n.*

Define sheaf of r-forms $\Omega^r_{X/k} := \wedge^r \Omega_{X/k}$ that is locally free of rank $\binom{n}{r}$. In particular, define canonical sheaf $\omega_X (= \omega_{X/k}) := \Omega^n_{X/k} = \det(\Omega_{X/k})$ to the sheaf of n-forms, which is invertible. Set K_X to any Cartier divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$.

If X is also projective (proper) over k, set $P_g(X) := \dim_k(\Gamma(X, \omega_X))$ to be the geometric genus.

Proposition 9.8. *If X, Y are smooth and projective over k and X, Y are k-birational, then $P_g(X) = P_g(Y)$ i.e. P_g is birational invariant.*

Proof. Let $f : X \longrightarrow Y$ be the birational map. As X is normal and Y is proper, there exists open subset $U \subseteq X$ such that $f\big|_{X\setminus U}$ is injective and $X \setminus U$ is of codimension ≥ 2. Get $f^*\Omega_{Y/k} \longrightarrow \Omega_{U/k}$ both locally free of rank $n = \dim(X) = \dim(Y)$, inducing map $f^*\omega_Y \longrightarrow \omega_U$. Thus there are two maps

$$\varphi : \Gamma(Y, \omega_Y) \longrightarrow \Gamma(U, \omega_U)$$

and

$$\psi : \Gamma(X, \omega_X) \longrightarrow \Gamma(U, \omega_U)$$

Claim that the first one is injective and the second one is bijective.

Note that there exists open subset $V \subseteq U \subseteq X$ such that $f\big|_V$ is isomorphic onto image, get $\omega_U\big|_V \cong \omega_Y\big|_{f(V)}$. For all $s \in \ker(\varphi)$, have that $s\big|_{f(V)} = 0$ so that $s = 0$. Then φ is injective.

For the second claim, suffices to show that for all affine open subsets $V \subseteq X$ trivializing ω_X, $\Gamma(V, \omega_X\big|_V) \longrightarrow \Gamma(U \cap V, \omega_X\big|_{U\cap V})$ is bijective. As X is normal and $X \setminus U$ is of codimension ≥ 2, this is a result form Algebraic Hartogs. $\square$

Proposition 9.9. *Let X, Y be smooth k-schemes, $f : X \longrightarrow Y$ k-morphism.*
(1)If f is smooth, then the following sequence is exact

$$0 \longrightarrow f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \xrightarrow{\beta} \Omega_{X/Y} \longrightarrow 0$$

(2)If f is closed (or locally closed) immersion with ideal sheaf $\mathcal{I}$, then the following sequence is exact

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow f^*\Omega_{Y/k} \xrightarrow{\alpha} \Omega_{X/k} \longrightarrow 0$$

Proof. In both (1) and (2), $\ker(\beta)$ and $\ker(\alpha)$ are locally free, thus exactness on the left follows by computing ranks, provided we know $\mathcal{I}/\mathcal{I}^2$ locally free of the expected rank. As $\ker(\alpha)$ is locally free of rank $r = \mathrm{codim}_Y(X)$, for all $x \in X$ closed point, can choose $f_1, \cdots, f_r \in \mathcal{I}$ in an open neighbourhood of x such that $df_1, \cdots, df_r$ form a basis of $\ker(\alpha)$. Set $\mathcal{I}' = (f_1, ..., f_r)$ corresponds to another closed immersion $X' \overset{f'}{\hookrightarrow} Y$. Then the following sequence is exact.

$$0 \longrightarrow \mathcal{I}'/\mathcal{I}'^2 \longrightarrow f'^*\Omega_{Y/k} \xrightarrow{\alpha'} \Omega_{X'/k} \longrightarrow 0$$

Thus X' is smooth over k of dimension $\dim(Y) - r$ in an open neighbourhood of x. But X is smooth over k of same dimension, get $X = X'$ and $\mathcal{I} = \mathcal{I}'$ in an open neighbourhood of x. □

Remark 9.7. *Taking dual and the associated vector bundle in (1), get exact sequence*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0$$

In (2), define normal sheaf $\mathcal{N}_{X/Y} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$, corresponding to normal bundle $N_{X/Y} := \mathrm{Spec}(Sym(\mathcal{I}/\mathcal{I}^2))$. Taking dual and the associated vector bundle in (2), get exact sequence

$$0 \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow N_{X/Y} \longrightarrow 0$$

Example 9.5. *Let $D \subseteq X$ be an effective (Cartier) divisor, with both X, D smooth over k. Then $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-D)\big|_D$, $\mathcal{N}_{D/X} \cong \mathcal{O}_X(D)\big|_D$ and the following sequence is exact*

$$0 \longrightarrow \mathcal{O}_X(-D)\big|_D \longrightarrow \Omega_{X/k}\big|_D \longrightarrow \Omega_{D/k} \longrightarrow 0$$

Proposition 9.10. *Let X be a scheme, $\mathcal{F}$ locally free sheaf on X of rank $n < \infty$. Define* $\det(\mathcal{F}) := \wedge^n\mathcal{F}$*, which is invertible. Then*
(1)$\det(\mathcal{F}^\vee) \cong (\det(\mathcal{F}))^\vee$ *is canonical isomorphism.*
(2)for all $f : Y \longrightarrow X$ morphism of schemes, $\det(f^*\mathcal{F}) \cong d^* \det(\mathcal{F})$.
(3)If there is an exact sequence of locally free sheaves of rank $< \infty$

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

then $\det(\mathcal{E}) \otimes_{\mathcal{O}_X} \det(\mathcal{G}) \cong \det(\mathcal{F})$.

Corollary 9.3 (***Adjunction Formula***)**.** *Let $D \subseteq X$ be an effective (Cartier) divisor, with both X, D smooth over k. Then $\omega_D \cong (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))\big|_D$ and $K_D \sim (K_X + D)\big|_D$.*

Example 9.6. *Recall that when $X = \mathbb{P}^n_k$, there is an exact sequence*

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Thus $\omega_X \cong \mathcal{O}_X(-n-1)$ and $\omega_{\mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2)$. If $D \subseteq \mathbb{P}^n_k$ is hypersurface smooth over k of degree d, then by Adjunction Formula, $\omega_D \cong \mathcal{O}_{\mathbb{P}^n_k}(-n-1+d)\big|_D$.

People classify hypersurfaces depending on degree d.

$$\begin{cases} d < n+1 & \textit{Fano} \\ d = n+1 & \textit{Calabi-Yau} \\ d > n+1 & \textit{general type} \end{cases}$$

When $d \geq n+1$, we see that $P_g(d) > 0$ so that D is not rational.

10 Blowing-up

10.1 Warm-up Construction

Let X be a noetherian scheme, $\mathcal{G}$ quasi-coherent sheaf on X which is a graded algebra object in $Mod_{\mathcal{O}_X}$ i.e. $\mathcal{G} = \oplus_{d\geq 0}\mathcal{G}_d$. Further assume that $\mathcal{G}_0 = \mathcal{O}_X$, $\mathcal{G}_1$ is coherent $\mathcal{O}_X$-module and $\mathcal{G}$ is locally generated by $\mathcal{G}_1$ as $\mathcal{G}_0 = \mathcal{O}_X$-algebra. Thus for all d, $\mathcal{G}_d$ is coherent.

(1)Can define $C = \operatorname{Spec}\mathcal{G}$ called the cone of $\mathcal{G}$ by gluing the various Spec over an affine open covering of X. Then there is a natural affine morphism $\pi : C \longrightarrow X$. For example, if $\mathcal{E}$ is a locally free sheaf over X, then $Sym(\mathcal{E}^\vee)$ is graded $\mathcal{O}_X$-algebra so that vector bundle $E = \operatorname{Spec}(Sym(\mathcal{E}^\vee))$ is the cone of $Sym(\mathcal{E}^\vee)$.

(2)Also can define $\mathbb{P}(C) = \operatorname{Proj}(\mathcal{G})$, called the projective cone of $\mathcal{G}$ by gluing the various Proj over an affine open covering of X. Then there is a natural proper morphism $p : \mathbb{P}(C) \longrightarrow X$ and a natural invertible sheaf $\mathcal{O}_{\mathbb{P}(C)}(1)$ on $\mathbb{P}(C)$. For example, if $\mathcal{E}$ is a locally free sheaf over X, then $Sym(\mathcal{E}^\vee)$ is graded $\mathcal{O}_X$-algebra so that projectiver bundle $\mathbb{E} = \operatorname{Proj}(Sym(\mathcal{E}^\vee))$ is the projective cone of $Sym(\mathcal{E}^\vee)$. Note that here $\mathbb{P}(E)$ is the $\mathbb{P}(\mathcal{E}^\vee)$ in Hartshorne.

(3)If $\mathcal{G} \longrightarrow \mathcal{G}'$ is a surjective morphism of graded $\mathcal{O}_X$-algebras, consider $C = \operatorname{Spec}\mathcal{G}$ and $C' = \operatorname{Spec}(\mathcal{G}')$, then there is closed immersions $C' \hookrightarrow C$ and $\mathbb{P}(C') \hookrightarrow \mathbb{P}(C)$ such that $\mathcal{O}_{\mathbb{P}(C)}(1)\big|_{\mathbb{P}(C')} \cong \mathcal{O}_{\mathbb{P}(C')}(1)$. In particular, as $\mathcal{G} \longrightarrow \mathcal{G}_0 = \mathcal{O}_X$ is surjective, get zero section $0 : X \hookrightarrow C$.

(4)(base change)Let $C = \operatorname{Spec}\mathcal{G}$ on X be the cone of $\mathcal{G}$ and $f : X' \longrightarrow X$ be a morphism with X' noetherian. Then $f^*C := C \times_X X' \cong \operatorname{Spec}(f^*\mathcal{G})$ and $\mathbb{C} \times_X X' \cong \operatorname{Proj}(f^*\mathcal{G})$, where $f^*\mathcal{G}$ is a graded $\mathcal{O}_{X'}$-algebra.

10.2 Blowing-up

Definition 10.1. *Let X be a noetherian scheme and $Y \hookrightarrow X$ be a closed immersion. Assume $\mathcal{I} \subseteq \mathcal{O}_X$ is the corresponding ideal sheaf of the closed immersion.*
(1)The normal cone of Y in X is the cone $C_{Y/X} := \operatorname{Spec}(\oplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1})$, where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^0/\mathcal{I}^1$ is isomorphic as the push forward of $\mathcal{O}_Y$ so that $\oplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1}$.

(2)The blow-up of X along Y is the projective $\widetilde{X} = Bl_Y(C) := \mathrm{Proj}(\oplus_{d\geq 0}\mathcal{I}^d)$ and $\mathcal{O}_{\widetilde{X}}(1)$ is an invertible sheaf on $\widetilde{X}$.

Proposition 10.1. *Let X be a noetherian scheme and $Y \hookrightarrow X$ be a closed immersion. Assume $\mathcal{I} \subseteq \mathcal{O}_X$ is the corresponding ideal sheaf of the closed immersion. Then we have that*
(1)$E = p^{-1}(Y) \cong \mathbb{P}(C_{Y/X})$ called the exceptional divisor, where $p : \widetilde{X} \longrightarrow X$ is the natural morphism
(2)ideal sheaf of E is isomorphic to $\mathcal{O}_{\widehat{X}}(1)$.
(3)$U = X \setminus Y$. Then $p : p^1(U) \longrightarrow U$ is isomorphic.

Proof. For (1), as $\oplus_{d\geq 0}\mathcal{I}^d \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} \cong \oplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1}$, take Proj, done!

For (2), ideal sheaf of E is isomorphic to $\mathcal{I}\mathcal{O}_{\widetilde{X}}$. As $\mathcal{I} \oplus_{d\geq 0} \mathcal{I}^d = \oplus_{d\geq 1}\mathcal{I}^d$, ideal sheaf is isomorphic to $\mathcal{O}_{\widetilde{X}}(1)$.

For (3), we have that $p^{-1}(U) = \mathrm{Proj}(\mathcal{O}_U \oplus \mathcal{O}_U \oplus \cdots) \cong \mathrm{Proj}(\mathcal{O}_U[t]) \cong U$ and $\mathcal{I}\big|_U \cong \mathcal{O}_X\big|_U$. ☐

Example 10.1. *(1)Let $X = \mathrm{Spec}\, A$, Y closed subscheme of X corresponding to ideal $I \subseteq A$. Assume that $f_1, \cdots, f_n$ are generators of I and $S = \oplus_{d\geq 0} I^d$. There is a surjective morphism of graded A-algebra $\phi : A[y_1, \cdots, y_n] \longrightarrow S \quad y_i \longmapsto t_i \in S_1 = I$. Here t_i corresponds to f_i. Set $J := \ker(\phi)$, then $f_i y_j - f_j y_i \in J_1$. Then ϕ induces a closed immersion $\widetilde{X} = \mathrm{Proj}(S) \hookrightarrow \mathbb{P}^{n-1}_A$.*
(2)in particular, if $I = (f)$, where f is not invertible and not zero-divisor, then $\phi : A[t] \longrightarrow S$ is isomorphic so that $\widetilde{X} \longrightarrow \mathbb{P}^0_A = \mathrm{Spec}\, A = X$ is isomorphic. More generally, for X noetherian scheme and $Y \hookrightarrow X$ closed immersion, $p : \widetilde{X} \longrightarrow X$ is isomorphic if and only if Y is an effective Cartier divisor. On the other hand, when $X = \mathrm{Spec}\, A$ is noetherian affine scheme, $\widetilde{X} = \varnothing$ if and only if I is nilpotent.
(3)Note that S is an integral domain (resp. reduced) if and only if A is an integral domain (resp. reduced). Thus for X noetherian scheme, if X is integral and $\mathcal{I} \neq 0$, then $\widetilde{X}$ is intgral and $p : \widetilde{X} \longrightarrow X$ is birational.
(4)Let $X = \mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \cdots, x_n])$ and $Y = \{(0, \cdots, 0)\}$. Then corresponding iedal $I = (x_1, \cdots, x_n)$ and $J = \ker(\phi) = (x_i y_j - x_i y_i)_{1\leq i\leq n, 1\leq j\leq n}$.
(5)Let $X = \mathrm{Spec}\, A$. Set $z_i := \frac{y_i}{y_1} \in \mathcal{O}_{\widetilde{X}(D_+(y_i))}$. Then $J_{(y_1)} = \{P \in A[z_2, \cdots, z_n] \big| \exists d \geq 0$ such that $f_1^d P \in (f_1 z_2 - f_2, \cdots, f_1 z_n - f_n)\}$.
(6)We have seen that $J' := (f_i z_j - f_j)_{1\leq i\leq n, q\leq j\leq n} \subseteq J$. Suppose that $\{f_i\}$ form a minimal set of generators and $Z := V_+(J^{'})$ is integral. Then $\widehat{X} = V_+(J) \hookrightarrow Z = V_+(J')$ is isomorphic. It suffices to check on each $D_+(y_i) \cap Z$. Note that $f_i \notin (f_j)_{j\neq i}$ so that $f_i \neq 0$ on $D_+(y_i) \cap Z$. As Z is integral, f_i is not zero-divisor on $D_+(y_i) \cap Z$. Then apply (5).
(7)Let $A = k[x, y]/(y^2 - x^3 - x^2)$ and $X = \mathrm{Spec}\, A$, $I = (x, y) \subseteq A$, $Y = \{(0, 0)\}$. Then $C_{Y/X} = \mathrm{Spec}(k[x, y]/(y^2 - X^2))$. That's why it looks like two lines when we stand at point $(0, 0)$.
(8)Let $A = k[x, y]/(y^2 - x^3)$ and $X = \mathrm{Spec}\, A$, $I = (x, y) \subseteq A$, $Y = \{(0, 0)\}$. Then $C_{Y/X} = \mathrm{Spec}(k[x, y]/(y^2))$. That's why it looks like two coincided lines when we stand at point $(0, 0)$.
(9)Let $A = k[x, y, z]/(xy - z^2)$ and $X = \mathrm{Spec}\, A$, $I = (y, z) \subseteq A$, $Y = \mathrm{Spec}(k[x])$. Then

$C_{Y/X} = \mathrm{Spec}(k[x][y,z]/(xy))$. *Thus when* $x \neq 0$, $C_{Y/X}$, *the blow up is still one point, while when* $x = 0$, *the blow up is a line.*

Proof of (5). "$\subseteq$": Let $P \in J_{(y_1)} \subseteq A[z_2, \cdots, z_n]$. Can write for some $d \geq 0$ that $f_1^d P = \sum_{i=2}^n Q_i(f_1 z_i - f_i) + a$, where $a \in A$. As $P \in J_{(y_1)}$ and $f_1 z_i - f_i \in J_{(y_1)}$, get $a \in J_{(y_1)}$ i.e. $\phi(a) = 0 \in S_{(t_1)}$. Thus there exists $r \geq 0$ such that $at_1^r = 0$ in S so that $af_1^r = 0$ in A. Replace d by $d+r$, get $a = 0$ and $f_1^d P \in (f_1 z_2 - f_2, \cdots, f_1 z_n - f_n)$.

"$\supseteq$": Let $P \in A[z_2, \cdots, z_n]$ such that $f_1^d P \in (f_1 z_2 - f_2, \cdots, f_1 z_n - f_n)$. Write $P = \frac{Q(y_1,\ldots,y_n)}{y_1^r}$ with $Q(y_1, \cdots, y_n)$ homogeneous of degree r. Thus $f_1^d y_1^e Q(y_1, \cdots, y_n) \subseteq J$ for some e. Get $t_1^{d+e} Q(t_1, \cdots, t_n) = 0$ in S so that $y_1^{d+e} Q(y_1, \cdots, Y_n) = 0$ in J. Thus $P = \frac{y_1^{d+e} Q(y_1, \cdots, y_n)}{y_1^{d+e+r}} \in J_{(y_1)}$. $\square$

Proposition 10.2 (***Base Change***)**.** *Let X be a noetherian scheme, $Y \hookrightarrow X$ closed immersion, $f : X' \hookrightarrow X$ morphism with X' noetherian. Then there is a closed immersion $Bl_{Y'}(X') \hookrightarrow Bl_Y(X) \times_X X'$. Let $\widetilde{f} : Bl_{Y'}(X') \longrightarrow Bl_Y(X)$ be the composition of the closed immersion with projection, then $\widetilde{f}^{-1}(E) = E'$.*

Proof. Reduce to affine case that $X = \operatorname{Spec} A$, $Y = \mathrm{Spec}(A/I)$ and $X' = \mathrm{Spec}(A')$. Then we have exact sequence

$$I \otimes_A A' \longrightarrow A' \longrightarrow A/IA' \longrightarrow 0$$

Note that $I \otimes_A A' \longrightarrow IA'$ is surjective, get $(\oplus_{d \geq 0} I^d) \otimes_A A' \longrightarrow \oplus_{d \geq 0} (IA')^d$ surjective, inducing the wanted closed immersion. What's more, by commutative diagram, it is clear that $E' = \widetilde{f}^{-1}(E)$. $\square$

Remark 10.1. *There are two special cases*
(1)If $d : X' \longrightarrow X$ is flat, then $Bl_{Y'}(X') \cong Bl_Y(X) \times_X X'$.
(2)If $Y \hookrightarrow X$ and $Z \hookrightarrow X$ are both closed immersions, then with the following commutative diagram

$$\begin{tikzcd}
Y \cap Z \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\
Y \arrow[r, hook] & X
\end{tikzcd}$$

get that

$$\begin{tikzcd}
Bl_{Y \cap Z}(Z) \arrow[rr, hook] \arrow[dr, hook] & & Bl_Y(X) \\
& Bl_Y(X) \times_X Z \arrow[ur, hook] &
\end{tikzcd}$$

where $Bl_{Y \cap Z}(Z)$ is called the strict transform and $Bl_Y(Z) \times_X Z$ is called the total transform.

Theorem 10.1. *Let X be a noetherian scheme, $Y \hookrightarrow X$ closed immersion, $p : \tilde{X} = Bl_Y(X) \longrightarrow X$ natural morphism. For all $f : Z \longrightarrow X$ morphism with Z noetherian such that the pull back W of Z along $Y \hookrightarrow X$ is an effective Cartier divisor, there exists unique morphism $g : Z \longrightarrow \widetilde{X}$ such that the following diagram commutes*

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "p"] \\
& X
\end{tikzcd}$$

Proof. As discuss in Example 10.1 (2), since W is an effective divisor, we have that $Bl_W(Z) \cong Z$. Thus the existence of g follows form Proposition 10.2. For uniqueness, reduce to the affine case that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec}(A/I)$. Then the image of $I \longrightarrow \mathcal{O}_Z(Z)$ generates the ideal sheaf $\mathcal{J}$ corresponding to W. Let $f_1, \cdots, f_n$ be generators of I, mapping to global sections $s_1, \cdots, s_n$ of $\mathcal{J}$ which also generate $\mathcal{J}$. In fact, such $\mathcal{J}$ is invertible. Thus by Proposition 6.10, there exists unique morphism $h : Z \longrightarrow \mathbb{P}^{n-1}_A$ such that $\mathcal{J} \cong h^*\mathcal{O}_{\mathbb{P}^{n-1}_A}(1)$ and $s_i = h^*(x_i)$. Easy to show that h factors through $\widetilde{X}$ so that g is uniquely determined by h. □

Proposition 10.3. *Let X be a scheme of finite type ober field k, $Y \hookrightarrow X$ closed immersion. If X is pure of dimension n, then $C_{Y/X}$ is also pure of dimension n.*

Proof. Consider $Y \times_k \{0\} \hookrightarrow X \times_k \{0\} \hookrightarrow X \times_k \mathbb{A}^1_k$, where $\{0\}$ is the one-point closed subset of $\mathbb{A}^1_k$. Then $C_{Y\times_k\{0\}/X\times_k\mathbb{A}^1_k} \cong C_{Y/X} \oplus \mathbb{1}$, here the definition of $C_{Y/X} \oplus \mathbb{1}$ can be seen in the following remark. Consider $Bl_{Y\times_k\{0\}}(X \times_k \mathbb{A}^1_k) \overset{birational}{\longrightarrow} X \times_k \mathbb{A}^1_k$. Thus $\dim(Bl_{Y\times_k\{0\}}(X \times_k \mathbb{A}^1_k)) = \dim(X \times_k \mathbb{A}^1_k) = \dim(X) + 1$. As $E = \mathbb{P}(C_{Y/X} \oplus \mathbb{1})$ is an effective Cartier divisor which is pure of codimension 1 and $C_{Y/X}$ is an open subset of E, get $\dim(C_{Y/X}) = \dim(E) = \dim(X) + 1 - 1 = \dim(X)$. □

Remark 10.2. *For $\mathcal{G}$ graded $\mathcal{O}_Y$-algebra, we define graded $\mathcal{O}_Y$-algebra $\mathcal{G}[t]$, whose graded structure is given by $\mathcal{G}[t]_d := \mathcal{G}_d \oplus \mathcal{G}_{d-1}t \oplus \cdots \oplus \mathcal{G}_0 t^d$. If $C = \operatorname{Spec}\mathcal{G}$, then define $C \oplus \mathbb{1} :=$* $\operatorname{Spec}(\mathcal{G}[t])$.

Let X be a scheme of finite type over field k, $Y \hookrightarrow X$ closed immersion. Our goal is to construct closed immersion $Y \times_k \mathbb{A}^1_k \hookrightarrow M^o$ such that the following diagram commutes

$$\begin{tikzcd} Y \times_k \mathbb{A}^1_k \arrow[rr, hook] \arrow[dr] & & M^o \arrow[dl, "f"] \\ & \mathbb{A}^1_k & \end{tikzcd}$$

where f is flat, satisfying that over $\mathbb{A}^1_k \setminus \{0\}$, $M^o|_{\mathbb{A}^1_k\setminus\{0\}} \cong X \times_k (\mathbb{A}^1_k \setminus \{0\})$ and over $\{0\}$, $M^o_0 \cong C_{Y/X}$.

Construction. Step 1: Consider $Y \times_k \{0\} \hookrightarrow X \times_k \mathbb{A}^1_k$ with ideal sheaf $\mathcal{J}$. Define $M := Bl_{Y\times_k\{0\}}(X \times_k \mathbb{A}^1_k)$. Want to show that $M \longrightarrow X \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ is flat. Note that flatness is equivalent to torsion-freeness over PID. As $\mathcal{J}$ is torsion-free, get it is flat over $\mathcal{O}_{\mathbb{A}^1_k}$ so that $\oplus_{d\geq 0}\mathcal{J}^d$ is flat over $\mathcal{O}_{\mathbb{A}^1_k}$ and so is M.

On the other hand, $Y \times_k \{0\}$ is an effective Cartier divisor of $Y \times_k \mathbb{A}^1_k$, as Example 10.1 (2), get $Bl_{Y\times_k\{0\}}(Y \times_k \mathbb{A}^1_k) \cong Y \times_k \mathbb{A}^1_k \hookrightarrow M$. Thus $M|_{\mathbb{A}^1_k\setminus\{0\}} \cong X \times_k (\mathbb{A}^1_k \setminus \{0\})$. But what's about the M_0? Note $E = \mathbb{P}(C_{Y/X} \oplus \mathbb{1}) \subseteq M_0$, $\widetilde{X} = Bl_{Y\times_k\{0\}}(X \times_k \{0\}) = Bl_Y(X) \subseteq M_0$ and $\mathbb{P}(C_{Y/X})$ is contatined in both of them. Claim that $M_0 = E \cup \widetilde{X}$, $M_0 \setminus \widetilde{X} = C_{Y/X}$ and $E \cap \widetilde{X} = \mathbb{P}(C_{Y/X})$.

Step 2: With the above claim, take $M^o = M \setminus \widetilde{X}$, then it is clear that $M^o_0 = C_{Y/X}$ so that it is what we want. To prove the claim, it suffices to do some local calculation. Let

$X = \operatorname{Spec} A$, $Y = \operatorname{Spec}(A/I)$, $\mathbb{A}^1_k = \operatorname{Spec}(k[t])$ and $Bl_{Y\times_k\{0\}}(X\times_k \mathbb{A}^1_k) = \operatorname{Proj}(S)$, where $S_d = J^d = (I,t)^d = I^d \oplus I^{d-1}t \oplus \cdots \oplus At^d \oplus At^{d+1} \oplus \cdots$. Then $\operatorname{Proj}(S)$ can be covered by $\operatorname{Spec}(S_{(a)})$, where $a \in S_1 = (I,t)$. In $\operatorname{Spec}(S_{(a)})$, get $E = V(\frac{a}{1})$ and $\widetilde{X} = V(\frac{t}{a})$. As $\frac{t}{1} = \frac{a}{1}\frac{t}{a}$, get $M_0\big|_{Spec(S_{(a)})} = E \cup \widetilde{X}\big|_{Spec(S_{(a)})}$.

On the other hand, since $\operatorname{Spec}(S_{(t)}) = M \setminus \widetilde{X}$ and $S_{(t)} = \cdots \oplus I^2t^{-2} \oplus It^{-1} \oplus A \oplus At \oplus \cdots$, we have that $S_{(t)}/tS_{(t)} = \oplus_{d\geq 0} I^d/T^{d+1}$ so that $M_0 \setminus \widetilde{X} = C_{Y/X}$. And by base change, get $E \cap \widetilde{X} = \mathbb{P}(C_{Y/X})$ which is the exceptional divisor of $\widetilde{X}$. □

11 Serre Duality

Definition 11.1 (*δ-functor*). *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. A (covairant) δ-functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a collection of additive functors $(T^i : \mathcal{C} \to \mathcal{C}')_{i\geq 0}$ together with morphisms $(\delta^i : T^i(C) \to T^{i+1}(A))_{i\geq 0}$ for all exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}$ satisfying that*

(1)There is a long exact sequence

$$\cdots \longrightarrow T^i(A) \longrightarrow T^i(B) \longrightarrow T^i(C) \xrightarrow{\delta^i} T^{i+1}(A) \longrightarrow \cdots$$

(2)(functoriality) For all commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

we have commutative diagram

$$\begin{array}{ccc} T^C & \xrightarrow{\delta^i} & T^{i+1}(A) \\ \downarrow & & \downarrow \\ T^i(C') & \xrightarrow{\delta^i} & T^{i+1}(A') \end{array}$$

The δ-functor $T = ((T^i)_{i\geq 0}, (\delta^i)_{i\geq 0})$ is called universal if for all δ-functor $T' : \mathcal{C} \longrightarrow \mathcal{C}'$ and natural transformation $f^0 : T^0 \longrightarrow T'^0$, there exists unique natural transformation $T \longrightarrow T'$ extending f^0 i.e. $\{f^i : T^i \to T'^i\}_{i\geq 0}$ commuting with δ.

Remark 11.1. *Given $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ (covariant) left exact functor, then there exists at most one universal δ-functor T with $T^0 = \mathcal{F}$ up to unique morphism.*

Theorem 11.1. *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories, $T : \mathcal{C} \longrightarrow \mathcal{C}'$ a δ-functor. If for all $A \in \mathcal{C}$, there exists monomorphism $A \hookrightarrow J$ such that $T^i(J) = 0$ for all $i > 0$, then T is universal.*

Example 11.1. *If $\mathcal{C}$ has enough injective objects, then for all $\mathcal{F}$ left exact, $(R^i\mathcal{F})_{i\geq 0}$ is the wanted universal δ-functor with $R^0\mathcal{F} \cong \mathcal{F}$. Conversely, if $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a universal δ-functor, then T^0 is left exact and $T^i \cong R^iT^0$ for all $i \geq 0$.*

Recall that for ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$-module $\mathcal{F}$, we have that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ are covariant left exact.

Definition 11.2 (***Extension Functor***)**.** *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module. Define* $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \cdot) := R^i \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ *and* $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \cdot) := R^i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$.

Remark 11.2. *If $\mathcal{F}, \mathcal{H}$ are two $\mathcal{O}_X$-modules, then $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{H}, \mathcal{F})$ is one-to-one corresponding to $\{$extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0\}/\sim$.*

Lemma 11.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}\big|_U$ is injective $\mathcal{O}_U$-module for all open subset $U \subseteq X$.*

Proof. For injection $\mathcal{F} \hookrightarrow \mathcal{G}$ in $Mod(\mathcal{O}_X)$ and morphism $\mathcal{F} \longrightarrow \mathcal{I}\big|_U$, consider $j_!\mathcal{F} \hookrightarrow j_!\mathcal{G}$, where $j : U \hookrightarrow X$ is the inclusion and $j_!\mathcal{F}, j_!\mathcal{G}$ extend $\mathcal{F}, \mathcal{G}$ by 0 outside U respectively. Then there is a morphism $j_!\mathcal{G} \longrightarrow \mathcal{I}$ such that the following diagram commutes

$$\begin{tikzcd} j_!\mathcal{F} \arrow[r, hook] \arrow[d] & j_!\mathcal{G} \arrow[d] \\ j_!(\mathcal{I}\big|_U) \arrow[r, hook] & \mathcal{I} \end{tikzcd}$$

take restriction of $j_!\mathcal{G} \longrightarrow \mathcal{I}$ to U, done! □

Proposition 11.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. Then for all $U \subseteq X$ open subset, we have that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big|_U \cong \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}\big|_U, \mathcal{G}\big|_U)$.*

Proof. Both sides $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \cdot)\big|_U$ and $\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}\big|_U, \cdot\big|_U)$ are δ-functor. They agree for $i = 0$ and both vanish for $i > 0$ and $\mathcal{G}$ injective. Apply Theorem 11.1, they both are universal, thus they are identical up to isomorphism. □

Remark 11.3. *In fact, we have that $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \cong \mathcal{G}$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = 0$ for $i > 0$ since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \cdot) \cong id$. In addition, $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O})$ In addition, $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$ for all $i \geq 0$ since $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \cdot) \cong \Gamma(X, \cdot)$.*

Proposition 11.2. *Let $(X, \mathcal{O}_X)$ be a ringed space, $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$ exact sequence in $Mod(\mathcal{O}_X)$. Then for all $\mathcal{O}_X$-module $\mathcal{G}$, there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \longrightarrow \cdots$$

similar result holds for $\mathcal{E}xt$ too.

Proof. Take injective resolution $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}^*$. Since $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{I}^i)$ is exact, by Lemma 11.1, get $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \mathcal{I}^i)$ also exact. Thus there is an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^*) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^*) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^*) \longrightarrow 0$$

take associated long exact sequence, done! Same argument for $\mathcal{H}om$ gives the long exact sequence for $\mathcal{E}xt$. □

Proposition 11.3. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. It there exists exact sequence*

$$\cdots \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\mathcal{L}_i$ locally free of finite rank, which is called a locally free resolution $\mathcal{L}_* \longrightarrow \mathcal{F} \longrightarrow 0$, then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong h^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_*, \mathcal{G}))$.

Proof. Both sides $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\cdot)$ and $h^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_*,\cdot))$ are δ-functors. They agree for $i=0$ and both vanish for $i>0$ and $\mathcal{G}$ injective. Apply Theorem 11.1, done! $\square$

Corollary 11.1. *Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{L}$ loally free of finite rank, $\mathcal{G}$ an $\mathcal{O}_X$-module. Then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L},\mathcal{G})=0$ for $i>0$.*

Remark 11.4. *(1)For X quasi-projective over* $\operatorname{Spec} A$ *with A noetherian, any coherent $\mathcal{F}$ is quotient of locally free of finite rank. Thus to compute $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$, where $\mathcal{F},\mathcal{G}$ are coherent, can stay within $Coh(X)$.*
(2)Proposition 11.3 does not mean $\mathcal{E}xt^i_{\mathcal{O}_X}(\cdot,\mathcal{G})$ is a derived functor in a naive sense. In fact, $Mod(\mathcal{O}_X)$ and $Qcoh(X)$ rarely have enough projective objects.
(3)About $\mathcal{F}\otimes_{\mathcal{O}_X}\cdot$, for $\mathcal{O}_X$-module $\mathcal{G}$, we can take a flat resolution $\mathcal{K}_*\longrightarrow\mathcal{G}\longrightarrow 0$ and define $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}):=h_i(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{K}_*)$. Need to check this definition is well defined.

Lemma 11.2. *Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{L}$ locally free sheaf of finite rank, $\mathcal{L}^\vee=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)$. Then for all $\mathcal{I}$ injective, $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{I}$ is injective.*

Proof. Note that $\operatorname{Hom}_{\mathcal{O}_X}(\cdot,\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{I})\cong\operatorname{Hom}_{\mathcal{O}_X}(\cdot\otimes_{\mathcal{O}_X}\mathcal{L}^\vee,\mathcal{I})$, since $\cdot\otimes_{\mathcal{O}_X}\mathcal{L}^\vee$ is exact, get $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{I}$ is injective. $\square$

Proposition 11.4. *Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{L}$ locally free sheaf of finite rank, $\mathcal{L}^\vee=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)$. Then for all $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have that $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L},\mathcal{G})\cong\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{L}^\vee\otimes_{\mathcal{O}_X}\mathcal{G})$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L},\mathcal{G})\cong\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\otimes_{\mathcal{O}_X}\mathcal{L}^\vee$.*

Proof. Both sides as functors in $\mathcal{G}$ are δ-functors. They agree for $i=0$ and both vanish for $i>0$ and $\mathcal{G}$ injective. Apply Theorem 11.1, done! $\square$

Proposition 11.5. *Let X be a noetherian scheme, $x\in X$, $\mathcal{F}$ coherent sheaf on X, $\mathcal{G}$ an $\mathcal{O}_X$-module. Then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x\cong\operatorname{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$.*

Proof. Can assume X affine. Take a free resolution $\mathcal{L}_*\longrightarrow\mathcal{F}\longrightarrow 0$ exact. Take stalks at x, get $(\mathcal{L}_x)_*\longrightarrow\mathcal{F}_x\longrightarrow 0$ exact. Use the two resolutions to compute both sides, as $\mathcal{L}$ is free and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{G})_x\cong\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x,\mathcal{G}_x)$, done! $\square$

Remark 11.5. *When $i=0$, it shows that when $\mathcal{F}$ is coherent, hom functor commutes with directed limit.*

Proposition 11.6. *Let X be a noetherian scheme, $\mathcal{F},\mathcal{G}$ quasi-coherent sheaves on X. Then we have that*
(1)if $\mathcal{F}$ is coherent, then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is quasi-coherent for all $i\geq 0$
(2)if both $\mathcal{F},\mathcal{G}$ are coherent, then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is coherent for all $i\geq 0$.

Reason 11.1. *Can assume X is affine. Take a free resolution $\mathcal{L}_*\longrightarrow\mathcal{F}\longrightarrow 0$ and compute.*

Proposition 11.7. *Let X be a projective scheme over* $\operatorname{Spec} A$*, A noetherian, $\mathcal{F},\mathcal{G}$ coherent sheaves on X. Then there exists $n_0>0$ relative to $\mathcal{F}$, $\mathcal{G}$ and i such that $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}(n))=\Gamma(X,\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}(n)))$ for all $n\geq n_0$.*

Proof. If $i = 0$, by definition, we get it for all $\mathcal{F}$, $\mathcal{G}$ and n. If $i > 0$ and $\mathcal{F} = \mathcal{O}_X$, then the left side is isomorphic to $H^i(X, \mathcal{G}(n))$ while the right side is 0. By Serre Theorem, $H^i(X, \mathcal{G}(n))$ vanish for large enough n so that result is ok for $\mathcal{F} = \mathcal{O}_X$. If $i > 0$ and $\mathcal{F}$ is locally free (of finite rank), can reduce to $\mathcal{F} = \mathcal{O}_X$.

For arbitrary $\mathcal{F}$, write it as a quotient of locally free sheaf $\mathcal{E}$ of finite rank. Set $\mathcal{R} = \ker(\mathcal{E} \to \mathcal{F})$, then we have the following exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longleftarrow \mathcal{F} \longrightarrow 0$$

By Proposition 11.2, get

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}(n)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{R}, \mathcal{G}(n)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) \longrightarrow 0 \longrightarrow \cdots$$

Similar sequence for $\mathcal{E}xt$ functor also holds. By Serre Theorem, as $\mathcal{E}$ is locally free of finite rank, for large enough n, $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{R}, \mathcal{G}(n)) \cong \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{R}, \mathcal{G}(n)) \cong \mathcal{E}xt^{i+1}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))$ for all $i > 0$. In addition, by Proposition 11.3, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}(n)) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{R}, \mathcal{G}(n)) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & & & \\
\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}(n))) & \longrightarrow & \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{G}(n))) & \longrightarrow & \Gamma(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

By 5 Lemma, get $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) \longrightarrow \Gamma(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)))$ is isomorphic. Note $\mathcal{R}$ is also coherent, by induction, done! ☐

11.1 Serre Duality for Projective Space

Theorem 11.2. *Let k be a field and $X = \mathbb{P}^n_k$, $\omega_{X/k} = \wedge^n \Omega_{X/k}$ canonical sheaf. Then we have that*
(1)$H^n(X, \omega_{X/k}) \cong k$.
(2)For all coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_{X/k}) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_{X/k})$$

is a perfect pairing of finite-dimensional vector spaces over k.
(3)For all coherent sheaf $\mathcal{F}$ on X and $i \geq 0$, there exists isomorphism $\mathrm{Ext}^i(\mathcal{F}, \omega_{X/k}) \cong H^{n-i}(X, \mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F})^\vee$ functorial in $\mathcal{F}$.

Proof. For (1), by Euler sequence, we have that $\omega_{X/k} \cong \mathcal{O}_X(-n-1)$ so that $H^n(X, \omega_{X/k}) \cong H^n(X, \mathcal{O}_X(-n-1)) \cong k$.

For (2), first consider the case when $\mathcal{F} = \mathcal{O}_X(q)$. Then we get $\mathrm{Hom}(\mathcal{O}_X(q), \omega_{X/k}) \cong H^0(X, \mathcal{O}_X(-n-1-q))$. By Theorem 7.6, the following pairing is a perfect pairing

$$H^0(X, \mathcal{O}_X(-n-1-q)) \times H^n(X, \mathcal{O}_X(q)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)).$$

If $\mathcal{F}$ is a direct sum of $\mathcal{O}_X(q)$, as cohomology commutes with direst sum, it is also ok. For arbitrary $\mathcal{F}$, by Corollary 6.1, there is an exact sequence

$$\mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_0$ are direct sum of $\mathcal{O}_X(q)$. Then by Grothendieck Vanishing Theorem, there is a commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\mathcal{F},\omega_{X/k}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{E}_0,\omega_{X/k}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(\mathcal{E}_1,\omega_{X/k}) \arrow[d, "\sim"] \\ 0 \arrow[r] & H^n(X,\mathcal{F})^\vee \arrow[r] & H^n(X,\mathcal{E}_0)^\vee \arrow[r] & H^n(X,\mathcal{E}_1)^\vee \end{tikzcd}$$

By 5 Lemma, $\mathrm{Hom}(\mathcal{F},\omega_{X/k}) \cong H^n(X,\mathcal{F})^\vee$.

For (3), both sides $\mathrm{Ext}^i(\cdot,\omega_{X/k})$ and $H^{n-i}(X,\cdot)^\vee$ are contravariant δ-functors from $Coh(X)$ to $Vect(k)$. They agree for $i = 0$ by (2). Remains to show that for all coherent sheaf $\mathcal{F}$ on X, there exists exact sequence $\mathcal{E} = \oplus\mathcal{O}_X(q) \longrightarrow \mathcal{F} \longrightarrow 0$ for some q. Can choose q much less that 0 such that $\mathrm{Ext}^i(\mathcal{E},\omega_{X/k}) = 0$ and $H^{n-i}(X,\mathcal{E})^\vee = 0$ for all $i > 0$. Then apply Theorem 11.1, done. □

11.2 Serre Duality in General

Want to know what is the Serre duality for general X projective scheme over k of dimension n.

Remark 11.6. *Version (1): there exists coherent sheaf ω_X^o on X together with linear map $t : H^n(X,\omega_X^o) \longrightarrow k$ called the trace homomorphism such that for all coherent sheaf $\mathcal{F}$ on X, the natural pairing*

$$\mathrm{Hom}(\mathcal{F},\omega_X^o) \times H^n(X,\mathcal{F}) \longrightarrow H^n(X,\omega_X^o) \xrightarrow{t} k$$

is a perfect pairing.

Version (1′): there exists coherent sheaf ω_X^o on X such that for all coherent sheaf $\mathcal{F}$ on X, there exists isomorphism $\mathrm{Hom}(\mathcal{F},\omega_X^o) \xrightarrow{\sim} H^n(X,\mathcal{F})^\vee$ functorial in $\mathcal{F}$. In fact, for given $\mathcal{F} \longrightarrow \omega_X^o$, with the following commutative diagram , it is clear that version (1) is equivalent to version (1′).

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{F},\omega_X^o) \arrow[r, "\sim"] & H^n(X,\mathcal{F})^\vee \\ \mathrm{Hom}(\omega_X^o,\omega_X^o) \arrow[r, "\sim"] \arrow[u] & H^n(X,\omega_X^o)^\vee \arrow[u] \end{tikzcd}$$

Version (2): ω_X^o in version (1) also satisfies that for all coherent sheaf $\mathcal{F}$ on X and $i \geq 0$, there exists isomorphism $\mathrm{Ext}^i(\mathcal{F},\omega_X^o) \xrightarrow{\sim} H^{n-1}(X,\mathcal{F})^\vee \cong \mathrm{Ext}^{n-i}(\mathcal{O}_X,\mathcal{F})^\vee$ functorial in $\mathcal{F}$.

Version (2-): ω_X^o in version (1) also satisfies that for all locally free sheaf $\mathcal{E}$ of finite rank on X and $i \geq 0$, there exists isomorphism $H^i(X,\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X^o) \cong H^{n-i}(X,\mathcal{E})^\vee$ functorial in $\mathcal{E}$. This is what used in the proof of Riemann-ROch Theorem for curves.

Version (2+): ω_X^o in version (1) also satisfies that for all coherent sheaf $\mathcal{F}$, locally free sheaf $\mathcal{E}$ of finite rank on X and $i \geq 0$, there exists isomorphism

$$\mathrm{Ext}^i(\mathcal{F} \otimes_{\mathcal{O}_X} \omega_X^o) \cong \mathrm{Ext}^i(\mathcal{F},\mathcal{E} \otimes_{\mathcal{O}_X} \omega_X^o) \cong \mathrm{Ext}^i(\mathcal{E},\mathcal{F})^\vee \cong H^{n-i}(X,\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F})$$

functorial in both $\mathcal{F}$ and $\mathcal{E}$.

Version (2'): ω_X^o in version (1) also satisfies that for all coherent sheaf $\mathcal{F}$ on X, there exists a natural pairing

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^o) \times H^{n-i}(X, \mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}) \longrightarrow H^n(X.\omega_X^o) \cong \mathrm{Ext}^n(\mathcal{O}_X, \omega_X^o) \xrightarrow{t} k$$

called the Yoneda pairing, which is a perfect pairing for X under assumptions.

Version (3): duality generalized to the derived category.

Finally, if X is smooth and projective over k of pure dimension, want $\omega_X^o \cong \omega_X$ is the canonical sheaf of X.

Example 11.2. *Version (2) needs more assumption on X, here is a counterexample. Let X be two projective planes meeting at one point x as a subscheme of $\mathbb{P}_k^4$. Suppose that there exists ω_X^o such that $H^1(X, \omega_X^o(q)) \cong H^1(X, \mathcal{O}_X(-q))^\vee$. By Serre Vanishing Theorem, $H^1(X, \mathcal{O}_X(-q))$ for large enough q.*

But if let $H \subseteq \mathbb{P}_k^4$ be a hyperplane not containing x, then $H \cap X$ is union of two disjoint lines. Set $Z := qH \cap X$ where q is large enough. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Z) = \mathcal{O}_X(-q) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

Then consider long exact sequence of cohomology groups, get

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{O}_X(-q)) = 0$$

While $H^0(X, \mathcal{O}_X) \cong k$ is one-dimensional and $H^0(Z, \mathcal{O}_Z)$ is two-dimensional, it is impossible that there is a surjection from $H^0(X, \mathcal{O}_X)$ to $H^0(Z, \mathcal{O}_Z)$, contradiction!

Let X be a projective scheme over field k, $\dim(X) = n$.

Definition 11.3 (***Dualizing Sheaf***)**.** *A dualizing sheaf on X is a coherent sheaf ω_X^o on X together with a trace homomorphism $t : H^n(X, \omega_X^o) \longrightarrow k$ such that for all coherent sheaf $\mathcal{F}$ on X, the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \omega_X^o) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X^o) \xrightarrow{t} k$$

is a perfect pairing.

Proposition 11.8. *If a dualizing sheaf exists, then it is unique i.e. if ω_X' together with trace homomorphism t' is another one, then there exists unique isomorphism $\phi : \omega_X^o \xrightarrow{\sim} \omega_X'$ such that $t = t' \circ H^n(X, \phi)$.*

Reason 11.2. *Since ω_X' is dualizing sheaf, we have that $\mathrm{Hom}(\omega_X^o, \omega_X') \cong H^n(X, \omega_X')^\vee$, take ϕ to be the preimage of t'. By functoriality, $t' \circ H^n(X, \phi) = t$.*

Remark 11.7. *In fact, we can say that ω_X^o together with t represents the functor $\mathcal{F} \in Coh(X) \longmapsto H^n(X, \mathcal{F})^\vee \in Vect(k)$.*

Lemma 11.3. *Let $X \subseteq \mathbb{P}_k^N$ be a closed subscheme of codimension r. Then $\mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}_k^N}) = 0$ for all $i < r$.*

Proof. Set $\mathcal{F}^i := \mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$, which is a coherent sheaf on $\mathbb{P}^N_k$. After twisting by $\mathcal{O}_{\mathbb{P}^N_k}(q)$ for large enough q, by Theorem 6.3, we get $\mathcal{F}^i(q)$ is generated by global sections. Suffices to prove $\Gamma(\mathbb{P}^N_k, \mathcal{F}^i(q)) = 0$ for all $i < r$ and large enough q. By Proposition 11.7, $\Gamma(\mathbb{P}^N_k, \mathcal{F}^i(q)) \cong \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k}(q))$ for large enough q.

By Serre duality for $\mathbb{P}^N_k$ and Grothendieck Vanishing Theorem, get

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k}(q)) \cong H^{N-i}(\mathbb{P}^N_k, \mathcal{O}_X(-q))^\vee \cong H^{N-i}(X, \mathcal{O}_X(-q))^\vee = 0$$

for $i < r$ since $N - i > \dim(X)$. □

Lemma 11.4. *Let $\mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ be a complex of injective objects in an abelian category $\mathcal{C}$ such that $h^i(\mathcal{I}^\bullet) = 0$ for $0 \le i < r$. Then can write $\mathcal{I}^\bullet \cong \mathcal{I}_1^\bullet \oplus \mathcal{I}_2^\bullet$ such that $\mathcal{I}_1^\bullet$ is in degrees $0 \le i \le r$ and exact and $\mathcal{I}_2^\bullet$ is in degree $i \ge r$.*

Lemma 11.5. *Let $X \subseteq \mathbb{P}^N_k$ be a closed subscheme of codimension r. Set wanted dualizing sheaf $\omega_X^o := \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$ viewed as $\mathcal{O}_X$-module Then for all $\mathcal{O}_X$-module $\mathcal{F}$, there exists isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^o) \cong \mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$ functorial in $\mathcal{F}$.*

Proof. Let $0 \to \omega_{\mathbb{P}^N_k} \to \mathcal{I}^\bullet$ be an injective resolution of $\omega_{\mathbb{P}^N_k}$. Then $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) = h^i(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \mathcal{I}^\bullet))$. As $\mathcal{F}$ is an $\mathcal{O}_X$-module, $\mathcal{F} \to \mathcal{I}^i$ factors through $\mathcal{J}^i := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \mathcal{I}^i)$ so that $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \mathcal{I}^i) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^i)$. Thus $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) \cong h^i(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet))$. Moreover, each $\mathcal{J}^i$ is an injective $\mathcal{O}_X$-module since $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{J}^i) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N_k}}(\cdot, \mathcal{I}^i)$ as functors from $Mod(\mathcal{O}_X)$ to Ab. By Lemma 11.3, $h^i(\mathcal{J}^*) = \begin{cases} 0 & \forall i < r \\ \omega_X^o & i = r \end{cases}$. By Lemma 11.4, can write $\mathcal{J}^* \cong \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet$ such that $\mathcal{J}_1^\bullet$ is in degrees $0 \le i \le 1$ and exact and $\mathcal{J}_2^\bullet$ is in degrees $i \ge r$. Then $\omega_X^o = h^r(\mathcal{J}^\bullet) \cong \ker(d_2^r : \mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$ and for all $\mathcal{O}_X$-module $\mathcal{F}$, we have that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^o) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \ker(d_2^r)) \cong h^r(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)) \cong \mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$$

done! □

Remark 11.8. *In fact, same argument as Lemma 11.5 shows that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) = 0$ and $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) = 0$ for all $i < r$. Also, we have that*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^o) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \ker(d_2^r)) \cong h^r(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)) \cong \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$$

which gives that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^o) \cong \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$ functorial in $\mathcal{F}$.

Proposition 11.9. *Let X be a projective scheme over field k of dimension n. Then X has a dualizing sheaf.*

Proof. Since $X \subseteq \mathbb{P}^N_k$ of codimension r for some N, take ω_X^o in the Lemma 11.4. Then for all $\mathcal{O}_X$-module $\mathcal{F}$, there exists isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^o) \cong \mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k})$. As $H^n(X, \mathcal{F})^\vee \cong H^{N-r}(\mathbb{P}^N_k, \mathcal{F})^\vee$, suffice to show that $\mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{F}, \omega_{\mathbb{P}^N_k}) \cong H^{N-r}(\mathbb{P}^N_k, \mathcal{F})^\vee$, which is just Serre duality. Thus there exists isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_X^o) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$ functorial in $\mathcal{F} \in Coh(X)$. This is version $(1')$ of generalization of Serre duality. □

11.3 Some Algebra

Definition 11.4 (***Regular Sequence***)**.** *Let A be a ring, $M \neq 0 \in Mod_A$. A sequence $\{a_1, \cdots, a_r\} \subseteq A$ is called a regular sequence for M if a_1 is not a zero-divisor in M, a_2 is not a zero divisor in M/a_1M and so on with $M/(a_1, \cdots, a_r)M \neq 0$.*

Definition 11.5 (***Depth of Module***)**.** *Let A be a local ring, $\mathfrak{m} \subseteq A$ maximal ideal, $M \in Mod_A$. Define the depth of M is the maximal length of regular sequence for M, denoted by* $\mathrm{depth}(M)$.

Definition 11.6 (***Cohen-Macaulay Ring***)**.** *Let A be a noetherian local ring. A is called Cohen-Macaulay if $\mathrm{depth}(A) = \dim(A)$ as A-module.*

Proposition 11.10. *Let A be a noetherian local ring, $\mathfrak{m} \subseteq A$ maximal ideal. Then we have that*
(1)if A is regular, then A is Cohen-Macaulay
(2)if A is Cohen-Macaulay, then $A_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$.
(3)if A is Cohen-Macaulay, then $\{a_1, \cdots, a_r\}$ form a regular sequence for A if and only if $\dim(A/(a_1, \cdots, a_r)) = \dim(A) - r$. *In particular, regular sequence for A is independent on order and $A/(a_1, \cdots, a_r)$ is also Cohen-Macaulay.*
(4)if $\{a_1, \cdots, a_r\}$ form a regular sequence for A, then A is Cohen-Macaulay if and only if $A/(a_1, \cdots, a_r)$ is Cohen-Macaulay.

Proposition 11.11. *Let A be a ring, $I = (a_1, \cdots, a_r) \subseteq A$ ideal. Then we have that*
(1)if $\{a_1, \cdots, a_r\}$ form a regular sequence for A, then the natural surjective homomorphism of graded rings

$$A/I[x_1, \cdots, x_r] \longrightarrow \oplus_{d \geq 0} I^d/I^{d+1}$$
$$x_i \longmapsto a_i$$

is an isomorphism. In particular, I/I^2 is free A/I-module of rank r.
(2)if A is noetherian local ring such that A/I is regular, then $\{a_1, \cdots, a_r\}$ form a regular sequence for A if and only if A is regular.

Definition 11.7 (***Projective Dimension***)**.** *Let A be a ring, $M \in Mod_A$. Define the projective dimension of M to be the minimal length of projective resolution of M, denoted by* $\dim_{\mathrm{proj}}(M)$.

Proposition 11.12. *Let A be a ring, $M \in Mod_A$. Then we have*
(1)M is projective $\iff$ $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and $N \in Mod_A$ $\iff$ $\mathrm{Ext}^1(M, N) = 0$ for all $N \in Mod_A$.
(2)$\dim_{\mathrm{proj}}(M) \leq n$ *if and only if* $\mathrm{Ext}^i(M, N) = 0$ *for all $i > n$ and $N \in Mod_A$.*
(3)if A is noetherian local ring and M is finitely generated with $\dim_{\mathrm{proj}}(M) < \infty$, *then* $\dim_{\mathrm{proj}}(M) + \mathrm{depth}(M) = \mathrm{depth}(A)$.
(3)if A is noetherian local ring and M is finitely generated with $\dim_{\mathrm{proj}}(M) < \infty$, *then* $\dim_{\mathrm{proj}}(M) \geq n$ *if and only if* $\mathrm{Ext}^i(M, A) = 0$ *for all $i > n$.*

11.4 Serre Duality in General (cont.)

Definition 11.8 (***Cohen-Macaulay Schemes***)**.** *Let X be a locally noetherian scheme. X is called Cohen-Macaulay if all its stalks are Cohen-Macaulay.*

Theorem 11.3 (***Serre Duality for Projective Schemes***)**.** *Let X be a projective scheme over k, $\dim(X) = n$, ω_X^o dualizing sheaf, $\mathcal{O}_X(1)$ very ample invertible sheaf on X. Then we have that*
(1)for all coherent sheaf $\mathcal{F}$ on X and $i > 0$, there exists natural homomorphism

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^o) \longrightarrow H^{n-i}(X, \mathcal{F})^\vee$$

functorial in $\mathcal{F}$ such that θ^0 is given by the definition of ω_X^o.
(2)the following conditions ar equivalent:
(i)X is pure and Cohen-Macaulay
(ii)for all locally sheaf $\mathcal{F}$ of finite rank on X, $i < n$ and q large enough, $H^i(X, \mathcal{F}(-q)) = 0$.
(iii)θ^i are isomorphic for all $i \geq 0$.

Proof. For (1), since X is projective, by Corollary 6.1, there is an exact sequence

$$\oplus \mathcal{O}_X(-q) = \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow o$$

for q large enough. Then by Serre Theorem, $\mathrm{Ext}^i(\mathcal{E}, \omega_X^o) \cong H^i(X, \omega_X^o(q)) = 0$ for all $i > 0$ and q large enough. By Theorem 11.1, $\mathrm{Ext}^i(\cdot, \omega_x^o)$ is universal contravariant δ-functor. By universal property, there exists unique morphism of δ-functors θ^i extending θ^0.

For (2), first prove (i) induce (ii). Assume that $X \subseteq \mathbb{P}_k^N$ for some N of codimension r. Then for all locally free sheaf $\mathcal{F}$ of finite rank on X and $x \in X$ closed point, we have that $\mathrm{depth}(\mathcal{F}_x) = \mathrm{depth}(\mathcal{O}_{X,x}) = n = \dim(\mathcal{O}_{X,x})$ since X is pure of dimension n and Cohen-Macaulay. Moreover, as $\mathcal{O}_{\mathbb{P}_k^N,x}$ is a regular local ring of dimension N, get
$depth(\mathcal{F}_x) = n$ also as $\mathcal{O}_{\mathbb{P}_k^N,x}$-module. By Proposition 11.12, $\dim_{\mathrm{proj}}(\mathcal{F}_x) = N - n = r$ for all closed point $x \in X$. Then also by Proposition 11.12, $(\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{F}, \mathcal{G}))_x \cong \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{F}_x, \mathcal{G}_x) = 0$ so that $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{F}, \mathcal{G}) = 0$ for all $i > N - n$. On the other hand, for q large enough, $H^i(X, \mathcal{F}(-q))^\vee \cong \mathrm{Ext}^{N-i}_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{F}, \omega_{\mathbb{P}_k^N}(q))$. Thus $H^i(X, \mathcal{F}(-q))^\vee = 0$ for $i < n$ and q large enough.

For $(ii) \Rightarrow (i)$, argue backwards with $\mathcal{F} = \mathcal{O}_X$ and get $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{O}_X, \omega_{\mathbb{P}_k^N}) = 0$ for all $i > N - n \geq r$. Then for all closed point $x \in X$, $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}_k^N,x}) = 0$ for all $i > N - n$. By Proposition 11.12, $\dim_{\mathrm{proj}}(\mathcal{O}_{X,x}) \leq N - n$ so that $\mathrm{depth}(\mathcal{O}_{X,x}) \geq n = \dim(X)$. While $\mathrm{depth}(\mathcal{O}_{X,x}) \leq n$, get $\mathcal{O}_{X,x}$ Cohen-Macaulay of dimension n for all closed point $x \in X$. By Proposition 11.10, $\mathcal{O}_{X,y}$ is Cohen-Macaulay for all $y \in X$ since $\mathcal{O}_{X,y}$ is localization of $\mathcal{O}_{X,x}$ for some closed poitn x.

For $(ii) \Rightarrow (iii)$, suffice to show $H^{n-i}(X, \cdot)^\vee$ is universal δ-functor. For all $\mathcal{F}$ coherent sheaf on X, we have exact sequence

$$\oplus \mathcal{O}_X(-q) = \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

for q large enough. By (ii), get $H^{n-i}(X,\mathcal{E})^\vee = \oplus H^{n-i}(X,\mathcal{O}_X(-q))^\vee = 0$ for all $i>0$ and q large enough. Then apply Theorem 11.1.

For $(iii) \Rightarrow (ii)$, for all locally free sheaf $\mathcal{F}$ of finite rank on X, by (iii) and Serre Theorem, get

$$H^i(X,\mathcal{F}(-q))^\vee \xrightarrow{\theta^{n-i}} \mathrm{Ext}^{n-i}(\mathcal{F}(-q),\omega_X^o) \cong H^{n-i}(X,\mathcal{F}^\vee\otimes\omega_X^o(q)) = 0$$

for all $i>0$ and large enough. □

Remark 11.9. *Let $X\subseteq \mathbb{P}_k^N$ be closed subscheme of codimension r as before. Then X is pure and Cohen-Macaulay if and only if* $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}_k^N}}(\mathcal{O}_X,\omega_{\mathbb{P}_k^N}) = \begin{cases} 0 & i\neq r \\ \omega_X^o & i=r \end{cases}$

Definition 11.9. *A closed immersion $X\hookrightarrow Y$ of locally noetherian schemes with ideal sheaf $\mathcal{I}$ is called a regular immersion (of codimension r) if for all $x\in X$, $\mathcal{I}_x\subseteq\mathcal{O}_{Y,x}$ is generated by a regular sequence (of r elements).*

Remark 11.10. *In the old language, regular immersion into smooth k-scheme is called "local complete intersection".*

Proposition 11.13. *All schemes discussed are locally noetherian.*
(1)Let $X\hookrightarrow Y$ be a regular immersion of codimension r with ideal sheaf $\mathcal{I}$. Then by Proposition 11.11, $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r and $C_{X/Y}\cong N_{X/Y}$.
(2)(composition) Let $f:X\hookrightarrow Y$ and $g:Y\hookrightarrow Z$ be regular immersions. Then by Proposition 11.10, $g\circ f: X\hookrightarrow Z$ is still a regular immersion, in which case, the following sequence is exact

$$0\longrightarrow N_{X/Y}\longrightarrow N_{X/Z}\longrightarrow f^*N_{Y/Z}\longrightarrow 0$$

(3)(flat base change) Let $f:X\hookrightarrow Y$ be a regular immersion and $g:Y'\longrightarrow Y$ be a flat morphism. Then $f':X'=X\times_Y Y'\hookrightarrow Y'$ is a regular immersion with $\mathcal{N}_{X'/Y'} = g'^*\mathcal{N}_{X/Y}$.
(4)Let $f:X\hookrightarrow Y$ be a regular immersion of S-schemes. Assume that X is smooth over S. Then by Proposition 11.11, f is regular immersion if and only if Y is smooth over S in a neighbourhood of X, in which case, the following sequence is exact

$$0\longrightarrow T_{X/S}\longrightarrow f^*T_{Y/S}\longrightarrow N_{X/Y}\longrightarrow 0$$

Example 11.3. *(1)Regular immersion of codimension 1 is equivalent to effective Cartier divisor.*
(2)Let $f:X\longrightarrow S$ be a (separated) flat morphism of finite type between locally noetherian schemes with fibers pure of dimension n. Then f is smooth if and only if $\Delta_{X/S}:X\longrightarrow X\times_S X$ is regular immersion of codimension n. "$\Rightarrow$" is by Proposition 11.11. For "$\Leftarrow$", assume that $\mathcal{I}$ is the ideal sheaf corresponds to $\Delta_{X/S}$. Note that $\mathcal{I}/\mathcal{I}^2\cong\Omega_{X/S}$. Then Proposition 9.6 tells us that X is smooth over S.
(3)Let $X\hookrightarrow Y$ be regular immersion of codimension r. Then by Proposition 11.10, X is pure and Cohen-Macaulay if Y is pure and Cohen-Macaulay.

Theorem 11.4. *Let $X \hookrightarrow \mathbb{P}^N_k$ be a regular immersion of codimension r with ideal sheaf $\mathcal{I}$. Since $\mathbb{P}^N_k$ is pure and Cohen-Macaulay, then X is pure and Cohen-Macaulay. Then $\omega_X^o \cong \omega_{\mathbb{P}^N_k}\big|_X \otimes_{\mathcal{O}_X} \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$. In particular, ω_X^o is an invertible sheaf.*

Corollary 11.2. *Let $X \subseteq \mathbb{P}^N_k$ be a smooth projective scheme over k pure of dimension n. Then $\omega_X^o \cong \omega_X$.*

Proof. Let $X \subseteq \mathbb{P}^N_k$. Can assume that X is pure of codimension r with ideal sheaf $\mathcal{I}$. As both X and $\mathbb{P}^N_k$ are smooth over k, by Proposition 11.11, $X \hookrightarrow \mathbb{P}^N_k$ is a regular immersion of codimension r. Get exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathbb{P}^N_k}\big|_X \longrightarrow \Omega_X \longrightarrow 0$$

Take determinant, get $\omega_{\mathbb{P}^N_k}\big|_X = \omega_X \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)$. Then by previous theorem, $\omega_X \cong \omega_{\mathbb{P}^N_k}\big|_X \otimes_{\mathcal{O}_X} \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee \cong \omega_X^o$. □

Corollary 11.3. *Take determinant, get $\omega_{\mathbb{P}^N_k}\big|_X = \omega_X \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)$. Let $X \subseteq \mathbb{P}^N_k$ be a smooth projective scheme over k pure of dimension n. Then $H^q(X, \Omega^p_X) \cong H^{n-q}(X, \Omega^{n-p}_X)^\vee$.*

Reason 11.3. *This comes from $\Omega^{n-p}_X \cong (\Omega^p_X)^\vee \otimes_{\mathcal{O}_X} \omega_x$ and $\Omega^p_X \cong (\Omega^{n-p}_X)^\vee \otimes_{\mathcal{O}_X} \omega_X$.*

Remark 11.11. *If further X is geometrically integral, then $H^n(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong k$.*

Definition 11.10 (***Koszul Complex***)**.** *Let A be a ring, $a_1, \cdots, a_r \in A$. Define the Koszul complex $K_\bullet = K_\bullet(a_1, \cdots, a_r)$ of A-modules like following*
(1)K_1 is a free A-module of rank r with basis $e_1, \cdots, e_r$.
(2)$K_i := \wedge^i K_1$ for all $i \geq 0$.
(3)boundary map $d_i : K_i \longrightarrow K_{i-1}$ is given by $e_{j_1} \wedge \cdots \wedge e_{j_i} \longmapsto \sum_{k=1}^i (-1)^{k-1} a_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}$.

For $M \in Mod_A$, set $K_\bullet(M) = K_\bullet \otimes_A M$. In particular, $h_0(K_\bullet(M)) \cong M/(a_1, \cdots, a_r)M$.

Proposition 11.14 (***Regular sequence is Koszul regular***)**.** *Let A be a ring, $M \in Mod_A$, $a_1, \cdots, a_r \in A$. If $a_1, \cdots, a_r$ form a regular sequence for M, then $h_i(K_\bullet(M)) = 0$ for all $i > 0$ so that $K_\bullet(M) \longrightarrow M/(a_1, \cdots, a_r)M \longrightarrow 0$ is exact.*

Proof of Theorem 11.4. Recall that $\omega_X^o = \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$. For all $x \in X$, $\mathcal{I}_x$ is generated by a regular sequence $a_1, \cdots, a_r \in \mathcal{O}_{\mathbb{P}^N_k, x}$. Then by Proposition 11.14, the Koszul complex $K_\bullet(\mathcal{O}_{\mathbb{P}^N_k, x})$ is a finite free resolution of $\mathcal{O}_{\mathbb{P}^N_k, x}/(a_1, \cdots, a_r) \cong \mathcal{O}_{X,x}$.

Since everything is noetherian, get a finite free resolution $K_\bullet(\mathcal{O}_{\mathbb{P}^N_k})$ of $\mathcal{O}_X$ over a neighbourhood U of x. Over U, we have that $\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) \cong h^r(\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N_k}}(K_\bullet(\mathcal{O}_{\mathbb{P}^N_k}, \omega_{\mathbb{P}^N_k}))) \cong \omega_{\mathbb{P}^N_k}/(a_1, \cdots, a_r)\omega_{\mathbb{P}^N_k}$ i.e. $\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) \cong \omega_{\mathbb{P}^N_k}\big|_X$. But this isomorphism depends on the choice of $a_1, \cdots, a_r$. If $b_i = \sum_j c_{ij} a_j$ is another choice, then get a functor of $\det(c_{ij})$ for $K_r = \wedge^r K_1$ and the isomorphism differs by $\det(c_{ij})$.

To fix this, consider $\mathcal{I}/\mathcal{I}^2$ which is locally free of rank r. In fact, it is free over U with different basis $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$. Then $\wedge^r(\mathcal{I}/\mathcal{I}^\in)$ is free of rank 1 over U with two different basis $a_1 \wedge \cdots \wedge a_r$ and $b_1 \wedge \cdots \wedge b_r$, which differ by $\det(c_{ij})$. In conclusion, over U, we have that $\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) \cong \omega_{\mathbb{P}^N_k}\big|_X \otimes_{\mathcal{O}_X} \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$. Glue these isomorphisms up to get $\omega_X^o \cong \omega_{\mathbb{P}^N_k}\big|_X \otimes_{\mathcal{O}_X} \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$. □

11.5 Second Approach to Serre Duality

Let X be a projective scheme over k of dimension n. Instead of $X \subseteq \mathbb{P}_k^N$, consider finite morphism $X \longrightarrow \mathbb{P}_k^n$. Then we need some starting points
(1)(Noetherian Normalization) there exists finite morphism $f : X \longrightarrow \mathbb{P}_k^n$.
(2)(Miracle Flatness) if further X is pure and Cohen-Macaulay, then f is flat.

Definition 11.11. *Let $f : X \longrightarrow Y$ be a finite morphism between locally noetherian schemes. Then*
(1)For all quasi-coherent sheaf $\mathcal{G}$ on Y, $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$ is canonically a quasi-coherent $\mathcal{O}_X$-module, which yields a functor $f^! : Qcoh(Y) \longrightarrow Qcoh(X)$ called upper shrick.
(2)$(f_*, f^!)$ form a left-right adjoint pair i.e.

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.
(3)there exists natural isomorphism

$$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Remark 11.12. *Same statement holds for coherent sheaves if X, Y are noetherian.*

Proof of Serre Duality. Let X be a projective scheme over k pure of dimension n and Cohen-Macaulay. Consider finite and flat morphism $f : X \longrightarrow \mathbb{P}_k^n$. Set $\omega_X^o := f^!\omega_{\mathbb{P}_k^n} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(f_*\mathcal{O}_X, \omega_{\mathbb{P}_k^n})$. First assume $\mathcal{F}$ locally free of finite rank on X. Then we have that

$$\begin{aligned}
\mathrm{Ext}^i(\mathcal{F}, \omega_X^o) &\cong H^i(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X^o) \\
&\cong H^i(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} f^!\omega_{\mathbb{P}_k^n}) \\
&\cong H^i(\mathbb{P}_k^n, f_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} f^!\omega_{\mathbb{P}_k^n})) \text{ as } f \text{ is affine} \\
&\cong H^i(\mathbb{P}_k^n, f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\omega_{\mathbb{P}_k^n})) \\
&\cong H^i(\mathbb{P}_k^n, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(f_*\mathcal{F}, \omega_{\mathbb{P}_k^n})) \\
&\cong H^i(\mathbb{P}_k^n, (f_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \omega_{\mathbb{P}_k^n}) \\
&\cong H^{n-i}(\mathbb{P}_k^n, f_*\mathcal{F})^\vee \text{ by Serre duality for projective space} \\
&\cong H^{n-i}(X, \mathcal{F})^\vee \text{ as } f \text{ is affine}
\end{aligned}$$

Then complete the proof by universal δ-functors and Theorem 11.1. □

Remark 11.13. *To discribe whar really happens, we need to use the language of derived category. For all $f : X \longrightarrow Y$ morphism between separated schemes of finite type over k, there exists $f^! : DCoh(Y) \longrightarrow DCoh(X)$ satisfying that*
(1)(composition) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms, there exists natural isomorphism $(g \circ f)^! \xrightarrow{\sim} f^! \circ g^!$.
(2)(adjointion) For $f : X \longrightarrow Y$ proper, $(Rf_*, f^!)$ form a left-right adjoint pair i.e.

$$\mathrm{Hom}_{DCoh(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \cong \mathrm{Hom}_{DCoh(Y)}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$.

Example 11.4. *(1)If $f : X \longrightarrow Y$ is a finite morphism, then $f^!\mathcal{G}^\bullet \cong R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}^\bullet)$ in $DCoh(X)$*
(2)If $f : X \longrightarrow Y$ is smooth of relative dimension d, then $f^!\mathcal{G}^\bullet \cong Lf^*\mathcal{G}^\bullet \otimes \otimes\omega_{X/Y}[d]$, where $\omega_{X/Y}$ is the relative canonical ideal.

Now for all X separated scheme of finite type over field k with morphism $p : X \longrightarrow \operatorname{Spec} k$. Set $\omega_x^\bullet := p^!\mathcal{O}_{Spec(k)}$ called dualizing complex.

Example 11.5. *(1)$\omega^\bullet_{\mathbb{P}^n_k} \cong \omega_{\mathbb{P}^n_k}[n]$*
(2)X is pure of dimension n and Cohen-Macaulay if and only if $\omega_x^\bullet \cong \omega_X^o[n]$.

Then adjointion $\operatorname{Hom}_{DCoh(X)}(\mathcal{F}^\bullet, \omega_X^\bullet) \cong \operatorname{Hom}_k(R\Gamma(X, \mathcal{F}^\bullet), k)$ induces Serre duality if X is proper over k.

Theorem 11.5 (***Grothendieck Duality***)**.** *Let $f : X \longrightarrow Y$ be a proper morphism. Then $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \cong R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ functorial in $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$.*

12 Cohomology and Base Change

Recall give cartesian diagram of noetherian schemes with $f : X \longrightarrow Y$ separated and $\mathcal{F}$ coherent sheaf on X.

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y \\ & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

We get base change morphism

$$u^*(R^if_*\mathcal{F}) \longrightarrow R^ig_*(v^*\mathcal{F}), \forall i \geq 0$$

If $f : X \longrightarrow Y$, then $u^*f_*\mathcal{F} \longrightarrow g_*v^*\mathcal{F}$ is isomorphic. Also, if $u : Y' \longrightarrow Y$ is flat, then all the base change morphisms are isomorphic. Consider cartesian diagram of fibre

$$\begin{array}{ccc} X_y & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(k(y)) & \longrightarrow & Y \end{array}$$

Then base change morphisms change to be

$$R^if_*\mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \longrightarrow H^i(X_y, \mathcal{F}\big|_{X_y})$$

Want to know when the base change morphisms are isomorphic and how does $H^i(X_y, \mathcal{F}\big|_{X_y})$ change as $y \in Y$ varies.

Example 12.1. *Let E be a curve of genus 1 over k, $P \in E(k)$, (E, P) elliptic curve. Let $X = E \times_k E$, $Y = E$ and f be the diagonal map, $\Delta \subseteq X$ diagonal set. Set $\mathcal{L} := \mathcal{O}_X(\Delta - \{P\} \times_k E)$. For $g \in Y - E$, $\mathcal{L}\big|_{X_y}$ is trivial if and only if $y = P$. Then $H^0(X_y, \mathcal{L}\big|_{X_y}) = \begin{cases} k & y = P \\ 0 & otherwise \end{cases}$.*

On the other hand, $R^0 f_*\mathcal{L} \cong f_*\mathcal{L}$ is torsion-free coherent sheaf on Y. Get $f_*\mathcal{L}$ is locally free of finite rank. To determine the rank, flat base change to the generic point η with $\operatorname{Spec}(k(\eta)) \longrightarrow Y$ *flat. Get $f_*\mathcal{L} \otimes_{\mathcal{O}_Y} k(\eta) \cong H^0(X_\eta, \mathcal{L}\big|_{X_\eta}) = 0$ so that $f_*\mathcal{L} = 0$.*

12.1 Hilbert Polynomial

Definition 12.1 (***Hilbert Polynomial***)**.** *Let X be a projective scheme over field k, $\mathcal{O}_X(1)$ very ample invertible sheaf on X, $\mathcal{F}$ coherent sheaf on X. Then the function $P_{\mathcal{F}}(m) := \chi(X, \mathcal{F}(m)) = \sum_{i=0}^{\infty} (-1)^i \dim_k(H^i(X\mathcal{F}(m)))$, where $\chi(X, \mathcal{F})$ is the Euler characteristic of $\mathcal{F}$, if a polynomial in m of degree equal to $\dim(Supp(\mathcal{F}))$, callled the Hilbert polynomial of $\mathcal{F}$.*

Remark 12.1. *By Serre Vanishing Theorem, $\dim_k(H^0(X, \mathcal{F}(m)))$ is a polynomial in m for m large enough.*

Proof. By flat base change, can assume k algebraically closed so that k is infinite. Also, by pushing forward, can assume $X = \mathbb{P}_k^N$. Then for all coherent sheaf $\mathcal{F}$ on X, there exists hyperplane $H \subseteq X$ together with exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

inducing exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_H \longrightarrow 0$$

This is because finitely generated module over noetherian ring has finitely many associated prime ideals. While k is infinite, can find H avoiding all these associated prime ideals. Denote $\mathcal{I}_H := \mathcal{O}_X(-1)$ and $\mathcal{G} := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_H$. Then $\chi(X, \mathcal{F}(m)) = \chi(X, \mathcal{F}(m-1)) + \chi(X, \mathcal{G}(m))$ so that $P_{\mathcal{F}}(m) - P_{\mathcal{F}}(m-1) = P_{\mathcal{G}}(m)$. Also $\dim(Supp(\mathcal{G})) = \dim(Supp(\mathcal{F})) - 1$. By following lemma, done! ☐

Lemma 12.1. *(1)Any numerical polynomial can be written as $P(m) = \sum_{i=0}^{n} a_i \binom{m}{i}$ where $a_i \in \mathbb{Z}$ and $n = \deg(P(m))$.*
(2)If $Q(m)$ is a numerical polynomial of degree $n-1$ and $P(m)$ is a function on the integers satisfying that $P(m+1) - P(m) = Q(m)$, then $P(m)$ is a numerical polynomial for degree n.

Remark 12.2. *For $\mathcal{F} = \mathcal{O}_X$, write $P_X(m) := P_{\mathcal{O}_X}(m)$ and $\deg(P_X(m)) = \dim(X)$. Define the degree of X to be $\deg(X) := n!(\text{leading coefficient of } P_X(m))$, where $n = \dim(X)$.*

Theorem 12.1. *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $P_y(m) := P_{\mathcal{F}\big|_{X_y}}(m)$ is locally constant as $y \in Y$ varies. If moreover Y is reduced, then the converse holds.*

Proof. Since question is local on Y, can assume $Y = \operatorname{Spec} A$. Also, by pushing forward, can assume $X = \mathbb{P}_A^N$. Can even assume A is a noetherian local ring since flatness is local. Consider the following statements.

(1)$\mathcal{F}$ is flat on $Y = \operatorname{Spec} A$

(2)$H^0(X, \mathcal{F}(m))$ is a free A-module of finite rank for m large enough

(3)$P_y(m)$ is independent of $y \in Y = \operatorname{Spec} A$.

Claim that (1) $\iff$ (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) if A is reduced. For (1) $\Rightarrow$ (2), compute $H^i(X, \mathcal{F}(m))$ using Čech cohomology. Take $\mathcal{U} = \{U_i\}$ standard open covering of X, then $H^i(X, \mathcal{F}(m)) \cong H^i(C^\bullet(\mathcal{U}, \mathcal{F}(m)))$. As $\mathcal{F}$ is flat, each $C^i(\mathcal{U}, \mathcal{F}(m))$ is flat A-module. Moreover $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$ and m large enough. Get exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}(m)) \longrightarrow C^0(\mathcal{U}, \mathcal{F}(m)) \longrightarrow C^1(\mathcal{U}, \mathcal{F}(m)) \longrightarrow \cdots$$

for m large enough. Note that for exact sequence in category of A-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

if M and N are flat, then L is flat. Repeatedly using the fact, get $H^0(X, \mathcal{F}(m))$ is finitely generated flat A-module for m large enough. As A is local, $H^0(X, \mathcal{F}(m))$ is free of finite rank for m large enough.

For (2) $\Rightarrow$ (1), set $M = \oplus_{m > m_0} H^0(X, \mathcal{F}(m))$ for large enough m_0. Then M is flat A-module so that $\widetilde{M}$ is flat over Y. For (2) $\Rightarrow$ (3), suffice to show $P_y(m) = \operatorname{rank}_A(H^0(X, \mathcal{F}(m)))$ for m large enough. We show that $H^0(X, \mathcal{F}(m)) \otimes_A k(y) \cong H^0(X_y, \mathcal{F}(m)\big|_{X_y})$ for $y \in Y$ and m large enough, which doesn't need flatness. First consider closed point $y \in Y = \operatorname{Spec} A$, then $k(y) = A/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal. There is an exact sequence

$$A^q \longrightarrow A \longrightarrow k(y) \longrightarrow 0$$

View this sequence in category of modules of sheaves over Y. After pulling back and tensored by $\mathcal{F}$, get exact sequence

$$\mathcal{F}^q \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} f^*\widetilde{k(y)} \longrightarrow 0$$

Then by Serre Vanishing Theorem, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^0(X, \mathcal{F}(m)) \otimes_A A^q & \longrightarrow & H^0(X, \mathcal{F}(m)) & \longrightarrow & H^0(X, \mathcal{F}(m)) \otimes_A k(y) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow & & \\
H^0(X, \mathcal{F}(m)^q) & \longrightarrow & H^0(X, \mathcal{F}(m)) & \longrightarrow & H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^*\widetilde{k(y)}) & \longrightarrow & 0
\end{array}$$

for m large enough. By universal property of cokernel, we get that $H^0(X, \mathcal{F}(m)) \otimes_A k(y) \cong H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^*\widetilde{k(y)})$. In fact, by some argument of graded module, we have that $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\widetilde{k(y)} \cong \mathcal{F}\big|_{X_y}$. Thus $H^0(X, \mathcal{F}(m)) \otimes_A k(y) \cong H^0(X_y, \mathcal{F}(m)\big|_{X_y})$ for m large enough.

For general point $y \in Y$, set $Y' = \operatorname{Spec}(\mathcal{O}_{Y,y})$ which is flat over Y and $g : X' \longrightarrow X$ is the pull back of $Y' \longrightarrow Y$ along f. Thus by flat base change, we can reduce Y to Y' so that now y is closed point. Then by same argument, get $H^0(X_y, \mathcal{F}(m)\big|_{X_y}) \cong H^0(X', g^*\mathcal{F}(m)) \otimes_{A'} k(y) \cong H^0(X, \mathcal{F}(m))$ for m large enough.

For (3) $\Rightarrow$ (2) with A reduced, recall that a finitely generated module M over reduced local ring A is free if and only if $\dim_{k(Y)}(M \otimes_A k(y))$ is independent of $y \in \operatorname{Spec} A$. Thus $H^0(X, \mathcal{F}(M))$ is free over A for m large enough if $P_y(m)$ is independent of $y \in \operatorname{Spec} A$, as base change morphisms are isomorphic for m large enough. $\square$

Corollary 12.1. *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X. Then $\mathcal{F}$ is flat over Y if and only if $f_*(\mathcal{F}(m))$ is locally free of finite rank for m large enough.*

Remark 12.3. *This is in fact* (1) $\iff$ (2).

The same proof also gives

Proposition 12.1 (***Base Change without Flatness***)**.** *Give a cartesian diagram of noetherian schemes.*

$$\begin{array}{ccc} X' & \xrightarrow{v} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

Let $f : X \longrightarrow Y$ be a projective morphism, $\mathcal{F}$ coherent sheaf on X. Then the base change morphism

$$u^* f_*(\mathcal{F}(m)) \longrightarrow f_* v^*(\mathcal{F}(m))$$

is isomorphic for m large enough.

Proof. Reduce to the case that $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec}(A')$ and $X = \mathbb{P}^N_A$. As $\mathcal{F}$ is coherent, there is an exact sequence

$$\mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}_i$ are bothe of the form $\oplus \mathcal{O}_X(q)$. By pulling back, get exact sequence

$$v^*\mathcal{E}_1 \longrightarrow v^*\mathcal{E}_0 \longrightarrow v^*\mathcal{F} \longrightarrow 0 \tag{3}$$

Get commutative diagram with exact rows

$$\begin{array}{ccccccc} H^0(X, \mathcal{E}_1(m)) \otimes_A A' & \longrightarrow & H^0(X, \mathcal{E}_0(m)) \otimes_A A' & \longrightarrow & H^0(X, \mathcal{F}(m)) \otimes_A A' & \longrightarrow & 0 \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow & & \\ H^0(X', v^*(\mathcal{E}_1(m))) & \longrightarrow & H^0(X', v^*(\mathcal{E}_0(m))) & \longrightarrow & H^0(X', v^*(\mathcal{F}(m))) & \longrightarrow & 0 \end{array}$$

for m large enough. Then by universal property of cokernel, done! $\square$

12.2 Three Important Theorems

Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X flat on Y. Recall that we have base change morphisms for all $y \in Y$ as following,

$$\phi^i_y : (R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \longrightarrow H^i(X_y, \mathcal{F}\big|_{X_y})$$

where by Nakayama, dimension of the left side is upper semicontinuous.

Theorem 12.2 (***Semicontinuousity***)**.** *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X flat on Y. Then the function $y \longmapsto \dim_{k(y)}(H^i(X_y, \mathcal{F}\big|_{X_y}))$ is upper semicontinuous for all $i \geq 0$.*

Theorem 12.3 (***Grauert***). *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X flat on Y. Further assume Y is reduced. Suppose $\dim_{k(y)}(H^i(X_y, \mathcal{F}\big|_{X_y}))$ is locally constant for some i. Then $R^i f_* \mathcal{F}$ is locally free and ϕ^i_y is isomorphic for all $y \in Y$.*

Remark 12.4. *The converse is obviously true with Y reduced. When Y is not reduced, there is a counterexample that $X = Y = \mathrm{Spec}(k[t]/(t^2))$ and $\mathcal{F} = \overline{(t)}$.*

By these two theorems, if Y is integral, then $\dim_{k(y)}(H^i(X_y, \mathcal{F}\big|_{X_y}))$ is constant in an open dense subset of Y relative to i where $R^i f_* \mathcal{F}$ is locally free.

Theorem 12.4 (***Cohomology and Base Change***). *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X flat on Y. Suppose that for some $y \in Y$ and i, base change morphism ϕ^i_y is surjective. Then*
(1)there exists open neighbourhood $U \ni y$ such that $\phi^i_{y'}$ is isomorphic for all $y' \in U$.
(2)ϕ^{i-1}_y is also surjective if and only if $R^i f_* \mathcal{F}$ is locally free over an open neighbourhood $U \ni y$.

Remark 12.5. *There is a stronger statement of (1)*
(1′)there exists open neighbourhood $U \ni Y$ such that for all cartesian diagram of noetherian schemes

$$\begin{array}{ccc} X' & \xrightarrow{v} & X_U \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & U \end{array}$$

there is an isomorphism $U^*(R^i f_* \mathcal{F}) \xrightarrow{\sim} R^i g_*(v^* \mathcal{F})$.

Example 12.2. *Let $f : X \longrightarrow Y$ be a projective morphism between noetherian schemes, $\mathcal{F}$ coherent sheaf on X flat on Y.*
(1)If $H^i(X_y, \mathcal{F}\big|_{X_y}) = 0$ for some i and all $y \in Y$, then by Theorem 12.4 and Nakayama Lemma, $R^i f_* \mathcal{F} = 0$ and $R^{i-1} f_* \mathcal{F}$ commutes with arbitrary base change. In particular, if $H^1(X_y, \mathcal{F}\big|_{X_y}) = 0$ for all $y \in Y$, then $R^1 f_* \mathcal{F} = 0$ and $f_* \mathcal{F}$ is locally free of finite rank.

(2)If $R^i f_* \mathcal{F} = 0$ for all $i \geq i_0$, then by Theorem 12.4 and Grothendieck Vanishing Theorem, $H^i(X_y, \{\big|_{X_y}) = 0$ for all $y \in Y$ and $i \geq i_0$.

We present the proof in Vakil, which relies on the following observation by Mumford.

Theorem 12.5 (***Mumford***). *Let A be a noetherian ring, $C^\bullet$ complex of A-modules bounded on the right i.e. there exists N large enough such that $C^i = 0$ for all $i > N$. Suppose that all cohomology groups $h^i(C^\bullet)$ are finite A-modules. Then there exists complex $K^\bullet$ of free A-modules of finite rank such that $K^i = 0$ for all $i > N$ and $\alpha : K^\bullet \longleftarrow C^\bullet$ morphism of complexes such that $h^i(\alpha) : h^i(K^\bullet) \xrightarrow{\sim} h^i(C^\bullet)$ is isomorphic for all i.*

Further, if C^i is flat A-module for all i, then for all ring homomorphism $A \longrightarrow B$, $h^i(\alpha \otimes_A B) : h^i(K^\bullet \otimes_A B) \xrightarrow{\sim} h^i(C^\bullet \otimes_A B)$ is isomorphic for all i.

Back to proof of theorems, can assume that $Y = \operatorname{Spec} A$, $X = \mathbb{P}^N_A$ and $f : \mathbb{P}^N_A \longrightarrow \operatorname{Spec} A$ is the projection. We apply Theorem 12.5 to Čech complex $C^\bullet := C^\bullet(\mathcal{U}, \mathcal{F})$, where $\mathcal{U} = \{U_i\}$ is standard open covering. Get complex $K^\bullet$ of free A-module of finite rank

$$\cdots \longrightarrow K^{i-1} \xrightarrow{\delta^{i-1}} K^i \xrightarrow{\delta^i} K^{i+1} \longrightarrow \cdots \longrightarrow K^N \longrightarrow 0$$

Proof of Semicontinuouosity. For $y \in Y = \operatorname{Spec} A$,

$$\begin{aligned}\dim_{k(y)} H^i(X_y, \mathcal{F}\big|_{X_y}) &= \dim_{k(y)}(\ker(\delta^i \otimes_A k(y))) - \dim_{k(y)}(\operatorname{im}(\delta^{i-1} \otimes_A k(y))) \\ &= \dim_{k(y)}(K^i \otimes_A k(y)) - \dim_{k(y)}(\operatorname{im}(\delta^i \otimes_A k(y))) \\ &\quad - \dim_{k(y)}(\operatorname{im}(\delta^{i-1} \otimes_A k(y)))\end{aligned}$$

Note that $\dim_{k(y)}(K^i \otimes_A k(y))$ is constant and $\dim_{k(y)}(\operatorname{im}(\delta^i \otimes_A k(y)))$ is lower semicontinuous, done! □

Definition 12.2. *Let X be a noetherian scheme, $\mathcal{E}, \mathcal{F}$ locally free sheaves on X of rank $r+a$ and $r+b$ respectively. A morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ is called of constant rank r (or say morphism of vector bundles) if there exists open covering $\{U_i\}$ on X such that for all i, there is a commutative diagram*

$$\begin{tikzcd} \mathcal{E}\big|_{U_i} \arrow[rr, "\phi|_{U_i}"] \arrow[d, "\sim"] & & \mathcal{F}\big|_{U_i} \arrow[d, "\sim"] \\ \mathcal{O}_{U_i}^{\oplus r+a} \arrow[r, "projection"] & \mathcal{O}_{U_i}^{\oplus r} \arrow[r, "inclusion"] & \mathcal{O}_{U_i}^{\oplus r+b} \end{tikzcd}$$

Remark 12.6. *If $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ is of constant rank r, then $\ker(\phi)$, $\operatorname{im}(\phi)$, $\operatorname{coker}(\phi)$ and $\operatorname{coim}(\phi)$ are all locally free of rank r. Moreover, for all $x \in X$, $\phi_x \otimes_{\mathcal{O}_{X,x}} \mathrm{id}_{k(x)} : \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} k(x) \longrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is of constant rank r as morphism of locally free sheaves on $\operatorname{Spec}(k(x))$. Also, the converse holds if X is reduced. In addition, being of constant rank commutes with base change.*

Lemma 12.2. *$\phi : \mathcal{E} \longrightarrow \mathcal{F}$ is of constant rank if and only if $\operatorname{coker}(\phi)$ is locally free of constant rank. In particular, if ϕ is surjective, then it is of constant rank.*

Remark 12.7. *This is a exercise using Nakayama Lemma.*

Lemma 12.3. *$\phi : \mathcal{E} \longrightarrow \mathcal{F}$ is of constant rank over an open neighbourhood of $x \in X$ if and only if $(\ker(\phi))\big|_x = \ker(\phi)_x \otimes_{\mathcal{O}_{X,x}} k(x) \longrightarrow \ker(\phi\big|_x) = \ker(\phi_x \otimes_{\mathcal{O}_{X,x}} \mathrm{id}_{k(x)})$ is surjective.*

Proof. "⇒": Obviously.

"⇐": choose bases $\mathcal{E}\big|_x \cong k(x)^{\oplus r+a}$ and $\mathcal{F}\big|_x \cong k(x)^{\oplus r+b}$ under which $\phi\big|_x : \mathcal{E}\big|_x \longrightarrow \mathcal{F}\big|_x$ given by projection and inclusion. Lift first r basis elements of $\mathcal{E}\big|_x$ to local sections $e_1, \cdots, e_r$ of $\rceil$. Lift last b basis elements of $\mathcal{F}\big|_x$ to local sections $f_1, \cdots, f_b$ of $\mathcal{F}$. Since $(\ker(\phi))\big|_x \longrightarrow \ker(\phi\big|_x)$ is surjective, we can lift last a basis elements of $\mathcal{E}\big|_x$ to local sections $k_1, \cdots, k_a$ of $\ker(\phi) \subseteq \mathcal{E}$. Then $e_1, \cdots, e_r, k_1, \cdots, k_a$ give $\mathcal{E} \cong \mathcal{O}_X^{\oplus r+s}$ and $\phi(e_1), \cdots, \phi(e_r), f_1, \cdots, f_b$ give $\mathcal{F} \cong \mathcal{O}_X^{\oplus r+b}$ over an open neighbourhood of x under which ϕ takes the desired form. □

Proof of Grauert. As $\dim_{k(y)}(H^i(X_y, \mathcal{F}|_{X_y}))$ is constant, both $\delta^{i-1}|_y$ and $\delta^i|_y$ are of constant rank for all $y \in Y = \operatorname{Spec} A$. As Y is reduced, both δ^{i-1} and δ^i are of constant rank. Consider exact sequence

$$0 \longrightarrow h^i(K^\bullet) \longrightarrow \operatorname{coker}(\delta^{i-1}) \longrightarrow \operatorname{im}(\delta^i) \longrightarrow 0$$

Note that $\operatorname{coker}(\delta^i)$ and $\operatorname{im}(\delta^i)$ are both locally free of finite rank and $\operatorname{coker}(\delta^{i-1}) \longrightarrow \operatorname{im}(\delta^i)$ is surjective, by Lemma 12.2, $h^i(K^\bullet)$ is locally free of finite rank and commutes with arbitrary base change. $\square$

Lemma 12.4. *Given a morphism of complexes*

$$\begin{tikzcd}
K^{i-1} \arrow[r, "\delta_K^{i-1}"] \arrow[d, two heads] & K^i \arrow[r, "\delta_K^i"] \arrow[d] & K^{i+1} \arrow[d] \\
L^{i-1} \arrow[r, "\delta_L^{i-1}"] & L^i \arrow[r, "\delta_L^i"] & L^{i+1}
\end{tikzcd}$$

We have that $h^i(K^\bullet) \longrightarrow h^i(L^\bullet)$ is surjective if and only if $\ker(\delta_K^i) \longrightarrow \ker(\delta_L^i)$ is surjective.

Reason 12.1. *This is an easy exercise by Snake Lemma.*

Consider the following commutative diagram

$$\begin{tikzcd}
K^{i-1} \arrow[r, "\delta^{i-1}"] \arrow[d] & K^i \arrow[r, "\delta^i"] \arrow[d] & K^{i+1} \arrow[d] \\
K_y^{i-1} \arrow[r] \arrow[d, two heads] & K_y^i \arrow[r] \arrow[d] & K_y^{i+1} \arrow[d] \\
K^{i-1}|_y \arrow[r, "\delta^{i-1}|_y"] & K^i|_y \arrow[r, "\delta^i|_y"] & K^{i+1}|_y
\end{tikzcd}$$

Proof of Cohomology and Base Change. For $y \in Y = \operatorname{Spec} A$, $h^i(K^\bullet) \longrightarrow h^i(K^\bullet|_y)$ is surjective, by Lemma 12.4, is equivalent to that $(\ker(\delta^i))|_y \longrightarrow \ker(\delta^i|_y)$ is surjective, and by Lemma 12.3, is equivalent to that δ^i is of constant rank near $y \in Y$. Thus $\ker(\delta^i)$ is locally free of finite rank and commutes with base change near y. Consider exact sequence

$$K^{i-1} \longrightarrow \ker(\delta^i) \longrightarrow h^i(K^\bullet) \longrightarrow 0$$

Note that K^{i-1} and $\ker(\delta^i)$ both are locally free and commute with base change near y, get $h^i(K^\bullet)$ commutes with base change near y. This proves (1) and (1′).

For (2), $h^{i-1}(K^\bullet)|_y \longrightarrow h^{i-1}(K^\bullet|_y)$ is also surjective if and only if δ^{i-1} is of constant rank near y. Obviously, if δ^{i-1} is of constant rank near y, then $h^i(K^\bullet)$ is locally free of finite rank near y. For the converse, if $h^i(K^\bullet)$ is locally free of finite rank near y, then by Lemma 12.2, $K^{i-1} \longrightarrow \ker(\delta^i)$ is of constant rank so that the composition δ^{i-1} is also of constant rank. $\square$

Remark 12.8 (***Derived Base Change***)**.** *We have discussed lots about base change morphisms, but where does the base change isomorphisms always hold? The answer is de-*

rived/homotopy cartesian diagram.

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

Locally $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec}(A')$ *and* $X' = \operatorname{Spec}(B \overset{L}{\otimes}_A A')$, *where* $\overset{L}{\otimes}$ *is the derived tensor product. Assume that all these derived schemes are qcqs. Then for all quasi-coherent sheaf* $\mathcal{F}$ *on* X, $Lu^*(R^i f_* \mathcal{F}) \xrightarrow{\sim} R^i g_*(Lv^* \mathcal{F})$ *is isomorphic in* $DQcoh(Y')$.

13 Case Study: Hilbert Schemes

Want to construct a scheme parametrizing all closed subschemes of $\mathbb{P}_k^n$. Naively, we have that

$$\{\text{closed subschemes of } \mathbb{P}_k^n\} \hookrightarrow \{\text{vector subspaces of } k[x_1, \cdots, x_n]\}$$

Problems are that RHS is infinite-dimensional Grassmannian manifold, this is not a one-to-one correspondence and the set $\{\text{finite-dimensional subspaces of } k[x_1, \cdots, x_n]\}/\sim$ is not clear to be a scheme. To solve these problems, we need to define Hilbert functor and representability of functors.

13.1 Hilbert Functor

By now, we assume all schemes discussed are noetherian.

Definition 13.1 (***Representable Functor***)**.** *Let* S *be a scheme,* Sch_S *category of* S*-schemes,* Set *category of sets. Suppose* $F : Sch_S \longrightarrow Set$ *is a contravariant functor. We say an* S*-scheme* X *and an element* $U \in F(X)$ *represent* F *if for all* S*-scheme* T, $\operatorname{Hom}_X(T, X) \longrightarrow F(T) \quad g \longmapsto g^*U := F(g)(U)$ *is isomorphic,* U *is called the universal element/family over* X*. In particular,* (X, U) *is unique up to isomorphism.*

Definition 13.2 (***Hilbert Functor***)**.** *Let* $f : X \longrightarrow S$ *be a projective morphism. The Hilbert functor* $\mathcal{H}ilb_{X/S} : Sch_S \longrightarrow Set$ *is given by*

$$T \longmapsto \{\textit{closed subschemes } V \subseteq X \times_S T \textit{ flat over } T\}$$

Remark 13.1. *More generally,* $X \longrightarrow S$ *is not necessarily projective. For* $X \longrightarrow S$ *strongly projective i.e. it factors through* $\mathbb{P}(E)$ *for some vector bundle on* S *as following*

$$\begin{tikzcd} X \arrow[r, hook, "closed"', "immersion"] \arrow[dr] & \mathbb{P}(E) \arrow[d] \\ & S \end{tikzcd}$$

we can also define similar Hilbert functor.

In addition, it is clear that closed subscheme $V \in \mathcal{H}ilb_{X/S}(T)$ is one-to-one corresponding to a quotient coherent sheaf $\mathcal{O}_{X\times_S T} \twoheadrightarrow \mathcal{F}$ for some $\mathcal{F}$ flat over T, which gives us a way to rewrite the Hilbert functor. And the new version can be viewed as a special case of Quot functor.

Recall flatness induces local constancy of Hilbert polynomial. Fix $P = P(m)$ a numerical polynomial. There is a finer Hilbert functor $\mathcal{H}ilb^P_{X/S} : Sch_S \longrightarrow Set$ given by

$$T \longmapsto \{V \in \mathcal{H}ilb_{X/S}(T)\big|P_{V_t}(m) = P, \forall t \in T\}.$$

Similarly, as Remark 13.1, we can also rewrite $\mathcal{H}ilb^P_{X/S}$. In particular, we have that $\mathcal{H}ilb_{X/S} = \sqcup_P \mathcal{H}ilb^P_{X/S}$.

Example 13.1. *Let $S = \operatorname{Spec} k$, $X = \mathbb{P}^n_k$ and $T = \operatorname{Spec} k$. Then $\mathcal{H}ilb_{\mathbb{P}^n_k/Spec(k)}(Spec(k)) = \{$closed subschemes of $\mathbb{P}^n_k\}$.*

The main theorem we wanted can be stated as following

Theorem 13.1. *Let $f : X \longrightarrow S$ be a strongly projective morphism, P a fixed numerical polynomial. The functor $\mathcal{H}ilb^P_{X/S}$ is represented by an S-scheme $\mathrm{Hilb}^P_{X/S}$ and a closed subscheme $U^P_{X/S} \subseteq X \times_S \mathrm{Hilb}^P_{X/S}$. Moreover, $\mathrm{Hilb}^P_{X/S} \longrightarrow S$ is strongly projective factoring through $\mathbb{P}(E)$ for some vector bundle E on S.*

Example 13.2. *(1)Let $P = 1$ be constant. Then $\mathrm{Hilb}^1_{X/S} = X$.*
(2)Let C be a curve smooth and projective over k and $P = n$ be constant. Then $\mathrm{Hilb}^n_C = C^{(n)} := (C \times_k \cdots \times_k C)/S_n$.
(3)Let S be a surface smooth and projective over k and $P = n$ be constant. Then $\mathrm{Hilb}^n_S = S^{[n]} \longrightarrow S^{(n)}$ is not in general isomorphic for $n > 1$. In particular, when $n = 2$, we have that $\mathrm{Hilb}^2_S = Blow_\Delta(S \times_k S)/S_2$.

13.2 Special Case: Grassmannians

Definition 13.3. *Let S be a scheme, E vect bundle on S, r natural number. The Grassmannian functor $\mathcal{G}r(r, E) : Sch_S \longrightarrow Set$ is given by*

$$\begin{aligned} T \longmapsto &\{\text{subbundles of rank } r \text{ of } E \times_S T\} \\ =&\{\text{projective subbundles of rank } r-1 \text{ of } \mathbb{P}(E) \times_S T\} \\ \subseteq &\mathcal{H}ilb_{\mathbb{P}(E)/S}(T) \end{aligned}$$

Lemma 13.1. *Let V be a finite-dimensional vector space, $W \subseteq V$ subspace of dimension r. Then under the wedge map $V \otimes \wedge^r V \xrightarrow{\wedge} \wedge^{r+1} V$, W and $\wedge^r W$ are annihilator of each other.*

Theorem 13.2. *Let S be a scheme, E vect bundle on S, r natural number. Then $\mathcal{G}r(r, E)$ is represented by an S-scheme $\mathrm{Gr}(r, E)$ and a subbundle $U \subseteq E \times_S \mathrm{Gr}(r, E)$.*

Example 13.3. *Let $r = 1$. Then $\mathrm{Gr}(1, E) = \mathbb{P}(E)$.*

Sketch Proof of Theorem 13.2. Via "Plucier embedding". Set $Y = \wedge^r E$, then there is a map $p : Y \longrightarrow S$. Consider $\sigma : Y \longrightarrow p^*(\wedge^r E) \quad y \longmapsto "y" \in (p^*(\wedge^r E))_y = (\wedge^r E)_s$, where $s \in S$ and $y \in Y_s = (\wedge^r E)_s$. And $E \otimes \wedge^r E \xrightarrow{\wedge} \wedge^{r+1} E$ on S induces $\phi := \wedge\sigma : p^*E \longrightarrow p^*(\wedge^{r+1}E)$ on Y. For all $y \in Y_s$, $\phi_y := (\wedge\sigma)_y : E_s = (p^*E)_y \longrightarrow (p^*(\wedge^{r+1}E))_y = \wedge^{r+1}E_s$.

Now, ϕ is locally given by a matrix (m_{ij}), where m_{ij} are local functions on Y. By Lemma 13.1, get $\mathrm{rank}(\phi) \geq n-r$ everywhere away from 0, where $n = \mathrm{rank}(E)$. Thus $\mathrm{rank}(\phi) = n-r$ at $y \in Y \setminus \{0\}$ if and only if $\ker(\phi_y)$ is r-dimensional and $y \in \wedge^r(\ker(\phi_y)) \setminus \{0\}$. Let $Y_r \subseteq Y$ be the closed subscheme locally defined by $(n-r+1) \times (n-r+1)$ minors of (m_{ij}). Then $K := \ker(\phi|_{Y_r \setminus \{0\}}) \subseteq p^*E|_{Y_r \setminus \{0\}}$ is locally free of rank r on $Y_r \setminus \{0\}$ i.e. a vector bundle.

Finally, set $\mathbb{P}(Y) := (\wedge^{\backslash}\mathbb{E}) \xrightarrow{q} S$. Y_r descends to closed subscheme $\mathrm{Gr}(r,E) \subseteq \mathbb{P}(Y)$ and K descends to subbundle $U \subseteq q^*E|_{\mathrm{Gr}(r,E)} \cong E \times_S \mathrm{Gr}(r,E)$. Claim that $(\mathrm{Gr}(r,E), U)$ represent $\mathcal{G}r(r,E)$. Suppose that $g : T \longrightarrow \mathrm{Gr}(r,E)$ and get subbundle $g^*U \subseteq E \times_S T$ of rank r. And $\wedge^r(g^*U) \subseteq \wedge^r E \times_S T$ is the unique line bundle which annihilates g^*U.

Conversely, for F subbundle of rank r of $E \times_S T$. Then line bundle $\wedge^r F \subseteq \wedge^r E \times_S T$ corresponds to $g : T \longrightarrow \mathbb{P}(Y)$ since projective cone represents the functor $\mathcal{G}r(1,Y)$. Consider $E_T \otimes \wedge^r F \longrightarrow \wedge^{r+1} E \times_S T$, whose kernel is locally free of rank r by Lemma 13.1. Thus $g : T \longrightarrow \mathbb{P}(Y)$ factors through $\mathrm{Gr}(1,E)$ and $F = g^*U$. $\square$

Corollary 13.1. $\mathrm{Gr}(r,E)$ *is strongly projective factoring through* $\mathbb{P}(Y)$.

13.3 Castelnuovo-Mumford Regularity

Theorem 13.3. *For all numerical polynomial P, there exists integer m_P relative to P such that for all ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ with Hilbert polynomial P and for all $q \geq m_P$, we have that*
(1)$H^i(\mathbb{P}^n_k, \mathcal{I}(q)) = 0$ for all $i > 0$.
(2)$\mathcal{I}(q)$ is generated by global sections.
(3)$H^0(\mathbb{P}^n_k, \mathcal{I}(q)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{I}(q+1))$ is surjective.

Example 13.4. *Consider $\mathcal{F}_a := \mathcal{O}_{\mathbb{P}^1_k}(-a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(a)$ for $a \geq 0$. Then $P_{\mathcal{F}_a}(m) = 2m+2$ which is independent on a. But for all a, there is no q such that $H^1(\mathbb{P}^1_k, \mathcal{F}_a(q)) = 0$ and there is no q such that $\mathcal{F}_a(q)$ is globally generated. Thus, the condition that $\mathcal{I}$ is an ideal sheaf is necessary.*

Recall the Hilbert functor $\mathcal{H}ilb^P_{X/S}$. For simplicity, assume that $S = \operatorname{Spec} k$, $X = \mathbb{P}^n_k$ and $T = \operatorname{Spec} k$. Let $V \subseteq \mathbb{P}^n_k$ be a closed subscheme with $P_V = P$, $\mathcal{I}_V$ ideal sheaf, $P_{\mathcal{I}_V} = Q := P_{\mathbb{P}^n_k} - P$. Choose m_Q in Theorem 13.3. Then $\mathcal{I}_V(m_Q)$ is globally generated. Consider exact sequence

$$0 \longrightarrow \mathcal{I}_V(m_Q) \longrightarrow \mathbb{P}^{\backslash}_{\|}(m_Q) \longrightarrow \mathcal{O}_V(m_Q) \longrightarrow 0$$

Get $H^0(\mathbb{P}^n_k, \mathcal{I}_V(m_Q)) \subseteq H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m_Q))$ gives a subspace. While $\mathcal{I}_V(m_Q)$ is globally generated, the correspondence is one-to-one.

Moreover, by Theorem 13.3, we also get the dimension $\dim_k(H^0(\mathbb{P}^n_k, \mathcal{I}_V(m_Q))) = Q(m_Q)$. Set-theorectically, this gives an injection from the set of closed subschemes of $\mathbb{P}^n_k$ with Hilbert polynomial P to $\mathrm{Gr}(Q(m_Q), H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m_Q)))$. We furthur need to put a scheme structure on the image.

Definition 13.4 (***Castelnuovo-Munford Regularity***). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. We say that $\mathcal{F}$ is m-regular if $H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0$ for all $i > 0$.*

Example 13.5. *$\mathcal{O}_{\mathbb{P}^n_k}(a)$ is m-regular for $m \geq -a$.*

Proposition 13.1. *If $\mathcal{F}$ is a m-regular coherent sheaf on $\mathbb{P}^n_k$. Then*
(1)$H^i(\mathbb{P}^n_k, \mathcal{F}(q)) = 0$ for all $i > 0$ and $q \geq m - i$ or say that $\mathcal{F}$ is q-regular for all $q \geq m$.
(2)$\mathcal{F}(q)$ is generated by global sections for all $q \geq m$.
(3)$H^0(\mathbb{P}^n_k, \mathcal{F}(q)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{F}(q+1))$ is surjective for all $q \geq m$.

Proof. Induction on n. For $n = 0$, obviously ok. Take a general hyperplane $H \subseteq \mathbb{P}^n_k$ such that the following sequence is exact

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\big|_H \longrightarrow 0$$

By twisting, get exact sequence

$$0 \longrightarrow \mathcal{F}(q-1) \longrightarrow \mathcal{F}(q) \longrightarrow \mathcal{F}\big|_H(q) \longrightarrow 0$$

Take cohomology group, get exact sequence

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \longrightarrow H^i(H, \mathcal{F}\big|_H(m-i)) \longrightarrow H^{i+1}(\mathbb{P}^n_k, \mathcal{F}(m-i-1))$$

As $\mathcal{F}$ is m-regular, the left and right sides are both 0 for all $i > 0$. Thus $\mathcal{F}\big|_H$ is m-regular on H. Then by induction hypothesis, $\mathcal{F}\big|_H$ is q-regular for all $q \geq m$.

For (1), consider another exact sequence

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \longrightarrow H^i(\mathbb{P}^n_k, \mathcal{F}(m-i+1)) \longrightarrow H^i(H, \mathcal{F}\big|_H(m-i+1))$$

Note that by Castelnuovo-Mumford regularity, the left and right sides are both 0, get $\mathcal{F}$ is $(m+1)$-regularity. Complete the proof by induction.

For (3), consider the following commutative diagram with lower row exact

$$\begin{array}{ccccc}
 & & H^0(\mathbb{P}^n_k, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) & \overset{v}{\twoheadrightarrow} & H^0(H, \mathcal{F}\big|_H(m)) \otimes H^0(H, \mathcal{O}_H(1)) \\
 & & \downarrow g & & \downarrow f \\
H^0(\mathbb{P}^n_k, \mathcal{F}(m)) & \xrightarrow{\quad w \quad} & H^0(\mathbb{P}^n_k, \mathcal{F}(m+1)) & \xrightarrow{\qquad u \qquad} & H^0(H, \mathcal{F}\big|_H(m+1))
\end{array}$$

As $\mathcal{F}$ is m-regular, $H^1(\mathbb{P}^n_k, \mathcal{F}(m-1)) = 0$ so that $H^0(\mathbb{P}^n_k, \mathcal{F}(m)) \longrightarrow H^0(H, \mathcal{F}\big|_H(m))$ is surjective. Thus v is also surjective by right exactness of tensor product. Since $\mathcal{F}\big|_H$ is also m-regular, by induction hypothesis, we get f is surjective. Now $v \circ g = f \circ v$ is surjective. Note that $\mathrm{im}(w) \subseteq im(g)$, by exactness of lower row, get g surjective. Complete the proof by induction. While (2) is a direct consequence of (3) since by Theorem 6.3, $\mathcal{F}(q)$ is globally generated for large enough q. □

Proof of Theorem. By Proposition 13.1, suffices to show that there exists integer m_P relative to P such that any ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ with Hilbert polynomial P is m_P-regular. Induction on n. For $n = 0$, obviously ok. Take a general hyperplane $H \subseteq \mathbb{P}^n_k$ such that the following sequence is exact

$$0 \longrightarrow \mathcal{I}(m) \longrightarrow \mathcal{I}(m+1) \longrightarrow \mathcal{I}\big|_H(m+1) \longrightarrow 0$$

Suppose that $P(m) = \sum_{i=0}^{n} a_i \binom{m}{i}$ where $a_i \in \mathbb{Z}$. Then

$$\begin{aligned}\chi(H, \mathcal{I}\big|_H(m+1)) &= \chi(\chi(H, \mathcal{I}\big|_H(m+1)), \mathcal{I}(m+1)) - \chi(\chi(H, \mathcal{I}\big|_H(m+1)), \mathcal{I}(m)) \\ &= \sum_{i=0}^{n} a_i(\binom{m+1}{i} - \binom{m}{i}) \\ &= \sum_{i=0}^{n-1} a_{i+1}\binom{m}{i}\end{aligned}$$

By induction hypothesis, there exist integer m_1 depending on $P_{\mathcal{I}|_H}$ such that $\mathcal{I}\big|_H$ is m_1-regular. Consider the following exact sequence of cohomology groups

$$H^{i-1}(H, \mathcal{I}\big|_H(m+1)) \longrightarrow H^i(\mathbb{P}_k^n, \mathcal{I}(m)) \longrightarrow H^i(\mathbb{P}_k^n, \mathcal{I}(m+1)) \longrightarrow H^i(H, \mathcal{I}\big|_H(m+1))$$

Thus if $i \geq 2$ and $m \geq m_1 - i$, then $H^i(\mathbb{P}_k^n, \mathcal{I}(m)) \cong H^i(\mathbb{P}_k^n, \mathcal{I}(m+1))$. By Serre Vanishing Theorem, get $H^i(\mathbb{P}_k^n, \mathcal{I}(m)) = 0$ for all $i \geq 2$ and $m \geq m_1 - i$.

For $i = 1$, we can only conclude that $H^1(\mathbb{P}_k^n, \mathcal{F}(m)) \longrightarrow H^1(\mathbb{P}_k^n, \mathcal{F}(m+1))$ is surjective for $m \geq m_1 - 1$. Claim that if the map is isomorphic for some $m \geq m_1 - 1$, then it is isomorphic for all $q \geq m$. Consider the following commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}_k^n, \mathcal{F}(m+1)) \otimes H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1)) \arrow[r, two heads, "v"] \arrow[d] & H^0(H, \mathcal{F}\big|_H(m+1)) \otimes H^0(H, \mathcal{O}_H(1)) \arrow[d, two heads, "f"] \\ H^0(\mathbb{P}_k^n, \mathcal{F}(m+2)) \arrow[r, "u"] & H^0(H, \mathcal{F}\big|_H(m+2)) \end{tikzcd}$$

As v and f are both surjective, get u also surjective. Thus the map is isomorphic for $m+1$. By induction, we prove the claim.

With the claim, by Serre Vanishing Theorem, get $H^1(\mathbb{P}_k^n, \mathcal{I}(m)) = 0$ for all $m \geq m_1 + \dim_k(H^1(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1)))$. Set $m_2 := m_1 + \dim_k(H^1(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1)))$. Then $\mathcal{I}$ is m_2-regular. What we need to do is to bound $\dim_k(H^1(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1)))$. Note that

$$\begin{aligned}\dim_k(H^1(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1))) &= \dim_k(H^0(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1))) - \chi(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1)) \\ &\leq \dim_k(H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m_1 - 1))) - \chi(\mathbb{P}_k^n, \mathcal{I}(m_1 - 1))\end{aligned}$$

Thus $\mathcal{I}$ is m_P-regular for some m_P depending on P. $\square$

13.4 Flattening Stratification

Recall that a loally closed immersion $X \longrightarrow Y$ is a composition

$$X \overset{closed}{\longrightarrow} U \overset{open}{\longrightarrow} Y$$

Definition 13.5 (***Stratification***)**.** *A stratification of a scheme S is a locally finite collection $\{S_n\}$ of locally closed subschemes of S such that $S = \sqcup_n S_n$.*

Remark 13.2. *Here, we don't need the countable version of stratification. In the following, when we talk about stratification, we always mean there is a finite disjoint union.*

Theorem 13.4 (***Flattening Stratification***)**.** *Let $f : X \longrightarrow S$ be a strongly projective morphism, $\mathcal{F}$ coherent sheaf on X. Then there exists stratification $S = \sqcup_P S_P$ satisfying that a morphism $g : T \longrightarrow S$ factors as $T \longrightarrow S_P hookrightarrow S$ if and only if the pull back $g^*\mathcal{F}$ on $X \times_S T$ is flat over T with fiberwise Hilbert polynomial P. In particular, such a stratification is unique.*

Example 13.6. *Let $X = S$, $f = \mathrm{id}_S : S \longrightarrow S$ and $\mathcal{F}$ be a coherent sheaf on S. Then there exists stratification $S = \sqcup_r S_r$ such that $\mathcal{F}\big|_{S_r}$ is locally free of rank r. Moreover, a morphism $g : T \longrightarrow S$ factors $T \longrightarrow S_r \hookrightarrow S$ if and only if $g^*\mathcal{F}$ on T is locally free of rank r.*

For the general case, recall that $\mathcal{F}$ is flat over S if and only if $f_*(\mathcal{F}(m))$ is locally free of finite rank for m large enough. To start our proof of the theorem, first consider a result from Commutative Algebra.

Proposition 13.2 (***Generic Flatness***)**.** *Let $f : X \longrightarrow Y$ be a morphism of finite type between noetherian schemes with Y integral, $\mathcal{F}$ coherent sheaf on X. Then there exists an open dense subset $U \subseteq Y$ such that $\mathcal{F}\big|_U$ is flat over U.*

This proposition easily reduces to the following lemma.

Lemma 13.2. *Let A be a noetherian integral domain, B finitely generated A-algebra, M finite B-module. Then there exists $f \in A$ such that $M_f = M \otimes_A A_f$ is a free A_f-module.*

Remark 13.3. *The process of reducing to this lemma has used Corollary 12.1, so we can replace flatness by freeness.*

Proof. Note that if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of B-modules, L_f is free over A_f and N_g is free over A_g, then M_{fg} is free over A_{fg}. As M is finite over B, we can take composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

where $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Then we reduce to $M = B$ is finitely generated A-algebra and integral domain.

Set $K = \mathrm{Frac}(A)$ and $L = \mathrm{Frac}(B)$. Induct on $n = \mathrm{trdeg}(L/K)$. For $n = 0$, $B \otimes_A K$ is a finite dimensional K-vector space. Thus there exists $f \in A$ which is a product of denominators such that B_f is a finite A_f-module. Conclude by taking composition series of B_f.

For $n > 0$, apply Noetherian Normalization Theorem to $B \otimes_A K$. Then there exist $f_1, \cdots, f_n \in B$ such that $B \otimes_A K$ is integral over $K[f_1, \cdots, f_n]$. Thus there exists $f \in A$ which is a product of denominators such that B_f is integral over $A_f[f_1, \cdots, f_n]$. Since finiteness is equivalent to finitely generated and integral, we get B_f is fintie over $A_f[f_1, \cdots, f_n]$. Can find $b_1, \cdots, b_m \in B_f$ generating a maximal free $A_f[f_1, \cdots, f_n]$-submodule. This gives an exact sequence

$$0 \longrightarrow A_f[f_1, \cdots, f_n]^m \longrightarrow B_f \longrightarrow C \longrightarrow 0$$

We are left to deal with C. Conclude by taking composition series of C as $A_f[f_1, \cdots, f_n]$-module and reducing to integral domain with $\mathrm{trdeg}(L/K) < n$. ☐

Proof of Theorem. Step 1: Want a stratification $S = V_1 \sqcup \cdots \sqcup V_t$ such that $\mathcal{F}\big|_{f^{-1}(V_i)}$ is flat over V_i. Can assume that S is reduced and $S_1, \cdots, S_n$ are irreducible components. Take $Y = S_1 \setminus (\cup_{i=2}^n S_i)$. Apply generic flatness, get open dense subset $U \subseteq Y \subseteq S$ such that $\mathcal{F}\big|_{f^{-1}(U)}$ is flat over U. Repeat this process with $V = S \setminus U$. By noetherian induction, we get $S = V_1 \sqcap \cdots \sqcup V_t$, where V_i are integral locally closed subschemes such that $\mathcal{F}\big|_{f^{-1}(V_i)}$ is flat over V_i.

For simplicity, set $f_i := f\big|_{f^{-1}(V_i)}$ and $\mathcal{F}_i := \mathcal{F}\big|_{f^{-1}(V_i)}$. By construction, $\mathcal{F}_i$ is flat over V_i with fiberwise Hilbert polynomial $P_i(m)$. Thus there exist finitely many numerical polynomial $P_1(m), \cdots, P_t(m)$ such that for all $s \in S$, $P_{\mathcal{F}\big|_s}(m) = P_i(m)$ for some i. By Serre Vanishing, for all $o = 1, \cdots, t$, there exists m_i such that $R^j(f_i)_*\mathcal{F}_i(m) = 0$ for all $j > 0$ and $m \geq m_i$. By cohomology and base change, get $H^j(X_i, \mathcal{F}\big|_s(m)) = 0$ for all $j > 0$, $m \geq m_i$ and $s \in V_i$, $(f_i)_*\mathcal{F}_i(m)$ is locally free of rank $P_i(m)$ for all $m \geq m_i$ and $((f_i)_*\mathcal{F}_i)_s \otimes_{\mathcal{O}_{V_i,s}} k(s) \xrightarrow{\sim} H^0(X_i, \mathcal{F}\big|_s(m))$ of dimension $P_i(m)$ is isomorphic for all $m \geq m_i$.

Taking $N >> \max\{m_i\}$, we have gotten some results

(1)There exist finitely many numerical polynomials $P_1, \cdots, P_t$ such that for all $s \in S$, $P_{\mathcal{F}\big|_s}(m) = P_i(m)$ for some i.

(2)$H^j(X_s, \mathcal{F}\big|_s(m)) = 0$ for all $j > 0$ and $m \geq N$.

(3)$(f_*\mathcal{F}(m))_s \otimes_{\mathcal{O}_{S,s}} k(s) \xrightarrow{\sim} H^0(X_s, \mathcal{F}\big|_s(m))$ is isomorphic for all $m \geq N$, which has dimension $P_i(m)$ for some i.

Step 2: Fix n such that $\deg(P_i) \leq n$ for all P_i in (1). Note that we can determine P_i by $n+1$ values $P_i(N), \cdots, P_i(N+n)$. Consider $\mathcal{E}_j := f_*\mathcal{F}(N+j)$ on S, where $j = 0, 1, \cdots, n$. Recall that there exists locally closed subscheme $W_{jr} \hookrightarrow S$ which is universal for the property that $\mathcal{E}_j\big|_{W_{ir}}$ is locally free of rank r. Consider for $P(m)$ numerical polynomial, define $W_P^0 := \cap_{j=0}^n W_{jP(N+j)}$. By definition, a morphism $g : T \longrightarrow S$ factors through W_P^0 if and only if $g^* f_* \mathcal{F}(N+j)$ is locally free of rank $P(N+j)$ for all j. In particular, $s \in W_P^0$ if and only if $P_{\mathcal{F}\big|_s}(m) = P(m)$. Then by (2) and (1), $S = \sqcup_P W_P^0$ but W_P^0 might not have the correct scheme structure.

Step 3: To correct scheme structure, we need to consider more values of P. Set $W_P^k := \cap_{j=0}^{n+k} W_{jP(N+j)}$. Get a sequence

$$W_P^0 \supseteq W_P^1 \supseteq \cdots$$

Note that $W_P^{k+1} \subseteq W_P^k$ is locally closed subscheme and they have same support, get $W_P^{k+1} \hookrightarrow W_P^k$ is closed immersion. Thus the sequence induces a sequence of ideal sheaves

$$\mathcal{I}_P^1 \subseteq \mathcal{I}_P^2 \subseteq \cdots$$

By noetherian condition, the sequence stabilizes to an ideal $\mathcal{I}_P$ cutting out a closed subscheme $S_P \subseteq W_P^0$.

Claim that $S = \sqcup_P S_P$ given the flattening stratification. As $S_P = \cap_{j=0}^{\infty} W_{jP(N+j)}$, by definition, a morphism $g : T \longrightarrow S$ factors through S_P if and only if $g^* f_* \mathcal{F}(N+j)$ is locally free of rank $P(N+j)$ for all $j \geq 0$. Consider the following commutative diagram

$$\begin{array}{ccc} X \times_S T & \longrightarrow & X \\ \Big\downarrow{\scriptstyle f_T} & & \Big\downarrow \\ T & \longrightarrow & S \end{array}$$

By base change without flatness, $g^* f_* \mathcal{F}(N+j) \cong (f_T)_* g^* \mathcal{F}(N+j)$. Thus by Corollary 12.1, a morphism $g : T \longrightarrow S$ factors through S_P if and only if $g^*\mathcal{F}$ on $X \times_S T$ is flat over T with fiberwise Hilbert polynomial P. And $S = \sqcup_P S_P$ gives the flattening stratification. $\square$

13.5 Construction of Hilbert Scheme

Let $S = \operatorname{Spec} k$, $X = \mathbb{P}^n_k$ and $T = \operatorname{Spec} k$. Assume $V \subseteq \mathbb{P}^n_k$ is a closed subscheme with $P_V = P$ and $\mathcal{I}_V \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ is the corresponding ideal sheaf with $P_{\mathcal{I}_V} = Q := P_{\mathbb{P}^n_k} - P$. Choose m_Q as in Castelnuovo-Munford Theorem. Consider the following exact sequence

$$0 \longrightarrow \mathcal{I}_V(m_Q) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(m_Q) \longrightarrow \mathcal{O}_V(m_Q) \longrightarrow 0$$

Get $H^0(\mathbb{P}^n_k), \mathcal{I}_V(m_Q) \subseteq H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m_Q))$ is a subspace of dimension $Q(m_Q)$, which determines $\mathcal{I}_V(m_Q)$ since $\mathcal{I}_V(m_Q)$ globally generated. As $\mathcal{I}_V = \mathcal{I}_V(m_Q) \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(-m_Q)$, the subspace also determines $\mathcal{I}_V$. Thus we have an injection

$$\mathcal{H}ilb^P_{\mathbb{P}^n_k}(k) \hookrightarrow \mathcal{G}r(Q(m_Q), H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m_Q)))(k)$$

Now, with this idea, we take 5 steps to complish our proof.

Construction of Hilbert Scheme. Step 1: For $X \longrightarrow S$ strongly projective, assume it factors through $\mathbb{P} = \mathbb{P}(E)$, where E is a vector bundle on S of rank $n+1$. As fiber product preserve closed immersion, it is clear that there is an injection from set of closed subschemes of $X \times_S T$ flat over T to the set of closed subschemes of $\mathbb{P} \times_S T$ flat over T for any $T \in Sch_S$. Thus we get $\mathcal{H}ilb_{X/S} \hookrightarrow \mathcal{H}ilb_{\mathbb{P}/S}$ as a subfunctor.

Step 2: Let $f : \mathbb{P} \longrightarrow S$ be a projection. For all morphism $g : T \longrightarrow S$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P} \times_S T & \longrightarrow & \mathbb{P} \\ \Big\downarrow{\scriptstyle f_T} & & \Big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Let $V \subseteq \mathbb{P} \times_S T$ be a closed subscheme flat over T with fiberwise Hilbert polynomial P. Denote $Q := P_{\mathbb{P}^n_k} - P$ and take m_Q as in Castelnuovo-Munford Theorem. Consider the following exact sequences

$$0 \longrightarrow \mathcal{I}_V \longrightarrow \mathcal{O}_{\mathbb{P} \times_S T} \longrightarrow \mathcal{O}_V \longrightarrow 0$$

and

$$0 \longrightarrow (f_T)_* \mathcal{I}_V(m_Q) \longrightarrow (f_T)_* \mathcal{O}_{\mathbb{P} \times_S T}(m_Q) \longrightarrow (f_T)_* \mathcal{O}_V(m_Q) \longrightarrow R^1 (f_T)_* \mathcal{I}_V(m_Q) \longrightarrow \cdots$$

By Castelnuovo-Mumford, we know that the long sequence is in fact a short sequence. Moreover, by cohomology and base change, $(f_T)_* \mathcal{I}_V(m_Q)$ and $(f_T)_* \mathcal{O}_V(m_Q)$ are locally free of finite rank, respectively $Q(m_Q)$ and $P(m_Q)$. Thus $(f_T)_*\mathcal{I}_V(m_Q) \hookrightarrow (f_T)_*\mathcal{O}_{\mathbb{P}\times_S T}(m_Q) \cong g^*f_*\mathcal{O}_{\mathbb{P}}(m_Q)$ is a subbundle of rank $Q(m_Q)$. With this fact, get morphism $\mathcal{H}ilb^P_{\mathbb{P}/S} \longrightarrow \mathcal{G}r(Q(m_Q), F)$, where F is the vector bundle associated with $f_*\mathcal{O}_{\mathbb{P}}(m_Q)$.

Step 3: Consider the following commutative diagram

$$\begin{tikzcd} \mathbb{P} \times_S \mathrm{Gr}(Q(m_Q), F) \arrow[r, "p_1"] \arrow[d, "p_1"] & \mathbb{P} \arrow[d, "f"] \\ \mathrm{Gr}(Q(m_Q), F) \arrow[r, "h"] & S \end{tikzcd}$$

Let $U_{Gr} \hookrightarrow h^*F$ be the universal subbundle. Then consider

$$\begin{aligned} p_2^*U_{Gr} &\longrightarrow p_2^*h^* = p_2^*h^*(f_*\mathcal{O}_{\mathbb{P}}(m_Q)) = p_1^*f^*f_*\mathcal{O}_{\mathbb{P}}(m_Q) \\ &\longrightarrow p_1^*\mathcal{O}_{\mathbb{P}(m_Q)} \end{aligned}$$

To some extent, we can view the image as "universal $\mathcal{I}_V(m_Q)$". Set coherent sheaf $\mathcal{G} :=$ coker as "universal $\mathcal{O}_V(m_Q)$". Apply Flattening Stratification Theorem to $\mathcal{G}(-m_Q)$ on $\mathbb{P} \times -S\,\mathrm{Gr}(Q(m_Q), F) \xrightarrow{p_2} \mathrm{Gr}(Q(m_Q), F)$. Get there exists "universal" locally closed subscheme $\mathrm{Hilb}^P_{\mathbb{P}/S} \xhookrightarrow{i} \mathrm{Gr}(Q(m_Q), F)$ such that $i^*\mathcal{G}(-m_Q)$ on $\mathbb{P} \times_S \mathrm{Hilb}^P_{\mathbb{P}/S}$ is flat over $\mathrm{Hilb}^P_{\mathbb{P}/S}$ with Hilbert polynomial P. And we have that $\mathcal{O}_{\mathbb{P}\times_S \mathrm{Gr}(Q(m_Q),F)} \twoheadrightarrow i^*\mathcal{G}(-m_Q)$, whose kernel is an ideal sheaf $\mathcal{I}$. Take the corresponding closed subscheme $U^P_{\mathbb{P}/S}$.

Claim that $(\mathrm{Hilb}^P_{\mathbb{P}/S}, U^P_{\mathbb{P}/S})$ represents $\mathcal{H}ilb^P_{\mathbb{P}/S}$. By construction, $U^P_{\mathbb{P}/S} \in \mathcal{H}ilb^P_{\mathbb{P}/S}(\mathrm{Hilb}^P_{\mathbb{P}/S})$. Conversely, let $V \subseteq \mathbb{P}\times_S T$ be a closed subscheme flat over T with fiberwise Hilbert polynomial P. By step 2, get subbundle $(f_T)_*\mathcal{I}_V(m_Q) \hookrightarrow g^*F$ of rank $Q(m_Q)$. For all morphism $g : T \longrightarrow S$, consider its lifting $\widetilde{g}$ such that the following diagram commutes

$$\begin{tikzcd} T \arrow[r, "\widetilde{g}"] \arrow[dr, "g"'] & \mathrm{Gr}(Q(m_Q), F) \arrow[d, "h"] \\ & S \end{tikzcd}$$

and $\widetilde{g}^*[U_{Gr} \hookrightarrow h^*F] \cong [(f_T)_*\mathcal{I}_V(m_Q) \hookrightarrow g^*F]$. Thus $\widetilde{g}^*\mathcal{G} \cong \mathcal{O}_V(m_Q)$ so that $\widetilde{g}^*\mathcal{G}(-m_Q) \cong \mathcal{O}_V$ flat over T with fiberwise Hilbert polynomial P. By Flattening Stratification, $\widetilde{g}$ factors as $T \longrightarrow \mathrm{Hilb}^P_{\mathbb{P}/S} \xhookrightarrow{i} \mathrm{Gr}(Q(m_Q), F)$.

Step 4: Want to represent $\mathcal{H}ilb^P_{X/S}$ by a subscheme of $\mathrm{Hilb}^P_{\mathbb{P}/S}$. Set $U' := U^P_{\mathbb{P}/S} \cap (X \times_S \mathrm{Hilb}^P_{\mathbb{P}/S}) \subseteq X \times_S \mathrm{Hilb}^P_{\mathbb{P}/S}$. By Flattening Stratification Theorem, there exists closed subscheme $\mathrm{Hilb}^P_{X/S} \xhookrightarrow{j} \mathrm{Hilb}^P_{\mathbb{P}/S}$ satisfying a morphism $T \longrightarrow \mathrm{Hilb}^P_{\mathbb{P}/S}$ factors through $\mathrm{Hilb}^P_{X/S}$ if and only if the pullback $U' \times_{\mathrm{Hilb}^P_{\mathbb{P}/S}} T$ is flat over T with fiberwise Hilbert polynomial P. Set $U^P_{X/S} := j^*U' \subseteq X \times_S \mathrm{Hilb}^P_{X/S}$.

Claim that $(\mathrm{Hilb}^P_{X/S}, U^P_{X/S})$ represents $\mathcal{H}ilb^P_{X/S}$. Similar to the previous claim, by construction, $U^P_{X/S} \in \mathcal{H}ilb^P_{X/S}(\mathrm{Hilb}^P_{X/S})$. Conversely, let $V \subseteq X \times_S T \subseteq \mathbb{P} \times_S T$ be a closed subscheme flat over T with fiberwise Hilbert polynomial P. For all morphism $g : T \longrightarrow S$, there exists a lifting $\widetilde{g}$ such that the following diagram commutes

$$\begin{tikzcd} T \arrow[r, "\widetilde{g}"] \arrow[dr, "g"'] & \mathrm{Hilb}^P_{\mathbb{P}/S} \arrow[d] \\ & S \end{tikzcd}$$

and $V = \widetilde{g}^* U^P_{\mathbb{P}/S}$. Thus $V = \widetilde{g}^*(U^P_{\mathbb{P}/S} \cap (X \times_S \mathrm{Hilb}^P_{\mathbb{P}/S})) = \widetilde{g}^* U' \subseteq X \times_S T$ is flat over T with fiberwise Hilbert polynomial P. By Flattening Stratification Theorem, $\widetilde{g}$ factors through $\mathrm{Hilb}^P_{X/S}$.

Step 5: Now, we are only left to show the properity of Hilbert Scheme. By Construction, $\mathrm{Hilb}^P_{X/S} \hookrightarrow \mathrm{Gr}(Q(m_Q), F)$ is loally closed immersion. Also, $\mathrm{Gr}(Q(m_Q), F)$ is strongly projective over S. Remains to show that $\mathrm{Hilb}^P_{X/S}$ is proper over S. Use valuative criterion

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\widetilde{g \circ a}"] \arrow[d, "a"] & \mathrm{Hilb}^P_{X/S} \arrow[d] \\ \operatorname{Spec}(R) \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & S \end{tikzcd}$$

where $\widetilde{g \circ a}$ is one-to-one corresponding to a closed subscheme $V \subseteq X \times_S \operatorname{Spec} K$ with Hilbert polynomial P. Want $\widetilde{g}$ which is one-to-one corresponding to a closed subscheme $\overline{V} \subseteq X \times_S \operatorname{Spec}(R)$ flat over R with fiberwise Hilbert polynomial P whose restriction to $X \times_S \operatorname{Spec} K$ is V. And this is given by a proposition in Hartshorne Chapter III 9.8. $\square$