代数几何进阶-2025 秋

Contents

1 Deformation Theory

Let's begin with a classical example.

Example 1.1 (***27 lines in smooth cubic hypersurface in*** $\mathbb{P}^3$)**.** *Step 1(moduli space): Consider the moduli space of* $\{$*lines in* $\mathbb{P}^3\}$。*When the base scheme is* $\operatorname{Spec}\mathbb{C}$*, the set is just* $\mathcal{G}r(2,4)(\mathbb{C})$*. Denote* $\operatorname{Gr}(2,4)$ *by* G*, then for any scheme* S

$$\begin{aligned} Mor(S,G) &= \text{ family of 2-dim subspaces in } \mathbb{C}^4 \text{ parametrized by } S \\ &= \mathcal{E} \overset{f}{\hookrightarrow} \mathcal{O}_S^{\oplus 4} \text{ injective as a bundle map, where } \mathcal{E} \text{ is bundle of rank } 2 \end{aligned} \tag{1}$$

In addition, the moduli space is also equal to Hilb^P *where Hilbert polynomial* $P = n+1$*. The natural isomorphism* $\mathcal{H}ilb^P \xrightarrow{\sim} \mathcal{G}r(2,4)$ *can be given as following*

$$\text{"}\longrightarrow\text{"}: \begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & \mathbb{P}^3 \times S \arrow[d, "\pi"] \\ & S \end{tikzcd} \text{ gives } \begin{tikzcd} \pi_*(\mathcal{O}_{\mathbb{P}^3_S}(1)) \arrow[r] \arrow[d, equal] & \pi_*(\mathcal{O}_{\mathbb{P}^3_S}(1)\big|_Z) \\ \mathcal{O}_S^{\oplus 4} \arrow[r, two heads] & \mathcal{F} \end{tikzcd}$$
$$\text{"}\longleftarrow\text{"}: \mathcal{E} \hookrightarrow \mathcal{O}_S^{\oplus 4} \twoheadrightarrow \mathcal{F} \text{ gives } \begin{tikzcd} \operatorname{Proj}(Sym\mathcal{O}_S^{\oplus 4}) \arrow[d, equal] & \operatorname{Proj}(Sym\mathcal{F}) \arrow[l, hook'] \arrow[d, equal] \\ \mathbb{P}^3 \times S & Z \end{tikzcd} \tag{2}$$

Step 2(intersection number): Assume $X \subset \mathbb{P}^3$ *is a cubic hypersurface defined by section* $F \in H^0(\mathbb{P}^3, \mathcal{O}(3))$*. Consider the following diagram*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "\rho"] & \mathbb{P}^3 \times \operatorname{Hilb}^P \arrow[r] \arrow[d] & \mathbb{P}^3 \\ & \operatorname{Hilb}^P & \end{tikzcd} \tag{3}$$

For each point $x \in \operatorname{Hilb}^P$*, we have that*

$$Z_x \subseteq X \iff i^*F\big|_{Z_x} = 0 \iff \rho_* i^* F \text{ vanishes at } x \text{ as a section of } \rho_* i^* \mathcal{O}(3) \tag{4}$$

In particular, when x *is a k-point. Then* Z_x *is a line in* $\mathbb{P}^3_k$*.*

Step 3(deformation theory): The key is that Zariski tangent space is 0*-dim everywhere if and only if 1st order deformations are trivial. Recall that for scheme* M *and k-point* $x \in M(k)$*, the Zariski tangent space is defined as following*

$$T_x M := \left\{ \begin{tikzcd} \operatorname{Spec}(k[t]/(t^2)) \arrow[rr] & & M \\ & \operatorname{Spec} k \arrow[ul, "t\mapsto 0"] \arrow[ur, "x"'] & \end{tikzcd} \right\} \tag{5}$$

It is clear that the set of 1st order deformations of L *defined below is one-to-one corresponding to* $T_{[L]}\mathcal{H}_X$*, where* $\mathcal{H}_X$ *is the moduli space of lines in* X *and* $[L]$ *is the k-point corresponding to* L*. In fact, by step 2,* $\mathcal{H}_X = (\rho_* i^* F)^{-1}(0) \subset \operatorname{Hilb}^P$*. And the correspondence can be precisely* *written as following*

$$\begin{tikzcd} L' \arrow[r, hook] \arrow[d, "flat"] & \mathbb{P}^3 \times \operatorname{Spec}(k[t]/(t^2)) \arrow[dl] \\ \operatorname{Spec}(k[t]/(t^2)) & \end{tikzcd} \rightsquigarrow \operatorname{Spec}(k[t]/(t^2)) \longrightarrow \mathcal{H}_X \tag{6}$$

and

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d] & \mathbb{P}^3 \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd} \rightsquigarrow \operatorname{Spec} k \longrightarrow \mathcal{H}_X \tag{7}$$

The commutativity in 1st order deformation is equivalent to the commutativity in Zariski tangent space.

Definition 1.1. *Given line $L \subset X$, where $X \subset \mathbb{P}^3$ is a cubic hypersurface. A 1st order deformation of L to L' is a commutative diagram*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L' \arrow[r, hook] \arrow[d, "flat"] & \mathbb{P}^3 \times \operatorname{Spec}(k[t]/(t^2)) \arrow[dl] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec}(k[t]/(t^2)) & \end{tikzcd} \tag{8}$$

1.1 1st deformation of closed subschemes

From now on, we would denote $k[t]/(t^2)$ by $k[\varepsilon]$, called dual numbers. Fix scheme X and closed subscheme $Z \subset X$.

Definition 1.2. *A 1st order deformation of Z to Z' is a commutative diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[r, hook] \arrow[d, "flat"] & X \times \operatorname{Spec} k[\varepsilon] \arrow[dl] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon] & \end{tikzcd} \tag{9}$$

Proposition 1.1. *Let $0 \to J \to A' \to A \to 0$ be an exact sequence, where $J \subseteq A'$ is an ideal such that $J^2 = 0$. Then for any A'-module M', M' is flat over A' if and only if $M := M' \otimes_{A'} A$ is flat over A and $J \otimes_A M = J \otimes_{A'} M' \hookrightarrow M'$ is injective.*

Theorem 1.1. *There is a one-to-one correspondence between 1st deformations of Z in X and $H^0(Z, \mathcal{N}_{Z/X})$, where $\mathcal{N}_{Z/X} := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_{Z/X}/\mathcal{I}^2_{Z/X}, \mathcal{O}_Z)$ is the normal sheaf.*

Proof. Reduce to affine case, assume that $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A/I)$. Denote $A' = A \otimes_k k[\varepsilon]$. Then a 1st deformation of Z can be expressed by following conditions

- $I' \subset A'$ ideal
- $A'/I' \otimes_{k[\varepsilon]} k \xrightarrow{\sim} A/I$
- A'/I' flat over $k[\varepsilon]$

By Proposition 1.1, we can rewrite the conditions

- $I' \subset A'$ ideal
- $A'/I' \otimes_{k[\varepsilon]} k \xrightarrow{\sim} A/I$
- A'/I' flat over $k[t]$ under map $t \longmapsto \varepsilon$
- $A/I \xrightarrow{\cdot\varepsilon} A'/I'$ injective

inducing a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\cdot\varepsilon"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\cdot\varepsilon"] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/I \arrow[r, "\cdot\varepsilon"] \arrow[d] & A'/I' \arrow[r] \arrow[d] & A/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{10}$$

By chasing diagram, we get an A-module homomorphism $\theta : I \longrightarrow A/I$. In fact, the correspondence between the diagram and θ is bijective. Thus the set of 1st deformations is just

$$\begin{aligned}
\mathrm{Hom}_A(I, A/I) &= \mathrm{Hom}_{A/I}(I/I^2, A/I) \\
&= \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2, \mathcal{O}_Z) \\
&= \Gamma(Z, \mathcal{N}_{Z/X})
\end{aligned} \tag{11}$$

□

1.2 1st deformation of vector bundles

Fix scheme X and vector bundle $\mathcal{E}$ on X. Set $X' = X \times_k k[\varepsilon]$.

Definition 1.3. *A 1st deformation of $\mathcal{E}$ is a vector bundle $\mathcal{E}'$ on X', flat over* $\operatorname{Spec} k[\varepsilon]$*, with isomorphism $\mathcal{E}'\big|_X \xrightarrow{\sim} \mathcal{E}$.*

Take affine open covering $\mathcal{U} = \{U_i = \operatorname{Spec} A_i\}$ trivializing $\mathcal{E}$. For any 1st deformation $\mathcal{E}'$, as $\mathcal{E}'\big|_X$ is isomorphic to $\mathcal{E}$, $\mathcal{E}'\big|_{U_i}$ are also trivial. Denote $\mathcal{E}'_0$ to be the trivial deformation. Given $\mathcal{E}'$, there are isomorphisms $\varphi_i : \mathcal{E}'\big|_{U_i} \to \mathcal{E}'_0\big|_{U_i}$. Then for any i, j, $\varphi_i^{-1}\big|_{U_i \cap U_j} \circ \varphi_j\big|_{U_i \cap U_j} \in Aut(\mathcal{E}'\big|_{U_i \cap U_j})$. Thus to figure out what the set of 1st deformations likes, we would firstly consider automorphisms of deformation.

Still consider locally, assume $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} A'$, where $A' = A \otimes_k k[\varepsilon]$. Note that there is an exact sequence

$$0 \longrightarrow (\varepsilon) \longrightarrow k[\varepsilon] \longrightarrow k \longrightarrow 0 \tag{12}$$

Tensor by M' over $k[\varepsilon]$, get exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot\varepsilon} M' \longrightarrow M \longrightarrow 0 \tag{13}$$

For an automorphism φ of $\mathcal{E}'$, consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\cdot\varepsilon"] \arrow[d, equal] & M' \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\cdot\varepsilon"] & M' \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \tag{14}$$

inducing a A-module homomorphism $\theta : M \to M$, given by $a \mapsto \varphi(a') - a'$, where a' is a lift of a along $M' \to M$. In fact, $\varphi \leftrightsquigarrow \theta$ is bijective. Thus the set of automorphisms of $\mathcal{E}'$ as deformation is just $\mathrm{Hom}_A(M, M)$. Globalized, get $H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$.

Back to deformation, for $\varphi_{ij} := \varphi_i^{-1} \circ \varphi_j\big|_{U_{ij}} \in Aut(\mathcal{E}'\big|_{U_i \cap U_j}) = H^0(U_i \cap U_j, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$. Assume that φ_{ij} corresponds to θ_{ij}. Then

- $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \leftrightsquigarrow \theta_{ij} + \theta_{jk} = \theta_{ik}$
- modify $\varphi_i = \varphi_i' \circ \psi_i \leftrightsquigarrow \theta_{ij} = \theta_{ij}' + \rho_i - \rho_j$

Conclude that 1st order deformations of $\mathcal{E}$ in X is $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$.

Remark 1.1. *If generalize vector bundles to coherent sheaves, the answer is* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ *since each element in* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ *gives an extension*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow 0 \tag{15}$$

1.3 1st order deformation of affine schemes smooth over k

Lemma 1.1. *Let* $\mathrm{Spec}\, A \to \mathrm{Spec}\, k$ *be a smooth morphism and* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *be any* k*-morphism. Given extension of* k*-algebras*

$$0 \longrightarrow J \longrightarrow B' \longrightarrow B \longrightarrow 0, \text{ where } J^2 = 0 \tag{16}$$

such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[r] \arrow[d, hook] & \mathrm{Spec}\, A \arrow[d] \\ \mathrm{Spec}\, B' \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\, k \end{tikzcd} \tag{17}$$

Then there exists morphism $\mathrm{Spec}\, B' \to \mathrm{Spec}\, A$ *making the diagram commutative.*

Proof. Choose $P = k[x_1, \cdots, x_n] \twoheadrightarrow A$, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\rho"] & P \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{18}$$

where φ exists since P is free over k and hence projective. By diagram chasing, we find that φ gives P-module structure on J and ϕ gives A-module structure on both B and J, which is independent of choosing of φ.

To get a map $A \to B'$, want $I \to J$ to be zero. So we need to change φ. For two different φ_1 and φ_2, set $\delta := \varphi_1 - \varphi_2$, which is in fact a homomorphism from P to J. We have that

$$\begin{aligned}\delta(ab) &= \varphi_1(a)\varphi_1(b) - \varphi_2(a)\varphi_2(b)\\ &= \varphi_1(a)\varphi_1(b) - \varphi_1(a)\varphi_2(b) + \varphi_1(a)\varphi_2(b) - \varphi_2(a)\varphi_2(b)\\ &= \varphi_1(a)\delta(b) - \delta(a)\varphi_2(b)\\ &= a\delta(b) - \delta(a)b\end{aligned} \tag{19}$$

Thus δ satisfies Laibniz's Law and hence $\delta \in Der_k(P, J)$ is a derivation.

As A is smooth over k, there is an exact sequence of locally free sheaves

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0 \tag{20}$$

Apply $\mathrm{Hom}_A(\cdot, J)$, get

$$\begin{tikzcd} \mathrm{Hom}_A(\Omega_{P/k} \otimes_P A, J) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Hom}_A(I/I^2, J) \arrow[d, equal] \\ \mathrm{Hom}_P(\Omega_{P/k}, J) \arrow[d, equal] & \mathrm{Hom}_P(I, J) \\ Der_k(P, J) & \end{tikzcd} \tag{21}$$

By surjectivity, we can find appropriate δ such that $I \to J$ induced by $\varphi + \delta$ is zero map. $\square$

Definition 1.4. *Let $X \in Sch_k$. A 1st order deformation of X is a fibered diagram*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d, "flat"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} k[\varepsilon] \end{tikzcd} \tag{22}$$

Two deformations are isomorphic if there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, hook] & \\ X_1' \arrow[rr, "\sim"] \arrow[dr] & & X_2' \arrow[dl] \\ & \operatorname{Spec} k[\varepsilon] & \end{tikzcd} \tag{23}$$

Remark 1.2. *To describe deformations, in fact we need a strengthened version of Lemma 1.1 that for $X \to Y$ smooth morphism of affine schemes and B, B' same as above, if there is a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \operatorname{Spec} B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{24}$$

then there exists $\operatorname{Spec} B' \to X$ making the diagram commutative.

Claim that deformations are locally trivial. Assume there is a deformation of X to X'. Locally, X and X' are affine. Take $X[\varepsilon] = X \times_{k[\varepsilon]} k$. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & X' \arrow[d, "smooth"] \\ X[\varepsilon] \arrow[r] \arrow[ur, dashed, "u"] & \operatorname{Spec} k[\varepsilon] \end{tikzcd} \tag{25}$$

By strengthened version of Lemma 1.1, there exists $u : X[\varepsilon] \to X'$ making the diagram commutative. Assume that $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$. Note that X' and $X[\varepsilon]$ are both extension, consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\cdot\varepsilon"] \arrow[d, equal] & B \arrow[r] \arrow[d, "u^\sharp"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "\cdot\varepsilon"] & A[\varepsilon] \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{26}$$

By diagram chasing, it is easy to show that $u^\sharp$ is isomorphic. Thus deformations are locally trivial.

Back to general case, now local extension always trivial. Take affine open covering $\{U_i\}$ with local extensions X'_i. To glue up a global extension, we need cocycle condition that the following diagram commutes

$$\begin{tikzcd} X'_i \arrow[r, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & X'_j \arrow[d, "\varphi_{jk}"] \\ & X'_k \end{tikzcd} \tag{27}$$

Thus we should firstly figure out the automorphisms of deformation. Locally set $X = \operatorname{Spec} B$ and $X' = \operatorname{Spec} B'$. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\cdot\varepsilon"] \arrow[d, equal] & B' \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[d, bend left, "\mathrm{id}+\delta"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r, "\cdot\varepsilon"] & B' \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{28}$$

where $\delta \in \operatorname{Hom}_B(\Omega_{B/k}, B)$ the tangent vector fields. Thus automorphisms are $H^0(X, \mathcal{T}_X)$

Given deformation of X to X'. Choose affine open covering $\mathcal{U} = \{U_i\}$ of X. On each U_i, choose isomorphism $\varphi_i : X'\big|_{U_i} \to X[\varepsilon]$. Then on each U_{ij}, we get different isomorphisms

$$\begin{tikzcd} X'\big|_{U_{ij}} \arrow[r, bend left, "\varphi_i"] \arrow[r, bend right, "\varphi_j"'] & X[\varepsilon]\big|_{U_{ij}} \end{tikzcd} \tag{29}$$

Their difference $(\delta_{ij})_{i,j}$ satisfying cocycle condition gives an element in $\check{H}^1(\mathcal{U}, \mathcal{T}_X)$.

In conclusion, infinitesimal automorphisms of X is $H^0(X, \mathcal{T}_X)$ and 1st order deformations of X is $H^1(X, \mathcal{T}_X)$.

Remark 1.3. *Given a flat family $\mathfrak{X} \longrightarrow S$ and k-point $x \in S$. Take fiber $X := \mathfrak{X}_x$ smooth over k. There is a Kodaira-Spencer map from tangent space $T_{x,S} \cong \operatorname{Hom}_X(\operatorname{Spec} k[\varepsilon], S)$ to deformations $H^1(X, \mathcal{T}_X)$.*

Example 1.2. *(1)Let* $X = \mathbb{P}^n$*, then* $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = 0$*. There are no deformations.*
(2)Let C *be a smooth curve of genus* g*. Then* $H^1(C, \mathcal{T}_C) = 3g - 3$ *for* $g \geq 2$*.*

2 Obstruction Theory

Let's begin with a bady example.

Example 2.1. *Let* V *be a vector space of finite dimension and* $T : V \to V$ *be a linear map,* $W \subseteq V$ T*-invariant subspace. Wonder if there exists* T*-invariant complement. Consider the following exact sequence*

$$0 \longrightarrow W \longrightarrow V \xrightarrow{\pi} V/W \longrightarrow 0 \tag{30}$$

Fix $\sigma : V/W \to V$ *such that* $\mathrm{id}_{V/W}$*. Thus there exists* T*-invariant complement if and only if there exists* σ *such that* $\sigma\overline{T} = T\sigma$*. We say that* $\sigma\overline{T} - T\sigma \in \mathrm{Hom}_k(V/W, W)$ *is an obstruction.*

For two different sections σ *and* σ'*, get difference* $\theta = \sigma - \sigma' \in \mathrm{Hom}_k(V/W, W)$*. Define* $\varphi : \mathrm{Hom}_k(V/W, W) \to \mathrm{Hom}_k(V/W, W) \quad \theta \mapsto \theta\overline{T} - T\theta$*. Then question change to if there exists* $\sigma + \theta$ *such that* $\sigma + \theta$ *is a splitting of* $k[T]$*-module.*

$$\begin{aligned} &\iff \sigma\overline{T} - T\sigma \in \mathrm{im}(\varphi) \\ &\iff [\sigma\overline{T} - T\sigma] = 0 \textit{ in } \mathrm{coker}(\varphi) \end{aligned} \tag{31}$$

which is called a obstruction class.

As its name, obstruction is something stopping us from extension. For higher order deformation, we globalize the place where we extend to. Here is an example.

Example 2.2 (***Extension of maps***)**.**

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "{(0,0)}"] \arrow[d] & V(y^2 - x^3) \subset \mathbb{A}^2_k \\ \mathrm{Spec}\, k[\varepsilon] \arrow[ur, "{(at,bt)}"'] & \end{tikzcd} \tag{32}$$

Each extension is determined by a tangent vector $v = (a, b)$*. If we moreover extend map to* $\mathrm{Spec}(k[t]/(t^3))$*, then*

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "{(0,0)}"] \arrow[d] & V(y^2 - x^3) \subset \mathbb{A}^2_k \\ \mathrm{Spec}(k[t]/(t^3)) \arrow[ur, "{(a_1t+a_2t^2, b_1t+b_2t^2)}"'] & \end{tikzcd} \tag{33}$$

satisfies that $(a_1t + a_2t^2)^3 - (b_1t + b_2t^2)^2 \equiv 0 \pmod{t^3}$*. Thus* $b_1 = 0$ *and hence tangent vect should be contained in* x*-axis.*

Replace $V(y^2 - X^3)$ *by general* $V(g(x, y))$ *singular at* $(0, 0)$*. Given extension*

$$\begin{tikzcd} \mathrm{Spec}(k[t]/(t^r)) \arrow[r, "{(x_r(t), y_r(t))}"] \arrow[d] & V(g(x,y)) \subset \mathbb{A}^2_k \\ \mathrm{Spec}(k[t]/(t^{r+1})) \arrow[ur, "{(x_{r+1}(t), y_{r+1}(t))}"'] & \end{tikzcd} \tag{34}$$

where $x_{r+1}(t) = x_r(t) + at^r$ *and* $y_{r+1} = y_r(t) + bt^r$*, satisfies that* $g(x_{r+1}(t), y_{r+1}(t)) \equiv 0 \pmod{t^{r+1}}$*. Take Taylor expansion at* $(0,0)$

$$g(x_{r+1}(t), y_{r+1}(t)) \equiv g(x_r(t), y_r(t)) + \frac{\partial g(0,0)}{\partial x} t^r + \frac{\partial g(0,0)}{\partial y} bt^r \pmod{t^{r+1}} \tag{35}$$

Hence we need $\frac{\partial g(0,0)}{\partial x} = \frac{\partial g(0,0)}{\partial y} = 0$ *and* $g(x_r(t), y_r(t)) \equiv 0 \pmod{t^{r+1}}$*.*

2.1 Higher order deformation

Set $k = \overline{k}$ a algebraically closed field. Consider k-algebras.

Definition 2.1. *An extension of artinian local k-algebras with residue field k,*

$$0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0 \tag{36}$$

is called a star extension if $\mathfrak{m}_{A'} J = 0$*.*

Remark 2.1. *With* $\mathfrak{m}_{A'} J = 0$*,* A'*-module structure on* J *descends to a* k*-vector space structure.*

Usually, our question is that given object over A, what's the obstruction to lift it to an object over A', there are how many deformationS and what are automorphisms of extension.

2.2 Higher order deformation of vector bundles

Definition 2.2. *Let* $X_0 \in Sch_k$ *be a* k*-scheme. Set* $X = X_0 \times_k A$*. Given* $\mathcal{E}$ *vector bundle on* X *and* X' *extension of* X *over* A'*. An extension of* $\mathcal{E}$ *is a vector bundle* $\mathcal{E}'$ *on* X' *such that* $\mathcal{E}'\big|_X \cong \mathcal{E}$*.*

For automorphism, locally consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \otimes_A J \arrow[r] \arrow[d, equal] & \mathcal{E}' \arrow[r] \arrow[d, "\varphi"] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{E} \otimes_A J \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E} \arrow[r] & 0 \end{tikzcd} \tag{37}$$

where φ one-to-one corresponds to elements in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes_A J)$. Generalized, the set of automorphisms is $H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$.

Definition 2.3. *Let* G *be a group,* S *a set.* S *is called a* G*-torsor if* S *is nonempty and* G *acts on* S *freely transitively.*

Definition 2.4. *Let* G *be a group,* S *a set.* S *is called a* G*-pseudo torsor if* S *is either a* G*-torsor or empty set.*

For how many deformations, as the argument of 1st order deformation, gluing imformation corresponds to element in $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$. Thus the set of deformations is a $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$-pseudo torsor.

For obstruction, as local extension exists, take affine open covering $\mathcal{U} = \{U_i\}$ of X' trivializing $\mathcal{E}$. Now on each U_i, we get an extension $\mathcal{E}'_i$. And on each U_{ij}, there is isomorphism

$\varphi_{ij} : \mathcal{E}'_i\big|_{U_{ij}} \to \mathcal{E}'_j\big|_{U_{ij}}$. To glue up, we need φ_{ij} satisfy cocycle condition. Set $\rho_{ijk} = \varphi_{ik}^{-1} \circ \varphi_{jk} \circ \varphi_{ij}$, corresponding to $\theta_{ijk} \in H^0(U_{ijk}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$. Thus we get an element $[\theta]$ in $H^2(U_{ij}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$ independent of choosing of $\mathcal{E}'_i$, called the obstruction.

If $[\theta] = [0]$, then there exists $(\delta_{ij})_{i,j} \in \prod_{i,j} \Gamma(U_{ij}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$ mapping to $[\theta]$. As $H^0(U_{ij}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_A J)$ is the set of automorphisms, each δ_{ij} gives an automorphism $\psi_{ij} : \mathcal{E}'_j\big|_{U_{ij}} \to \mathcal{E}'_j\big|_{U_{ij}}$. Consider $\psi_{ij}^{-1} \circ \varphi_{ij} : \mathcal{E}'_i\big|_{U_{ij}} \to \mathcal{E}'_j\big|_{U_{ij}}$. It is clear that these new maps satisfy cocycle condition so that we can glue up an extension.

Example 2.3. *(1)Let X_0 be a smooth projective variety, $\mathcal{L}_0$ line bundle on X_0. Even though obstruction space $H^2(X_0, \mathcal{E}nd_{\mathcal{O}_{X_0}}(\mathcal{L}_0) \otimes_k J) = H^2(X_0, \mathcal{O}_{X_0} \otimes_k J)$ needn't be zero, higher deformations of $\mathcal{L}$ are unobstructed.*
(2)Let X_0 be a smooth projective variety, $\mathcal{E}_0$ vector bundle on X_0. There is a trace map $H^2(X_0, \mathcal{E}nd_{\mathcal{O}_{X_0}}(\mathcal{E}_0)) \xrightarrow{tr} H^2(X_0, \mathcal{E}nd_{\mathcal{O}_{X_0}}(\det \mathcal{E}_0))$ mapping $ob(E)$ to $ob(\det \mathcal{E})$.

2.3 Higher order deformation of closed subschemes

Given X_0, X, X' as before. Let $Y \subset X$ be a closed subscheme and $Y_0 = Y \times_A k$. Local extension of Y may not exist in general.

Obstruction is in $H^1(Y_0, \mathcal{N}_{Y_0/X_0} \otimes_k J)$, deformations are a $H^0(Y_0, \mathcal{N}_{Y_0/X_0} \otimes_k J)$-pseudo torsor and identity map is the only automorphism.

Proposition 2.1. *If Y is a local complete intersection and X is smooth, then local extension exists.*

Remark 2.2. *For deformations for local complete intersection, refer to Vistoli's note.*

2.4 Higher order deformation of abstract smooth schemes

In this case, local extension exists.

Obstruction is in $H^2(X_0, \mathcal{T}_{X_0} \otimes_k J)$, deformations are a $H^1(X_0, \mathcal{T}_{X_0} \otimes_k J)$-pseudo torsor and automorphisms are $H^0(X_0, \mathcal{T}_{X_0} \otimes_k J)$.

Definition 2.5. *Let $f : X \to Y$ be a morphism of schemes. We say that f is formally smooth if for any morphism $S = \operatorname{Spec} A \hookrightarrow S' = \operatorname{Spec} A'$ with ideal $J^2 = 0$ and commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & X \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{38}$$

there exists morphism $S' \to X$ making the diagram commutative.

There are some facts about formal smoothness.

Proposition 2.2. *(1)smooth $\iff$ of finitely presented and formally smooth.*
(2)If X, Y are noetherian, suffices to check for cases that S' is spectrum of some artinian local ring.

Corollary 2.1. *Let $Y \subset \mathbb{P}^n$ be a local complete intersection. If $H^1(Y, \mathcal{N}_{Y/\mathbb{P}^n}) = 0$, then hilbert scheme is smooth at $[Y]$ with fiber of dimension $\dim_k H^0(Y, \mathcal{N}_{Y/\mathbb{P}^n})$.*

3 Obstruction Theory of Moduli Space

3.1 Obstruction of a local ring

Let S be a scheme. Recall that in a fine moduli space M, family over S is equivalent to a morphism $S \to S$.

Definition 3.1. *Let M be a fine moduli space, $x \in M$ a k-point. A deformation of object x is a commutative diagram*

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, dashed] & M \\
\operatorname{Spec} k \arrow[u, hook, "\mathfrak{m}_A"] \arrow[ur, "x"'] &
\end{tikzcd} \tag{39}$$

where A is an artinian local ring with residue field k.

As homomorphism between stalks is local, the deformation diagram factors through $\operatorname{Spec} \mathcal{O}_{M,x}$ as following

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, dashed] & \operatorname{Spec} \mathcal{O}_{M,x} \arrow[r] & M \\
\operatorname{Spec} k \arrow[u, hook, "\mathfrak{m}_A"] \arrow[ur, "x"'] & &
\end{tikzcd} \tag{40}$$

Hence deformation of object x is equivalent to deformation of a local ring.

Denote $C := \mathcal{O}_{M,x}$. For star extension of artinian local rings $0 \to J \to A' \to A \to 0$, consider if we can extend $\operatorname{Spec} A \to \operatorname{Spec} C$ to $\operatorname{Spec} A' \to \operatorname{Spec} C$.

Lemma 3.1. *If C is regular, then for any star extension of A to A', in the following commutative diagram*

$$\begin{tikzcd}
\operatorname{Spec} A' \arrow[r, dashed] & \operatorname{Spec} C \\
\operatorname{Spec} A \arrow[u, hook] \arrow[ur] & \operatorname{Spec} k \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} \tag{41}$$

$\operatorname{Spec} A' \to \operatorname{Spec} C$ always exists.

Remark 3.1. *In this case, we would say there is no obstruction.*

Proof. Consider completion $\widehat{C}$ of C. Since completion of regular local ring is a regular local ring (Atiyah Proposition 11.24), and by Corollary 28.2 in *Commutative Algebra*, Matsumura, $\widehat{C} \cong k[[x_1, \cdots, x_n]]$ for some n.

In addition, since A is artinian and $C \to A$ is local homomorphism, $C \to A$ factors through $C \twoheadrightarrow C/\mathfrak{m}_C^t$ for large enough t. Hence the following diagram commutes

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, two heads] & A & A' \arrow[l, two heads] & \\
& C/\mathfrak{m}_C^t \arrow[u] & \widehat{C} \arrow[l] \arrow[u, dashed] & k[x_1, \cdots, x_n] \arrow[l, hook'] \arrow[ul, dashed]
\end{tikzcd} \tag{42}$$

As $k[x_1, \cdots, x_n]$ is projective k-algebra, there exists $k[x_1, \cdots, x_n] \to A'$. Moreover, since A' is complete, the map extends to $\widehat{C} \to A'$. As A' is artinian local, factors through some $C/\mathfrak{m}^r$ for large enough r. Take $t = r$ so that we get $C \to A'$. $\square$

For C not regular, we need to define obstruction theory.

Definition 3.2. *An obstruction theory of C is a vector space and an assignment φ*

$$\left.\begin{array}{r} 0 \to J \to A' \to A \to 0 \\ C \xrightarrow{u} A \\ \mathfrak{m}_{A'} J = 0 \end{array}\right\} \xmapsto{\varphi} ob(A', u) \in V \otimes_k J \tag{43}$$

satisfies the following conditions

- $ob(A', u) = 0$ *iff* $\exists$ *lifting of* u,
$$\begin{tikzcd} C \arrow[r, "u"] \arrow[dr, dashed] & A \\ & A' \arrow[u, two heads] \end{tikzcd}$$

- *Functorial in the sense that* $\forall K \subset J$ *k-vector subspace, in the following commutative diagram*
$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A' & \longrightarrow & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J/K & \longrightarrow & A'/K & \longrightarrow & A & \longrightarrow & 0 \end{array}$$
$\varphi(u, A') \mapsto \varphi(u, A'/K)$ *under* $V \otimes J \to V \otimes J/K$

3.2 Canonical obstruction theory

There is also a canonical obstruction theory. Assume there exists $P \twoheadrightarrow C$, where P is regular local with residue k. Further assume $I := \ker(P \twoheadrightarrow C) \subset \mathfrak{m}_P^2$.

$$0 \longrightarrow I \longrightarrow P \longrightarrow C \longrightarrow 0 \tag{44}$$

Set $V_C := (I \otimes_P k)^\vee = (I/\mathfrak{m}_P I)^\vee$.

To give assignment φ, consider the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & P \arrow[r, two heads] \arrow[d, "\widetilde{u}"] & C \arrow[r] \arrow[d, "u"] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & J \arrow[r] & A' \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd} \tag{45}$$

As P is regular local ring, by Lemma 3.1, $\widetilde{u}$ always exist which is a local homomorphism. As $J^2 = 0$, f in fact is map $\widetilde{f} : I/\mathfrak{m}_P I \to J$ corresponding to an element $\varphi(A', u) \in (I/\mathfrak{m}_P)^\vee \otimes_k J$. Then $\widetilde{f}$ is zero map if and only if there exists lifting.

For diffrent $\widetilde{u_1}, \widetilde{u_2}$, set $\theta = \widetilde{u_1} - \widetilde{u_2}$. For any $h \in I$, consider $\theta(h) = \sum \theta(\alpha_i \beta_i)$ where $\alpha_i, \beta_i \in \mathfrak{m}_P$. As θ is a derivation, $\theta(h) = \sum \alpha_i \theta(\beta_i) + \theta(\alpha_i)\beta_i$. While $\theta(\alpha_i), \theta(\beta_i) \in J$, we get $\theta(h) = 0$ so that f is well defined.

3.3 Geometric point of view

Let W be a smooth variety over k, $E \to W$ vector bundle. Assume $s : W \to E$ is section of E. Set $X = V(s) \subset W$. For $x \in X$, in the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A' \arrow[rr, dashed, "\widetilde{u}"] & & W \\ \operatorname{Spec} A \arrow[u, hook] \arrow[rr, "u"] & & X \arrow[u, hook] \\ & x \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd} \tag{46}$$

$\widetilde{u}$ factors through X if and only if $\widetilde{u}^* s = 0$ as sections of $\widetilde{u}^* E$. Note that there is an exact sequence

$$0 \longrightarrow \widetilde{J} \longrightarrow \mathcal{O}_{\operatorname{Spec} A'} \longrightarrow \mathcal{O}_{\operatorname{Spec} A} \longrightarrow 0 \tag{47}$$

Tensor $\widetilde{u}^* E$ over $\mathcal{O}_{\operatorname{Spec} A'}$, get exact sequence

$$0 \longrightarrow \widetilde{J} \otimes \widetilde{u}^* E \longrightarrow \widetilde{u}^* E \xrightarrow{v} \mathcal{O}_{\operatorname{Spec} A} \otimes \widetilde{u}^* E \longrightarrow 0 \tag{48}$$

As $X = V(s)$, $v \circ \widetilde{u}^* s$ is zero map. Hence $\widetilde{u}^* s$ is in fact in $\Gamma(J \otimes \widetilde{u}^* E)$.

Lemma 3.2. *On X, we can well define map $TW\big|_X \xrightarrow{\nabla s} E\big|_X$.*

Reason 3.1. *Choose frame $\sigma_1, \cdots, \sigma_k$ for E. Write $s = \sum s_i \sigma_i$. Then for $v \in \Gamma(u, Tw)$, $\nabla_V(s) = \sum \nabla_V(s_i)\sigma_i$. Take another frame, for example $g_1\sigma_1, \cdots, g_k\sigma_k$. Then $s = \sum \frac{s_i}{g_i} g_i \sigma_i$ so that*

$$\begin{aligned} \nabla_v(S) &= \sum \nabla_v(\frac{s_i}{g_i}) g_i \sigma_i \\ &= \sum \nabla_v(s_i) g_i^{-1} g_i \sigma_i + \sum \nabla_v(g_i^{-1}) s_i g_i \sigma_i \\ &= \sum \nabla_v(s_i)\sigma_i \end{aligned} \tag{49}$$

since s_i vanish on X.

Remark 3.2. *Dually, there is a well defined map*

$$E^\vee\big| \longrightarrow \Omega^1_W\big|_X \quad \sigma \longmapsto d\langle \sigma, s \rangle \tag{50}$$

factoring through I/I^2, where $I = I_{X/W}$. And there is a complex

$$E^\vee\big|_X \xrightarrow{\langle \cdot, s \rangle} I/I^2 \xrightarrow{d} \Omega^1_W\big|_X \longrightarrow \Omega^1_X \longrightarrow 0 \tag{51}$$

When $I \not\subset \mathfrak{m}_x^2$, $\widetilde{u}^* s$ in $\Gamma(\widetilde{u}^* E \otimes J)$ is not well defined. In fact, $\widetilde{u}^* E \otimes J$ is supported on $\operatorname{Spec} k(x)$*. Can directly write $\widetilde{u}^* E \otimes J$ as a $k(x)$-vector space.*

For different liftings $\widetilde{u}_1, \widetilde{u}_2$, $\widetilde{u}_1 - \widetilde{u}_2 \in u^* TW \otimes J$ mapping into $\widetilde{u}^* E \otimes J$ along $u^* TW \otimes J \xrightarrow{\nabla_s|_x \otimes J} \widetilde{u}^* E \otimes J$. So similar as argument in subsection 3.2, we need $I \subset \mathfrak{m}_x^2$ to make $\nabla_S\big|_x \otimes J$ a zero map.

Example 3.1. *Let $W = \mathbb{A}^3_k$ with coordinate x, y, z, $E = \mathcal{O}^{\oplus 3}_{\mathbb{A}^3_k}$, $s = (s_1, s_2, s_3) = (xy, yz, xz)$. Hence $X = V(s)$ is just union of axes. For $TW\big|_X \cong \mathcal{O}^{\oplus X}_X$ and $E\big|_X \cong \mathcal{O}^{\oplus 3}_X$, the map ∇_s has a matrix representation $\begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}$, which is of rank 0 at origin and of rank 2 elsewhere.*

3.4 Smallestness of canonical obstruction theory

Now let's show the smallestness of canonical obstruction theory. Notations are same.

Theorem 3.1. *Given any obstruction theory (V, φ). There exists canonical injection $V_C \hookrightarrow V$ commutting with obstruction class assignment. In particular, I can be generated by at most $\dim V$ elements.*

Proof. Note that $\mathrm{Hom}_k(V_C, V) \cong I/\mathfrak{m}_P I \otimes V$. Want to find an obstruction class $ob \in V \otimes I/\mathfrak{m}_P I$ corresponding to an injection. Hence we need a star extension with $J = I/\mathfrak{m}_P I$.

Firstly consider the exact sequence

$$0 \longrightarrow I/\mathfrak{m}_P I \longrightarrow P/\mathfrak{m}_p I \longrightarrow C \longrightarrow 0 \tag{52}$$

but here both C and $P/\mathfrak{m}_P I$ are not necessary artinian local. Take quotient of some power $\mathfrak{m}_P$, get exact sequence

$$0 \longrightarrow I/\mathfrak{m}_P I + \mathfrak{m}_P^n \cap I \cong I + \mathfrak{m}_P^n/\mathfrak{m}_P I + \mathfrak{m}_P^n \longrightarrow P/\mathfrak{m}_P I + \mathfrak{m}_P^n \longrightarrow C/\mathfrak{m}_C^n \longrightarrow 0 \tag{53}$$

By Artin-Reez Lemma, for n large enough, $I \cap \mathfrak{m}_P^n \subseteq \mathfrak{m}_P I$ so that $I/\mathfrak{m}_P I + \mathfrak{m}_P^n = I/\mathfrak{m}_P I$. For each $\alpha \in V_C$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, hook] \arrow[d, two heads] & P \arrow[r, two heads] \arrow[d, two heads] & C \arrow[r] \arrow[d, two heads, "u"] \arrow[dl, dashed, "?"'] \arrow[ddl, dashed] & 0 \\
0 \arrow[r] & I/\mathfrak{m}_P I \arrow[r, hook] \arrow[d, two heads, "\alpha"] & P/\mathfrak{m}_P I + \mathfrak{m}_P^n \arrow[r, two heads, "?"] \arrow[d, two heads] & C/\mathfrak{m}_C^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r, hook] & P/\mathfrak{m}_P I + \mathfrak{m}_P^n + \ker(\alpha) \arrow[r, two heads] & C/\mathfrak{m}_C^n \arrow[r] & 0
\end{tikzcd} \tag{54}$$

where all surjections are quotient maps. Note that by definition of obstruction theory, $\mathrm{id}_V \otimes \alpha : V \otimes I/\mathfrak{m}_P I \to V \otimes k$ sends $\varphi(u, P/\mathfrak{m}_P I) + \mathfrak{m}_P^n$ to $\varphi(u, P/\mathfrak{m}_P I + \mathfrak{m}_P^n + \ker(\alpha))$.

For every $\alpha \neq 0$, by canonical obstruction theory, we know that u cannot by extended to $P/\mathfrak{m}_P I + \mathfrak{m}_P^n + \ker(\alpha)$. Hence the corresponding element of $\varphi(u, P/\mathfrak{m}_P I + \mathfrak{m}_P^n)$ in $\mathrm{Hom}_k(V_C, V)$ is an injection. $\square$

Remark 3.3. *Note that $\varphi(u, P/\mathfrak{m}_P I + \mathfrak{m}_P^n)$ is independent to choosing of n since for all large enough n, kernel ideals J of the star extension are all $I/\mathfrak{m}_P I$ with identity map of $I/\mathfrak{m}_P I$ corresponding to same obstruction class in $V_C \otimes I/\mathfrak{m}_P I$.*

In addition, for given obstruction theory (V, φ), the functorial property shows that $\varphi(u, P/\mathfrak{m}_P I + \mathfrak{m}_P^n)$ is mapping to $\varphi(u, P/\mathfrak{m}_P I + \mathfrak{m}_P^{n+1})$ under identity $V \otimes I/\mathfrak{m}_P I$. Hence the inclusion is canonical.

Corollary 3.1. *If C has an obstruction theory (V, φ), then* $\dim C \geq \dim(\mathfrak{m}_C/\mathfrak{m}_C^2) - \dim V$.

Reason 3.2. *Since P is regular local ring,* $\dim P = \dim(\mathfrak{m}_P/\mathfrak{m}_P^2) \geq \dim(\mathfrak{m}_C/\mathfrak{m}_C^2)$ *As* $\dim C \geq \dim P - \sharp$*generators of I, by Theorem 3.1, immediately get* $\dim C \geq \dim(\mathfrak{m}_C/\mathfrak{m}_C^2) - \dim V$.

3.5 Categories of artinian local rings

Example 3.2. *Let $x \in E \subset \mathbb{P}^1_k$ be a k-point, where E is an elliptic curve. Note that the stalks of x in E and $\mathbb{P}^1_k$ can be not isomorphic otherwise their fraction fields are isomorphic, inducing a birational map from $\mathbb{P}^1_k$ to E. However, the completion of their stalks are isomorphic so that their deformation problems are equivalent.*

Definition 3.3. *Let $k = \overline{k}$. Denote $\mathcal{C}$ to be the category of artinian local k-algebras. A functor of artinian local ring is a covariant functor from $\mathcal{C}$ to Sets.*

Definition 3.4. *Let $0 \to J \to A' \to A \to 0$ be a star extension. We say that the extension is a small extension if the ideal J is generated by one single element i.e. J is k-vector space of dimension* 1.

Example 3.3. *Let X_0 be a scheme over k. Set $\mathcal{F}(A) =$ deformation classes of X_0 over A.*

$$\begin{tikzcd} A_1 \arrow[r] & A_2 \\ \operatorname{Spec} A_1 & \operatorname{Spec} A_2 \arrow[l] \end{tikzcd} \xrightarrow{\mathcal{F}} \begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] & X_1 \arrow[d] & X_2 := X_1 \times_{A_1} A_2 \arrow[l] \arrow[d] \\ \operatorname{Spec} k \arrow[r] \arrow[rr, bend right] & \operatorname{Spec} A_1 & \operatorname{Spec} A_2 \arrow[l] \end{tikzcd} \tag{55}$$

Called deformation functor of X_0.

Let $\widehat{\mathcal{C}}$ be the category of complete local k-algebra with residue field k. For $R \in \widehat{\mathcal{C}}$, define contravariant functor $h_R : \mathcal{C} \to Sets$ mapping A to $\operatorname{Hom}(R, C) = \operatorname{Mor}_{Sch_k}(\operatorname{Spec} C, \operatorname{Spec} R)$.

Definition 3.5. *Let $\mathcal{F} : \mathcal{C} \to Sets$ be a functor of artinian local ring. We say $\mathcal{F}$ is pro-representable if $\mathcal{F} \cong h_R$ for some $R \in \widehat{\mathcal{C}}$.*

Example 3.4. *For $\mathcal{F}$ in Example 3.3, if $X_0 \subset \mathbb{P}^N_k$ is a closed subscheme of projective space, then $\mathcal{F} \cong h_R$ where $R = \widehat{\mathcal{O}_{\mathrm{Hilb},[X_0]}}$.*

Note that for a moduli space, we can naturally define a universal family. But here R needn't be artinian local ring so that we cannot give the universal family by $\mathrm{id}_R \in \operatorname{Hom}(R, R)$. To simulate the universal family of moduli space, we consider $\mathcal{F}(R/\mathfrak{m}^n)$ for all n.

Assume $\mathcal{F} \cong h_R$, then for each n, $\mathcal{F}(R/\mathfrak{m}^n) \cong \operatorname{Hom}(R, R/\mathfrak{m}^n)$. Hence we can find a

$\xi_n \in \mathcal{F}(R/\mathfrak{m}^n)$ corresponding to the quotient map. There is a sequence

$$\begin{tikzcd}
\mathrm{Hom}(R, R/\mathfrak{m}) \arrow[d, "\sim"] & \mathrm{Hom}(R, R/\mathfrak{m}^2) \arrow[l] \arrow[d, "\sim"] & \mathrm{Hom}(R, R/\mathfrak{m}^3) \arrow[l] \arrow[d, "\sim"] & \cdots \arrow[l] \\
\mathcal{F}(R/\mathfrak{m}) & \mathcal{F}(R/\mathfrak{m}^2) \arrow[l] & \mathcal{F}(R/\mathfrak{m}^3) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{56}$$

$$\xi_1 \longleftarrow\!\shortmid\; \xi_2 \longleftarrow\!\shortmid\; \xi_3 \longleftarrow\!\shortmid\; \cdots$$

Given $A \in \mathcal{C}$ and ring homomorphism $f : R \to A$. Since A is artinian local, f factors through $g : R/\mathfrak{m}^n \to A$ for some n. Now we can get a family in $\mathcal{F}(A)$ by the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(R, R/\mathfrak{m}^n) \arrow[r, "\sim"] \arrow[d, "g\circ"] & \mathcal{F}(R/\mathfrak{m}^n) \arrow[d] & \xi_m \arrow[d, mapsto] \\
\mathrm{Hom}(R, A) \arrow[r, "\sim"] & \mathcal{F}(A) & \mathcal{F}(g)(\xi_n)
\end{tikzcd} \tag{57}$$

Definition 3.6. *Every functor $\mathcal{F}$ of artinian local ring induces a covariant functor*

$$\widehat{\mathcal{F}} : \widehat{\mathcal{C}} \longrightarrow Sets \quad R \longmapsto \varprojlim \mathcal{F}(R/\mathfrak{m}^n) \tag{58}$$

In particular, element $(\xi_n)_n \in \varprojlim \mathcal{F}(R/\mathfrak{m}^n)$ is called a formal family.

For a functor $\mathcal{F}$ of artinian local ring, there are several questions

- When pro-representable?
- When algebraic?
- Existence of global moduli?

Example 3.5 (***No moduli of $\mathbb{P}^1_k$-bundles***)**.** *There does not exist moduli space M such that* $\mathrm{Mor}(S, M) \cong \{\mathbb{P}^1_k - \textit{bundles over } S\}$.

About the first question, there is a fact

Proposition 3.1. *Let X_0 be a (smooth) projective variety with $\mathrm{Aut}\, X_0 = 0$. Then the functor given by deformation classes of X_0 is pro-representable i.e. there exists $R \in \widehat{\mathcal{C}}$ such that the following commutative diagram commutes*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \\
\mathrm{Spec}\, R/\mathfrak{m} \arrow[r] & \mathrm{Spec}\, R/\mathfrak{m}^2 \arrow[r] & \mathrm{Spec}\, R/\mathfrak{m}^3 \arrow[r] & \cdots
\end{tikzcd} \tag{59}$$

About the second question, it asks if there is a scheme X such that the following diagram commutes for some n

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}\, R/\mathfrak{m}^n \arrow[r] & \mathrm{Spec}\, R
\end{tikzcd} \tag{60}$$

4 Formal Family

4.1 Formal family

Formal moduli is try to study deformations of a point in moduli space. However, just as Example 3.5, global moduli does not always exist. So we turn to another much more weaker problem that the existence of formal neighbourhood, which is also parametrized by something with some kind of "universal property".

Here we begin with a new definition of formal family.

Definition 4.1. *A formal family over $R \in \widehat{\mathcal{C}}$ is a natural transformation from h_R to a functor $\mathcal{F}$ of artinian local ring.*

Lemma 4.1. *The two definitions are in fact same since they contain same datum.*

Proof. Obviously, given a natural transformation $h_R \to \mathcal{F}$, by definition, we get a sequence $\{\xi_n\}$. Hence, we only need to show that given an element $(\xi_n)_n \in \varprojlim \mathcal{F}(R/\mathfrak{m}^n)$, we can recover a natural transformation.

For any $A \in \mathcal{C}$ and $f \in h_R(A)$, since A is artinian local ring, f factors through quotient map $R \twoheadrightarrow R/\mathfrak{m}^n$ for some n

$$f : R \twoheadrightarrow R/\mathfrak{m}^n \xrightarrow{\widetilde{f}} A \tag{61}$$

By Yoneda's Lemma, we have that $\mathrm{Mor}(h_{R/\mathfrak{m}^n}, \mathcal{F}) \cong \mathcal{F}(R/\mathfrak{m}^n)$. Hence $\xi_n \in \mathcal{F}(R/\mathfrak{m}^n)$ corresponds to a natural transformation $\varphi_n : h_{R/\mathfrak{m}^n} \to \mathcal{F}$. Consider the following diagram

$$\begin{array}{ccccc} h_R(A) & \xleftarrow[\circ f]{} & h_{R/\mathfrak{m}^n}(A) & \xrightarrow{\varphi_n(A)} & \mathcal{F}(A) \\ f & \longleftarrow\!\shortmid & \widetilde{f} & \longmapsto & \mathcal{F}(\widetilde{f})(\xi_n) \end{array} \tag{62}$$

It is natural to define a map $h_R(A) \to \mathcal{F}(A)$ by mapping f to $\mathcal{F}(\widetilde{f})(\xi_n)$. Now we remains to show that this map is well defined. That is, if f has two factorizations

$$f = R \twoheadrightarrow R/\mathfrak{m}^{n_1} \xrightarrow{\widetilde{f}_1} A = R \twoheadrightarrow R/\mathfrak{m}^{n_2} \xrightarrow{\widetilde{f}_2} A \tag{63}$$

then $\mathcal{F}(\widetilde{f}_1)(\xi_{n_1})$ should equal to $\mathcal{F}(\widetilde{f}_2)(\xi_{n_2})$.

We may assume that $n_1 \geq n_2$. Since $(\xi_n)_n \in \varprojlim \mathcal{F}(R/\mathfrak{m}^n)$, we have that ξ_{n_1} is mapping to ξ_{n_2} under $\mathcal{F}(R/\mathfrak{m}^{n_1}) \to \mathcal{F}(R/\mathfrak{m}^{n_2})$ so that

$$\varphi_{n_2} : h_{R/\mathfrak{m}^{n_2}} \longrightarrow \mathcal{F} = h_{R/\mathfrak{m}^{n_2}} \longrightarrow h_{R/\mathfrak{m}^{n_1}} \xrightarrow{\varphi_{n_1}} \mathcal{F} \tag{64}$$

In particular, we get

$$\begin{array}{ccccc} h_{R/\mathfrak{m}^{n_2}}(A) & \longrightarrow & h_{R/\mathfrak{m}^{n_1}}(A) & \xrightarrow{\varphi_{n_1}(A)} & \mathcal{F}(A) \\ \widetilde{f}_2 & \longmapsto & \widetilde{f}_1 & \longmapsto & \mathcal{F}(\widetilde{f}_1)(\xi_{n_1}) = \mathcal{F}(\widetilde{f}_2)(\xi_{n_2}) \end{array} \tag{65}$$

Thus $(\zeta_n)_n$ successfully recover all datum of the original natural transformation, done! □

Remark 4.1. *In addition, there is corollary of Yoneda's Lemma*

$$\mathrm{Mor}(h_R, \mathcal{F}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R) \tag{66}$$

Definition 4.2. *Given a formal family $\varphi : h_R \to \mathcal{F}$. We say that φ is*
• *versal (or say formally smooth), if φ is strongly surjective i.e. for all $A \in \mathcal{C}$, $\varphi(A)$ is surjective and for all surjection $B \twoheadrightarrow A$, $h_R(B) \to h_R(A) \times_{\mathcal{F}(A)} \mathcal{F}(B)$ is surjective.*
• *miniversal, if φ is versal and $\varphi(k[\varepsilon])$ is bijective.*
• *universal, if φ is a natural isomorphism of functors.*

Remark 4.2. *The map $h_R(B) \to h_R(A) \times_{\mathcal{F}(A)} \mathcal{F}(B)$ comes from the commutative diagram of natural transformation, and the strong surjectivity can be interpreted as a lifting diagram that for all $(\theta, \eta) \in h_R(A) \times_{\mathcal{F}(A)} \mathcal{F}(B)$ with commutative diagram*

$$\begin{tikzcd} h_A \arrow[r, "\theta"] \arrow[d, hook] & h_R \arrow[d, "\varphi"] \\ h_B \arrow[r, "\eta"] \arrow[ur, dashed, "\delta"] & \mathcal{F} \end{tikzcd} \tag{67}$$

there always exists a lifting $\delta : h_B \to h_R$ making the diagram commutative. Many of following propositions can also be interpreted as such a lifting diagram.

Recall that we have defined that a functor of artinian local ring $\mathcal{F}$ is said to be pro-representable if there exists $R \in \widehat{\mathcal{C}}$ such that $\mathcal{F} \cong h_R$. Thus $\mathcal{F}$ has universal family is equivalent to $\mathcal{F}$ is pro-representable.

Lemma 4.2. *Given versal family $\varphi : h_R \to \mathcal{F}$. Take $\widetilde{R} = R[[t]]$. Then composition $\widetilde{\varphi} : h_{\widetilde{R}} \to h_R \xrightarrow{\varphi} \mathcal{F}$ is also versal family.*

Proof. Note that $h_{\widetilde{R}}(A) \to h_R(A)$ is surjective for all $A \in \mathcal{C}$, we get $\widetilde{\varphi}(A)$ is still surjective as composition of surjections. And for commutative diagram

$$\begin{tikzcd} h_A \arrow[r, "\theta"] \arrow[d, hook] & h_{\widetilde{R}} \arrow[r, two heads] \arrow[d, "\widetilde{\varphi}"] & h_R \arrow[dl, "\varphi"] \\ h_B \arrow[r, "\eta"] \arrow[ur, dashed, "\widetilde{\delta}"] \arrow[urr, dashed, "\delta"] & \mathcal{F} & \end{tikzcd} \tag{68}$$

lifting $\delta : h_B \to h_R$ always exists. Assume that δ corresponds to ring homomorphism $R \to B$. Then the existence of lifting $\widetilde{\delta}$ making the diagram commutative is equivalent to existence of lifting $R[[t]] \to B$ in the following commutative diagram

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, hook] & B \arrow[d, two heads] \\ R[[t]] \arrow[r, "\theta"] \arrow[ur, dashed] & A \end{tikzcd} \tag{69}$$

In fact, we can construct the map by mapping t to any preimage of $\theta(t)$ in B. □

Proposition 4.1. *Given versal family $h_R \to \mathcal{F}$, then*
(a)$\sharp\mathcal{F}(k) = 1$
(b)For all $A' \twoheadrightarrow A$ and $A'' \twoheadrightarrow A$, map $\mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$ is surjective.

Further if $h_R \to \mathcal{F}$ *is miniversal, then*
(c)$\mathcal{F}(k[\varepsilon]) =: t_{\mathcal{F}}$ *is naturally a finite dimensional* k*-vector space, called the tangent space of* $\mathcal{F}$*.*
(d)For all $A \in \mathcal{C}$, $\mathcal{F}(A \times_k k[\varepsilon]) \to \mathcal{F}(A) \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon])$ *is bijective.*
(e)For small extension $A' \twoheadrightarrow A$ *and all* $\eta \in \mathcal{F}(A)$, $t_{\mathcal{F}} \circlearrowleft \{\eta' \in \mathcal{F}(A') \big| \eta' \mapsto \eta\}$ *is a transitive group action.*

Finally, if $h_R \to \mathcal{F}$ *is universal, then*
(f)Maps in (b) is bijective
(g)In (e), $\{\eta' \in \mathcal{F}(A') \big| \eta' \mapsto \eta\}$ *is* $t_{\mathcal{F}}$*-pseudo torsor.*

Remark 4.3. *If take* $\mathcal{F}$ *as Example 3.3 and* $A = k[\varepsilon]$ *in (c), then we get*

$$\begin{aligned} \mathcal{F}(k[\varepsilon]) \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon]) &\xleftarrow{\sim} \mathcal{F}(k[\varepsilon] \times_k k[\varepsilon]) \\ &= \mathcal{F}(k[\varepsilon_1, \varepsilon_2]) \\ &\longrightarrow \mathcal{F}(k[\varepsilon]) \end{aligned} \tag{70}$$

which is in fact the addition of $H^1(X, \mathcal{T}_X)$*.*

For versal case, $t_{\mathcal{F}}$ *need not be a vector space hence we don't have group structure on* $t_{\mathcal{F}}$*. For miniversal case, the group action can be not free i.e. there could exist stabilizer. For universal case, the group action is free.*

Proof. (a)Since $h_R(k) = \mathrm{Mor}_{Sch_k}(\operatorname{Spec} k, \operatorname{Spec} R)$ is one point set and $h_R(k) \twoheadrightarrow \mathcal{F}(k)$ is surjective, $\mathcal{F}(k)$ is also one point set.

(b)Given commutative diagram

$$\begin{tikzcd} h_{A'} \arrow[rr, "\theta"] \arrow[dr, dashed, "f"] & & \mathcal{F} \\ & h_R \arrow[ur, "\varphi"] & \\ h_A \arrow[uu] \arrow[ur] \arrow[rr] & & h_{A''} \arrow[uu, "\eta"'] \arrow[ul, dashed, "g"'] \end{tikzcd} \tag{71}$$

Since $h_R(A) \twoheadrightarrow \mathcal{F}_A$ is surjective, $h_A \to \mathcal{F}$ factor through some $h_A \to h_R$. As $A' \twoheadrightarrow A$ and $A'' \twoheadrightarrow A$ are surjections, by strong surjectivity, there exist liftings $f : h_{A'} \to h_R$ and $g : h_{A''} \to h_R$ making the diagram commutative. Hence we get

$$\begin{tikzcd} R \arrow[dr, dashed] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & A' \times_A A'' \arrow[r] \arrow[d] & A'' \arrow[d] \\ & A' \arrow[r] & A \end{tikzcd} \tag{72}$$

By universal property of fibered product, we get there exist $R \to A' \times_A A''$ inducing $h_{A' \times_A A''} \to h_R$. Taking composition, we get $h_{A' \times_A A''} \to \mathcal{F}$ making the following diagram commutative

$$\begin{tikzcd} h_{A'} \arrow[rr, "\theta"] \arrow[dr] & & \mathcal{F} \\ & h_{A' \times_A A''} \arrow[ur, "\varphi"] & \\ h_A \arrow[uu] \arrow[rr] & & h_{A''} \arrow[uu, "\eta"'] \arrow[ul] \end{tikzcd} \tag{73}$$

Conclude that $\mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$ is surjective.

(c)As $h_R(k[\varepsilon]) \xrightarrow{\sim} \mathcal{F}(k[\varepsilon])$, it suffices to give a k-vector space structure on $h_R(k[\varepsilon])$ as following

$$\begin{aligned} &(r \mapsto a + b_1\varepsilon) + (r \mapsto a + b_2\varepsilon) \\ =&r \mapsto a + (b_1 + b_2)\varepsilon \\ &k \cdot (r \mapsto a + b\varepsilon) \\ =&r \mapsto a + kb\varepsilon \end{aligned} \tag{74}$$

Easy to check these are well defined.

(d)By (b), we already have that $\mathcal{F}(A \times_k k[\varepsilon]) \twoheadrightarrow \mathcal{F}(A) \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon])$ is surjective. Suffice to show injectivity. Consider the following diagram.

$$\begin{tikzcd} & h_R \arrow[dr, "\varphi"] & & \\ h_A \arrow[ur] \arrow[rr] \arrow[dr, "\eta"'] & & \mathcal{F} & \\ & h_{A\times_k k[\varepsilon]} \arrow[uu, shift left, "f_1"] \arrow[uu, shift right, "f_2"'] \arrow[ur, shift left, "\varphi_1"] \arrow[ur, shift right, "\varphi_2"'] & & h_{k[\varepsilon]} \arrow[ll, "\pi"] \arrow[ul] \end{tikzcd} \tag{75}$$

If there exist two elements $\varphi_1, \varphi_2 \in \mathcal{F}(A \times_k k[\varepsilon])$ making the diagram commutative, then by strong surjectivity, $h_R(A \times_k k[\varepsilon]) \twoheadrightarrow h_R(A) \times_{\mathcal{F}(A)} \mathcal{F}(A \times_k k[\varepsilon])$ is surjective, there exist $f_1, f_2 \in h_R(A \times_k k[\varepsilon])$ making the diagram commutative. In particular, we get $f_1 \circ \eta = f_2 \circ \eta$.

Now we have that

$$\begin{aligned} \varphi \circ f_1 \circ \pi &= \varphi_1 \circ \pi \\ &= \varphi_2 \circ \pi \\ &= \varphi \circ f_2 \circ \pi \end{aligned} \tag{76}$$

While $h_R(k[\varepsilon]) \xrightarrow{\sim} \mathcal{F}(k[\varepsilon])$ is bijective, we get $f_1 \circ \pi = f_2 \circ \pi$. Thus there is a commutative diagram

$$\begin{tikzcd} R \arrow[dr, shift left, "f_1"] \arrow[dr, shift right, "f_2"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_k k[\varepsilon] \arrow[r, "\pi"] \arrow[d, "\eta"] & k[\varepsilon] \arrow[d] \\ & A \arrow[r] & k \end{tikzcd} \tag{77}$$

By universal property of fibered product, we get $f_1 = f_2$. Conclude that $\varphi_1 = \varphi_2$ so that $\mathcal{F}(A \times_k k[\varepsilon]) \xrightarrow{\sim} \mathcal{F}(A) \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon])$ is bijective.

(e)For small extension $A' \twoheadrightarrow A$, assume the kernel ideal is (t). Then there is an isomorphism $A' \times_A A' \xrightarrow{\sim} A' \times_k k[\varepsilon]$ defined by

$$(x, y) \longmapsto (x, \overline{x} + \overline{\frac{x-y}{t}}\varepsilon) \tag{78}$$

And we get a diagram

$$\begin{tikzcd}
\mathcal{F}(A') \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon]) \arrow[r, "\psi"] \arrow[d, "\sim"] & \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A') \\
\mathcal{F}(A' \times_k k[\varepsilon]) \arrow[r, "\sim"] & \mathcal{F}(A' \times_A A') \arrow[u, two heads]
\end{tikzcd} \tag{79}$$

where ψ comes from composition. Thus we can define a group action that for all $v \in \mathcal{F}(k[\varepsilon])$ and $\eta' \in \mathcal{F}(A)$ mapping to η, $v(\eta')$ is the second entry in $\psi(\eta, v)$ corresponding to the following natural transformation

$$h_{A'} \longrightarrow h_{A' \times_A A'} \longrightarrow h_{A' \times_k k[\varepsilon]} \xrightarrow{(\eta, v)} \mathcal{F} \tag{80}$$

(f)Similarly consider diagram

$$\begin{tikzcd}
 & h_R \arrow[dr, "\varphi"] & & \\
h_{A'} \arrow[ur] \arrow[rr] \arrow[dr, "\eta"'] & & \mathcal{F} & \\
 & h_{A' \times_A A''} \arrow[u, shift left, "f_1"] \arrow[u, shift right, "f_2"'] \arrow[ur, shift left, "\varphi_1"] \arrow[ur, shift right, "\varphi_2"'] & & h_{A''} \arrow[ul] \arrow[ll, "\pi"]
\end{tikzcd} \tag{81}$$

If there exist two elements $\varphi_1, \varphi_2 \in \mathcal{F}(A' \times_A A'')$ making the diagram commutative, then $h_R(A' \times_A A'') \twoheadrightarrow \mathcal{F}(A' \times_A A'')$ is surjective, there exist $f_1, f_2 \in h_R(A' \times_A A'')$ such that $\varphi \circ f_i = \varphi_i$.

Now we have that

$$\begin{aligned}
\varphi \circ f_1 \circ \pi &= \varphi_1 \circ \pi & \qquad \varphi \circ f_1 \circ \eta &= \varphi_1 \circ \eta \\
&= \varphi_2 \circ \pi \quad \textit{and} & &= \varphi_2 \circ \eta \\
&= \varphi \circ f_2 \circ \pi & &= \varphi \circ f_2 \circ \eta
\end{aligned} \tag{82}$$

While $h_R(A') \xrightarrow{\sim} \mathcal{F}(A')$ and $h_R(A'') \xrightarrow{\sim} \mathcal{F}(A'')$ is bijective, we get $f_1 \circ \pi = f_2 \circ \pi$ and $f_1 \circ \eta = f_2 \circ \eta$. Thus there is a commutative diagram

$$\begin{tikzcd}
R \arrow[dr, shift left, "f_1"] \arrow[dr, shift right, "f_2"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & A' \times_A A'' \arrow[r, "\pi"] \arrow[d, "\eta"] & A'' \arrow[d] \\
 & A' \arrow[r] & A
\end{tikzcd} \tag{83}$$

By universal property of fibered product, we get $f_1 = f_2$. Conclude that $\varphi_1 = \varphi_2$ so that $\mathcal{F}(A' \times_A A'') \xrightarrow{\sim} \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$ is bijective.

(g)By (f), we immediately get the map ψ defined in (e) is bijective hence $\{\eta' \in \mathcal{F}(A') \big| \eta' \mapsto \eta\}$ is $t_{\mathcal{F}}$-pseudo torsor. $\square$

Remark 4.4. *For (b), in fact we only need one of the two surjections since we can get $h_A \to h_R$ by compositing with $h_{A'} \to h_R$ and $h_{A''} \to h_R$ at first, so that the other surjection would similarly give diagram 72.*

4.2 Schelessinger's Criterion

Theorem 4.1. *Let $\mathcal{F}$ be a functor of artinian local ring. Then $\mathcal{F}$ has miniversal family i.e. there exists $R \in \widehat{\mathcal{C}}$ such that $\varphi : h_R \to \mathcal{F}$ is miniversal, if and only if the following conditions hold*

(H_0)$\sharp\mathcal{F}(k) = 1$

(H_1)For any small extension $A'' \twoheadrightarrow A$ and any morphism $A' \to A$, the natural map $\mathcal{F}(A' \times_A A'') \twoheadrightarrow \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$ is surjective.

(H_2)For any morphism $A' \to A$, the natural map $\mathcal{F}(A' \times_k k[\varepsilon]) \twoheadrightarrow \mathcal{F}(A') \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon])$ is bijective.

(H_3)$\mathcal{F}(k[\varepsilon])$ is a finite dimensional k-vector space.

Further, $\mathcal{F}$ has universal family if and only if H_0, H_1, H_2, H_3 and following H_4 hold

(H_4)For any small extension $A' \twoheadrightarrow A$ and $\eta \in \mathcal{F}(A)$, $\{\eta' \in \mathcal{F}(A') \big| \eta' \text{ mapping to } \eta\}$ is a $t_{\mathcal{F}}$-pseudo torsor under the group .

Example 4.1. *Let X_0 be a scheme over k. Consider the deformation functor $\mathcal{F}$ of X_0 defined in Example 3.3. If X satisfies either of the two following hypothesis*

- *X_0 is projective*
- *X_0 is affine with isolated singularities*

then $\mathcal{F}$ has miniversal family. In fact, H_0, H_1 and H_2 hold for all k-scheme X_0 and the two hypothesis only play a role in H_3.

In particular, let X_0 be a smooth projective variety. There is an obvious question that why we get pseudo torsor condition in previous section but here we only have miniversal family. The difference occurs in the equivalence relations. When we talk about deformation of X/A to X'/A' for star extension $A' \twoheadrightarrow A$, an isomorphism is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, hook] \arrow[dr, hook] & \\ X_1' \arrow[rr, "\sim"] \arrow[dr] & & X_2' \arrow[dl] \\ & \operatorname{Spec} A' & \end{tikzcd} \tag{84}$$

But here in $\mathcal{F}(A')$, an isomorphism is a commutative diagram

$$\begin{tikzcd} & X_0 \arrow[dl, hook] \arrow[dr, hook] & \\ X_1' \arrow[rr, "\sim"] \arrow[dr] & & X_2' \arrow[dl] \\ & \operatorname{Spec} A' & \end{tikzcd} \tag{85}$$

Clearly, the second equivalence relation is much weaker causing that isomorphism classes would be less and hence there exist nontrivial stabilizer.

Enlightened by this example, we could give a condition equivalent to universal.

Proposition 4.2. *Let X_0 be a scheme over k, $\mathcal{F}$ the deformation functor of X_0. Assume X satisfies either of the two following hypothesis*

- *X_0 is projective*
- *X_0 is affine with isolated singularities*

Then $\mathcal{F}$ has universal family if and only if for all deformation X'/A', the natural map $\mathrm{Aut}_{A'}(X'/X_0) \to \mathrm{Aut}_A(X/X_0)$ is surjective, where $X = X'\big|_{\mathrm{Spec}\,A}$.

Proof. "$\Rightarrow$": Suppose there is an automorphism $\varphi : X \xrightarrow{\sim} X$ which cannot be lifted to $\mathrm{Aut}_{A'}(X'/X_0)$. Denote $i : X_0 \hookrightarrow X$ and $j : X \hookrightarrow X'$. Then $j : X \hookrightarrow X'$ and $j \circ \varphi : X \hookrightarrow X'$ are not equivalent as deformation of X. While $\varphi \circ i = i$, they are equivalent in $\mathcal{F}(A')$.

If denote equivalence classes under the two different equivalence relations by $[\cdot]_1$ and $[\cdot]_2$, then since $[(X', j)]_1 \neq [(X', j \circ \varphi)]_1$ and the group action is transitive, there exists nonzero $v \in t_{\mathcal{F}}$ such that $v[(X', j)]_1 = [(X', j \circ \varphi)]_1$ but $v[(X', j \circ i)]_2 = [(X', j \circ i)]_2$. Thus stabilizer of $[(X', j \circ i)]_2$ is nontrivial, contradicting to universal.

"$\Leftarrow$": Suffice to show that the two equivalence relations are same. Assume X_1' and X_2' are two deformations of X over A' satisfying that $[X_1']_2 = [X_2']_2$. Then there is an commutative diagram

$$\begin{tikzcd} & X_0 \arrow[dl, hook] \arrow[dr, hook] & \\ X_1' \arrow[rr, "\sim", "\varphi"'] \arrow[dr] & & X_2' \arrow[dl] \\ & \mathrm{Spec}\,A' & \end{tikzcd} \tag{86}$$

Now φ restricts to an automorphism $\varphi\big|_X : X \xrightarrow{\sim} X$. By assumption, there exists $\psi \in \mathrm{Aut}_{A'}(X_1'/X_0)$ extending $\varphi\big|_X^{-1}$. Hence we get commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi|_X^{-1}", "\sim"'] \arrow[d, hook, "j_1"] & X \arrow[r, "\varphi|_X", "\sim"'] \arrow[d, hook, "j_1"] & X \arrow[d, hook, "j_2"] \\ X_1' \arrow[r, "\sim", "\psi"'] \arrow[dr] & X_1' \arrow[r, "\sim", "\varphi"'] \arrow[d] & X_2' \arrow[dl] \\ & \mathrm{Spec}\,A' & \end{tikzcd} \tag{87}$$

which shows that $[X_1']_1 = [X_2']_1$, done! $\square$

Corollary 4.1. *Let X_0 be a projective scheme over k with $H^0(X_0, \mathcal{T}_{X_0}) = 0$, $\mathcal{F}$ the deformation functor of X_0. Then $\mathcal{F}$ has universal family.*

Reason 4.1. *With $H^0(X_0, \mathcal{T}_{X_0}) = 0$, by induction on length of A, we can prove that for all deformation X/A, $\mathrm{Aut}_A(X/A) = \{\mathrm{id}\}$. Hence by Proposition 4.2, we are done.*

Example 4.2. *For curve X_0 of genus at least 2, as $H^0(X_0, \mathcal{T}_{X_0}) = 0$, $\mathcal{F}$ has universal family. For curve X_0 of genus 1, even though $H^0(X_0, \mathcal{T}_{X_0}) = k \neq 0$, $\mathcal{F}$ still has universal family since*

$$\mathrm{Aut}_{A'}(X'/X_0) = A' \twoheadrightarrow \mathrm{Aut}_A(X/X_0) = A \tag{88}$$

is just the surjection.

Example 4.3. *Consider vector bundles on $\mathbb{P}^1_k$ of rank 2, which should of the form $\mathcal{O}(a)\oplus\mathcal{O}(b)$. For example, the automorphism group of $\mathcal{O}^{\oplus 2}$ is $GL_2(k)$ and the automorphism group of $\mathcal{O}(-1)\oplus\mathcal{O}(1)$ is $G := \{\begin{pmatrix}\mu & \\ f & \lambda\end{pmatrix} \big| \mu,\lambda \in k^\times$ and $f \in H^0(\mathcal{O}(2))\}$. Note dimensions are respectively 4 and 5. So intuitively, $\mathrm{Aut}(\mathcal{O}(-1)\oplus\mathcal{O}(1))$ not always lift to $\mathrm{Aut}(\mathcal{O}^{\oplus 2})$.*

4.3 Crude deformation functor

Definition 4.3. *Let X_0 be a k-scheme. Define the crude deformation functor $\mathcal{F}_1$ of X_0 as following*

$$A \longmapsto \mathcal{F}_1(A) = \{X \overset{flat}{\to} \big| X \times_A k \cong X_0\}/\sim_1 \tag{89}$$

where the equivalence relation is given by the following commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, "\sim"] \arrow[dr] & & X_2 \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd} \tag{90}$$

Remark 4.5. *Comparing to deformation functor, here the equivalence relation does not ask the isomorphism to be compatible with closed immersions $X_0 \hookrightarrow X_1$ and $X_0 \hookrightarrow X_2$, resulting that $\mathcal{F}_1(A)$ is much smaller and $\mathcal{F}_1$ is not well behaved in general.*

Lemma 4.3. *Let X_0 be a k-scheme, $\mathcal{F}$ deformation functor of X_0 and $\mathcal{F}_1$ crude deformation functor of X_0. Then there is a strongly surjective natural transformation $\psi : \mathcal{F} \to \mathcal{F}_1$ defined by*

$$\psi(A) : \mathcal{F}(A) \longrightarrow \mathcal{F}_1(A) \quad X \longmapsto [X]_1 \tag{91}$$

where $[X]_1$ denotes the equivalence class of X in $\mathcal{F}_1(A)$.

Proof. Obviously, for all $A \in \mathcal{C}$, $\psi(A) : \mathcal{F}(A) \to \mathcal{F}_1(A)$ is surjective. For any surjection $B \twoheadrightarrow A$ and commutative diagram

$$\begin{tikzcd} h_A \arrow[r, "X"] \arrow[d, hook] & \mathcal{F} \arrow[d, "\psi"] \\ h_B \arrow[r, "{[Y]_1}"'] \arrow[ur, dashed, "?"] & \mathcal{F}_1 \end{tikzcd} \tag{92}$$

where $X \in \mathcal{F}(A)$ and $[Y]_1 \in \mathcal{F}_1(B)$, want to show that there exists lifting $h_B \to \mathcal{F}$.

As the diagram is commutative, we get $Y \sim_1 X'$ for some deformation of X over B. Thus we can replace $[Y]_1$ by $[X']_1$. Take $h_B \to \mathcal{F}$ corresponding to $X' \in \mathcal{F}(B)$ and clearly this is our desired lifting. ☐

Proposition 4.3. *Let X_0 be a k-scheme satisfying one of the two hypothesis in Proposition 4.2, $\mathcal{F}_1$ crude deformation functor of X_0. Then*
(1)$\mathcal{F}_1$ has a versal family.
(2)$\mathcal{F}_1$ satisfies (H_0) and (H_1).
(3)$\mathcal{F}_1$ has a miniversal family if and only if for any deformation $X \to \operatorname{Spec} k[\varepsilon]$, the natural map $\mathrm{Aut}_{k[\varepsilon]}(X) \twoheadrightarrow \mathrm{Aut}(X_0)$ is surjective.

Proof. (1)Consider deformation functor $\mathcal{F}$ of X_0 and natural transformation $\psi : \mathcal{F} \to \mathcal{F}_1$ defined in previous lemma. Since $\mathcal{F}$ has a versal family $\varphi : h_R \to \mathcal{F}$, what we need to show is that composition of strongly surjective natural transformations is still strongly surjective.

For all $A \in \mathcal{C}$, $\psi \circ \varphi(A) : h_R \to \mathcal{F}_1(A)$ is composition of surjections hence surjective. For any surjection $B \twoheadrightarrow A$ and commutative diagram

$$\begin{tikzcd} h_A \arrow[r] \arrow[d, hook] & h_R \arrow[r, "\varphi"] & \mathcal{F} \arrow[d, "\psi"] \\ h_B \arrow[ur, dashed, "\delta'"] \arrow[urr, dashed, "\delta"'] \arrow[rr] & & \mathcal{F}_1 \end{tikzcd} \tag{93}$$

As ψ is strongly surjective, there exists lifting $\delta : h_B \to \mathcal{F}$ making the diagram commutative. Then again as φ is strongly surjective, there exists lifting $\delta' : h_B \to h_R$ making the diagram commutative, done!

(2)Now $\psi \circ \varphi : h_R \to \mathcal{F}_1$ is a versal family. Then immediately $\mathcal{F}_1(k)$ is one-point set and (H_0) holds. For any morphism $f : A' \to A$ and any small extension $g : A'' \twoheadrightarrow A$, consider the following commutative diagram

$$\begin{tikzcd} \mathcal{F}(A' \times_A A'') \arrow[r, two heads] \arrow[d, two heads] & \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'') \arrow[d, "\psi(A') \times_{\psi(A)} \psi(A'')"] \\ \mathcal{F}_1(A' \times_A A'') \arrow[r] & \mathcal{F}_1(A') \times_{\mathcal{F}_1(A)} \mathcal{F}_1(A'') \end{tikzcd} \tag{94}$$

Hence we only need to show that $\psi(A') \times_{\psi(A)} \psi(A'')$ is surjective. For $([X']_1, [X'']_1) \in \mathcal{F}_1(A') \times_{\mathcal{F}_1(A)} \mathcal{F}_1(A'')$, as $\mathcal{F}(A') \twoheadrightarrow \mathcal{F}_1(A')$ is surjective, we can lift $[X']_1$ to $X' \in \mathcal{F}(A')$. Then there is a commutative diagram

$$\begin{tikzcd} h_A \arrow[r, "X'|_{\operatorname{Spec} A}"] \arrow[d, hook] & \mathcal{F} \arrow[d, "\psi"] \\ h_{A''} \arrow[ur, dashed, "Y''"] \arrow[r, "{[X'']_1}"'] & \mathcal{F}_1 \end{tikzcd} \tag{95}$$

as ψ is strongly surjective, there exists lifting $h_{A''} \to \mathcal{F}$ corresponding to $Y'' \in \mathcal{F}(A'')$. Hence $Y''\big|_{\operatorname{Spec} A} \sim X'\big|_{\operatorname{Spec} A}$ and $[Y'']_1 = [X'']_1$ so that $(X', Y'') \mapsto ([X']_1, [X'']_1)$. Thus $\psi(A') \times_{\psi(A)} \psi(A'')$ surjective and (H_1) holds.

(3)As ϕ is miniversal, $\psi \circ \varphi$ is miniversal if and only if $\psi(k[\varepsilon])$ is bijective. Assume the right side holds. For $X_1, X_2 \in \mathcal{F}(k[\varepsilon])$, if $X_1 \sim_1 X_2$, then there exists isomorphism ϕ with commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, "\sim", "\phi"'] \arrow[dr] & & X_2 \arrow[dl] \\ & \operatorname{Spec} k[\varepsilon] & \end{tikzcd} \tag{96}$$

Restrict ϕ to X_0. As $\operatorname{Aut}_{k[\varepsilon]}(X_1) \twoheadrightarrow \operatorname{Aut}(X_0)$ is surjective, then we can extend $\phi\big|_{X_0}^{-1}$ to an automorphism $\widetilde{\phi} : X_1 \xrightarrow{\sim} X_1$.

$$\begin{tikzcd}
X_0 \arrow[r, "\phi|_{X_0}^{-1}", "\sim"'] \arrow[d, hook, "j_1"] & X_0 \arrow[r, "\phi|_{X_0}", "\sim"'] \arrow[d, hook, "j_1"] & X_0 \arrow[d, hook, "j_2"] \\
X_1 \arrow[r, "\sim", "\widetilde{\phi}"'] \arrow[dr] & X_1 \arrow[r, "\sim", "\phi"'] \arrow[d] & X_2 \arrow[dl] \\
& \operatorname{Spec} k[\varepsilon] &
\end{tikzcd} \tag{97}$$

Hence $X_1 \sim X_2$ so that $\psi(k[\varepsilon])$ is bijective.

Conversely, assume $\psi(k[\varepsilon])$ is bijective. Suppose there exists $\phi : X_0 \xrightarrow{\sim} X_0 \in \operatorname{Aut}(X_0)$ cannot be extended to element in $\operatorname{Aut}_{k[\varepsilon]}(X)$ for some $(X, i) \in \mathcal{F}(k[\varepsilon])$, where $i : X_0 \hookrightarrow X$ is the closed immersion. Then (X, i) must be not equivalent to $(X, \phi \circ i)$, while $[(X, i)]_1$ and $[(X, \phi \circ i)]_1$ are obviously same, contradiciton! $\square$

Example 4.4. *Set $k = \mathbb{C}$. Consider maps from scheme S to $[\mathbb{C}^2/\mathbb{C}^\times]$. By definition, a map is equivalent to datum $(\mathcal{L}, x, y)$ where $\mathcal{L} \in \operatorname{Pic}(S)$ is line bundle and $x, y \in H^0(S, \mathcal{L})$. Given $\mathcal{L}_0 \in \operatorname{Pic}(\operatorname{Spec}\mathbb{C})$ and $x_0 = y_0 = 0$, define deformation functor as*

$$\mathcal{F}(A) = \{\mathcal{L} \in \operatorname{Pic}(\operatorname{Spec} A), x, y \in H^0(\operatorname{Spec} A, \mathcal{L}) \big| x, y \equiv 0 \pmod{\mathfrak{m}} \text{ and } \mathcal{L} \otimes_A \mathbb{C} \underset{\theta}{\xrightarrow{\sim}} \mathcal{L}_0\}/\sim$$

where $(\mathcal{L}_1, x_1, y_1, \theta_1) \sim (\mathcal{L}_2, x_2, y_2, \theta_2)$ if there exists $\varphi : \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ such that $\varphi(x_1) = x_2$, $\varphi(y_1) = y_2$ and there is a commutative diagram

$$\begin{tikzcd}
\mathcal{L}_1 \otimes_A \mathbb{C} \arrow[rr, "\varphi\otimes\mathrm{id}"] \arrow[dr, "\theta_1"'] & & \mathcal{L}_2 \otimes_A \mathbb{C} \arrow[dl, "\theta_2"] \\
& \mathcal{L}_0 &
\end{tikzcd} \tag{98}$$

As line bundle on $\operatorname{Spec} A$ can be trivialized, we can rewrite $\mathcal{F}(A)$ as $\{(x, y) \big| x, y \in \mathfrak{m}_A\}/\sim$, where $x \sim y$ if there exists $h \in 1 + \mathfrak{m}$ such that $x_1 = hx_2$ and $y_1 = hy_2$.

Consider the tangent space of $\mathcal{F}$. We have that $\mathcal{F}(\mathbb{C}[\varepsilon]) = \{(a\varepsilon, b\varepsilon) \big| a, b \in \mathbb{C}\}/\sim$. Suppose $(a_1\varepsilon, b_1\varepsilon) \sim (a_2\varepsilon, b_2\varepsilon)$, then there exists $c \in \mathbb{C}$ such that

$$\begin{cases} a_1\varepsilon = (1 + c\varepsilon)a_2\varepsilon \\ b_1\varepsilon = (1 + c\varepsilon)b_2\varepsilon \end{cases} \Rightarrow \begin{cases} a_1 = a_2 \\ b_1 = b_2 \end{cases} \tag{99}$$

Hence the equivalence relation is just identity.

In fact, $\mathcal{F}$ has miniversal family over $R = \widehat{\mathcal{O}_{\mathbb{C}^2,(0,0)}}$. Given $(x, y) \in \mathcal{F}(A)$, we have a spectrum map $\operatorname{Spec} A \overset{(x,y)}{\to} \operatorname{Spec} R$. Take $A = \mathbb{C}[t]/(t^3)$, then $\mathcal{F}(A) = \{(a_1t + a_2t^2, b_1t + b_2t^2)\}/\sim$. Suppose $(a_1t + a_2t^2, b_1t + b_2t^2) \sim (\widetilde{a}_1t + \widetilde{a}_2t^2, \widetilde{b}_1t + \widetilde{b}_2t^2)$, then there exists $c_1, c_2 \in \mathbb{C}$ such that

$$\begin{cases} a_1t + a_2t^2 = (1 + c_1t + c_2t^2)(\widetilde{a}_1t + \widetilde{a}_2t^2) \\ b_1t + b_2t^2 = (1 + c_1t + c_2t^2)(\widetilde{b}_1t + \widetilde{b}_2t^2) \end{cases} \Rightarrow \begin{cases} a_1 = \widetilde{a}_1 \quad a_2 = \widetilde{a}_1c + \widetilde{a}_2 \\ b_1 = \widetilde{b}_1 \quad b_2 = \widetilde{b}_1c + \widetilde{b}_2 \end{cases} \tag{100}$$

Hence two different spectrum maps from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ can mapping to same equivalence class in $\mathcal{F}(A)$ so that $h_R \to \mathcal{F}$ is not universal.

Remark 4.6. *In GTM257 written by Hartshorne, Example 18.4.1 gives two example with similar pathology above. One shows that crude deformation functor is not necessarily miniversal, the other shows that deformation functor is not necessarily universal.*

5 Algebraization of Formal Moduli

Example 5.1. *Let X_0 be a smooth projective k-scheme, $\mathcal{F}$ deformation functor of X_0. Algebraization of formal moduli is the following process*

$$\begin{array}{ccccccc} \textit{formal family} & & \textit{formal scheme} & & \textit{scheme} & & \textit{family} \\ h_R & & \mathfrak{X} & & \widetilde{X} & & X \\ \downarrow & \rightsquigarrow & \downarrow & \rightsquigarrow & \downarrow & \rightsquigarrow & \downarrow \\ \mathcal{F} & & \operatorname{Spf} R & & \operatorname{Spec} R & & S \ni s_0 \end{array}$$

where $\widetilde{X} \to \operatorname{Spec} R$ is flat and of finite type such that $X \times_{\operatorname{Spec} R} \operatorname{Spf} R \cong \mathfrak{X}$ in category of formal schemes, and S is of finite type over k such that $R = \widehat{\mathcal{O}_{S,s_0}}$, $X \times_S \operatorname{Spec} R \cong \widetilde{X}$ and $X \times_S s_0 \cong X_0$.

Definition 5.1. *Let A be a noetherian, $I \subseteq A$ ideal, $\widehat{A}$ completion of A under I-adic topology. Define formal scheme $\operatorname{Spf} \widehat{A} = (\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ to be a locally ringed space, where $\mathfrak{X} = \operatorname{Spec}(\widehat{A}/\widehat{I})$ and structure sheaf $\mathcal{O}_\mathfrak{X}$ is defined by*

$$\mathcal{O}_\mathfrak{X}(U) = \varprojlim \mathcal{O}_{\operatorname{Spec}(\widehat{A}/\widehat{I}^n)}(U) \tag{101}$$

This is also called completion of A along I. Generally, we say a locally ringed space $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is a formal scheme if there exists an open covering U_i such that U_i is completion of noetherian ring A_i along ideal I_i.

Example 5.2. *(1)Let $A_0 = k[x,y]$, $I = (x,y)$ and $A = \widehat{A_0} = k[[x,y]]$. Then underlying topological space of $\operatorname{Spf} A$ is a one-point set with structure sheaf given by A.*
(2)Let $A_0 = k[x,y]$, $I = (x)$ and $A = \widehat{A_0} = k[y][[x]]$. Then underlying topological space of $\operatorname{Spf} A$ is y-axis and global sections are $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) = k[y][[x]]$. For $U = D(y)$, $\Gamma(U, \mathcal{O}_\mathfrak{X}) = \widehat{k[y][[x]]_y} = k[y,y^{-1}][[x]]$. Note that $x+y$ is also invertible in $\Gamma(U, \mathcal{O}_\mathfrak{X})$ since

$$\frac{1}{x+y} = \frac{y^{-1}}{1+xy^{-1}} = y^{-1}(\sum_i (-xy^{-1})^i) \tag{102}$$

This coincides with our geometric intuition that $D(y)$ and $D(x+y)$ intersecting with y-axis are both y-axis without origin.

Remark 5.1. *We should be careful with these formal series rings. For example, $k[[x]][y] \subsetneq k[y][[x]]$ since elements in $k[[x]][y]$ would have a bound for degree of y, while elements in $k[y][[x]]$ is a convergent power series with coeffcients in $k[y]$. Thus $\sum_i y^i x^i \in k[y][[x]] \setminus k[[x]][y]$.*

5.1 Formal family to formal scheme

Definition 5.2. *Let $\{A_n\}$ be an inverse system together with homomorphisms $\varphi_{j,i} : A_j \to A_i$. We say that $\{A_n\}$ satisfies Mittag-Leffler condition, (ML) for short, if for all n, the decreasing chain*

$$\varphi_{n+1,n}(A_{n+1}) \supseteq \varphi_{n+2,n}(A_{n+2}) \supseteq \cdots \tag{103}$$

is stationary.

Remark 5.2. *By Proposition 9.1 in Algebraic Geometry, Hartshorne, we know that if $0 \to A_n \to B_n \to C_n \to 0$ is an exact sequence of inverse systems with $\{A_n\}$ satisfying (ML), then the sequence of inverse limits is also exact*

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0 \tag{104}$$

Proposition 5.1. *Let $(R, \mathfrak{m})$ be a complete local ring with residue field k, X_0 of finite type over k. Given formal family $\{X_n\}$, where X_n is deformation of X_0 over $R/\mathfrak{m}^n$. Then there exists unique formal scheme $\mathfrak{X}$ with morphism of locally ringed spaces $\mathfrak{X} \to \operatorname{Spf} R$ such that $\mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec}(R/\mathfrak{m}^n) \cong X_n$.*

Idea. Given affine open covering U^α of X_0, $B_n^\alpha := \Gamma(U^\alpha, \mathcal{O}_{X_n})$ and $B^\alpha := \varprojlim B_n^\alpha$, where B_n^α can be viewed as deformation of B_0^α over $R/\mathfrak{m}^n$. We need to check that B^α is noetherian and $B^\alpha/\mathfrak{m}^n B^\alpha \cong B_n^\alpha$.

Since B_0^α is of finite type over k, there is a surjection $A_0 := k[x_1, \cdots, x_N] \overset{\varphi_0}{\twoheadrightarrow} B_0^\alpha$. Similarly, set $A_i := R/\mathfrak{m}^{i+1}[x_1, \cdots, x_N]$. Denote $\psi_i : B_i^\alpha \twoheadrightarrow B_{i-1}^\alpha$. Choose one preimage of $\varphi_0(x_i)$ in B_1^α for each x_i, define homomorphism $\varphi_1 : A_1 \to B_1^\alpha$ by mapping x_i to corresponding preimage. Inductively, we get a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, two heads] & A_2 \arrow[r, two heads] \arrow[d, "\varphi_2"] & A_1 \arrow[r, two heads] \arrow[d, "\varphi_1"] & A_0 \arrow[d, "\varphi_0"] \\ \cdots \arrow[r, two heads] & B_2^\alpha \arrow[r, two heads, "\psi_2"] & B_1^\alpha \arrow[r, two heads, "\psi_1"] & B_0^\alpha \end{tikzcd} \tag{105}$$

Consider $A := \varprojlim A_n = R\{x_1, \cdots, x_n\}$ and $I := \varprojlim I_n$ where $I_n = \ker(\varphi_i)$, want to show that $B^\alpha \cong A/I$. For all i, by flatness of B_n^α over $R/\mathfrak{m}^{n+1}$, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}^{n+1}} A_n \arrow[r, hook] \arrow[d, "\mathrm{id} \otimes \varphi_n"] & A_n \arrow[r, two heads] \arrow[d, "\varphi_n"] & A_{n-1} \arrow[r] \arrow[d, two heads, "\varphi_{n-1}"] & 0 \\ 0 \arrow[r] & \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}^{n+1}} A_n \otimes_{R/\mathfrak{m}^{n+1}} B_n^\alpha \arrow[r, hook] & B_n^\alpha \arrow[r, two heads, "\psi_n"] & B_{n-1}^\alpha \arrow[r] & 0 \end{tikzcd} \tag{106}$$

Inductively, assume that we have known φ_i is surjective for all $i \le n-1$, want to show that φ_n is surjective. In fact, there is a commutative diagram

$$\begin{tikzcd} \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}^{n+1}} A_n \arrow[r, "\sim"] \arrow[d, "\mathrm{id} \otimes \varphi_n"] & \mathfrak{m}^n \otimes_R A_0 \arrow[d, "\mathrm{id} \otimes \varphi_0"] \\ \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}^{n+1}} A_n \otimes_{R/\mathfrak{m}^{n+1}} B_n^\alpha \arrow[r, "\sim"] & \mathfrak{m}^n \otimes_R B_0^\alpha \end{tikzcd} \tag{107}$$

Hence $\mathrm{id}\otimes\varphi_n = \mathrm{id}\otimes\varphi_0$ is surjective. And by 5 Lemma, we get φ_n is a surjection. Now we have a commutative diagram with exact rows and columns

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \mathfrak{m}^n\otimes_R A_0 \arrow[r, two heads, "\mathrm{id}\otimes\varphi_0"] \arrow[d, hook] & \mathfrak{m}^n\otimes_R B_0^\alpha \arrow[d, hook] & \\ 0 \arrow[r] & I_n \arrow[r, hook] \arrow[d, "\phi_n"] & A_n \arrow[r, two heads, "\varphi_n"] \arrow[d, two heads] & B_n^\alpha \arrow[r] \arrow[d, two heads, "\psi_n"] & 0 \\ 0 \arrow[r] & I_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, hook, "\varphi_{n-1}"] & B_{n-1}^\alpha \arrow[r] & 0 \end{tikzcd} \tag{108}$$

By Snake Lemma, the following sequence is exact

$$\mathfrak{m}^n\otimes_R A_0 \xrightarrow{\mathrm{id}\otimes\varphi_0} \mathfrak{m}^n\otimes_R B_0^\alpha \longrightarrow \operatorname{coker}\phi_n \longrightarrow 0 \tag{109}$$

which implies that ϕ_n is surjective for all n. Thus inverse system $\{I_n\}$ satisfies (ML) and hence we get exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B^\alpha \longrightarrow 0 \tag{110}$$

so that B^α is quotient ring of A hence northerian too. By diagram chasing, we also get $B^\alpha/\mathfrak{m}^n B^\alpha \xrightarrow{\sim} B_n^\alpha$. $\square$

5.2 Formal scheme to scheme

Definition 5.3. *Let $\mathfrak{X}\to\operatorname{Spf}R$ be the noetherian formal scheme given by previous subsection. We sat that $\mathfrak{X}\to\operatorname{Spf}R$ is effective if there exists scheme $\widetilde{X}$ flat and of finite type over $\operatorname{Spec}R$ such that completion of $\widetilde{X}$ along fiber at $\mathfrak{m}$ is $\mathfrak{X}$.*

Remark 5.3. *This is not always possible. Intuitively, if we take formal completion open covering of $\mathfrak{X}$, denoted by $\{U_i=\widehat{X_i}\}$, then U_i and U_j are compatible at $U_i\cap U_j$.*

However, note that underlying topological spaces of X_i and X_j are bigger than $\widehat{X_i}$ and $\widehat{X_j}$, it is not necessary for X_i and X_j to be compatible at $X_i\cap X_j$ so that we cannot always glue up X_i to get our desired $\widetilde{X}$.

Theorem 5.1 (***Grothendieck***)**.** *Let $X\to\operatorname{Spec}R$ be a proper morphism with $R\in\widehat{\mathcal{C}}$. Take $\widehat{X}$ to be the completion of X along the fiber at $\mathfrak{m}$. Then $Coh(X)\to Coh(\widehat{X})$ mapping $\mathcal{F}$ to $\widehat{\mathcal{F}}$ gives equivalence of the two categories.*

Remark 5.4. *This theorem tells us that for deformation of coherent sheaf is always effective. In addition, properness is necessary. For example, if take $R=k[[x]]$ and $X=\operatorname{Spec}k[[x]][y]$, then $\widehat{X}=\operatorname{Spf}k[y][[x]]$. While*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X)=k[[x]][y] \quad \mathrm{Hom}_{\mathcal{O}_{\widehat{X}}}(\mathcal{O}_{\widehat{X}},\mathcal{O}_{\widehat{X}})=k[y][[x]] \tag{111}$$

are not same as Remark 5.1, which implies $\mathrm{Mor}_{Coh(X)}(\mathcal{O}_X,\mathcal{O}_X)\neq\mathrm{Mor}_{Coh(\widehat{X})}(\mathcal{O}_{\widehat{X}},\mathcal{O}_{\widehat{X}})$ so that the two categories are not equivalent.

Theorem 5.2 (***Grothendieck***)**.** *Let $\mathfrak{X} \to \operatorname{Spf} R$ be a proper morphism between formal schemes with $R \in \widehat{\mathcal{C}}$, $X_0 = \mathfrak{X} \otimes_{\operatorname{Spf} R} \operatorname{Spec} k$. Assume there exists line bundle $\mathcal{L}$ on $\mathfrak{X}$ such that $\mathcal{L}_0 := \mathcal{L}\big|_{X_0}$ is ample. Then $\mathfrak{X} \to \operatorname{Spec} R$ is effective i.e. there exists of finite type morphism $\widetilde{X} \to \operatorname{Spec} R$ and line bundle L on $\widetilde{X}$ such that $\widetilde{X} \times_{\operatorname{Spec} R} \operatorname{Spf} R \cong \mathfrak{X}$ and $\widehat{L} \cong \mathcal{L}$.*

Reason 5.1. *Since $\mathcal{L}_0$ is ample, we can embed X into a projective space $\mathbb{P}^N_k$, then the problem of deformation of scheme X_0 is equivalent to the problem of deformation of closed subscheme $X_0 \subseteq \mathbb{P}^N_k$. While the deformation of subscheme is equivalent to deformation of ideal sheaf, hence by Theorem 5.1, we get $\mathfrak{X} \to \operatorname{Spf} R$ is effective.*

Example 5.3. *Let X_0 be a projective k-scheme. Assume that deformation of line bundles on X_0 is unobstructed (for example if $H^2(X_0, \mathcal{O}_{X_0}) = 0$). Then given ample line bundle $\mathcal{L}_0$ on X_0, we can extend it to compatible $\mathcal{L}_n$ to X_n. Taking inevrse limit, we get our desired $\mathcal{L}$ so that in this case, $\mathfrak{X} \to \operatorname{Spf} R$ is effective.*

There is also an noneffective example.

Example 5.4. *Let $X_0 \subseteq \mathbb{P}^3_k$ be a smooth quartic surface with $\operatorname{Char} k = 0$. Intuitively, since $H^2(X_0, \mathcal{O}_{X_0}) = k$, deformation of line bundle on X_0 is not always unobstructed.*

In addition, computing by Hodge theory, we would see that $H^0(X_0, \mathcal{N}_{X_0/\mathbb{P}^3_k}) \to H^1(\mathcal{T}_{X_0})$ is not surjective, and hence not all deformations of X_0 comes from deformations of X_0 as subscheme of $\mathbb{P}^3_k$. Thus there exists deformation of X_0 where $\mathcal{L}_0 := \mathcal{O}_{\mathbb{P}^3_k}\big|_{X_0}$ does not deform to.

There is a fact that for all line bundles $\mathcal{L} \in \operatorname{Pic}(X_0)$, there exists some 1st order deformation X' of X_0 over $\operatorname{Spec} k[\varepsilon]$ such that $\mathcal{L}_0$ does not lift to X'. While $H^0(X_0, \mathcal{T}_{X_0}) = 0$, $H^2(X_0, \mathcal{T}_{X_0}) = 0$ and $H^1(X_0, \mathcal{T}_{X_0}) = k^{20}$ is finite dimensional, we get that deformation of X_0 over R is universal, where R is formal series ring over k with 20 variants.

Suppose that $\mathfrak{X} \to \operatorname{Spf} R$ is effective. Assume that $\widetilde{X} \to \operatorname{Spec} R$ is our desired morphism. As $\widetilde{X} \to \operatorname{Spec} R$ is flat and proper with geometric fibers smooth, we get $\widetilde{X} \to \operatorname{Spec} R$ itself smooth so that $\widetilde{X}$ is regular.

Take affine open subset $U \subseteq X$ meeting X_0. Assume that D is a prime Cartier divisor on U and $D \cap X_0 \neq \varnothing$. Since $\widetilde{X}$ is regular, $\overline{D}$ is a Cartier divisor and $\overline{D} \cap X_0$ is an effective Cartier divisor.

Take $\mathcal{L}_0 := \mathcal{O}_{X_0}(\overline{D} \cap X_0)$ which can be lifted to $\widehat{\mathcal{L}_0} := \mathcal{O}_X(\overline{D})$. While by the fact, there exists 1st deformation X_1 of X such that $\mathcal{L}_0$ does not lift to X_1. But by universal property, $\mathcal{L}_0$ can be lifted to $\mathcal{L}_1 := \widehat{\mathcal{L}_0}\big|_{X_1}$ on X_1, contradiciton!

In the following, we are going to give a proof of the fact we used in the previous example. Let X be a smooth k-scheme, $\mathcal{L}$ line bundle on X. Want to study deformations of pair $(X, \mathcal{L})$.

Definition 5.4. *Let X be a smooth k-scheme, $\mathcal{L}$ line bundle on X. Consider commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}"] \arrow[dr, "\Delta"] \arrow[dd, "\mathrm{id}"'] & & X \arrow[dd] \\
& X \times_k X \arrow[ur, "q"'] \arrow[dl, "p"] & \\
X \arrow[rr] & & k
\end{tikzcd} \tag{112}$$

Define bundle of 1st jets to be $J^1(\mathcal{L}) := q_*(p^*\mathcal{L} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{X\times X}/\mathcal{I}^2_{\Delta_X})$.

Remark 5.5. *Since X is smooth, we have exact sequence*

$$0 \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\iota} J^1(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0 \tag{113}$$

And for all $x \in X$, $J^1(\mathcal{L})\big|_x \cong \mathcal{L}/(\mathfrak{m}_x)^2 \cong k^{\dim X+1}$.

There exists a canonical section $\mathcal{L} \xrightarrow{\sigma} J^1(\mathcal{L})$ which is not $\mathcal{O}_X$-linear. For $\alpha \in H^0(U,\mathcal{L})$ and $f \in H^0(U,\mathcal{O}_X)$, $\sigma(\alpha) := q_*p^*(\alpha)$ and

$$\begin{aligned} \sigma(f\alpha) &= q_*(f\alpha \otimes 1) \\ &= q_*(\alpha \otimes (f \otimes 1)) \\ &= q_*(\alpha \otimes (df + 1 \otimes f)) \\ &= \iota(\alpha \otimes df) + \sigma(\alpha)f \end{aligned} \tag{114}$$

satisfying the Leibniz's Law.

Remark 5.6. *Intuitively, the definition of $J^1(\mathcal{L})$ is similar to giving a bimodule structure on $\mathcal{O}_{X\times X}/\mathcal{I}_{\Delta^2_X}$, where tensoring with $p_1^*\mathcal{L}$ gives a left module structure and pushing forward gives a right module structure.*

Definition 5.5. *Let $X \to S$ be a morphism of locally ringed spaces, $\mathcal{M}$ an $\mathcal{O}_X$-module. A connection of $\mathcal{M}$ over S is an $\mathcal{O}_S$-module homomorphism $\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{X/S}$ such that $\nabla(am) = a\nabla(m) + m \otimes da$.*

Hence there is a one-to-one correspondence

$$\begin{array}{ccccc} \{\mathcal{O}_X\text{-linear sections } \mathcal{L} \to J^1(\mathcal{L})\} & \ni & \varphi & & \sigma - \iota \circ \nabla \\ \updownarrow & & \downarrow & & \uparrow \\ \{\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \text{ connections}\} & \ni & \sigma - \varphi & & \nabla \end{array}$$

Note that existence of $\mathcal{O}_X$-linear section is equivalent to that exact sequence 113 splits. Hence the extension class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}) \cong H^1(X, \Omega^1_X)$, called the Atiyah class, is the obstruction to existence of connection.

Remark 5.7. *If consider $H^1(X,\Omega^1_X) \to H^2(X,\mathbb{C}) = H^2(X,\mathcal{O}_X) \oplus H^1(X,\Omega^1_X) \oplus H^0(X,\Omega^2_X)$, then in fact the Atiyah class is mapping to the 1st Chern class of X. Hence if the 1st Chern class of X is not zero, then there is no connection.*

Definition 5.6. *Let X be a smooth scheme, $\mathcal{L}$ line bundle on X. Define the sheaf of principal parts $\mathcal{P}_{\mathcal{L}}$ as an extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{P}_{\mathcal{L}} \longrightarrow \mathcal{T}_X \longrightarrow 0 \tag{115}$$

defined by the cohomology class $c(\mathcal{L}) \in H^1(X,\Omega^1_X) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X,\mathcal{O}_X)$, where $c(\mathcal{L})$ is the image of $[\mathcal{L}]$ under map $H^1(X,\mathcal{O}^*_X) \to H^1(X,\Omega^1_X)$ induced by $d\log : \mathcal{O}^*_X \to \Omega^1_X$ sending f to $\frac{df}{d}$.

Remark 5.8. *One way to see that this definition given by Hartshorne is same to our definition of bundle of 1st jets is that the Atiyah class is in fact just $c(\mathcal{L})$.*

Apply functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{L})$ to exact sequence 113 and we get a long exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(J^1(\mathcal{L}), \mathcal{L}) \to \Gamma(X, \mathcal{T}_X) \to H^1(X, \mathcal{O}_X) \to \cdots \tag{116}$$

In fact, $\mathrm{Ext}^i_{\mathcal{O}_X}(J^1(\mathcal{L}), \mathcal{L}) = H^i(X, \mathcal{P}_{\mathcal{L}})$. And intuitively, automorphism groups of deformations of $(X, \mathcal{L})$ is just $H^0(X, \mathcal{P}_{\mathcal{L}} \otimes J)$, deformations are classified by $H^1(X, \mathcal{P}_{\mathcal{L}} \otimes J)$ and the obstruction is in $H^2(X, \mathcal{P}_{\mathcal{L}} \otimes J)$.

Want to prove that for all $\mathcal{L} \neq \mathcal{O}_X$, there exists deformation X' of X such that deformation of $\mathcal{L}$ to X' is obstructed i.e. $H^1(X, \mathcal{T}_X) \stackrel{c(\mathcal{L})}{\to} H^2(X, \mathcal{O}_X)$ is not a zero map, where $c(\mathcal{L}) \in H^1(X, \Omega^1_X)$ induces the map by contraction $\mathcal{T}_X \otimes \Omega^1_X \to \mathcal{O}_X$.

By Serre duality, map $c(\mathcal{L})$ above is dual to $\mathbb{C} \cong H^0(X, \Omega^2_X) \stackrel{\widetilde{c}}{\to} H^1(X, \Omega^1_X \otimes \Omega^2_X) \cong H^1(X, \Omega^1_X)$ sending 1 to $c(\mathcal{L})$. When $\mathrm{Char}\, k = 0$, we have that $H^1(X, \mathcal{O}_X) = 0$ so that $H^1(X, \mathcal{O}^*_X) \to H^1(X, \Omega^1_X)$ is injective. Hence nontrivial line bundle gives nonzero $c(\mathcal{L})$ and $\widetilde{c}$ is not zero map. Thus $c(\mathcal{L})$ as dual of $\widetilde{c}$ is not either zero map.

5.3 Scheme to universal family

Theorem 5.3 (***M.Artin***)**.** *Let X_0 be a projective smooth scheme over k. Assume that X_0 admits an effective formal versal deformation $\widetilde{X}$ over complete local k-algebra R. Then there exist flat morphism $X \to S$ and $s_0 \in S$ such that S is of finite type over k, $R = \widehat{\mathcal{O}_{S,s_0}}$, $X \times_S \mathrm{Spec}\, R \cong \widetilde{X}$ and $X \times_S s_0 \cong X_0$.*

Example 5.5. *Let X_0 be smooth projective curve of genus $g \geq 2$. We have that $H^0(X_0, \mathcal{T}_{X_0}) = 0$, $H^1(X_0, \mathcal{T}_{X_0}) = k^{3g-3}$, $H^2(X_0, \mathcal{T}_{X_0}) = 0$ and $H^2(X_0, \mathcal{O}_{X_0}) = 0$. Then the first three conditions guarantee that deformation functor of X_0 has a universal family, and the last condition implies that there is no obstruction to deformation of line bundle. Hence deformation of X_0 is algebraic.*

Remark 5.9. *The algebraization (X, S, s_0) given by Noether's Theorem is only unique locally around s_0 in the etale topology, meaning that if (X', S', s'_0) is another such pair, then there exists S'' with a point s''_0 and etale morphisms $S' \to S$ and $S'' \to S'$ sending s''_0 to s_0 and s'_0 respectively such that $X \times_s S'' \cong X' \times_{S'} S''$.*

In fact, if given X, Y of finite type over k and $x \in X$, $y \in Y$ are k-points with $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{Y,y}}$, then

$$\begin{array}{ccc} & z \in Z & \\ \stackrel{etale}{\swarrow} & & \stackrel{etale}{\searrow} \\ x \in X & & y \in Y \end{array} \tag{117}$$

give same etale cover. However, we should be careful about that the isomorphism induced by the same etale cover would not in general be the original one.

Example 5.6. *Let $X = V(y^2 - x^3 - x^2)$ and $Y = V(uv)$, $x = y = (0,0)$. Then $\widehat{\mathcal{O}_{X,x_0}} = \widehat{\mathcal{O}_{Y,y_0}}$ and we have a diagram*

$$\begin{array}{ccc} & k[x,y,t]/(y^2 - x^3 - x^2, t^2 - 1 - x) & \\ \nearrow & & \nwarrow \\ k[x,y]/(y^2 - x^3 - x^2) & & k[u,v]/(uv) \end{array} \tag{118}$$

where u is mapping to $y - xt$ and v is mapping to $y + xt$.

The remark about local uniqueness up to etale cover inspires us to take isomorphisms into consideration when describing the universal property of algebraization. Let X_0 be a smooth projective curve of genus ≥ 2. Given another (X', S', s_0') such that S' is of finite type over k and $X' \times_{S'} s_0' \cong X_0$, define Isom scheme as following.

Consider projections

$$\begin{tikzcd} & S \times S' \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S & & S' \end{tikzcd} \tag{119}$$

For $T \to S \times S'$, we can also define a functor $\mathcal{I}som(p^*X, q^*X')$

$$T \mapsto \{(\alpha, \beta, \gamma) \big| (\alpha, \beta) \in S \times S'(T) \text{ and } \gamma : \alpha^* X \xrightarrow{\sim} \beta^* X'\}$$

In fact, $\mathcal{I}som(p^*X, q^*X')$ is representable by some scheme $\mathrm{Isom}(p^*X, q^*X')$. Note that for any field K, K-valued points of $\mathrm{Isom}(p^*X, q^*X')$ would correspond to $\mathcal{I}som(p^*X, q^*X')(\mathrm{Spec}\, K)$. Hence the set of closed points of $\mathrm{Isom}(p^*X, q^*X')$ is $\{(y, \theta) \big| x \in S \times S' \text{ closed point}, p(y) = s, q(y) = s' \text{ and } \theta : X_s \xrightarrow{\sim} X'_{s'}\}$.

Take θ_0 to be the given isomorphism $X_{s_0} \xrightarrow{\sim} X'_{s_0'}$. There is a one-to-one correspondence for all $A \in \mathscr{C}$

$$\{\mathrm{Spec}\, A \to (S', s_0')\} \leftrightsquigarrow \{\mathrm{Spec}\, A \to (\mathrm{Isom}(p^*X, q^*X'), \theta_0)\}$$

For each $\alpha' : \mathrm{Spec}\, A \to S'$, consider pull back $\mathrm{Spec}\, A \times_{S'} k(s_0')$ which is a deformation of X_0 over A. By universal property of $\widetilde{X}$, we get there exists $\alpha : \mathrm{Spec}\, A \to S$ such that $\mathrm{Spec}\, A \times_S k(s_0) \cong \mathrm{Spec}\, A \times_{S'} k(s_0')$. Denote the isomorphism by γ and we get an element $(\alpha, \alpha', \delta) \in \mathcal{I}som(p^*X, q^*X')(\mathrm{Spec}\, A)$. By property of universal family, we see that this corresponds to $\mathrm{Spec}\, A \to (\mathrm{Isom}(p^*X, q^*X'), \theta_0)$.

Now with the correspondence, as the deformation functor of X_0 has universal family, $\mathrm{Isom}(p^*X, q^*X') \to S'$ is etale near $\theta_0 \mapsto s_0'$. When S' also has this universal property, then we also get $\mathrm{Isom}(p^*X, q^*X') \to X'$ is etale near $\theta_0 \mapsto s_0$ so that the Isom scheme gives our desired locally etale cover.

6 Derived Category

6.1 Derived category

Definition 6.1. *Let $\mathscr{A}$ be an abelian category. Denote $\mathcal{C}(\mathscr{A})$ to be the set of cochain complexes in $\mathscr{A}$. For $A, B \in \mathcal{C}(\mathscr{A})$, a morphism $f : A \to B$ is called a quasi-isomorphism if $H^\bullet(f) : H^\bullet(A) \xrightarrow{\sim} H^\bullet(B)$ are isomorphic.*

Definition 6.2. *Let $\mathscr{A}$ be an abelian category. Define derived category of $\mathscr{A}$ to be $\mathcal{D}(\mathscr{A}) := \{$cochain complexes in $\mathscr{A}\}/$quasi-isomorphisms.*

Remark 6.1. *If $\mathscr{A}$ has enough injective objects, then for each $A \in \mathscr{A}$, we can take an injective resolution $0 \to A \to I^\bullet$ and this would give a quasi-isomorphism*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, hook] & 0 & \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd} \tag{120}$$

In addition, recall that for additive functor $\mathcal{F}$, we can define $R^i\mathcal{F}(A) := H^i(\mathcal{F}(I^\bullet))$. But taking cohomology would give up a lot of imformation, so we prefer to remember the whole complex.

For $A^\bullet \in \mathcal{C}^{\geq 0}(\mathscr{A})$, there is a Cartan-Eilenburg resolution

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^0 \arrow[r] \arrow[d, hook] & A^1 \arrow[r] \arrow[d, hook] & A^2 \arrow[r] \arrow[d, hook] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[d] & I^{1,0} \arrow[r] \arrow[d] & I^{2,0} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[d] & I^{1,1} \arrow[r] \arrow[d] & I^{2,1} \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd} \tag{121}$$

where for all i, $0 \to A^i \to I^{i,*}$ is injective resolution of A^i. We have that $A^\bullet \to Tot(I^{\bullet,*})$ is a quasi-isomorphism and hyper cohomology is defined as $R^i\mathcal{F}(A^\bullet) := H^i(\mathcal{F}(Tot(I^{\bullet,*})))$.

Example 6.1. *Let X be a smooth projective variety over $\mathbb{C}$, X^{an} analytification of X. There is a resolution of $\mathbb{C}$*

$$0 \to \mathbb{C}_{X^{\text{an}}} \to \mathcal{O}_{X^{\text{an}}} \to \Omega^1_{X^{\text{an}}} \to \Omega^2_{X^{\text{an}}} \to \cdots \tag{122}$$

Even though $\Omega^\bullet_{X^{\text{an}}}$ is not a injective resolution, we can apply hyper cohomology to compute cohomology of $\mathbb{C}$. In fact, spectral sequence $E_1^{p,q} := H^q(X, \Omega^p_X)$ converges to $H^1(X, \mathbb{C})$.

Intuitively, we can think that $ob(\mathcal{D}(\mathscr{A})) = ob(\mathcal{C}(\mathscr{A}))$, while quasi-isomorphisms in $\mathcal{C}(\mathscr{A})$ are identified as isomorphisms in $\mathcal{D}(\mathscr{A})$.

Lemma 6.1. *Let $\mathscr{A}$ be an abelian category and $\mathscr{B}$ be any category, $\mathcal{F} : \mathcal{C}(\mathscr{A}) \to \mathscr{B}$ functor. Assume that for all f quasi-isomorphism, $\mathcal{F}(f)$ is isomorphism in $\mathscr{B}$. Then $\mathcal{F}$ factor through unique $\mathcal{G} : \mathcal{D}(\mathscr{A}) \to \mathscr{B}$.*

$$\begin{tikzcd}
\mathcal{C}(\mathscr{A}) \arrow[r] \arrow[dr, "\mathcal{F}"'] & \mathcal{D}(\mathscr{A}) \arrow[d, dashed, "\exists ! \mathcal{G}"] \\
 & \mathscr{B}
\end{tikzcd} \tag{123}$$

Example 6.2. *Let $0 \to A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \to 0$ be a short exact sequence in $\mathcal{C}(\mathscr{A})$. Then we get connecting maps $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$. And there really exists a morphism $C^\bullet \to A^\bullet[1]$ in $\mathcal{D}(\mathscr{A})$ given as following*

$$\begin{tikzcd}
B^\bullet \arrow[r] \arrow[dr] & \text{cone}(u) \arrow[r] \arrow[d] & A^\bullet \\
 & C^\bullet \arrow[ur, dashed] &
\end{tikzcd} \tag{124}$$

where $\text{cone}(u) = B^\bullet \oplus A^\bullet[1]$ and $\text{cone}(u) \to C^\bullet$ sending (b, a) to $v(b)$ is a quasi-isomorphism.

Given a morphism $E \to F$ in $\mathcal{D}(\mathscr{A})$, by definition, it should be of the following form

$$\begin{tikzcd}[column sep=small] E & & F \\ & \bullet \arrow[ul, "qis"] \arrow[ur, "chain"'] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd}[column sep=small] E \arrow[dr, "chain"'] & & F \arrow[dl, "qis"] \\ & \bullet & \end{tikzcd}$$

so that it is hard to explicitly write composition of morphisms in $\mathcal{D}(\mathscr{A})$ as some chain maps. To solve this problem, we'd like to adapt a better way to define derived category.

Definition 6.3. *Let $\mathscr{A}$ be an abelian category. Define homotopy category $\mathcal{K}(\mathscr{A})$ of $\mathscr{A}$ with same objects as $\mathcal{C}(\mathscr{A})$ and morphism $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{C}(\mathscr{A})}(A^\bullet, B^\bullet)/homotopy$.*

Definition 6.4. *A triangle in an additive category with shift functors $A \mapsto A[n]$ is a sequence*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \tag{125}$$

inducing a complex

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow A[1] \longrightarrow B[1] \longrightarrow \cdots \tag{126}$$

A morphism of of triangles is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} \tag{127}$$

A standard triangle is a triangle of the form

$$A \xrightarrow{f} B \longrightarrow \mathrm{cone}(f) \longrightarrow A[1] \tag{128}$$

An exact triangle is a triangle isomorphic to a standard triangle.

Lemma 6.2. *Let $\mathscr{A}$ be an abelian category, $\mathcal{K}(\mathscr{A})$ homotopy category of $\mathscr{A}$. Then for all chain map $f : A^\bullet \to B^\bullet$, the following sequence is a triangle in $\mathcal{K}(\mathscr{A})$*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} A^\bullet[1] \tag{129}$$

Proof. Firstly, it is clear that $p \circ i$ is zero map hence null-homotopic. Suffices to show $i \circ f \simeq 0$ and $f[1] \circ p \simeq 0$. Take $D : A^n \to \mathrm{cone}(f)^{n-1}$ sending a to $(0, a)$. Then $\partial \circ D + D \circ \partial(a) = (f(a), -\partial a) + (0, \partial a) = i \circ f(a)$ and hence $i \circ f$ is null-homotopic.

Take $\widetilde{D} : \mathrm{cone}(f)^n \to B^\bullet[1]$ sending (b, a) to b. Then $\partial \circ \widetilde{D} + \widetilde{D} \circ \partial(b, a) = -\partial b + f(a) + \partial b = f[1] \circ p(b, a)$ and hence $f[1] \circ p$ is null-homotopic. □

Lemma 6.3. *Let $\mathscr{A}$ be an abelian category, $\mathcal{K}(\mathscr{A})$ homotopy category of $\mathscr{A}$. Assume that $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{i} C^\bullet \xrightarrow{p} A^\bullet[1]$ is an exact triangle in $\mathcal{K}(\mathscr{A})$. Then for any $X^\bullet$, there are two long exact sequences*

$$\to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, B^\bullet) \to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, C^\bullet) \to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, A^\bullet[1]) \to$$

and

$$\leftarrow \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, X^\bullet) \leftarrow \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(B^\bullet, X^\bullet) \leftarrow \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(C^\bullet, X^\bullet) \leftarrow \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet[1], X^\bullet) \leftarrow$$

Proof. Here we only show the exactness of the first sequence at $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, B^\bullet)$. In fact, we suffice to prove for standard triangles. Assume $C = \mathrm{cone}(f)$. By Lemma 6.2, the first sequence is obviously a complex. For all $\alpha \in \ker(\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, B^\bullet) \to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, C^\bullet))$, $g \circ \alpha$ is null-homotopic so that there exists $D : X^n \to C^{n-1}$ such that $g \circ \alpha = \partial \circ D + D \circ \partial$.

Assume that $D = (D', D'')$ i.e. $D(x) = (D'(x), D''(x))$. Then we have $(\alpha(x), 0) = (\partial D'(x) + f(D''(x)), -\partial D''(x)) + (D'(\partial x), D''(\partial x))$. Hence

$$\begin{cases} \alpha(x) = f(D'(x)) + \partial D'(x) + D'(\partial x) \\ \partial D''(x) = D''(\partial x) \end{cases} \tag{130}$$

It is natural to take $\beta : X^\bullet \to A^\bullet$ sending x to $D''(x)$. And we get $\alpha - f \circ \beta = D' \circ \partial + \partial \circ D'$ so that $\alpha \in \mathrm{im}(\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(X^\bullet, B^\bullet))$. $\square$

Definition 6.5. *Let $\mathscr{C}$ be a category, Q a class of morphisms in $\mathscr{C}$ is a left ore system if it satisfies the following conditions*
(1)Q is multiplicative i.e. $Q \circ Q \subseteq Q$ and $\mathrm{id}_C \in Q$ for all object $C \in \mathscr{C}$.
(2)Every pair of morphism $A' \xleftarrow{q} A \xrightarrow{f} B$ with $q \in Q$ can be completed to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"] & B \arrow[d, "r"] \\ A' \arrow[r, "g"] & B' \end{tikzcd} \tag{131}$$

with $r \in Q$.
(3)If $f \circ q = 0$ with $q \in Q$, then there exists $r \in Q$ such that $r \circ f = 0$.

Dually, we can also define right ore system.

Lemma 6.4. *Let $\mathscr{A}$ be an abelian category. Then for all f having homotopic inverse, f is a quasi-isomorphism.*

Remark 6.2. *This lemma tells us that quasi-isomorphism is well defined in $\mathcal{K}(\mathscr{A})$.*

Lemma 6.5. *Let $\mathscr{A}$ be an abelian category, $\mathcal{K}(\mathscr{A})$ homotopy category of $\mathscr{A}$. Then the set of quasi-isomorphisms is both left and right ore system of $\mathcal{K}(\mathscr{A})$.*

Proof. Here we only show that the set of quasi-isomorphisms is a left ore system. Given $E' \xleftarrow{q} E \xrightarrow{g} F$, take $F' = \mathrm{cone}((q, g))$. Then there is a diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "q"] & F \arrow[d, "r"] \\ E' \arrow[r, "-h"] & F' \end{tikzcd} \tag{132}$$

where r and g are natural inclusions. Want to check that r is quasi-isomorphism and the diagram is commutative in the sense up to homotopy.

For $[f] \in H^n(F)$, $[f]$ is mapping to $[(0, f, 0)]$. If $[(0, f, 0)] = [0]$, there exists (e_0', f_0, e_0) such that $(0, f, 0) = \partial(e_0', f_0, e_0)$. Hence

$$\begin{cases} \partial e_0' + q(e_0) = 0 \\ \partial f + g(e_0) = f \\ \partial e_0 = 0 \end{cases} \tag{133}$$

so that $[q(e_0)] = 0$. Since q is quasi-isomorphism, get $[e_0] = 0$ and hence $[f] = 0$. Thus r^* is injective. For all $[(e', f, e)] \in H^n(F')$, we have that

$$\begin{cases} \partial e' + q(e) = 0 \\ \partial f + g(e) = 0 \\ \partial e = 0 \end{cases} \tag{134}$$

so that $[q(e)] = 0$. Since q is quasi-isomorphism, get $[e] = 0$ and there exists e_0 such that $\partial e_0 = e$. Hence $[e' + q(e_0)] \in H^n(E')$ and as q is quasi-isomorphism, there exists e_1 such that $[e' + q(e_0)] = [q(e_1)]$. Thus there exists e'_0 such that $\partial e'_0 = e' + q(e_0) - q(e_1)$ and $\partial(e'_0, 0, e_1 - e_0) = (e', g(e_1) - g(e_0), e)$. Conclude that $[(e', f, e)] = [(0, f + g(e_0) - g(e_1), 0)]$ and r^* is surjective.

Want to show that $r \circ g \simeq -h \circ q$. For all e, $r \circ g + h \circ q(e) = (q(e), g(e), 0)$. Take $D : E^n \to {F'}^{n-1}$ sending e to $(0, 0, e)$. Then $D \circ \partial + \partial \circ D(e) = (0, 0, \partial e) + (q(e), g(e), -\partial e) = (q(e), g(e), 0)$. Thus $r \circ g \simeq -h \circ q$.

Given $E' \xrightarrow{q} E \xrightarrow{g} F$ with composition $g \circ q$ null-homotopic. Consider $E \xrightarrow{i} \text{cone}(q)$, by Lemma 6.3, g factors through some $h : \text{cone}(q) \to F$. Take $F' = \text{cone}(h)$ and $j : F \to F'$. Claim that j is quasi-isomorphism and $j \circ g$ is null-homotopic.

For all $[f] \in H^n(F)$, $[f]$ is mapping to $[(f, 0, 0)]$. If $[(f, 0, 0)] = 0$, there exists (f_0, e_0, e'_0) such that $\partial(f_0, e_0, e'_0) = (f, 0, 0)$. Hence

$$\begin{cases} \partial f_0 + h(e_0, e'_0) = f \\ \partial e_0 + q(e'_0) = 0 \\ \partial e'_0 = 0 \end{cases} \tag{135}$$

so that $[q(e'_0)] = 0$. Since q is quasi-isomorphism, $[e'_0] = 0$ so that there exists e'_1 such that $\partial e'_1 = e'_0$. As $\partial(e_0 + q(e'_1)) = 0$, there exists e'_2 and e_1 such that $\partial e_1 = e_0 + q(e'_1) - q(e'_2)$. Thus $\partial(e_1, e'_2 - e'_1) = (e_0, e'_0)$ so that $[f] = 0$ and hence j^* is injective.

For all $[(f, e, e')] \in H^n(F')$, we have that

$$\begin{cases} \partial f + h(e, e') = 0 \\ \partial e + q(e') = 0 \\ \partial e' = 0 \end{cases} \tag{136}$$

so that $[q(e')] = 0$. Since q is quasi-isomorphism, $[e'] = 0$ so that there exists e'_0 such that $\partial e'_0 = e'$. As $\partial(e + q(e'_0)) = 0$, there exists e'_1 and e_0 such that $\partial e_0 = e + q(e'_0) - q(e'_1)$. Thus $\partial(0, e_0, e'_1 - e'_0) = (h(e_0, e'_1 - e'_0), -e, -e')$ so that $[(f, e, e')] = [(f + h(e_0, e'_1 - e'_0), 0, 0)]$. Conclude that j^* is surjective.

Want to show that $j \circ g$ is null-homotopic. For all e, $j \circ g(e) = (g(e), 0, 0)$. Take $D : E^n \to {F'}^{n-1}$ sending e to $(0, e, 0)$. Then $D \circ \partial + \partial \circ D(e) = (0, \partial e, 0) + (h(e, 0), -\partial e, 0) = (h(e, 0), 0, 0)$. Note that $g(e) = h(e, 0)$, done! $\square$

Remark 6.3. *Now with this lemma, we have a more concise way to interpret composition of* *morphisms in $\mathcal{D}(\mathscr{A})$ as following*

$$\begin{tikzcd}
E_1 & & E_2 & & E_3 \\
& \bullet \arrow[ul, "qis"'] \arrow[ur, "chain"] & & \bullet \arrow[ul, "qis"'] \arrow[ur, "chain"] & \\
& & \bullet \arrow[ul, "qis"] \arrow[ur, "chain"'] & &
\end{tikzcd} \tag{137}$$

And for chain map $f : E \to F$, $f = 0$ in $\mathcal{D}(\mathscr{A})$ if and only if there exists quasi-isomorphism $q : E' \to E$ such that $f \circ q \simeq 0$.

In addition, the following corollary tells us there is a more easy understanding way to say two morphisms in derived category are same.

Corollary 6.1. *Let $\mathscr{A}$ be an abelian category, $A^\bullet, B^\bullet \in \mathcal{C}(\mathscr{A})$. Given two different interpretations of same morphism in $\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A^\bullet)$*

$$\begin{tikzcd}
& C^\bullet \arrow[ddl, "g_1"'] \arrow[ddr, "f_1"] & \\
& D^\bullet \arrow[dl, "g_2"] \arrow[dr, "f_2"'] & \\
B^\bullet & & A^\bullet
\end{tikzcd} \tag{138}$$

where g_1 and g_2 are quasi-isomorphisms, there exists a quasi-isomorphism $h : C^\bullet \to D^\bullet$ such that commutative diagram up to homotopy

$$\begin{tikzcd}
& C^\bullet \arrow[ddl, "g_1"'] \arrow[ddr, "f_1"] \arrow[d, "h"] & \\
& D^\bullet \arrow[dl, "g_2"] \arrow[dr, "f_2"'] & \\
B^\bullet & & A^\bullet
\end{tikzcd} \tag{139}$$

Proof. By Lemma 6.5, if we take $E^\bullet = \mathrm{cone}((f_2, g_2))$, there is a commutative diagram up to homotopy

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "g_1"'] \arrow[dr, "f_1"] & \\
B^\bullet \arrow[dr] & & A^\bullet \arrow[dl] \\
& E^\bullet &
\end{tikzcd} \tag{140}$$

Then we have chain homotopy $D = D_1 \oplus D_2 \oplus D_3 : C^\bullet \to E^\bullet$. By definition, we get

$$\begin{cases} f_1 = D_1 \circ \partial + \partial \circ D_1 + f_2 \circ D_3 \\ g_2 = D_2 \circ \partial + \partial \circ D_2 + g_2 \circ D_3 \\ D_3 \circ \partial - \partial \circ D_3 = 0 \end{cases} \tag{141}$$

Hence D_3 is a chain map and just our desired h. Since $g_2 \circ h \simeq g_1$, we get $g_{2\sharp} \circ h_\sharp = g_{1\sharp}$ as cohomology group homomorphism. Note that g_1 and g_2 are quasi-isomorphisms, we conclude that h is also a quasi-isomorphism. □

Remark 6.4. *In Corollary 6.1, what we have is only existence. Even though by symmetry, there would also exist chain map $h' : D^\bullet \to C^\bullet$ having similar property. The two chain maps in general would not be homotopic inverse of each other.*

Example 6.3. *Recall in Example 6.2, for a short exact sequence of complexes $0 \to A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \to 0$, there is a triangle up to quasi-isomorphism*

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] & B^\bullet \arrow[r, "v"] \arrow[dr] & C^\bullet \arrow[r] & A^\bullet[1] \\ & & \mathrm{cone}(u) \arrow[u, "qis"] \arrow[ur] & \end{tikzcd} \tag{142}$$

If we view $A \in \mathscr{A}$ as a complex $A[0] = [0 \to A \to 0]$ where A is at degree 0, then for a short sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ in $\mathscr{A}$, similarly we have an exact triangle up to quasi-isomorphism

$$\begin{tikzcd} A[0] \arrow[r, "u"] & B[0] \arrow[r, "v"] \arrow[dr] & C[0] \arrow[r] & A[1] \\ & & \mathrm{cone}(u) \arrow[u, "qis"] \arrow[ur] & \end{tikzcd} \tag{143}$$

In particular, $\mathrm{cone}(u) = [0 \to A \to B \to 0]$ *where B is at degree 0, and the map* $\mathrm{cone}(u) \to A[1]$ *is zero map in $\mathcal{D}(\mathscr{A})$ if and only if it is zero map in $\mathcal{K}(\mathscr{A})$ if and only if the given short exact sequence splits.*

I would like to give a lemma generalizing the last sentence in this example.

Lemma 6.6. *Let $\mathscr{A}$ be an abelian category, morphism $\pi : A \to B$ in $\mathscr{A}$. Then the following chain map is zero in $\mathcal{D}(\mathscr{A})$ if and only if it is zero in $\mathcal{K}(\mathscr{A})$.*

$$\begin{tikzcd} {[A} \arrow[r, "\pi"] \arrow[d, "f"] & {B]} \arrow[d] \\ {[C} \arrow[r] & {0]} \end{tikzcd} \tag{144}$$

In addition, dually, the following chain map is zero in $\mathcal{D}(\mathscr{A})$ if and only if it is zero in $\mathcal{K}(\mathscr{A})$.

$$\begin{tikzcd} {[0} \arrow[r] \arrow[d] & {C]} \arrow[d, "f"] \\ {[A} \arrow[r, "\pi"] & {B]} \end{tikzcd} \tag{145}$$

Proof. Suffice to show the $\Rightarrow$ arrow. Assume chain map 144 is zero in $\mathcal{D}(\mathscr{A})$, then there exists quasi-isomorphism $X^\bullet \xrightarrow{\varphi} [A \xrightarrow{\pi} B]$ such that their composition is null-homotopic. By canonical truncation, we may assume that for all $i \geq 1$, $X^i = 0$. Hence there exists morphism $D : X_0 \to A$ such that the following diagram commutes

$$\begin{tikzcd} X^{-1} \arrow[r, "\partial_X"] \arrow[d, "\varphi_{-1}"] & X^0 \arrow[r] \arrow[d, "\varphi_0"] \arrow[ddl, "D"'] & 0 \\ A \arrow[r, "\pi"] \arrow[d, "f"'] & B \arrow[r] \arrow[d] & 0 \\ A \arrow[r] & 0 & \end{tikzcd} \tag{146}$$

Note that φ is a quasi-isomorphism, for all $b \in B$, there exists some $x \in X^0$ such that $\overline{\varphi_0(x)} = \bar{b}$ in B/A. Hence there exists a such that $\pi(a) = \varphi_0(x) - b \in A$. Define $s(b) = D(x) - f(a)$.

First to check that s is well defined. Assume that $x_1, x_2 \in X^0$ satisfies that $\overline{x_1} = \bar{b} = \overline{x_2}$. Then $x_1 - x_2 \in \operatorname{im} \partial_X$ so that there exists $x' \in X^{-1}$ mapping to $x_1 - x_2$. Suppose $\pi(a_1) = \varphi_0(x_1) - b$ and $\pi(a_2) = \varphi_0(x_2) - b$. Note that $\pi(a_1 - a_2) = \varphi_0(x_1 - x_2) = \varphi_0 \circ \partial_X(x') = \pi \circ \varphi_{-1}(x')$. We get $a_1 - a_2 - \varphi_{-1}(x') \in \ker \pi$.

As φ is a quasi-isomorphism, there exists $x'' \in \ker \partial_X$ such that $\varphi_{-1}(x'') = a_1 - a_2 - \varphi_{-1}(x')$. Thus $D(x_1) - f(a_1) - (D(x_2) - f(a_2)) = D(x_1 - x_2) - f(a_1 - a_2) = f \circ \varphi_{-1}(x') - f(a_1 - a_2) = f \circ \varphi_{-1}(-x'') = D \circ \partial_X(-x'') = 0$ so that s is well defined. And $s(a) = 0 - f(-a) = f(a)$. Conclude that chain map 144 is null-homotopic. $\square$

Remark 6.5. *Note that when π is injective, the cohomology group homomorphism induced by chain map 144 is always a zero map, this is also an example that nonzero morphism in $\mathcal{K}(\mathscr{A})$ induces trivial cohomology group homomorphism.*

Lemma 6.7. *Let $\mathscr{A}$ be an abelian category, $A \in \mathscr{A}$. Then for any $B^\bullet \in \mathcal{C}^{\leq 0}(\mathscr{A})$, we have that $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A[0]) = \operatorname{Hom}_{\mathcal{C}(\mathscr{A})}(B^\bullet, A[0])$. Similarly, for any $B^\bullet \in \mathcal{C}^{\geq 0}(\mathscr{A})$, we have that $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(A[0], B^\bullet) = \operatorname{Hom}_{\mathcal{C}(\mathscr{A})}(A[0], B^\bullet)$.*

Proof. We only prove for the first case, given a morphism in $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A[0])$, we can interpret it as $B^\bullet \overset{qis}{\leftarrow} C^\bullet \xrightarrow{f} A[0]$. By truncation $\tau^{\geq 0}$, we may assume that $C^\bullet \in \mathcal{C}^{\leq 0}(\mathscr{A})$ Consider the commutative diagram

$$\begin{tikzcd} C^{-1} \arrow[r, "\partial_{C^\bullet}"] \arrow[d] & C^0 \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{147}$$

As $f \circ \partial_{C^\bullet} = 0$, f factors through $\operatorname{coker} \partial_{C^\bullet} = H^0(C^\bullet)$. Note that $C^\bullet \to B^\bullet$ is a quasi-isomorphism, we get $\operatorname{coker} \partial_{B^\bullet} = H^0(B^\bullet) = H^0(C^\bullet)$. Hence f induces an element in $\operatorname{Hom}_{\mathcal{C}(\mathscr{A})}(B^\bullet, A[0])$.

Want to show that this map is well defined. Given two different interpretations of same morphism in $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A[0])$, by Corollary 6.1, we have a commutative diagram up to homotopy

$$\begin{tikzcd} & C^\bullet \arrow[d, "h"] \arrow[ddl, "g_1"'] \arrow[ddr, "f_1"] & \\ & D^\bullet \arrow[dl, "g_2"] \arrow[dr, "f_2"'] & \\ B^\bullet & & A[0] \end{tikzcd} \tag{148}$$

where h, g_1 and g_2 are quasi-isomorphisms. Since $\operatorname{Hom}_{\mathcal{K}(\mathscr{A})}(C^\bullet, A[0]) = \operatorname{Hom}_{\mathcal{C}(\mathscr{A})}(C^\bullet, A[0])$, we immediately get that $f_1 = f_2 \circ h$. Hence by definition, we have a commutative diagram

$$\begin{tikzcd} C^0 \arrow[r, "h"] & D^0 \arrow[r, "f_2"] & A \\ \ker \partial^0_{C^\bullet} \arrow[u, hook] \arrow[r, "h"] & \ker \partial^0_{D^\bullet} \arrow[u, hook] \arrow[ur] & \\ C^{-1} \arrow[u, "\partial^{-1}_{C^\bullet}"] \arrow[r, "h"] & D^{-1} \arrow[u, "\partial^{-1}_{D^\bullet}"] & \end{tikzcd} \tag{149}$$

By universal property of cokernel, we get $H^0(C^\bullet) \xrightarrow{h_\sharp} H^0(D^\bullet) \to A$. Again by commutativity of the diagram, we get cohomology group homomorphism $g_{1_\sharp} = g_{2_\sharp} \circ h_\sharp$. Hence the map is well defined.

Obviously, the map is surjective. Remains to show that the map is injective. Assume that $B^\bullet \overset{qis}{\leftarrow} C^\bullet \xrightarrow{f} A[0]$ is mapping to zero map. Then $H^0(C^\bullet) \to A$ is zero map so that f is zero map. Conclude that the map is bijective. ☐

Corollary 6.2. *Let $\mathscr{A}$ be an abelian category, $A, B \in \mathscr{A}$. Then* $\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A[0], B[0]) = \mathrm{Hom}_{\mathscr{A}}(A, B)$.

Remark 6.6. *This immediately comes from the previous lemma, implying that $\mathcal{D}(\mathscr{A})$ is good since it preserves what it should preserve.*

We end this subsection with an important proposition, which deeply characterizes quasi-isomorphisms.

Proposition 6.1. *Let $f : A^\bullet \to B^\bullet$ be a chain map in $\mathcal{C}(\mathscr{A})$. Then f is quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is acyclic.*

Proof. "$\Rightarrow$": Assume $[(b, a)] \in H^n(\mathrm{cone}(f))$, then $\partial(b, a) = (\partial b + f(a), -\partial a) = 0$. Hence $[f(a)] = 0$ in $H^{n+1}(B^\bullet)$. Since f is quasi-isomorphism, $[a] = 0$ in $H^{n+1}(A^\bullet)$ so there exists $a' \in A^n$ mapping to a under boundary map. Take $(0, -a') \in \mathrm{cone}(f)^{n-1}$, we get

$$(b, a) - \partial(0, -a') = (b, a) - (-f(a'), a) = (b + f(a'), 0) \tag{150}$$

Note that $\partial(b + f(a')) = \partial b + f(a) = 0$, we get $[b + f(a')] \in H^n(B^\bullet)$. Again since f is quasi-isomorphism, there exists $[a''] \in H^n(A^\bullet)$ such that $[f(a'')] = [b + f(a')]$. Hence there exists $b' \in B^{n-1}$ such that $\partial b' = b + f(a') - f(a'')$. Take $(b', a'' - a')$, we get

$$\partial(b', a'' - a') = (\partial b' + f(a'') - f(a'), \partial a' - \partial a'') = (b, a) \tag{151}$$

Hence $[(b, a)] = 0$ in $H^n(\mathrm{cone}(f))$ and conclude that $\mathrm{cone}(f)$ is acyclic.

"$\Leftarrow$": If $[f(a)] = 0$ in $H^n(B^\bullet)$ for some $[a] \in H^n(A^\bullet)$, then there exists $b \in B^{n-1}$ such that $\partial b = f(a)$. Then $\partial(b, -a) = 0$. As $\mathrm{cone}(f)$ is acyclic, there exists $(b', a') \in \mathrm{cone}(f)^{n-2}$ such that $\partial(b', a') = (b, -a)$. Then we get $\partial a' = a$ so that $[a] = 0$ and f^* is injective.

For all $[b] \in H^n(B^\bullet)$, we get $[(b, 0)] \in H^n(\mathrm{cone}(f))$. As $\mathrm{cone}(f)$ is acyclic, there exists $(b', a) \in \mathrm{cone}(f)^{n-1}$ such that $\partial(b', a) = (b, 0)$. Hence $[a] \in H^n(A^\bullet)$ and $[f(a)] = [b]$. Conclude that f is quasi-isomorphism. ☐

Remark 6.7. *With this proposition, somehow we can much more simplify proof of Lemma 6.5.*

6.2 Derived functor

Definition 6.6. *Let $\mathscr{A}$ be an abelian category. A cochain complex $A^\bullet \in \mathcal{C}(\mathscr{A})$ is said to be strictly bounded-below (resp. above) if there exists some n_0 such that for all $i < n_0$ (resp. $i > n_0$), $A^i = 0$. Denote $\mathcal{C}^+(\mathscr{A})$ (resp. $\mathcal{C}^-(\mathscr{A})$) to be the full subcategory of $\mathcal{C}(\mathscr{A})$ of strictly bounded-below (resp. above) complexes.*

A cochain complex $A^\bullet \in \mathcal{C}(\mathscr{A})$ is said to be bounded-below (resp. above) if there exists some n_0 such that for all $i < n_0$ (resp. $i > n_0$), $H^i(A^\bullet) = 0$. Denote $\mathcal{D}^+(\mathscr{A})$ (resp. $\mathcal{D}^-(\mathscr{A})$) to be the set of equivalence classes of bounded-below (resp. above) complexes.

Remark 6.8. *Clearly, we have that $\mathcal{C}^+(\mathscr{A}) = \cup_n \mathcal{C}^{\geq n}(\mathscr{A})$ and $\mathcal{C}^-(\mathscr{A}) = \cup_n \mathcal{C}^{\leq n}(\mathscr{A})$.*

Definition 6.7. *Let $\mathscr{A}$ be an abelian category. Define canonical truncation $\tau^{\geq n}$ (resp. $\tau^{\leq n}$) to be a functor from $\mathcal{C}(\mathscr{A})$ to $\mathcal{C}(\mathscr{A})$*

$$A^\bullet \longmapsto [0 \to A^n / \operatorname{im} \partial_{n-1} \to A^{n+1} \to \cdots]$$

and respectively

$$A^\bullet \longmapsto [\cdots \to A^{n-1} \to \ker \partial_n \to 0]$$

Remark 6.9. *There are also notions of naive truncations (or say stupid truncations) which sending $A^\bullet$ to $[0 \to A^n \to A^{n+1} \to \cdots]$ or $[\cdots \to A^{n-1} \to A^n \to 0]$.*

Proposition 6.2. *Let $\mathscr{A}$ be an abelian category, $A \in \mathscr{A}$ and $B^\bullet \in \mathcal{C}(\mathscr{A})$. Then for all i, we have that $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A[i]) = \operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^{\geq -i}, A[i])$ and $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(A[i], B^\bullet) = \operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(A[i], B^{\leq -i})$.*

Proof. Here we only prove for the first case. We may assume that $i = 0$. The natural map $B^\bullet \to B^{\geq 0}$ would induces a homomorphism $\varphi : \operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^{\geq 0}, A[0]) \to \operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^\bullet, A[0])$. Want to show that φ is bijective.

For injectivity, suppose that $B^{\geq 0} \to C^\bullet \overset{qis}{\leftarrow} A[0]$ is mapping to zero under φ. Then there is a commutative diagram

$$\begin{tikzcd}
B^\bullet \arrow[d] & D^\bullet \arrow[l, "qis"'] \arrow[d] \arrow[dd, bend left, "0"] \\
B^{\geq 0} \arrow[d] & D^{\geq 0} \arrow[l, "qis"'] \\
C^\bullet & A[0] \arrow[l, "qis"']
\end{tikzcd} \tag{152}$$

Hence there exists chain homotopy $\psi : D^\bullet \to C^\bullet$ such that $\psi \circ \partial + \partial \circ \psi = D^\bullet \overset{qis}{\to} B^\bullet \to B^{\geq 0} \to C^\bullet$. Note that $A[0] \to C^\bullet$ is a quasi-isomorphism, for all $j < 0$, $C^\bullet$ is exact at degree j. Hence natural map $C^\bullet \to C^{\geq 0}$ is a quasi-isomorphism and we get another interpretation of the morphism $B^{\geq 0} \to A[0]$. Consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & D^{-1} \arrow[r] \arrow[d, "0"] & D^0 \arrow[r] \arrow[d] \arrow[dl, "\psi_0"] & D^1 \arrow[r] \arrow[d] \arrow[dl, "\psi_1"] & \cdots \\
\cdots \arrow[r] & C^{-1} \arrow[r] \arrow[d] & C^0 \arrow[r] \arrow[d, two heads] & C^1 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C^0/\operatorname{im}\partial \arrow[r] & C^1 \arrow[r] & \cdots
\end{tikzcd} \tag{153}$$

As the zero map $D^\bullet \to A[0]$ factors through $D^{\geq 0}$, we get

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & D^0/\operatorname{im}\partial \arrow[r] \arrow[d] \arrow[dl] & D^1 \arrow[r] \arrow[d] \arrow[dl, "\widetilde{\psi}_1"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C^0/\operatorname{im}\partial \arrow[r] & C^1 \arrow[r] & \cdots
\end{tikzcd} \tag{154}$$

Check that $\widetilde{\psi}$ is a chain homotopy between $D^{\geq 0} \to C^{\geq 0}$ and zero map. For $j \geq 2$, $\widetilde{\psi}_j = \psi_j$ hence ok. For $j = 1$, $\partial_{C^{\geq 0}} \circ \widetilde{\psi}_1 = \partial_{C^\bullet} \circ \psi_0$ hence ok. For $j = 0$, as $\operatorname{im}(\partial \circ \psi) \subseteq \operatorname{im}\partial$, for all $\overline{a} \in D^0/\operatorname{im}\partial$, $\widetilde{\psi}_1 \circ \partial(\overline{a}) = \overline{\psi_1(a)}$ is just the the image of $[a]$ under the chain map hence ok. Conclude that the morphism $B^\bullet \to A[0]$ is zero in $\operatorname{Hom}_{\mathcal{D}(\mathscr{A})}(B^{\geq 0}, A[0])$.

For surjectivity, for any morphism $B^\bullet \overset{qis}{\leftarrow} D^\bullet \to A[0]$, since $D^\bullet \to A[0]$ would factor through $D^{\geq 0}$, we get commutative diagram

$$\begin{tikzcd}
B^\bullet \arrow[d] & D^\bullet \arrow[l, "qis"'] \arrow[d] \arrow[dr] & \\
B^{\geq 0} \arrow[dr] & D^{\geq 0} \arrow[l, "qis"'] \arrow[r] & A[0] \arrow[dl, "qis"] \\
& C^\bullet &
\end{tikzcd} \tag{155}$$

where $C^\bullet$ is given by Lemma 6.5. By commutativity, we get $B^\bullet \overset{qis}{\leftarrow} D^\bullet \to A[0]$ is the image of $B^{\geq 0} \overset{qis}{\leftarrow} D^{\geq 0} \to A[0]$ under φ, done! $\square$

Lemma 6.8. *Let $\mathscr{A}$ be an abelian category. Then $\mathcal{D}^+(\mathscr{A})$ (resp. $\mathcal{D}^-(\mathscr{A})$) can be identified with the localization of $\mathcal{C}^+(\mathscr{A})$ (resp. $\mathcal{C}^-(\mathscr{A})$) by quasi-isomorphisms.*

Reason 6.1. *Linguistically, for all $A^\bullet \in \mathcal{D}^+(\mathscr{A})$, we only need to find a quasi-isomorphism $A^\bullet \to B^\bullet$ with B^{bullet} strictly bounded-below. Since $A^\bullet \in \mathcal{D}^+(\mathscr{A})$, there exists some n_0 such that for all $i < n_0$, $H^i(A^\bullet) = 0$. It is natural to consider canonical truncation $\tau^{\geq n_0 - 1}$. There is a natural chain map*

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n_0-2} \arrow[r] \arrow[d] & A^{n_0-1} \arrow[r] \arrow[d] & A^{n_0} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A^{n_0-1}/\operatorname{im}\partial_{n_0-2} \arrow[r] & A^{n_0} \arrow[r] & \cdots
\end{tikzcd} \tag{156}$$

Clearly, this is a quasi-isomorphism and $\tau^{\geq n_0-1}(A^\bullet)$ is strictly bounded-below.

Recall that in Remark 6.1, when $\mathscr{A}$ has enough injective objects, we have defined derived functors both for objects in $\mathscr{A}$ and complexes in $\mathcal{C}^+(\mathscr{A})$. And same idea works for strictly bounded-below complex in $\mathcal{C}^+(\mathscr{A})$. But what's about the case that $\mathscr{A}$ does not have enough injective objects? In the following, we would give a sufficient condition for existence of derived functor.

Definition 6.8. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories, $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ an additive functor, $\mathscr{A}' \subseteq \mathscr{A}$ full additive subcategory. We say that $\mathscr{A}'$ satisfies $\mathcal{F}$-condition if*
(1)for all $A \in \mathscr{A}$, there is an injection $A \hookrightarrow A'$ with $A' \in \mathscr{A}'$.
(2)for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, if A and B are both in $\mathscr{A}'$, then C is also in $\mathscr{A}'$.
(3)$\mathcal{F}$ is an exact functor on $\mathscr{A}'$.

Now given $\mathcal{F}$ and $\mathscr{A}' \subseteq \mathscr{A}$ satisfying $\mathcal{F}$-condition, by definition, for all $A \in \mathscr{A}$, we can find a $\mathcal{F}$-resolution $J^\bullet$ of A. Similar to the case when $\mathscr{A}$ has enough injective objects, we can just define $R^i\mathcal{F}(A) = H^i(\mathcal{F}(J^\bullet))$.

For complexes, by same strategy of construction of Cartan-Eilenburg resolution, there exists resolution $\mathcal{J}^{\bullet,*}$ for all $A^\bullet \in \mathcal{C}^+(\mathscr{A})$. Hence we can also define hyper cohomology $R^i\mathcal{F}(A^\bullet) = H^i(\mathrm{Tot}(\mathcal{J}^{\bullet,*}))$. And in fact, we would get a quasi-isomorphism $A^\bullet \to \mathrm{Tot}(\mathcal{J}^{\bullet,*})$. To define derived functor on derived category, consider the following diagram

$$\begin{tikzcd} \mathcal{K}^+(\mathscr{A}) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{K}^+(\mathscr{B}) \arrow[d] \\ \mathcal{D}^+(\mathscr{A}) \arrow[r, dashed, "R\mathcal{F}"] & \mathcal{D}^+(\mathscr{B}) \end{tikzcd} \tag{157}$$

where $\mathcal{K}^+(\mathscr{A})$ (resp. $\mathcal{K}^-(\mathscr{A})$) is the homotopy category of strictly bounded-below (resp. above) complexes. And it is natural to define $R\mathcal{F}(A^\bullet) = \mathcal{F}(\mathrm{Tot}(\mathcal{J})^{\bullet,*})$. To show this definition is well defined and really a functor, we need following lemmas.

Lemma 6.9. *Let $A^\bullet \to B^\bullet$ be a chain map in $\mathcal{C}^+(\mathscr{A})$. By argument above, there are two quasi-isomorphisms $A^\bullet \to A'^\bullet$ and $B^\bullet \to B'^\bullet$ with $A'^\bullet, B'^\bullet \in \mathcal{C}(\mathscr{A}')$. Then there exists morphism $A'^\bullet \to B'^\bullet$ in $\mathcal{D}(\mathscr{A}')$ making the following diagram commutes*

$$\begin{tikzcd} A^\bullet \arrow[r, "qis"] \arrow[d] & A'^\bullet \arrow[d] \\ B^\bullet \arrow[r, "qis"] & B'^\bullet \end{tikzcd} \tag{158}$$

Moreover, the morphism is independent to the choice of quasi-isomorphisms.

Proof. Applying Lemma 6.5 (1), there exists $C^\bullet \in \mathcal{C}(\mathscr{A})$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} A^\bullet \arrow[r, "qis"] \arrow[d] & A'^\bullet \arrow[d] \\ B^\bullet \arrow[r, "qis"] \arrow[d, "qis"] & C^\bullet \\ B'^\bullet & \end{tikzcd} \tag{159}$$

Note that $C^\bullet$ is also bounded-below, by Cartan-Eilenburg resolution, we can replace $C^\bullet$ by $C'^\bullet \in \mathcal{C}(\mathscr{A}')$ with $C^\bullet \overset{qis}{\to} C'^\bullet$. Again apply Lemma 6.5 (1), there exists $D^\bullet \in \mathcal{C}(\mathscr{A}')$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} A^\bullet \arrow[r, "qis"] \arrow[d] & A'^\bullet \arrow[d] \\ B^\bullet \arrow[r, "qis"] \arrow[d, "qis"] & C^\bullet \arrow[d] \\ B'^\bullet \arrow[r, "qis"] & D^\bullet \end{tikzcd} \tag{160}$$

Hence we get a morphism $A'^\bullet \to B'^\bullet$ in $\mathcal{D}(\mathscr{A}')$. Independentness is clear. $\square$

Lemma 6.10. *Let $A^\bullet \to B^\bullet$ be a quasi-isomorphism in $\mathcal{C}(\mathscr{A}')$. Then $\mathcal{F}(A^\bullet) \to \mathcal{F}(B^\bullet)$ is also a quasi-isomorphism.*

Reason 6.2. *As $\mathcal{F}$ is exact on $\mathscr{A}'$, it preserves kernel, image and hence cohomology group.*

Remark 6.10. *This lemma tells us we can well define $\mathcal{F}$ on $\mathcal{D}(\mathscr{A}')$.*

Lemma 6.11. *$R\mathcal{F}$ is well defined.*

Proof. Assume that there are two quasi-isomorphisms $f : A^\bullet \to A'^\bullet$ and $g : A^\bullet \to A''^\bullet$ with $A'^\bullet, A''^\bullet \in \mathcal{C}(\mathscr{A}')$. Similar to proof of Lemma 6.5 (1), taking $A'''^\bullet = \mathrm{cone}((f,g))$, we get a commutative diagram

$$\begin{tikzcd} A^\bullet \arrow[r, "g"] \arrow[d, "f"] & A''^\bullet \arrow[d, "\beta"] \\ A'^\bullet \arrow[r, "-\alpha"] & A'''^\bullet \end{tikzcd} \tag{161}$$

where α and β are natural inclusions. By same argument as proof of Lemma 6.5 (1), α and β are both quasi-isomorphism. By Lemma 6.10, $\mathcal{F}(A'^\bullet) = \mathcal{F}(A'''^\bullet) = \mathcal{F}(A''^\bullet)$ in $\mathcal{D}(\mathscr{A})$, $\square$

Recall that between triangulated categories, a triangulated functor (or say exact functor) is a functor respecting the triangulated structure i.e it commutes with shift functor and preserves exact triangles. Then for any additive functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, the induced functor $\mathcal{F} : \mathcal{K}(\mathscr{A}) \to \mathcal{K}(\mathscr{B})$ is triangulated.

With lemmas above, we can define a natural transformation

$$\begin{tikzcd} \mathcal{K}^+(\mathscr{A}) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{K}^+(\mathscr{B}) \arrow[d] \arrow[dl, Rightarrow, "\varepsilon"'] \\ \mathcal{D}^+(\mathscr{A}) \arrow[r, "R\mathcal{F}"] & \mathcal{D}^+(\mathscr{B}) \end{tikzcd} \tag{162}$$

For all $A^\bullet \in \mathcal{K}^+(\mathscr{A})$, take quasi-isomorphism $A^\bullet \to A'^\bullet$ with $A'^\bullet \in \mathcal{C}(\mathscr{A}')$. The morphism $\varepsilon(A^\bullet)$ in $\mathcal{D}(\mathscr{A})$ is just the morphism in $\mathcal{K}(\mathscr{A})$ induced by $A^\bullet \overset{qis}{\to} A'^\bullet$. And for all morphism $f : A^\bullet \to B^\bullet$ in $\mathcal{C}^+(\mathscr{A})$, there is a commutative diagram

$$\begin{tikzcd} \mathcal{F}(A^\bullet) \arrow[r, "\varepsilon(A^\bullet)"] \arrow[d, "\mathcal{F}f"] & R\mathcal{F}(A^\bullet) = \mathcal{F}(A'^\bullet) \arrow[d, "\mathcal{F}\tilde{f}"] \\ \mathcal{F}(B^\bullet) \arrow[r, "\varepsilon(B^\bullet)"] & R\mathcal{F}(B^\bullet) = \mathcal{F}(B'^\bullet) \end{tikzcd} \tag{163}$$

where $\widetilde{f}$ is the morphism in $\mathcal{D}(\mathscr{A}')$ given by Lemma 6.9.

In addition, $R\mathcal{F}$ is an exact functor and $(R\mathcal{F}, \varepsilon)$ has universal property as following. Given exact functor $\mathcal{G} : \mathcal{D}^+(\mathscr{A}) \to \mathcal{D}^+(\mathscr{B})$ and a natural transformation

$$\begin{tikzcd} \mathcal{K}^+(\mathscr{A}) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{K}^+(\mathscr{B}) \arrow[d] \arrow[dl, Rightarrow, "\varepsilon'"'] \\ \mathcal{D}^+(\mathscr{A}) \arrow[r, "\mathcal{G}"] & \mathcal{D}^+(\mathscr{B}) \end{tikzcd} \tag{164}$$

there exists unique natural transformation $\pi : R\mathcal{F} \to \mathcal{G}$ such that $\varepsilon' = \pi \circ \varepsilon$.

Example 6.4. *Let X be a smooth projective complex manifold. As Example 6.1, there is a resolution of $\mathbb{C}$*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X = \Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots \tag{165}$$

Hence $\mathcal{C}_X[0] \to \Omega_X^\bullet$ is a quasi-isomorphism and $R\Gamma(X, \mathbb{C}_X[0]) = R\Gamma(X, \Omega_X^\bullet)$.

Algebraically, the same complex is not still exact. However, by Serre's GAGA, we claim that $R\Gamma(X^{\mathrm{alg}}, \Omega^\bullet_{X^{\mathrm{alg}}}) = R\Gamma(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$.

6.3 Total hom functor

Definition 6.9. *Let $\mathscr{A}$ be an abelian category, $A^\bullet, B^\bullet \in \mathcal{C}(\mathscr{A})$. Define total hom complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$, where $\mathrm{Hom}^p(A^\bullet, B^\bullet) = \prod_n \mathrm{Hom}_{\mathscr{A}}(A^n, B^{n+p})$. In fact, this is the total complex of the following diagram*

$$\begin{tikzcd} & \cdots \arrow[d] & \cdots \arrow[d] & \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A^n, B^{n+p}) \arrow[r, "\circ\partial_A^{n-1}"] \arrow[d, "\partial_B^{n+p}\circ"] & \mathrm{Hom}_{\mathscr{A}}(A^{n-1}, B^{n+p}) \arrow[r] \arrow[d, "\partial_B^{n+p}\circ"] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A^n, B^{n+p+1}) \arrow[r, "\circ\partial_A^{n-1}"] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(A^{n-1}, B^{n+p+1}) \arrow[r] \arrow[d] & \cdots \\ & \cdots & \cdots & \end{tikzcd} \tag{166}$$

with boundary map given by $f \mapsto \partial_B^{n+p} \circ f - (-1)^p f \circ \partial_A^{n-1}$.

Fixed $A^\bullet$, we would get a covariant functor $\mathrm{Hom}^\bullet(A^\bullet, \cdot)$, called total hom functor. While total hom functor does not comes from some functor $\mathcal{F} : \mathscr{A} \to \mathbf{Ab}$, to define derived functor of it, we need generalize our argument in subsection 6.2.

Definition 6.10. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories, $\mathcal{F} : \mathcal{K}(\mathscr{A}) \to \mathcal{K}(\mathscr{B})$ a triangulated functor, $\mathcal{K}' \subseteq \mathcal{K}(\mathscr{A})$ full triangulated subcategory. We say that $\mathcal{K}'$ satisfies $\mathcal{F}$-condition if*
(1)for all $A^\bullet \in \mathcal{K}(\mathscr{A})$, there is a quasi-isomorphism $A^\bullet \to A'^\bullet$ with $A'^\bullet \in \mathcal{K}'$.
(2)for all $A^\bullet \in \mathcal{K}'$ acyclic, $\mathcal{F}A^\bullet$ is also acyclic.

Remark 6.11. *Here, we directly ask quasi-isomorphism exist. Hence we do not need the strictly bounded-below condition, which guarantees that we can take resolution in the past.*

Now given $\mathcal{F}$ and $\mathcal{K}' \subseteq \mathcal{K}(\mathscr{A})$ satisfying $\mathcal{F}$-condition, by definition, for all $A \in \mathcal{K}(\mathscr{A})$, there is a quasi-isomorphism $A^\bullet \to A'^\bullet$ with $A'^\bullet \in \mathcal{K}'$. To define derived functor on derived category, consider the following diagram

$$\begin{tikzcd} \mathcal{K}(\mathscr{A}) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{K}(\mathscr{B}) \arrow[d] \\ \mathcal{D}(\mathscr{A}) \arrow[r, dashed, "R\mathcal{F}"] & \mathcal{D}(\mathscr{B}) \end{tikzcd} \tag{167}$$

And it is natural to define $R\mathcal{F}(A^\bullet) = \mathcal{F}(A'^\bullet)$. To show this definition is well defined and really a functor, we need following generalized lemmas, and most of the proofs are same as their original version and we only provide proof of Lemma 6.13 here.

Lemma 6.12. *Let $A^\bullet \to B^\bullet$ be a chain map in $\mathcal{C}(\mathscr{A})$. By definition, there are two quasi-isomorphisms $A^\bullet \to A'^\bullet$ and $B^\bullet \to B'^\bullet$ with $A'^\bullet, B'^\bullet \in \mathcal{K}'$. Then there exists morphism $A'^\bullet \to B'^\bullet$ in $\mathcal{D}(\mathcal{K}')$ making the following diagram commutes*

$$\begin{tikzcd} A^\bullet \arrow[r, "qis"] \arrow[d] & A'^\bullet \arrow[d] \\ B^\bullet \arrow[r, "qis"] & B'^\bullet \end{tikzcd} \tag{168}$$

where $\mathcal{D}(\mathcal{K}')$ is the localization of $\mathcal{K}'$ over quasi-isomorphisms. Moreover, the morphism is independent to the choice of quasi-isomorphisms.

Lemma 6.13. *Let $A^\bullet \to B^\bullet$ be a quasi-isomorphism in $\mathcal{K}'$. Then $\mathcal{F}(A^\bullet) \to \mathcal{F}(B^\bullet)$ is also a quasi-isomorphism.*

Proof. Firstly, by Proposition 6.1, $\operatorname{cone}(f)$ is acyclic. By definition, we know $\mathcal{F}\operatorname{cone}(f)$ is still acyclic. Consider exact triangle

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \operatorname{cone}(f) \longrightarrow A^\bullet[1] \tag{169}$$

As $\mathcal{F}$ is a triangulated functor, applying $\mathcal{F}$, we get an exact triangle

$$\mathcal{F}A^\bullet \xrightarrow{\mathcal{F}f} \mathcal{F}B^\bullet \longrightarrow \mathcal{F}\operatorname{cone}(f) \longrightarrow \mathcal{F}A^\bullet[1] \tag{170}$$

By axiom $(TR3)$ of triangulated category, there is a commutative diagram in $\mathcal{K}'$

$$\begin{tikzcd} \mathcal{F}A^\bullet \arrow[r, "\mathcal{F}f"] \arrow[d, equal] & \mathcal{F}B^\bullet \arrow[r] \arrow[d, equal] & \operatorname{cone}(\mathcal{F}f) \arrow[r] \arrow[d, "\varphi"] & \mathcal{F}A^\bullet[1] \arrow[d, equal] \\ \mathcal{F}A^\bullet \arrow[r, "\mathcal{F}f"] & \mathcal{F}B^\bullet \arrow[r] & \mathcal{F}\operatorname{cone}(f) \arrow[r] & \mathcal{F}A^\bullet[1] \end{tikzcd} \tag{171}$$

By 5 Lemma, φ is an isomorphism and hence $\operatorname{cone}(\mathcal{F}f)$ is acyclic. Again by Proposition 6.1, we get $\mathcal{F}f$ is quasi-isomorphism. $\square$

Lemma 6.14. *$R\mathcal{F}$ is well defined.*

Assume that $\mathscr{A}$ has enough injective objects. Take $\mathcal{K}'$ to be the homotopy category of bound-below complexes of injective objects. To define derived functor of total hom functor, it remains to show that $\mathcal{K}'$ satisfies $\operatorname{Hom}^\bullet(A^\bullet, \cdot)$-condition for all $A^\bullet$. Here we only prove for the acyclic condition.

Proposition 6.3. *For all complexes $A^\bullet, B^\bullet \in \mathcal{C}(\mathscr{A})$, we have that $H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) = \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, B^\bullet[n])$.*

Proof. For chain map $f : A^\bullet \to B^\bullet$ in $\ker(\partial^n_{\mathrm{Hom}^\bullet})$, we have that $-(-1)^n f_i \circ \partial^{i-1}_{A^\bullet} + \partial^{i+n-1}_{B^\bullet} \circ f_{i-1} = 0$ so that $f_i \circ \partial^{i-1}_{A^\bullet} = \partial^{i-1}_{B^\bullet[n]} \circ f_{i-1}$. Hence each f defines a chain map $\widetilde{f} : A^\bullet \to B^\bullet[n]$.

For $[f] = [g]$ in $H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$, we get that $f - g = \partial h$ for some $h \in \mathrm{Hom}^{n-1}(A^\bullet, B^\bullet)$. While

$$\begin{aligned} \partial h &= (-(-1)^{n-1} h_i \circ \partial^{i-1}_{A^\bullet} + \partial^{i+n-1}_{B^\bullet} \circ g_{i-1}) \\ &= ((-1)^n (g_i \circ \partial^{i-1}_{A^\bullet} + \partial^{i+n-1}_{B^\bullet} \circ g_{i-1})) \end{aligned} \tag{172}$$

so that $\widetilde{f}$ is homotopic to $\widetilde{g}$. Thus we get a well defined map from $H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$ to $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, B^\bullet[n])$. Clearly, the map is bijective. $\square$

Recall that in homological algebra, we say a complex $A^\bullet$ is contractible if $\mathrm{id}_{A^\bullet}$ is null-homotopic and all acyclic bounded-below complex of injective objects is contractible. We have the following lemma.

Lemma 6.15. *Let $I^\bullet \in \mathcal{K}'$ be acyclic bounded-below complex of injective objects. Then for all complex $A^\bullet$, $\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)$ is still acyclic.*

Proof. By Proposition 6.3, this is equivalent to $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, I^\bullet[n]) = \{0\}$ for all n. Hence, we suffice to show that any chain map $f : A^\bullet \to I^\bullet$ is null-homotopic, which immediately comes from the fact that $I^\bullet$ is contractible. $\square$

Hence $\mathcal{K}'$ satisfies $\mathrm{Hom}^\bullet(A^\bullet, \cdot)$-condition for all $A^\bullet$. And we can define derived functor, denoted by $R\,\mathrm{Hom}(A^\bullet, \cdot)$.

Lemma 6.16. *Let $I^\bullet \in \mathcal{K}'$ be bounded-below complex of injective objects. Then for all complex $A^\bullet$, we have that $\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, I^\bullet)$.*

Proof. Only need to show that for any morphism $A^\bullet \to B^\bullet \overset{qis}{\leftarrow} I^\bullet$, there exists chain map $A^\bullet \to I^\bullet$ making the diagram commutative up to homotopy

$$\begin{tikzcd} & B^\bullet & \\ A^\bullet \arrow[ur] \arrow[rr] & & I^\bullet \arrow[ul, "qis"'] \end{tikzcd} \tag{173}$$

While here we prove a stronger fact that there exists chain map $B^\bullet \to I^\bullet$, whose composition with the quasi-isomorphism is homotopy to $\mathrm{id}_{I^\bullet}$.

In fact, since $q : I^\bullet \to B^\bullet \to B'^\bullet$ is quasi-isomorphism. By Proposition 6.1, $\mathrm{cone}(q) \in \mathcal{K}'$ is acyclic and hence contractible. Hence just as proof of Lemma 6.15, the natural map $\mathrm{cone}(q) \to I^\bullet[1]$ is null-homotopic. Consider the chain homotopy $D_n = (\alpha_n, \beta_n) : \mathrm{cone}(q)^n \to I^n$. Then

$$\begin{cases} \partial_{I^\bullet} \circ \alpha_n = \alpha_{n+1} \circ \partial_{\mathrm{cone}(q)} \\ \mathrm{id}_{I^\bullet} = \alpha_{n+1} \circ q - \partial_{I^\bullet} \circ \beta_n - \beta_{n+1} \circ \partial_{I^\bullet} \end{cases} \tag{174}$$

Hence α is a chain map and $\alpha \circ q$ is homotopy to $\mathrm{id}_{I^\bullet}$. $\square$

Lemma 6.17. *For all quasi-isomorphism $E \to E'$ in $\mathcal{C}(\mathscr{A})$, the induced map $\mathrm{Hom}^\bullet(E', F) \to \mathrm{Hom}^\bullet(E, F)$ is also a quasi-isomorphism for all $F \in \mathcal{K}'$.*

Proof. By Lemma 6.16, we immediately get $H^n(\mathrm{Hom}^\bullet(E', F)) = \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E', F[n])$ and $H^n(\mathrm{Hom}^\bullet(E, F)) = \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E, F[n])$. Hence

$$\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E', F[n]) \overset{\circ(E' \to E)}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E, F[n]) \tag{175}$$

is an isomorphism. Conclude that $\mathrm{Hom}^\bullet(E', F) \to \mathrm{Hom}^\bullet(E, F)$ is a quasi-isomorphism. $\square$

Remark 6.12. *With the previous two lemmas, when consider a total hom functor, we can directly take $A^\bullet \in \mathcal{D}(\mathscr{A})$. In addition, if we denote $\mathrm{Ext}^n(A^\bullet, B^\bullet) = H^n(R\,\mathrm{Hom}(A^\bullet, B^\bullet))$, then*

$$\begin{aligned} \mathrm{Ext}^n(A^\bullet, B^\bullet) &= H^n(R\,\mathrm{Hom}(A^\bullet, B^\bullet)) \\ &= H^n(\mathrm{Hom}^\bullet(A^\bullet, B'^\bullet)) \\ &= \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, B'^\bullet[n]) \\ &= \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A^\bullet, B'^\bullet[n]) \\ &= \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A^\bullet, B^\bullet[n]) \end{aligned} \tag{176}$$

so that we can directly write out extension classes without annoying $A'^\bullet$ or $B'^\bullet$.

Let $\mathscr{A} = \mathrm{Mod}(\mathcal{O}_X)$. For short, we denote homotopy category and derived category by $\mathcal{K}(X)$ and $\mathcal{D}(X)$ respectively. Consider total tensor product $\mathcal{F}^\bullet \otimes^n_{\mathcal{O}_X} \mathcal{G}^\bullet = \oplus_{p+q=n} \mathcal{F}^p \otimes_{\mathcal{O}_X} \mathcal{G}^q$. Replacing injective objects by flat $\mathcal{O}_X$-modules, we can also define derived functor of total tensor functor. In particular, we have derived pull back Lf^*.

7 Cotangent Complex

7.1 Some properties of cotangent complex

Let $f : X \to Y$ be a morphism of scheme. Denote the cotangent complex of f by $\mathbb{L}_f$, or when there is no confusion by $\mathbb{L}_{X/Y}$. At most of time, we would view it as an element in $\mathcal{D}^{\leq 0}(X)$. Firstly, let's state some properties about cotangent complex.

Proposition 7.1. *Given commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \tag{177}$$

there is an exact triangle in $\mathcal{D}(X)$

$$Lf^*\mathbb{L}_{Y/S} \longrightarrow \mathbb{L}_{X/S} \longrightarrow \mathbb{L}_f \longrightarrow Lf^*\mathbb{L}_{Y/S}[1] \tag{178}$$

Remark 7.1. *In fact, $H^0(\mathbb{L}_{X/Y}) = \Omega^1_{X/Y}$ and when taking sheaf cohomology, we would get a long exact sequence*

$$\cdots \longrightarrow H^{-1}(\mathbb{L}_f) \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{179}$$

which is a generalization of classic short exact sequence of Kahler differentials. Hence to some extent, $H^{-1}(\mathbb{L}_f)$ measures the singularities of f.

Lemma 7.1. *Let $X \to Y$ be an etale morphism. Then $\mathbb{L}_{X/Y} = 0$.*

Lemma 7.2. *Let $X \to Y$ be a smooth morphism. Then $\mathbb{L}_{X/Y} = \Omega^1_{X/Y}[0]$.*

Lemma 7.3. *Let $X \to Y$ be a morphism of schemes. Assume $X \to Y$ factors W such that $X \hookrightarrow W$ is a regular embedding with ideal sheaf $\mathcal{I}$ and $W \to Y$ is smooth. Then $\mathbb{L}_{X/Y} = [\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{W/Y}\big|_X]$.*

Definition 7.1. *Let $X \to Y$ be a morphism of schemes. Define naive cotangent complex of f to be $\mathbb{L}^{\geq -1}_{X/Y} = \tau^{\geq -1}\mathbb{L}_{X/Y}$. In some references, this would also be denoted as $N\mathbb{L}_{X/Y}$.*

Lemma 7.4. *Let $X \to Y$ be a morphism of schemes. Assume $X \to Y$ factors W such that $X \hookrightarrow W$ is a closed embedding with ideal sheaf $\mathcal{I}$ and $W \to Y$ is smooth. Then $\mathbb{L}^{\geq -1}_{X/Y} = [\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{W/Y}\big|_X]$.*

Remark 7.2. *In affine case, we can always find such a W. Let $f : A \to B$ be a ring homomorphism. We can define naive cotangent complex of f to be $N\mathbb{L}_{B/A} = [I/I^2 \to \Omega_{P/A} \otimes_P B]$, where $P = A[B]$ and I is kernel of $P \to B$.*

Proposition 7.2. *Let $X \hookrightarrow Y$ be a closed immersion with ideal sheaf $\mathcal{I}$. Then $\mathbb{L}^{\geq -1}_{X/Y} = \mathcal{I}/\mathcal{I}^2[1]$.*

Reason 7.1. *Consider $X \hookrightarrow Y \xrightarrow{\mathrm{id}_Y} Y$. By Lemma 7.4, we get $\mathbb{L}^{\geq -1}_{X/Y} = [\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y/Y}\big|_X] = \mathcal{I}/\mathcal{I}^2[1]$.*

7.2 Deformation of map

Consider deformation problem of map over K-scheme. Here K is a ring, not necessarily a field. Given closed immersion $T \hookrightarrow T'$ with ideal sheaf $\mathcal{J}$ of square 0. Assume there is a morphism $f : T \to X$. Question if there exists extension $T' \to X$ making the diagram commutative

$$\begin{tikzcd} T' \arrow[rr, dashed] & & X \\ & T \arrow[ul] \arrow[ur, "f"'] & \end{tikzcd} \tag{180}$$

In fact, we have that

- no automorphism
- extensions are pseudo torsor under $\mathrm{Ext}^0_T(Lf^*\mathbb{L}_X, \mathcal{J})$
- obstruction lies in $\mathrm{Ext}^1_T(Lf^*\mathbb{L}_X, \mathcal{J})$.

For extensions of map, assume that there are two extensions

$$\begin{tikzcd} T' \arrow[rr, shift left, "g"] \arrow[rr, shift right, "h"'] & & X \\ & T \arrow[ul] \arrow[ur, "f"'] & \end{tikzcd} \tag{181}$$

then locally we have that

$$\begin{tikzcd} A' \arrow[dr, two heads] & & B \arrow[ll, shift right, "g^\sharp"'] \arrow[ll, shift left, "h^\sharp"] \arrow[dl, "f^\sharp"] \\ & A & \end{tikzcd} \tag{182}$$

By commutativity, $g^\sharp - h^\sharp$ defines a K-module homomorphism $B \to J$. Thus

$$\begin{aligned} \{\text{extensions}\} &\leftrightsquigarrow \mathrm{Der}_K(B, J) \\ &\leftrightsquigarrow \mathrm{Hom}_B(\Omega_B, J) \end{aligned} \tag{183}$$

where $\mathrm{Hom}_B(\Omega_B, J) = \mathrm{Hom}_B(\Omega_B, \mathrm{Hom}_A(A, J)) = \mathrm{Hom}_A(\Omega_B \otimes_B A, J)$. Hence globally, extensions are classified by $\mathrm{Hom}_T(f^*\Omega^1_X, \mathcal{J})$.

Lemma 7.5. *In fact,* $\mathrm{Hom}_T(f^*\Omega^1_X, \mathcal{J}) = \mathrm{Ext}^0_T(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}) = \mathrm{Ext}^0_T(Lf^*\mathbb{L}_X, \mathcal{J})$.

Proof. We firstly show the second equality. Consider flat resolution of cotangent complex $\mathbb{L}_X$

$$\begin{tikzcd} & \cdots \arrow[d] & \cdots \arrow[d] & \\ \cdots \arrow[r] & \mathcal{F}^{-1,-2} \arrow[r] \arrow[d] & \mathcal{F}^{0,-2} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & \mathcal{F}^{-1,-1} \arrow[r] \arrow[d] & \mathcal{F}^{0,-1} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & \mathcal{F}^{-1,0} \arrow[r] \arrow[d] & \mathcal{F}^{0,0} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & \mathbb{L}_X^{-1} \arrow[r] & \mathbb{L}_X^{0} \arrow[r] & 0 \end{tikzcd} \tag{184}$$

Take total complex, we get quasi-isomorphism $\mathrm{Tot}(\mathcal{F}^{\bullet,*}) \to \mathbb{L}_X$. Note that by truncation $\tau^{\geq -1}$, $\mathrm{Tot}(\mathcal{F}^{\geq -1,*}) \to \mathbb{L}_X^{\geq -1}$ also a quasi-isomorphism, we have that $\partial^{-1}_{Lf^*\mathbb{L}_X} = \partial^{-1}_{Lf^*\mathbb{L}_X^{\geq -1}}$. While by Lemma 6.7,

$$\mathrm{Hom}_{\mathcal{D}(T)}(Lf^*\mathbb{L}_X, \mathcal{J}[0]) = \mathrm{Hom}_{\mathcal{C}(T)}(Lf^*\mathbb{L}_X, \mathcal{J}[0]) = \mathrm{Hom}_T(\mathrm{coker}\, \partial^{-1}_{Lf^*\mathbb{L}_X}, \mathcal{J}) \tag{185}$$

and

$$\mathrm{Hom}_{\mathcal{D}(T)}(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}[0]) = \mathrm{Hom}_{\mathcal{C}(T)}(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}[0]) = \mathrm{Hom}_T(\mathrm{coker}\, \partial^{-1}_{Lf^*\mathbb{L}_X^{\geq -1}}, \mathcal{J}) \tag{186}$$

Hence the second equality holds. For the first equality, as f^* is right exact functor, we get that $\mathrm{coker}\, \partial^{-1}_{Lf^*\mathbb{L}_X^{\geq -1}} = H^0(\mathrm{Tot}(\mathcal{F}^{\geq -1,*})) = H^0(\mathbb{L}_X^{\geq -1}) = \Omega_X$, done! $\square$

Remark 7.3. *Similarly, we also have* $\mathrm{Ext}^1_T(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}) = \mathrm{Ext}^1_T(Lf^*\mathbb{L}_X, \mathcal{J})$.

For obstruction, by Proposition 7.1, we have a diagram in $\mathcal{D}(T)$

$$\begin{tikzcd} Lf^*\mathbb{L}_X \arrow[r] & \mathbb{L}_T \arrow[r] \arrow[d, equal] & \mathbb{L}_f \arrow[r] & Lf^*\mathbb{L}_X[1] \\ Li^*\mathbb{L}_{T'} \arrow[r] & \mathbb{L}_T \arrow[r] & \mathbb{L}_{T/T'} \arrow[r] & Li^*\mathbb{L}_{T'}[1] \end{tikzcd} \tag{187}$$

Hence we get a morphism $\varphi : Lf^*\mathbb{L}_X \to \mathbb{L}^{\geq -1}_{T/T'}$. Note that by Proposition 7.2, $\mathbb{L}^{\geq -1}_{T/T'} = \mathcal{J}[1]$, the morphism φ is an element in $\mathrm{Hom}_{\mathcal{D}(T)}(Lf^*\mathbb{L}_X, \mathcal{J}[1]) = \mathrm{Ext}^1_T(Lf^*\mathbb{L}_X, J)$. In fact φ is just our desired obstruction.

More explicitly, assume that X is a closed subscheme of some W smooth over $\mathrm{Spec}\, K$ and there exists morphism $g : T' \to W$ making the following diagram commutative

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, hook, "i"] & X \arrow[d, hook, "j"] \\ T' \arrow[r, "g"] & W \end{tikzcd} \tag{188}$$

Denote the ideal sheaf of j by $\mathcal{I}$, locally we have a commutative diagram

$$\begin{tikzcd} A/J & B/I \arrow[l] \\ A \arrow[u, two heads] & B \arrow[l] \arrow[u, two heads] \end{tikzcd} \tag{189}$$

Hence there is natural map $I/I^2 \to J/J^2 = J$. And $B \to A$ factors through B/I if and only if $I/I^2 \to J$ is a zero map. Globally we get $g^* : \mathcal{I}/\mathcal{I}^2\big|_T \to \mathcal{J}$. Suffices to show that g^* can be modified to zero map if and only if $\varphi = 0$. Note that there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2\big|_T \longrightarrow \Omega^1_W\big|_T \longrightarrow \Omega^1_X\big|_T \longrightarrow 0 \tag{190}$$

By Lemma 7.4, $\mathbb{L}^{\geq -1}_X = [\mathcal{I}/\mathcal{I}^2 \to \Omega^1_W\big|_X]$. And g^* gives a morphism in $\mathrm{Hom}_{\mathcal{K}(T)}(f^*\mathbb{L}^{\geq -1}_X, \mathcal{J}[1])$

$$\begin{tikzcd} {[\mathcal{I}/\mathcal{I}^2\big|_T} \arrow[r] \arrow[d, "g^*"] & {\Omega^1_W\big|_T]} \arrow[dl, dashed, "h"] \\ {[\mathcal{J}} \arrow[r] & {0]} \end{tikzcd} \tag{191}$$

Hence g^* can be modified to zero map if and only if $g^* = 0$ in $\mathrm{Hom}_{\mathcal{K}(T)}(f^*\mathbb{L}^{\geq -1}_X, \mathcal{J}[1])$. By Lemma 6.6, this is also equivalent to that $g^* = 0$ in $\mathrm{Hom}_{\mathcal{D}(T)}(f^*\mathbb{L}^{\geq -1}_X, \mathcal{J}[1])$.

Lemma 7.6. *In the setting above,* $\mathrm{Ext}^1_T(Lf^*\mathbb{L}^{\geq -1}_X, \mathcal{J}) = \mathrm{Hom}_{\mathcal{D}(T)}(f^*\mathbb{L}^{\geq -1}_X, \mathcal{J}[1])$.

Proof. Take flat resolution $\mathcal{F}^\bullet$ of $\mathcal{I}/\mathcal{I}^2$. As $\Omega^1_W\big|_X$ is locally free, we get a quasi-isomorphism with the first row a complex of flat $\mathcal{O}_X$-module sheaves

$$\begin{tikzcd} \mathcal{F}^{-1} \arrow[r] & \mathcal{F}^0 \arrow[r] \arrow[d, two heads] & \Omega^1_W\big|_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega^1_W\big|_X \arrow[r] & 0 \end{tikzcd} \tag{192}$$

Hence $Lf^*\mathbb{L}^{\geq -1}_X = [\cdots \to f^*\mathcal{F}^{-1} \to f^*\mathcal{F}^0 \to f^*\Omega^1_W\big|_X \to 0]$. Applying pull back f^* to chain map 192 , we get

$$\begin{tikzcd} f^*\mathcal{F}^{-1} \arrow[r] & f^*\mathcal{F}^0 \arrow[r] \arrow[d] & f^*\Omega^1_W\big|_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & f^*\mathcal{I}/\mathcal{I}^2 \arrow[r] & f^*\Omega^1_W\big|_X \arrow[r] & 0 \end{tikzcd} \tag{193}$$

As pull back is right exact, the cohomology group homomorphisms at degree -1 and 0 are still isomorphic. Hence $\tau^{\geq -1} Lf^*\mathbb{L}_X^{\geq -1} \to f^*\mathbb{L}_X^{\geq -1}$ is a quasi-isomorphism. Thus by Proposition 6.2,

$$\begin{aligned} \operatorname{Ext}^1_T(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}) &= \operatorname{Hom}_{\mathcal{D}(T)}(Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}[1]) \\ &= \operatorname{Hom}_{\mathcal{D}(T)}(\tau^{\geq -1}Lf^*\mathbb{L}_X^{\geq -1}, \mathcal{J}[1]) \\ &= \operatorname{Hom}_{\mathcal{D}(T)}(f^*\mathbb{L}_X^{\geq -1}, \mathcal{J}[1]) \end{aligned} \tag{194}$$

done! □

Remark 7.4. *However, we do not always have such a setting. Firstly, in general, for T not affine, there is not always a morphism $T \to W$. In addition, unlike that in the case when $X = \operatorname{Spec} A$ is affine, we can take $W = \operatorname{Spec} K[A]$, for general scheme X, we cannot find such a scheme W.*

To deal with general X, we need to consider it as a ringed space. Define $\widetilde{X}$ with same underlying topological space as X and structure sheaf $\pi^{-1}\mathcal{O}_{\operatorname{Spec} K}[\mathcal{O}_X]$ defined as sheafification of the following presheaf of sets

$$U \longmapsto (\pi^{-1}\mathcal{O}_{\operatorname{Spec} K}(U))[\mathcal{O}_X(U)] \tag{195}$$

7.3 Local to global spectral sequence

Let X be a scheme. Consider total hom functor on the category of $\mathcal{O}_X$-module sheaves, denoted by $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$. Then we get derived total sheaf hom $R\mathcal{H}om$. As $\Gamma(X, \cdot) \circ \mathcal{H}om = \operatorname{Hom}$, it is natural to question if $R\Gamma(X, \cdot) \circ R\mathcal{H}om = R\operatorname{Hom}$. In fact, this comes from the following lemma.

Lemma 7.7. *Let X be a scheme, $\mathcal{F}$ sheaf of flat module. Then for all injective $\mathcal{I} \in \operatorname{Mod}(\mathcal{O}_X)$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$ is still injective.*

Proof. Want to show for all injection $\mathcal{A} \hookrightarrow \mathcal{B}$, the induced map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}))$ is surjective. In fact, as sheaf hom and tensor product are adjoint pair, we get commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})) \arrow[d, equal] \\ \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{I}) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{I}) \end{tikzcd} \tag{196}$$

Since $\mathcal{F}$ is flat and $\mathcal{I}$ is injective, $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{F}$ is still injective so that the induced map is surjective. □

Recall the following theorem by Grothendieck in homological algebra.

Theorem 7.1 (***Grothendieck***)**.** *Let $\mathscr{A} \xrightarrow{G} \mathscr{B} \xrightarrow{F} \mathscr{C}$ be covariant additive functors, where $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ are abelian categories with enough injective objects. Assume that F is left exact* *and for I injective, GI is F-acyclic. Then for all object $A \in \mathscr{A}$, there is a first quadrant spectral sequence with*

$$E_2^{p,q} = (R^pF)(R^qG)A \underset{p}{\Rightarrow} R^{p+q}(FG)A \tag{197}$$

Here we take $\mathscr{A} = \mathscr{B} = \mathrm{Mod}(\mathcal{O}_X)$ and $\mathscr{C} = \mathbf{Ab}$ with $F = \Gamma(X, \cdot)$ and $G = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$, where F is a flat module sheaves. Then Lemma 7.7 tells us that conditions of Grothendieck Theorem hold. Hence we get $H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \underset{p}{\Rightarrow} \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. And by diagram chasing (cf. Weibel, exercise 5.1.3), we have an exact sequence of low degree terms

$$\begin{aligned} 0 &\to H^1(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \\ &\to H^2(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \end{aligned} \tag{198}$$

Take injective resolution $\mathcal{I}^\bullet$ of $\mathcal{G}$. Similar to idea of exact sequence of low degree items, for all $\mathcal{F}^\bullet \in \mathcal{C}^{\leq 0}(X)$ complex of flat module sheaves, consider the following commutative diagram with all rows and columns complexes

$$\begin{array}{ccccccc} & & \cdots & & \cdots & & \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-2}, \mathcal{I}^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-2}, \mathcal{I}^1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-2}, \mathcal{I}^2) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-1}, \mathcal{I}^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-1}, \mathcal{I}^1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{-1}, \mathcal{I}^2) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{0}, \mathcal{I}^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{0}, \mathcal{I}^1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^{0}, \mathcal{I}^2) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array} \tag{199}$$

we can also get an exact sequence of low degree items

$$\begin{aligned} 0 &\to H^1(X, \mathcal{E}xt^0_X(\mathcal{F}^\bullet, \mathcal{G})) \to \mathrm{Ext}^1_X(\mathcal{F}^\bullet, \mathcal{G}) \to H^0(X, \mathcal{E}xt^1_X(\mathcal{F}^\bullet, \mathcal{G})) \\ &\to H^2(X, \mathcal{E}xt^0_X(\mathcal{F}^\bullet, \mathcal{G})) \to \mathrm{Ext}^2_X(\mathcal{F}^\bullet, \mathcal{G}) \end{aligned} \tag{200}$$

Note that for all $\mathcal{A}^\bullet \in \mathcal{C}^{\leq 0}(X)$, we can take such an $\mathcal{F}^\bullet$ with quasi-isomorphism $\mathcal{F}^\bullet \to A^\bullet$, we can replace $\mathcal{F}^\bullet$ by $\mathcal{A}^\bullet$ in exact sequence 200. In particular, if we take $A^\bullet = Lf^*\mathbb{L}_X$, then we would get

$$\begin{aligned} 0 &\to H^1(X, \mathcal{E}xt^0_X(Lf^*\mathbb{L}_X, \mathcal{G})) \to \mathrm{Ext}^1_X(Lf^*\mathbb{L}_X, \mathcal{G}) \to H^0(X, \mathcal{E}xt^1_X(Lf^*\mathbb{L}_X, \mathcal{G})) \\ &\to H^2(X, \mathcal{E}xt^0_X(Lf^*\mathbb{L}_X, \mathcal{G})) \to \mathrm{Ext}^2_X(Lf^*\mathbb{L}_X, \mathcal{G}) \end{aligned} \tag{201}$$

Note that $\mathrm{Ext}^1_X(Lf^*\mathbb{L}_X, \mathcal{G})$ is just where the obstruction lies, the exact sequence tells us that intuitively obstruction consists of two parts. One part is obstruction of locally extension lying in $\ker(H^0(X, \mathcal{E}xt^1_X(Lf^*\mathbb{L}_X, \mathcal{G})) \to H^2(X, \mathcal{E}xt^0_X(Lf^*\mathbb{L}_X, \mathcal{G})))$. When local obstruction vanishes, local extension always exists. And the obstruction to gluing up such local extensions lies in $H^1(X, \mathcal{E}xt^0_X(Lf^*\mathbb{L}_X, \mathcal{G}))$.

7.4 Deformation of scheme

Given closed immersion $S \hookrightarrow S'$ with ideal sheaf $\mathcal{J}$ of square zero and X flat over S, we question if there exists X' flat over S' making the following commutes

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\pi"] & X' \arrow[d, dashed] \\ S \arrow[r, hook] & S' \end{tikzcd} \tag{202}$$

such that $X \cong X' \times_{S'} S$. In fact, we have that

- automorphisms are $\mathrm{Ext}^0_X(\mathbb{L}_{X/S}, \pi^*\mathcal{J})$
- extensions are pseudo torsor under $\mathrm{Ext}^1_X(\mathbb{L}_{X/S}, \pi^*\mathcal{J})$
- obstruction lies in $\mathrm{Ext}^2_X(\mathbb{L}_{X/S}, \pi^*\mathcal{J})$

In the following, we would explain these results in the ring case.

Definition 7.2. *Let $0 \to I \to A' \to A \to 0$ be an exact sequence of $\mathrm{Mod}_{A'}$. We say that it is a 1st order thickening if $I^2 = 0$.*

Now, given a diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, dashed] & B' \arrow[r, dashed] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u, dashed] & A \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{203}$$

with the last row a 1st order thickening, where N is a B-module and $I \to N$ is A-module homomorphism. Question if there exists B' fitting into the diagram and making the first row also a 1st order thickening.

Remark 7.5. *If we ask $A \to B$ to be flat, then by the local criterion of flatness Proposition 1.1, given a solution B' to this problem, the homomorphism $A' \to B'$ is flat if and only if $I \otimes_A B \cong N$. Hence the new problem contains the original deformation problem.*

Lemma 7.8. *Given a commutative diagram with each row a 1st order thickening*

$$\begin{tikzcd} & 0 \arrow[rr] & & N_2 \arrow[rr] & & B_2' \arrow[rr] & & B_2 \arrow[rr] & & 0 \\ 0 \arrow[rr] & & N_1 \arrow[rr] \arrow[ur] & & B_1' \arrow[rr] \arrow[ur, dashed] & & B_1 \arrow[rr] \arrow[ur] & & 0 & \\ & 0 \arrow[rr] & & I_2 \arrow[rr] \arrow[uu] & & A_2' \arrow[rr] \arrow[uu] & & A_2 \arrow[rr] \arrow[uu] & & 0 \\ 0 \arrow[rr] & & I_1 \arrow[rr] \arrow[uu] \arrow[ur] & & A_1' \arrow[rr] \arrow[uu] \arrow[ur] & & A_1 \arrow[rr] \arrow[uu] \arrow[ur] & & 0 & \end{tikzcd} \tag{204}$$

question if there exists $B_1' \to B_2'$ fitting into the diagram. Then we have that

- *extensions are pseudo torsor under* $\mathrm{Ext}^0_{B_1}(\mathbb{L}_{B_1/A_1}, N_2)$
- *obstruction lies in* $\mathrm{Ext}^1_{B_1}(\mathbb{L}_{B_1/A_1}, N_2)$

Proof. Given two extensions φ_1 and φ_2, $\varphi_1 - \varphi_2$ gives a map from B_1' to N_2. For all $b \in B_1$, take $b_1' \in B_1'$ mapping to b_1. Define $\delta : b_1 \mapsto \varphi_1 - \varphi_2(b_1')$. By commutativity, it is clear that δ is well defined derivation of B_1 to N_2 over A_1. Hence extensions are classified by $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$, by Lemma 6.7, this is just $\mathrm{Ext}^0_{B_1}(\mathbb{L}_{B_1/A_1}, N_2)$.

Take $P = A_1[B_1]$ and $P' = A_1'[B_1]$. Denote kernel of $P \twoheadrightarrow B_1$ by J. Then $\mathbb{L}^{\geq -1}_{B_1/A_1} = [J/J^2 \to \Omega_{P/A_1} \otimes_P B_1]$. For each $b_1 \in B_1$, we take a b_1' in B_1' mapping to b_1 and so we get a map $P' \to B_1'$. Similarly, we can also construct a map $P' \to B_2'$.

Denote kernel of $P' \twoheadrightarrow B_1$ by J'. Then we get homomorphisms $J' \to N_1$ and $J' \to N_2$. Want to show that B_i' is push out of N_i along $J' \to P'$. Assume that there is a commutative diagram

$$\begin{tikzcd} N_i \arrow[r, "f"] & C \\ J' \arrow[u] \arrow[r] & P' \arrow[u, "g"'] \end{tikzcd} \tag{205}$$

Then for all $b_i'' \in B_i'$, b_i'' is mapping to some b_i in B_i. Hence $b_i'' - b_i' \in N_i$. Define $B_i' \to C$ by sending b_i'' to $f(b_i'' - b_i') + g([b_i])$. It is easy to check that this map is well defined and is unique to make the following diagram commutes

$$\begin{tikzcd} & & C \\ N_i \arrow[r] \arrow[urr, bend left, "f"] & B_i' \arrow[ur, dashed, "\exists!"] & \\ J' \arrow[u] \arrow[r] & P' \arrow[u] \arrow[uur, bend right, "g"'] & \end{tikzcd} \tag{206}$$

so that B_i' is push out. By universal property of push out, the obstruction to this extension problem is equivalent to the obstruction to the commutativity of the following diagram

$$\begin{tikzcd} N_1 \arrow[r] & N_2 \arrow[r] & B_2' \\ J' \arrow[u] \arrow[rr] & & P' \arrow[u] \end{tikzcd} \tag{207}$$

Denote $\alpha : J' \to N_1$, $\beta : J' \to N_2$ and $c : N_1 \to N_2$. Now consider map $J/J^2 \to N_2$ sending $\overline{f} \to c \circ \alpha(f') - \beta(f')$, where f' is a lift of f in J'. For any two lifts f_1' and f_2', $f_1' - f_2'$ would be in $I_1[B_1]$. As N_2 is of square zero. The map is well defined. And the diagram 207 is commutative if and only if the map can be modified to a zero map.

Now modify each b_i' to b_i'' and the modified map sends $\overline{f}$ to $c \circ \alpha'(f') - \beta'(f')$. Write $f = \sum_k a_k'[b_k]$. Then the difference is $\sum_k a_k'(c(b_{k1}'' - b_{k1}') + (b_{k2}'' - b_{k1}'))$. Hence the modification is given by $J/J^2 \to \Omega_{P/A_1} \otimes_P B_1 \to N_2$ where the second map sends $[b_1] \otimes 1$ to $c(b_1'' - b_1') + b_2'' - b_2$. Conclude that the obstruction lies in $\mathrm{Ext}^1_{B_1}(\mathbb{L}_{B_1/A_1}, N_2)$. $\square$

Lemma 7.9. *Let $\mathscr{A}$ be an abelian category with enough injective objects. Assume $A \xrightarrow{f} B$ is a morphism in $\mathscr{A}$ with B projective. Denote $[0 \to A \xrightarrow{f} B \to 0]$ by E. Then for all $C \in \mathscr{A}$, $\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E, C[1]) \cong \mathrm{Hom}_{\mathscr{A}}(A, C)/\mathrm{Hom}_{\mathscr{A}}(B, C)$. In particular, $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N) = \mathrm{Hom}_B(J/J^2, N)/\mathrm{Hom}_B(\Omega^1_{P/A} \otimes_P B, N)$.*

Proof. Take injective resolution $I^\bullet$ of C, then $\mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E, C[1]) = \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(E, I^\bullet[1]) = \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(E, I^\bullet[1])$. For any chain map φ

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi_{-1}"'] & B \arrow[r] \arrow[d, "\varphi_0"] \arrow[dl, dashed, "\psi"'] & 0 \arrow[d] & \\ 0 \arrow[r] & I^0 \arrow[r, "g"] & I^1 \arrow[r, "h"] & I^2 \arrow[r] & \cdots \end{tikzcd} \tag{208}$$

as $h \circ \varphi_0 = 0$, φ_0 factors through $\ker h$. Since $I^0 \twoheadrightarrow \operatorname{im} g = \ker h$ and B is projective, there exists $\psi : B \to I^0$ such that $g \circ \psi = \varphi_0$. Now for all $a \in A$, $g(\varphi_{-1} - \psi \circ f(a)) = g \circ \varphi_{-1} - \varphi_0 \circ f(a) = 0$, hence $\varphi_{-1} - \psi \circ f$ factors through $\ker g = C$.

Define map s sending φ to $\overline{\varphi_{-1} - \psi \circ f}$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)/\mathrm{Hom}_{\mathscr{A}}(B, C)$. Clearly, s is independent to choice ψ and homotopy and hence well defined. Remains to show that s is bijective. For surjectivity, for any $\overline{\pi}$, take lift $\pi : A \to C$. Then obviously $\overline{\pi}$ is the image of the following chain map

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\pi"'] & B \arrow[r] \arrow[d, "0"] & 0 \arrow[d] & \\ 0 \arrow[r] & I^0 \arrow[r, "g"] & I^1 \arrow[r, "h"] & I^2 \arrow[r] & \cdots \end{tikzcd} \tag{209}$$

so that s is surjective.

For injectivity, assume that φ is mapping to $\overline{0}$. Then $\varphi_{-1} - \psi \circ f = \phi \circ f$ for some $\phi : B \to C$. Consider ϕ as a map from B to I^0, then $\varphi_{-1} = (\psi + \phi) \circ f$. Take chain homotopy $\psi + \phi$ so that φ is null-homotopic, done! ☐

Lemma 7.10. *Solutions to original problem 203 is pseudo torsor under* $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$.

Proof. Given two solutions B_1' and B_2', we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[r] & N \arrow[rr] & & B_2' \arrow[rr] & & B \arrow[rr] & & 0 \\ 0 \arrow[r] & N \arrow[ur, equal] \arrow[rr] & & B_1' \arrow[ur, dashed] \arrow[rr] & & B \arrow[ur, equal] \arrow[rr] & & 0 & \\ & 0 \arrow[r] & I \arrow[uu] \arrow[rr] & & A' \arrow[uu] \arrow[rr] & & A \arrow[uu] \arrow[rr] & & 0 \\ 0 \arrow[r] & I \arrow[uu] \arrow[ur, equal] \arrow[rr] & & A' \arrow[uu] \arrow[ur, equal] \arrow[rr] & & A \arrow[uu] \arrow[ur, equal] \arrow[rr] & & 0 & \end{tikzcd} \tag{210}$$

Hence by Lemma 7.8, the obstruction to existence of $B_1' \to B_2'$ fitting into the diagram lies in $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$. Thus $B_1' \cong B_2'$ if and only if the obstruction vanishes.

On the other hand, given $\delta \in \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$. Note that by Lemma 7.9, we can lift δ to some $\varphi \in \mathrm{Hom}_B(J/J^2, N)$. Take $P = A'[B]$, given a commutative diagram

$$\begin{tikzcd} o \arrow[r] & J \arrow[r] \arrow[d, "\theta"] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & B_1' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{211}$$

$\theta + \varphi$ would give a pushout B_2' as following

$$\begin{tikzcd} J \arrow[r] \arrow[d, "\theta+\varphi"] & P \arrow[d] \\ N \arrow[r] & B_2' \end{tikzcd} \tag{212}$$

Explicitly, $B_2' = \{(n,p)\}/(\theta + \varphi(j) - j)$ and the multiplication is defined by

$$(n_1, p_1) \cdot (n_2, p_2) = (n_1 p_1 + n_2 p_2, p_1 p_2) \tag{213}$$

where np is well defined since N is a B-module. Conclude that solutions are classified by $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$. $\square$

For trivial case

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, dashed] & B' \arrow[r, dashed] & B \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & A \arrow[r, "\mathrm{id}_A"] \arrow[u, dashed] & A \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{214}$$

solutions are just $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$. In particular, $0 \in \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$ corresponds to $N \oplus B$ with ring structure similar to B_2' above. Consider

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A' \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & A' \arrow[r, "\mathrm{id}_{A'}"] \arrow[u, equal] & A' \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{215}$$

corresponding to some $\xi \in \mathrm{Ext}_A^{\mathbb{L}_{A/A'}, I}$. Taking push out of N along $I \to A'$ and pull back of A along $B' \to$, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \arrow[r, squiggly] & \xi \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, I) \arrow[d, mapsto] \\ 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & A'' \arrow[r] \arrow[d, equal, "?"] & A \arrow[r] \arrow[d, equal] & 0 \arrow[r, squiggly] & \xi'' \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N) \arrow[d, equal, "?"] \\ 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & B'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r, squiggly] & \xi''' \in \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N) \\ 0 \arrow[r] & N \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \arrow[r, squiggly] & \xi' \in \mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N) \arrow[u, mapsto] \end{tikzcd} \tag{216}$$

Hence by 5 Lemma and universal properties of push out and pull back, B' is a solution to original problem if and only if $A'' = B''$ if and only if $\xi'' = \xi'''$. If denote $\mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N) \to \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N)$ by ρ, then

- solutions are $\rho^{-1}(\xi'')$ a pseudo torsor under $\ker \rho$.
- obstruction class is $[\xi''] \in \operatorname{coker} \rho$.

For commutative diagram

$$\begin{tikzcd} B & A \arrow[l] \\ & A' \arrow[u] \arrow[ul] \end{tikzcd} \tag{217}$$

by Proposition 7.1, there is an exact triangle

$$\mathbb{L}_{A/A'} \otimes_A B \longrightarrow \mathbb{L}_{B/A'} \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{A/A'}[1] \otimes_A B \tag{218}$$

Applying $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}_B)}(\cdot, N)$, by Lemma 6.3, there is a long exact sequence

$$\cdots \to \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N) \to \mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N) \xrightarrow{\rho} \mathrm{Ext}^1_B(\mathbb{L}_{A/A'} \otimes_A B, N) \to \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, N) \tag{219}$$

Note that $\mathrm{Ext}^1_B(\mathbb{L}_{A/A'} \otimes_A B, N) = \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}, N)$, we can consider the image of ξ'' in $\mathrm{Ext}^2_B(\mathbb{L}_{B/A}, N)$, denoted ζ. As the sequence is exact, there exists some ξ' mapping to ξ'' if and only if $\zeta = 0$. Hence the obstruction class is just ζ. In addition, $\ker \rho = \mathrm{im}(\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N) \to \mathrm{Ext}^1_B(\mathbb{L}_{B/A'}, N))$ so that extensions are classified by $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$.

7.5 Deformation of sheaf

Let $X \hookrightarrow X'$ with ideal sheaf $\mathcal{I}$ be a 1st order thickening. Given $\mathcal{F}, \mathcal{G}$ sheaves of $\mathcal{O}_X$-modules on X and map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \overset{c}{\mathcal{G}}$, we question if there exists $\mathcal{O}_{X'}$-module sheaf $\mathcal{F}'$ fitting into

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow 0 \tag{220}$$

such that $\mathcal{F}\big|_X = \mathcal{F}$ and induced map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G}$ is the given map c. In fact, we have that

- automorphisms are $\mathrm{Ext}^0_X(\mathcal{F}, \mathcal{G})$.
- extensions are pseudo torsor under $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$.
- obstruction lies in $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{G})$.

7.6 Construction of cotangent complex

For ring homomorphism $A \to B$, consider simplicial resolution

$$\cdots \longrightarrow A^3[B] \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} A^2[B] \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} A[B] \xrightarrow{\varepsilon} B \longrightarrow 0 \tag{221}$$

Those parallel arrows are given by different way of adding or removing brakets. For example, the two maps from $A^2[A]$ to $A[B]$ is respectively given by $[[b_1] + [b_2]] \mapsto [b_1] + [b_2]$ and $[[b_1] + [b_2]] \mapsto [b_1 + b_2]$. Apply $\Omega_{\cdot/A} \otimes_{\cdot} B$, we get that

$$\cdots \longrightarrow \Omega_{A^3[B]/A} \otimes_{A^3[B]} B \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \Omega_{A^2[B]/A} \otimes_{A^2[B]} B \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \Omega_{A[B]/A} \otimes_{A[B]} B \xrightarrow{\varepsilon} \Omega_{B/A}$$

Taking alternating sums of such maps, we get that

$$\cdots \longrightarrow \Omega_{A^3[B]/A} \otimes_{A^3[B]} B \xrightarrow{\partial_0 - \partial_1 + \partial_2} \Omega_{A^2[B]/A} \otimes_{A^2[B]} B \xrightarrow{\partial_0 - \partial_1} \Omega_{A[B]/A} \otimes_{A[B]} B \xrightarrow{\partial} \Omega_{B/A} \tag{222}$$

And $[\cdots \to \Omega_{A^3[B]/A} \otimes_{A^3[B]} B \xrightarrow{\partial_0 - \partial_1 + \partial_2} \Omega_{A^2[B]/A} \otimes_{A^2[B]} B \xrightarrow{\partial_0 - \partial_1} \Omega_{A[B]/A} \otimes_{A[B]} B]$ is our desired cotangent complex.