一些常见结论

Contents

1 Abstract Algebra 抽象代数

Proposition 1.1. *Let S be a monoid. If an element $a \in S$ satisfies that there exists $b, c \in S$ s.t. $ba = 1, ac = 1$, then $b = c$.*

Proposition 1.2. *Let R be an integral domain. Then $R = \cap_\alpha R_{\mathfrak{m}_\alpha}$, where $\mathfrak{m}_\alpha$ varies over all the maximal ideals of R.*

Theorem 1.1 (***Schur' Theorem***)**.** *Let G be a group. If $Z(G)$ is of finite index in G, then $G' = [G, G]$ is finite.*

Proposition 1.3. *G is a infinite group. If every nontrivial proper subgroup of G is of finite index, then G is cyclic.*

Proposition 1.4. *G is a finite group, $H \leq G$ subgroup. Show there is a complete set of left cosets being a complete set of right cosets at the same time.*

Proposition 1.5. *All groups of order pq are either isomorphic to $\mathbb{Z}_{pq}$ (abelian case) or $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ (nonabelian case).*

Proposition 1.6. *G is a p-group. Then all maximal subgroups of G is normal and of index p.*

Proposition 1.7. *G is a finite group. Assume that $v_p(G) = n$. Then the number of subgroups of order p^i, n_1, satisfies that $n_1 \equiv 1 \pmod p$, for $\forall 1 \leq i \leq n$.*

Proposition 1.8. *G is a nonabelian simple group of order 60. Then $G \cong A_5$.*

Proposition 1.9. *G is a finite group, $N \trianglerighteq G$ a normal group, $p \mid \sharp(N)$. Assume that P is a Sylow p-group of N. Then $G = N_G(P)N$.*

2 Topology 拓扑

Proposition 2.1. *Let X be a topological space., $S \subseteq X$ subset. Then S is closed in X if and only if X can be covered by open subsets U_α s.t. $S \cap U_\alpha$ is closed in U_α.*

Proposition 2.2. *Let X be a topological space. Then every maximal irreducible subspace of X is closed.*

Remark 2.1. *For convenience, we call maximal irreducible subspace irreducible component.*

Proposition 2.3. *Let X be a noetherian topological space. Then X has only finitely many irreducible components.*

2.1 Dimension theory

Proposition 2.4. *Let X be a irreducible topological space. Then for all nonempty open subset $U \subseteq X$, U is dense.*

3 Commutative Algebra 交换代数

In this section, all rings are commutative.

Proposition 3.1. *Let R be a ring. The nilradical $\mathfrak{N}$ is the intersection of all prime ideals of R.*

Proposition 3.2. *Let R be a ring. Define the Jacobson radical $\mathfrak{R}$ is the intersection of all maximal ideals of R. We have $x \in \mathfrak{R} \iff 1 - xy$ is a unit in R for all $y \in R$.*

Proposition 3.3. *Let R be a ring, $I, J \subseteq R$ ideals. If $\sqrt{I}$ and $\sqrt{J}$ are coprime, then I and J are coprime.*

Proposition 3.4. *Let R be a ring. Then every idempotent of $R/\mathfrak{N}$ lifts to some idempotent of R. In fact, the lifting is unique.*

Proposition 3.5. *Let R be a noetherian ring. Then submodule of finitely generated R-module is still finitely generated.*

Proposition 3.6. *Let R be an artinian integral domain. Then R is a field.*

Proposition 3.7. *Let R be a local ring, $\mathfrak{m} \subseteq R$ maximal ideal. If $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$, then $\mathfrak{m}$ is principal.*

Proposition 3.8. *Let R be UFD. Then R is integrally closed.*

Proposition 3.9. *R is a ring. Then $R = \cap_{\mathfrak{m} \in Max(R)} R_{\mathfrak{m}}$.*

Proposition 3.10. *Let $B \in Alg_A$ with ring homomorphism $f : A \longrightarrow B$. For $\mathfrak{p} \in \operatorname{Spec} A$, if $\mathfrak{p}^{ec} = \mathfrak{p}$, then $\mathfrak{p} \in im(f^*)$.*

Proposition 3.11. *Let A be a graded ring and M, N, L graded A-modules. Then $Hom(M \otimes_A N, L) \cong Hom(M, Hom^*(N, L))$.*

Proposition 3.12. *Let $\varphi : A \longrightarrow B$ be a ring homomorphism with A, B noetherian. Then for each minimal prime ideal $\mathfrak{p}$ containing $\ker(\varphi)$, we can find $\mathfrak{P} \in \operatorname{Spec} B$ lying over $\mathfrak{p}$.*

4 Linear Algebra 线性代数

Proposition 4.1. *If U, V are subspaces of dimension r,s of a vector space of W of dimension n, then $U \cap V$ is a subspace of dimension more than r + s - n.*

5 Representation Theory 表示论

Theorem 5.1 (***Bornside $p^a q^b$ Theorem***)**.** *G is a finite group of order $p^a q^b$, where p, q are prime numbers. Then G is solvable group.*

Theorem 5.2 (***Clifford Theorem***)**.** *G is a finite group, $N \trianglerighteq G$ a normal subgroup. Assume that $\chi \in Irr(G)$ and $\varphi \in Irr(N)$ satisfy that φ is a direct summand of $Res_N^G(\chi)$. Then $Res_N^G(\chi) = e \sum_{i=1}^n \varphi^{g_i}$, where $e \in \mathbb{N}$, φ^{g_i} is given by $x \longmapsto \varphi(g_i x g_i^{-1})$ and $g_1, \cdots, g_n$ is a transversal of N in G.*

6 Algebraic Geometry 代数几何

6.1 Classic examples

Proposition 6.1. *Let $\mathcal{L}$ be an invertible sheaf on projective line $\mathbb{P}^1_k$. Then $\mathcal{L}$ must be isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(n)$ for some n.*

6.2 Properties of morphisms

Proposition 6.2. *Let X be a reduced scheme and Y be a separated scheme. Assume f and g are two morphisms from X to Y, agreeing on an open dense subset of X. Then $f = g$.*

Remark 6.1. *With this proposition, the equivalence classes of rational classes from reduced scheme to separated scheme are well behaved. This is why we always assume schemes are reduced and separated when we consider birational maps.*

6.3 Dimension theory

Proposition 6.3. *Let X be an irreducible affine scheme. Then for any nonempty open subset $U \subseteq X$, we have that $\dim U = \dim X$.*

Remark 6.2. *This is not true for general topological space.*

Proposition 6.4. *Let X be a noetherian topological space, $Y \subseteq X$ constructible set. Then $\dim Y = \dim \overline{Y}$.*

6.4 Universal injectivity

Proposition 6.5. *Let $f : \operatorname{Spec} L \to \operatorname{Spec} K$ be a morphism of schemes. Then f is universally injective if and only if field extension L/K is purely inseparable.*

Proposition 6.6. *Let $f : X \to Y$ be a morphism of schemes. Then f is universally injective if and only if $\Delta_{X/Y} : X \to X \times_Y X$ is surjective.*

6.5 Exceptional locus

Proposition 6.7. *Let $f : X \to Y$ be a birational morphism between reduced and separated schemes, $g : Y \dashrightarrow X$ the birational inverse of f. Then $\{x \in X \big| g \text{ is not defined at } f(x)\}$ and $X \setminus U_X$ are the same set, denoted by $\operatorname{Ex}(f)$, where U_X denotes the maximal open dense subset of X such that $f\big|_{U_X}$ is an isomorphism.*

Proposition 6.8. *Let $f : X \dashrightarrow Y$ be a birational map between reduced, noetherian and separated schemes, $g : Y \dashrightarrow X$ the birational inverse of f. Then for any prime divisor D on Y not contained in $\operatorname{Sing} Y$, the generic point η of D is contained in U_Y.*

7 Homological Algebra

Proposition 7.1. *Let $\mathscr{A}$ be an abelian category. Assume $I^\bullet$ be an acyclic bounded-below complex of injective objects. Then $I^\bullet$ is contractible i.e. $\mathrm{id}_{I^\bullet}$ is null-homotopic. Dually, acyclic bounded-above complex of projective objects is contractible.*

8 Proofs

8.1 Abstract Algebra 抽象代数

Proposition 1.1. $b = b1 = bac = 1c = c$ $\square$

Proposition 1.2. If there exists $z \notin R$, while $z \in \cap_\alpha R_{\mathfrak{m}_\alpha}$, consider the ideal $I = \{x \in R \big| xz \in R\}$. Since $1 \notin I$, we have I is a proper ideal of R, thus there is a maximal ideal $\mathfrak{m}$ s.t. $I \subset \mathfrak{m}$. However, it is obvious that $z \notin R_\mathfrak{m}$, contradiction! $\square$

Proposition 1.1. See in the website, Schur's Theorem on commutator subgroup. $\square$

Proposition 1.3. Take normal cyclic subgroup H s.t. the index of H is minimal. Consider the centralizer C of H, and use Theorem 1.1 to show $C = H$. Finally, suppose there is an element $g \notin H$, try to make contradiction. $\square$

Proposition 1.4. Consider the double coset for H. $\square$

Proposition 1.5. Abelian case by Fundamental Theorem of Finitely Generated Abelian Groups and nonabelian case by Sylow Theorem. $\square$

Proposition 1.6. By induction. $\square$

Proposition 1.7. It suffices to prove the case for p-group. G is a p-group of order p^n, denote the number of maximal subgroups of G by m_G. By induction, we assume that $m_G \equiv 1 \pmod p$ for $\forall 1 \leq n \leq k$.

Now consider the $k+1$ case. If $m_G = 1$, then done. Otherwise, assume P, P' are two different maximal subgroups of G. Note that by Proposition 1.6, we have $P \trianglelefteq G$, $P' \trianglelefteq G$ and $PP' = G$. In addition, $PP'/P \cong P'/P \cap P'$. Thus $\sharp(P \cap P') = p^{k-1}$. For each maximal subgroup H of P, P' such that $P' \cap P = H$ one-to-one corresponds to a subgroup of G/H of order p except P/H. The correspondence gives a bijection. Thus $m_G \equiv 1 + \sum_{\text{H maximal subgroup of P}} (m_{G/H} - 1) \equiv 1 \pmod p$. $\square$

Proposition 1.8. Obviously, G has no proper subgroup of index less than 5. Similarly, by Sylow's Theorem, the number of Sylow 3-groups and Sylow 5-groups are 10 and 6 respectively. And the number of Sylow 2-groups is 5 or 15. It is obvious to show $G \cong A_5$ in the first case. For the other case, there must exists two different Sylow 2-groups P and Q such that $P \cap Q \neq \{1\}$. Assume that $\langle a \rangle = P \cap Q$, then $o(a) = 2$. Consider the order of $C_G(a)$. $\square$

Proposition 1.9. For any $g \in G$, $g^{-1}Pg \subseteq g^{-1}Ng = N$. Thus $g^{-1}Pg$ is still some Sylow p-group of N. By Sylow's Theorem, there exists $n \in N$ such that $g^{-1}Pg = n^{-1}Pn$. Then $g = (gn^{-1})n$, where $gn^{-1} \in N_G(P)$. □

8.2 Topology 拓扑

Proposition 2.1. "⇒": obviously. "⇐": consider $U_\alpha \setminus S$ □

Proposition 2.2. Suppose V is a maximal irreducible subspace of X. Consider the closure of V. □

Proposition 2.3. Assume there is infinitely many irreducible components. Consider the descending chain of closed subsets

$$V_1 \supseteq V_1 \cap V_2 \supseteq \cdots \supseteq V_1 \cap \cdots \cap V_n \supseteq \cdots \tag{1}$$

where V_i are distinct irreducible components. □

8.2.1 dimension theory

Proposition 2.4. Suppose that U is not dense, then there exist $x \in X$ and open neighbourhood $W \ni x$ such that $W \cap U = \varnothing$, contradiciting to irreducibility. Hence U is dense in X. □

8.3 Commutative Algebra 交换代数

Proposition 3.1. See in the Atiyah. □

Proposition 3.2. See in the Atiyah. □

Proposition 3.3. $r(I \cup J) = r(r(I) \cup r(J)) = r(1) = (1)$ □

Proposition 3.4. Assume $\overline{x}$ is the idempotent in $A/\mathfrak{N}$, we have $x^2 - x$ is nilpotent. Thus $\exists n \in \mathbb{N}_+$ s.t. $x^n(1-x)^n = 0$. As (x) and (1 - x) are coprime, we have r(x) and r(1 - x) are coprime. Note that $r(x^n) = r(x)$ and $r((1-x)^n) = r(1-x)$, by Proposition 3.3, get (x^n) and $((1-x)^n)$ are coprime. By Chinese Remainder Theorem, we have

$$A = A/(0) = A/(x^n(1-x)^n) \cong A/(x^n) \times A/((1-x)^n) \tag{2}$$

Take e be the preimage of $(0,1) \in A/(x^n) \times A/((1-x)^n)$, e is the lifting of x. □

Proposition 3.5. Only need to prove for A^n-module. Consider the exact sequence

$$0 \longrightarrow A^{n-1} \longrightarrow A^n \longrightarrow A \longrightarrow 0 \tag{3}$$

from A^n-module M, we can get $N \subseteq A^{n-1}$ module and $I \subseteq A$ ideal. And from N and I, we can generate M. □

Proposition 3.6. For any $x \neq 0 \in R$, consider the descending chain of ideals

$$(x) \supseteq (x^2) \supseteq \cdots \supseteq (x^n) \supseteq \cdots \tag{4}$$

By Artinian property, we can get that x is invertible. $\square$

Proposition 3.7. Assume that $\mathfrak{m}/\mathfrak{m}^2 = R/\mathfrak{m}\overline{x}$. Consider the correspondence between ideals of $R/\mathfrak{m}^2$ and ideals of R containing $\mathfrak{m}^2$. Get $\mathfrak{m} = (x) + \mathfrak{m}^2$. By Nakayama's Lemma, we have $\mathfrak{m} = (x)$. $\square$

Proposition 3.8. For any $\frac{a}{b} \in \mathrm{Frac}(R) \setminus R$ integral over R, assume that $\gcd(a, b) = 1$. There exists a monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ where $a_i \in R$ s.t. $(\frac{a}{b})^n + a_1(\frac{a}{b})^{n-1} + \cdots + a_n = 0$. Thus $a^n = -b(a_1 a^n + \cdots + a_n b^{n-1})$, get $b \mid a^n$ which contradicts to our assumption, $\gcd(a, b) = 1$. $\square$

Proposition 3.9. Consider natural map $f : R \longrightarrow \cap_{\mathfrak{m} \in Max(R)} R_{\mathfrak{m}}$. Want to show that f is surjective. It suffices to prove that $f_{\mathfrak{m}} : R_{\mathfrak{m}} \longrightarrow (\cap_{\mathfrak{m} \in Max R} R_{\mathfrak{m}})_{\mathfrak{m}}$ is surjective for each maximal ideal $\mathfrak{m}$. $\square$

Proposition 3.10. Consider $f : A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$. Then the maximal ideal of $B_{\mathfrak{p}}$ lies over $\mathfrak{p}$. $\square$

Proposition 3.11. By definition. $\square$

Proposition 3.12. For any minimal prime ideal $\mathfrak{p}$ of A containing kernel, consider set $V = V(\varphi(\mathfrak{p}))$ in $\operatorname{Spec} B$. As A, B are both noetherian, V is finite. Suppose that preimages of all $\mathfrak{P} \in V(\varphi(\mathfrak{p}))$ are not $\mathfrak{p}$, then each $\mathfrak{P}_i$ gives an $a_i \notin \mathfrak{p}$. Set $a = \prod_i a_i$, then $a \notin \mathfrak{p}$ and $\varphi(a) \in \cap_i \mathfrak{P}_i$ so that $\varphi(a) \in \sqrt{\varphi(\mathfrak{p})}$. Note that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$, we get $a \in \mathfrak{p}$, contradiciton. $\square$

8.4 Linear Algebra 线性代数

Proposition 4.1. Consider the maximal linearly independent system $\square$

8.5 Representation Theory 表示论

Theorem 5.1. See in the note. $\square$

Theorem 5.2. Note that $\chi^{g_i} = \chi$, we have

$$\begin{aligned} \langle Res_N^G(\chi), \varphi^{g_i} \rangle &= \langle Res_N^G(\chi^{g_i}), \varphi^{g_i} \rangle \\ &= \langle Res_N^G(\chi)^{g_i}, \varphi^{g_i} \rangle \\ &= \langle Res_N^G(\chi), \varphi \rangle \\ &= e \end{aligned}$$

For the proof of showing φ^{g_i} are all the direct summand of $Res_N^G(\chi)$, see in the book Representation Theory of Finite Group Extensions. $\square$

8.6 Algebraic Geometry 代数几何

8.6.1 Classic examples

Proposition 6.1. See in the website, invertible sheaves of the projective line. $\square$

8.6.2 Separatedness

Proposition 6.2. Since the problem is locally, we may assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd} \tag{5}$$

By universal property of fibered product, we get a lifting $(f, g) : X \to Y \times Y$. Note that the set $\{x \in X \big| f(x) = g(x)\}$ is the preimage of Δ_Y under (f, g), which is a closed subset of X since Y is separated. As f and g agree on an open dense subset U of X, we get $U \subset \{x \in X \big| f(x) = g(x)\}$ and hence $\{x \in X \big| f(x) = g(x)\} = X$.

Now we have two ring homomorphisms $f^\sharp : B \to A$ and $g^\sharp : B \to A$ satisfying that for all $\mathfrak{p} \in \operatorname{Spec} A$, $f(\mathfrak{p}) = g(\mathfrak{p})$. Remains to show $f^\sharp = g^\sharp$. As X is reduced, it suffices to show that for all $b \in B$, $V(f^\sharp(b) - g^\sharp(b)) = X$. Since U is dense in X, it suffices to show that $U \subseteq V(f^\sharp(b) - g^\sharp(b))$.

In fact, since for all $\mathfrak{p} \in U$, we can take an affine open neighbourhood $D(s) \subseteq U$. Then $B \xrightarrow{f^\sharp} A \to A_s$ and $B \xrightarrow{g^\sharp} A_s$ are same. Hence we get $\frac{f^\sharp(b)}{1} = \frac{g^\sharp(b)}{1}$ in A_s and $(f^\sharp(b) - g^\sharp(b))s^n = 0$ for some n. As $s \notin \mathfrak{p}$, $f^\sharp(b) - g^\sharp(b) \in \mathfrak{p}$ and we are done! $\square$

8.6.3 Dimension Theory

Proposition 6.3. See in the website, dimension of irreducible affine variety is same as any open subset. $\square$

Proposition 6.4. Since constructible can be written as finite union of locally closed subsets, assume that $Y = \cap_{i=1}^n U_i \cap V_i$. In particular, by taking irreducible components of each V_i, we may assume that V_i are irreducible. Then $\dim \overline{Y} \geq \dim Y \geq \max_i\{\dim(U_i \cap V_i)\}$. Note that $\overline{Y} = \cup_{i=1}^n \overline{U_i \cap V_i}$ and by Proposition 6.3, we immediately get

$$\dim \overline{Y} = \max_i\{\dim(\overline{U_i \cap V_i})\} = \max_i\{\dim V_i\} = \max_i\{\dim(U_i \cap V_i)\} \tag{6}$$

Thus $\dim Y = \dim \overline{Y}$. $\square$

8.6.4 Universal injectivity

Proposition 6.5. I don't know $\square$

Proposition 6.6. As composition of projection map and $\Delta_{X/Y}$ is identity map of X, if f is universally injective, then projection map is injective and hence $\Delta_{X/Y}$ is surjective.

Conversely, for any morphism $Z \to Y$, it suffices to show that $\Delta_{X\times_Y Z/Z}$ is surjective. Note that there is a fibered diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\Delta_{X\times_Y Z/Z}"] \arrow[d] & X \times_Y Z \times_Z X \times_Y Z \cong X \times_Y X \times_Y Z \arrow[d] \\ X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X \end{tikzcd} \tag{7}$$

Then as base change preserves surjectivity, we get $\Delta_{X\times_Y Z/Z}$ is surjective. $\square$

8.6.5 Exceptional locus

Proposition 6.7. Obviously, $\{x \in X \big| g \text{ is not defined at } f(x)\} \subseteq X \setminus U_X$. We only need to show the other containment. Assume g is defined at $f(x)$. Then by definition of rational map, there are open neighbourhoods $U \ni x$ and $V \ni x$ such that $f(U) \subseteq V$ and g is defined on V.

As U_X is open dense subset of X, $U_X \cap U$ is nonempty and dense in U. Hence $g \circ f\big|_U$ and $U \hookrightarrow X$ agree on $U_X \cap U$. By Proposition 6.2, $g \circ f\big|_U$ is just the open immersion.

Hence we can replace V by $V \cap g^{-1}(U)$, which is still an open neighbourhood of $f(x)$. Applying same argument for $f \circ g\big|_V$, we conclude that $U \subseteq U_X$ and $V \subseteq U_Y$, done! $\square$

Proposition 6.8. As D is not contained in $\operatorname{Sing} Y$, Y is regular at η so that $\mathcal{O}_{Y,\eta}$ is a DVR. Assume Z is an irreducible component of Y such that $D \subseteq Z$ and ξ is the generic point of Z, then $\mathcal{O}_{Y,\xi} = \operatorname{Frac}(\mathcal{O}_{Y,\eta})$.

Since U_Y is dense in Y, $\xi \in U_Y$ so that g induces a map $\operatorname{Spec} \mathcal{O}_{Y,\xi} \to U_X$. And we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{Y,\xi} \arrow[r] \arrow[d] & U_X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}_{Y,\eta} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{8}$$

Applying valuative criterion of properness, there exists unique lifting $\operatorname{Spec} \mathcal{O}_{Y,\eta} \to X$. Hence there exists $\delta \in U_X$ such that $f(\delta) = \eta$. By definition, we know $\eta \notin \operatorname{Ex}(g)$. $\square$

8.7 Homological Algebra 同调代数

Proposition 7.1. Here we only prove for the injective version. As $I^\bullet$ is acyclic, denote $\ker \partial_n = Z_n$. Then there are short exact sequences

$$0 \longrightarrow Z_n \longrightarrow I^n \longrightarrow Z_{n+1} \longrightarrow 0 \tag{9}$$

Since $I^\bullet$ is bounded-below, we may assume for all negative i, $I^i = 0$. Hence $Z_2 = I^1$. Consider the short exact sequence

$$0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow Z_3 \longrightarrow 0 \tag{10}$$

As I^1 is injective, the exact sequence splits so that Z_3 is a summand of I^2 and hence injective. By induction, we get that all those short exact sequences split. Now we have projections

$p_n : I^n \to Z_n$ and inclusions $s_{n+1} : Z_{n+1} \to I^n$. Consider $D_n := s_n \circ p_n : I^n \to I^{n-1}$. In fact we have that

$$\partial \circ D_n + D_{n+1} \circ \partial = p_n + s_{n+1} = \mathrm{id}_{I^n} \tag{11}$$

Hence $I^\bullet$ is contractible. $\square$