

Remark

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1 Rising Sea

Remark about 7.4.8. For the equivalence, consider the following commutative diagram

$$\begin{array}{ccccc}
 A[x_1, x_2, \dots, x_n] & \xrightarrow{\alpha} & A/\mathfrak{p}[x_1, x_2, \dots, x_n] & \xrightarrow{\beta} & k(\mathfrak{p})[x_1, x_2, \dots, x_n] \\
 \uparrow & & \uparrow & & \uparrow \\
 A & \longrightarrow & A/\mathfrak{p} & \longrightarrow & k(\mathfrak{p})
 \end{array}$$

Note that α is a quation map by $\mathfrak{p}[x_1, x_2, \dots, x_n]$ and β is a localization at $A/\mathfrak{p} \setminus \{0\}$. Thus the equivalence is clear.

Remark about 11.3.C(a). For affine case, we can also apply Krull's Principal Ideal Theorem. But there we get $\dim(k[x_{0/i}, \dots, x_{n/i}]/(\mathfrak{p}, f)) \geq \dim(k[x_{0/i}, \dots, x_{n/i}]/\mathfrak{p}) - 1 \geq 0$.

When the equality fails, we can find some prime ideal containing f so that H meets X . However, when the equality holds, it is possible that $(\mathfrak{p}, f) = (1)$ inducing $H \cap X = \emptyset$.

Remark about 13.3.L. We say two closed subschemes transversally intersect if generators of their ideal sheaf form a regular sequence in the stalk. Hence two closed subschemes intersect at point p with multiplicity at least 2 if and only if generators of their ideal sheaf do not form a regular sequence in the stalk at p .

For the case in 13.3.L, assume that line L is cut out by $n - 1$ linear equations which are linearly independent, denoted by f_1, \dots, f_{n-1} . For convenience, we may assume that $p = [1, a_1, \dots, a_n] \in D_+(x_0)$ and $f, f_i \in k[x_{1/0}, \dots, x_{n/0}]$. Then $\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1}$ automatically form a regular sequence in $\mathcal{O}_{\mathbb{P}_k^n, p}$. As X meets L at p with multiplicity at least 2, we get $\frac{f}{1}$ is a zero divisor in $\mathcal{O}_{\mathbb{P}_k^n, p}/(\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1})$.

Note that p is a regular point, we have $\mathcal{O}_{\mathbb{P}_k^n, p}$ is a regular local ring and so is quotient one. Since regular local ring is an integral domain, we get $\frac{f}{1} \in (\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1})$. Assume that $\frac{f}{1} = \frac{\sum_i g_i f_i}{s}$ so that $sf = \sum_i g_i f_i$. Multiplying the tangent space $T_p X$ by $s(p) \neq 0$, the evaluation of s at p , we have that

$$\begin{aligned}
 \sum_j s(p) \frac{\partial f}{\partial x_j}(p)(x_j - a_j) &= \sum_j (s(p) \frac{\partial f}{\partial x_j}(p) - f(p) \frac{\partial s}{\partial x_j}(p))(x_j - a_j) \\
 &= \sum_j \frac{\partial s f}{\partial x_j}(p)(x_j - a_j) \\
 &= \sum_j \frac{\partial \sum_i g_i f_i}{\partial x_j}(p)(x_j - a_j) \\
 &= \sum_i \sum_j g_i(p) \frac{\partial f_i}{\partial x_j}(p)(x_j - a_j)
 \end{aligned} \tag{1}$$

since all f_i are linear equations, we immediately get $\sum_j \frac{\partial f_i}{\partial x_j}(p)x_j = f_i$ and $\sum_j \frac{\partial f_i}{\partial x_j}(p)a_j = f_i(p) = 0$. Thus $s(p)T_p X = \sum_i g_i(p)f_i \in (f_1, \dots, f_{n-1})$ i.e. line L is contained in tangent space $T_p X$.

Remark about 13.6.D. Since $\pi : X \rightarrow Y$ and $\rho : Y \rightarrow Z$ are smooth of relative dimensions m and n respectively, by definition, there are affine open covering pairs $(\{V_i\}, \{U_i\})$ and

$(\{\widetilde{U}_j\}, \{\widetilde{W}_j\})$ such that for all i, j , there are commutative diagrams

$$\begin{array}{ccc} V_i = D(F_i) & \hookrightarrow & \text{Spec } B_i[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r) \\ \pi|_{V_i} \downarrow & \swarrow & \\ U_i = \text{Spec } B_i & & \end{array} \quad (2)$$

and

$$\begin{array}{ccc} \widetilde{U}_j = D(G_j) & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}]/(g_1, \dots, g_t) \\ \rho|_{\widetilde{U}_j} \downarrow & \swarrow & \\ \widetilde{W}_j = \text{Spec } C_j & & \end{array} \quad (3)$$

where $F_i = \det(\frac{\partial f_a}{\partial x_b})_{a,b \leq r}$ and $G_j = \det(\frac{\partial g_{a'}}{\partial y_{b'}})_{a',b' \leq t}$. Take localization of B_i , we can get affine open covering pair $(\{V_{ijk}\}, \{U_{ijk}\})$ such that U_{ijk} cover $U_i \cap \widetilde{U}_j$ as standard open subsets both in U_i and \widetilde{U}_j .

Assume $U_{ijk} = D(H_{ijk})$ in \widetilde{U}_j , then we get a commutative diagram

$$\begin{array}{ccc} U_{ijk} & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1) \\ \downarrow & & \downarrow \\ \widetilde{U}_j & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}]/(g_1, \dots, g_t) \\ \rho|_{\widetilde{U}_j} \downarrow & \swarrow & \\ \widetilde{W}_j & & \end{array} \quad (4)$$

Hence if take $W_{ijk} = \widetilde{W}_j$, then $(\{U_{ijk}\}, \{W_{ijk}\})$ is also an affine open covering pair representing ρ . Moreover, there is a commutative diagram

$$\begin{array}{ccc} V_{ijk} & \hookrightarrow & \text{Spec } B_{ijk}[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r) \\ \pi|_{V_{ijk}} \downarrow & \swarrow & \\ U_{ijk} & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1) \\ \rho|_{U_{ijk}} \downarrow & \swarrow & \\ W_{ijk} & & \end{array} \quad (5)$$

Note that $B_{ijk} = C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1)_{G_j}$, we get

$$\begin{array}{ccc} V_{ijk} & \hookrightarrow & \text{Spec } C_j[x_1, \dots, x_{n+r}, y_1, \dots, y_{m+t+1}]/(f_1, \dots, f_r, g_1, \dots, g_m, y_{m+t+1}H_{ijk} - 1) \\ \downarrow & \swarrow & \\ W_{ijk} & & \end{array} \quad (6)$$

so that $\pi \circ \rho$ is a smooth morphism of relative dimension $n + m$.

Remark about 14.1.D. The transition functions of \mathcal{E} can be interpreted as a commutative

diagram

$$\begin{array}{ccc}
 & \mathcal{O}_{U_i \cap U_j}^{\oplus n} & \\
 \varphi_i|_{U_i \cap U_j} \nearrow & \downarrow T_{ij} & \\
 \mathcal{E}|_{U_i \cap U_j} & & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \\
 \varphi_j|_{U_i \cap U_j} \searrow & &
 \end{array} \quad (7)$$

Applying sheaf hom functor, we get

$$\begin{array}{ccc}
 & \mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j}) & \\
 \circ \varphi_i|_{U_i \cap U_j}^{-1} \nearrow & \downarrow \circ T_{ji} & \\
 \mathrm{Hom}(\mathcal{E}|_{U_i \cap U_j}, \mathcal{O}_{U_i \cap U_j}) & & \mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j}) \\
 \circ \varphi_j|_{U_i \cap U_j}^{-1} \searrow & &
 \end{array} \quad (8)$$

Hence for basis e_1, \dots, e_n of $\mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j})$, under transition map, e_s is mapping to $\sum_k a_{sk} e'_k$, where a_{sk} is the (s, k) -entry of T_{ji} . Conclude the new transition map is $T'_{ij} := (T_{ij}^{-1})^T$.

2 Higher Dimensional Algebraic Geometry

Remark about birational equivalence class of smooth projective curves. Let X be a smooth projective curve. Assume that $f : X \dashrightarrow Y$ is a birational map with Y projective. Since the question is local, we may assume that X is affine. Also, as X is smooth, it is disjoint union of irreducible component, we may also assume X is irreducible. Take some nonempty affine open subset U containing in the definition of domain. Embedding Y into \mathbb{P}^n for some n , then $f|_U$ is given by rational functions f_0, \dots, f_n in $\Gamma(U, \mathcal{O}_X) \subseteq K(X)$.

Suppose that there exists $p \in X$ of codimension 1 where f is not defined. Note that X is smooth, we get $\mathcal{O}_{X,p}$ is a discrete valuation ring with fraction field $K(X)$. Hence we can write $f_i = \varpi^{s_i} g_i$ where ϖ is uniformizer of $\mathcal{O}_{X,p}$ and g_i is invertible in $\mathcal{O}_{X,p}$.

Take $s = \min_i \{s_i\}$. And we have that $[f_0, \dots, f_n] = [\varpi^{s_0-s} g_0, \dots, \varpi^{s_n-s} g_n]$. As $s_0 - s \geq 0$ for all i , we can take affine open neighbourhood U' of p such that $g_i \in \Gamma(U', \mathcal{O}_X)$ for all i . Define morphism $g : U' \rightarrow Y$ by $\varpi^{s_0-s} g_0, \dots, \varpi^{s_n-s} g_n$. It is clear f and g coincide on $U \cap U'$. Hence f is defined over p .

Conclude that the indeterminacy locus of f is of codimension at least 2. While X is one-dimensional, we get f is in fact a morphism. What we prove is a more general proposition that indeterminacy locus of a rational map from smooth scheme to projective scheme is of codimension at least 2.

Now assume that $f : X \dashrightarrow Y$ is a birational map between smooth projective curves. By argument above, we would get both f and f^{-1} are in fact morphisms so that X is isomorphic

to Y . Conclude that the birational equivalence class of smooth projective curve is itself up to isomorphism.

3 Singularities of the Minimal Model Program

Remark about Serre's Condition. Let X be a locally noetherian scheme and $n \geq 0$.

(1) We say that X is regular in codimension n , or we say X has R_n property if for every $x \in X$, we have

$$\dim(\mathcal{O}_{X,x}) \leq n \Rightarrow \mathcal{O}_{X,x} \text{ regular} \quad (9)$$

i.e. regular in codimension n .

(2) We say that X has property S_n if for every $x \in X$, we have

$$\text{depth}(\mathcal{O}_{X,x}) \geq \min(n, \dim(\mathcal{O}_{X,x})) \quad (10)$$

Remark about relatively ample divisor. Let $f : X \rightarrow Y$ be a proper morphism of schemes, $D \subset X$ divisor. We say that D is f -ample if $\mathcal{O}_X(D)$ is f -ample i.e. for large enough m , $\mathcal{O}_X(D)^{\otimes m}$ is f -very ample.

Let \mathcal{L} be an invertible sheaf on X . We say that \mathcal{L} is f -very ample if the induced map $X \rightarrow \mathbb{P}(f_*\mathcal{L})$ is a closed embedding.

Remark about local complete intersection. Let $f : V \hookrightarrow X$ be a locally closed immersion factoring through open subscheme $U \subset X$. We say that V is a local complete intersection if there exists an affine open covering $\mathcal{U} = \{U_i = \text{Spec } A_i\}$ of U such that for all i , ideal sheaf of $V \cap U_i$ in U_i is generated by a Koszul-regular sequence f_1, \dots, f_r in A_i .

Let A be a ring. We say that $f_1, \dots, f_r \in A$ is an H_1 -regular sequence if the Koszul complex $K_\bullet(f_1, \dots, f_r)$ is exact at $K_1(f_1, \dots, f_r)$. Moreover, we say that $f_1, \dots, f_r \in A$ is a Koszul-regular sequence if the Koszul complex is exact.

Let $\varphi : M \rightarrow A$ be an A -module homomorphism. Define the Koszul complex $K_\bullet(\varphi)$ with underlying graded algebra $\wedge M$ as following

$$\cdots \longrightarrow K_n(\varphi) \xrightarrow{d_n} \cdots \longrightarrow K_1(\varphi) \xrightarrow{d_1} K_0(\varphi) = A \longrightarrow 0 \quad (11)$$

where each generator of the form $e_1 \wedge \cdots \wedge e_n$ is mapping to $\sum_i (-1)^{i+1} \varphi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$ under d_n . In particular, if take $M = A^{\oplus r}$ and $\varphi = (\varphi_i : A \xrightarrow{f_i} A)_i$ where $f_i \in A$, then $K_\bullet(\varphi)$ is also called the Koszul complex of sequence f_1, \dots, f_r .

The definition above is from Stacks Project. It seems that sometimes we would use the other definition where Koszul-regular condition is simply replaced by regular condition.

Remark about well-definedness of definition of canonical sheaf. Recall definition of canonical sheaf in *Singularities of the Minimal Model Program* that for an open immersion $i : X^0 \hookrightarrow X$ and a locally closed immersion $\ell : X^0 \hookrightarrow \mathbb{P}_B^N$ satisfying that

- $X \setminus X^0$ is of codimension at least 2 in X
- ℓ is a regular embedding

we define

$$\omega_{X^0/B} := \ell^*(\omega_{\mathbb{P}_B^N/B} \otimes \det^{-1}(\mathcal{I}/\mathcal{I}^2)) \quad (12)$$

and $\omega_{X/B} := i_*\omega_{X^0/B}$, where $\omega_{\mathbb{P}_B^N/B}$ is the canonical sheaf defined by differentials and \mathcal{I} is the ideal sheaf corresponding to $\overline{\ell(X^0)}$.

Here we need to show that this definition is independent on the choice of ℓ . Assume there are two immersions $\ell_1 : X^0 \hookrightarrow \mathbb{P}_B^{N_1}$ and $\ell_2 : X^0 \hookrightarrow \mathbb{P}_B^{N_2}$. Consider $Y := \mathbb{P}_B^{N_1} \times_B \mathbb{P}_B^{N_2}$ with two projections $p : Y \rightarrow \mathbb{P}_B^{N_1}$ and $q : Y \rightarrow \mathbb{P}_B^{N_2}$ and locally closed immersion $j : X^0 \hookrightarrow Y$ given by $\ell_1 \times \ell_2$. We are supposed to show that $\ell_1^*(\omega_{\mathbb{P}_B^{N_1}} \otimes \det^{-1}(\mathcal{I}_1/\mathcal{I}_1^2))$ and $\ell_2^*(\omega_{\mathbb{P}_B^{N_2}} \otimes \det^{-1}(\mathcal{I}_2/\mathcal{I}_2^2))$ are same.

Note that there is an exact sequence of Kahler differentials

$$0 \longrightarrow p^*\Omega_{\mathbb{P}_B^{N_1}/B} \longrightarrow \Omega_{Y/B} \longrightarrow \Omega_{Y/\mathbb{P}_B^{N_1}} \longrightarrow 0 \quad (13)$$

By taking highest exterior power, we get $\omega_{Y/B} \cong \omega_{Y/\mathbb{P}_B^{N_1}} \otimes p^*\omega_{\mathbb{P}_B^{N_1}/B}$. In addition, there is an exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{X^0/Y} \longrightarrow \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \longrightarrow j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}} \longrightarrow 0 \quad (14)$$

Again by taking highest exterior power, we get $\det \mathcal{N}_{X^0/Y} \cong \det \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \otimes j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}}$. Then we have that

$$\begin{aligned} \ell_1^*(\omega_{\mathbb{P}_B^{N_1}} \otimes \det^{-1}(\mathcal{I}_1/\mathcal{I}_1^2)) &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes \det j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes j^*\det \Omega_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes j^*\omega_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*\omega_{Y/B} \otimes \det \mathcal{N}_{X^0/B} \end{aligned} \quad (15)$$

By symmetry, we are done!

Remark about etale topology. Let X be a scheme. Define an etale covering of X to be a family of etale morphisms $\{\varphi_i : U_i \rightarrow X\}$ such that $X = \cup \varphi_i(U_i)$. With this "topology", we can say what an etale sheaf is.

An etale sheaf \mathcal{F} of sets on X is the data

- for each etale morphism $\varphi : U \rightarrow X$, there is a set $\mathcal{F}(U)$
- for each pair U, U' and morphism $U \rightarrow U'$, there is a restriction map $\rho_U^{U'} : \mathcal{F}(U') \rightarrow \mathcal{F}(U)$

where $\rho_U^U = id$ in case of $id_U : U \rightarrow U$ and $\rho_U^{U''} \circ \rho_U^{U'} = \rho_U^{U''}$ when we have $U \rightarrow U' \rightarrow U''$. In addition, for every etale covering $\{\varphi_i : U_i \rightarrow U\}$, the following diagram is a equalizer

$$\mathcal{F}(U) \xrightarrow{(\rho_{U_i}^U)_i} \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \quad (16)$$

Similarly, we can define etale sheaf of abelian groups, rings and so on.

4 Birational Geometry of Algebraic Varieties

Remark about Theorem 0.1. By Grothendieck's Theorem in *An Introduction to Homological Algebra*, Rotman, there is a spectral sequence with

$$E_2^{p,q} = (R^p\Gamma(Y, \cdot))(R^q f_*)\mathcal{F} \Rightarrow_p R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} \quad (17)$$

Hence it suffices to show that $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$. Take injective resolution of \mathcal{F} in $\text{Mod}(X)$

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \quad (18)$$

Acting by functor $\Gamma(Y, f_*\cdot)$, get complex

$$0 \longrightarrow \Gamma(Y, f_*\mathcal{F}) \longrightarrow \Gamma(Y, f_*I^0) \longrightarrow \Gamma(Y, f_*I^1) \longrightarrow \dots \quad (19)$$

Note that for quasi-coherent \mathcal{F} , we have that $f_*\mathcal{F} = \widetilde{H^0(X, \mathcal{F})}$. Hence we get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, I^0) \longrightarrow \Gamma(X, I^1) \longrightarrow \dots \quad (20)$$

which is just the complex $\Gamma(X, I^*)$ so that $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$.

Remark about transversal intersection. Let X be a smooth variety, D_1, D_2 prime divisors. We say that D_1 and D_2 intersect transversally at point $x \in D_1 \cap D_2$ if local equations defining D_1 and D_2 around x form a regular sequence in $\mathcal{O}_{X,x}$.

5 Note on Derived Categories and Derived Functors

Remark about Triangulated Category. A triangulated category consists of three parts of datum

- An additive category \mathcal{A}
- An automorphism $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$
- A class of \mathcal{T} -triangle in \mathcal{A} satisfying following axioms, whose elements are called exact (or distinguished) triangle

(TR0): Any triangle isomorphic to exact triangle is also exact triangle.

(TR1): Every morphism $f : A \rightarrow B$ in \mathcal{A} is the base of some exact triangle $A \xrightarrow{f} B \rightarrow C \rightarrow \mathcal{T}A$. And for all object $A \in \mathcal{A}$, $A \xrightarrow{\text{id}_A} A \rightarrow 0 \rightarrow \mathcal{T}A$ is exact triangle.

(TR2): For each exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A$, its rotations are also exact triangles

$$B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A \xrightarrow{\mathcal{T}f} \mathcal{T}B \text{ and } \mathcal{T}^{-1}C \xrightarrow{\mathcal{T}^{-1}h} A \xrightarrow{f} B \xrightarrow{g} C \quad (21)$$

(TR3): Given two exact triangles $A \rightarrow B \rightarrow C \rightarrow \mathcal{T}A$ and $A' \rightarrow B' \rightarrow C' \rightarrow \mathcal{T}A'$, every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \quad (22)$$

would extend to a morphism of triangles

$$\begin{array}{ccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \mathcal{T} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \mathcal{T}A'
 \end{array} \tag{23}$$

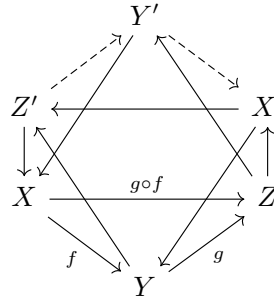
(TR4): If there are three exact triangles

$$\begin{array}{lcl}
 X & \xrightarrow{f} & Y \longrightarrow Z' \longrightarrow \mathcal{T}X \\
 Y & \xrightarrow{g} & Z \longrightarrow X' \longrightarrow \mathcal{T}Y \\
 X & \xrightarrow{g \circ f} & Z \longrightarrow Y' \longrightarrow \mathcal{T}X
 \end{array} \tag{24}$$

Then there is an exact triangle $Z' \rightarrow Y' \rightarrow X' \rightarrow \mathcal{T}Z'$ such that the following diagram commutes

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \mathcal{T}X \\
 \downarrow \text{id}_X & & \downarrow g & & \downarrow & & \downarrow \text{id}_{\mathcal{T}X} \\
 X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & \mathcal{T}X \\
 \downarrow f & & \downarrow \text{id}_Z & & \downarrow & & \downarrow \mathcal{T}f \\
 Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & \mathcal{T}Y \\
 \downarrow & & \downarrow & & \downarrow \text{id}_{X'} & & \downarrow \\
 Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \mathcal{T}Z'
 \end{array} \tag{25}$$

People always prefer another interpretation of the above diagram



$$\tag{26}$$

This is why axiom (TR4) is called octahedral axiom.

6 Stable Reduction

Remark about Scheme over Spectrum of a DVR. Let R be a DVR, $f : V \rightarrow \text{Spec } R$ a morphism. As there are only two points (0) and \mathfrak{m} in $\text{Spec } R$, we call their fibers general (or say generic) fiber and closed (or say special) fiber respectively.

Denote the two fibers by W and X respectively. For a Cartier divisor D on V , if $\text{Supp}(D)$ is contained in X , then we say it is a vertical divisor and otherwise a horizontal divisor.

As a generalization, when base is replaced by disc Δ or a smooth curve, we can also define generic fiber and special fiber to be the fibers at the generic point and 0 respectively. However, in these cases, special fiber is not the unique closed fiber. And for a Cartier divisor D on V , we say it is a horizontal divisor if any component of its support is not contained in any closed fiber.

Remark about Ramification. Let $f : X \rightarrow Y$ be a finite morphism of smooth algebraic curves over algebraically closed field k . Then for each closed point $P \in X$ with $f(P) = Q \in Y$, there is an inclusion $f^\# : \mathcal{O}_{Y,Q} \hookrightarrow \mathcal{O}_{X,P}$ of DVRs. Pick a uniformizer π at Q , then $f^\#(\pi) = s \cdot \varpi^{e_P}$, where $s \in \mathcal{O}_{X,P}^\times$ and $e_P \geq 1$ called the ramification index.

We say f is unramified at P if $e_P = 1$ or ramified at P if $e_P > 1$. And for a ramified point P , there is further distinction when $\text{Char } k = p > 0$. We say f is tamely ramified at P if e_P is not divisible by p and the residue field extension is separable, otherwise we say f is wildly ramified at P . Obviously, when $\text{Char } k = 0$, all ramified points are tame.

There is also a definition of ramification for arbitrary morphism. Let $f : X \rightarrow Y$ be a morphism of schemes. We say f is unramified at $x \in X$ if

- f is locally of finite type.
- the residue field extension $k(x)/k(f(x))$ is separable.
- $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$.

Equivalently, $\Omega_{X/Y}$ vanishes in a neighbourhood of x .

Remark about Riemann-Hurwitz Formula. Let $f : X \rightarrow Y$ be a finite morphism of smooth algebraic curves over algebraically closed field k . When $\text{Char } k = 0$ or f is tamely ramified, there is a Riemann-Hurwitz formula

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{P \in X} (e_P - 1) \quad (27)$$

where χ denotes the topological Euler characteristic $2 - 2g$. Equivalently, in terms of canonical divisors,

$$K_X \sim f^* K_Y + R \quad (28)$$

where $R = \sum_{P \in X} (e_P - 1)P$ is the ramification divisor. In addition, when f is wildly ramified, we still have a Riemann-Hurwitz formula where e_P are replaced by the different exponents d_P determined by higher ramification groups.

Remark about Stable Marked Curve. A stable marked curve (or say stable n -pointed curve) consists of

- A connected, projective curve C .
- C is smooth except for nodes (ordinary double points)
- n distinct marked points lying in C^{sm} .
- The automorphism group of C fixing all marked points is finite.

The key point is the finiteness of automorphism group. For genus 0 irreducible components, there should be at least 3 special points (marked points or nodes). For genus 1 irreducible components, there should be at least 1 special points. And for irreducible components of genus ≥ 2 , the finiteness automatically holds.

7 Others

Remark about the Relation between Closed Points and k -valued Points. Let X be a scheme over algebraically closed field k . In general, there is no containment relation between the set of closed points of X and the set of k -valued points of X . But when X is locally of finite type over k , by Hilbert's Nullstellensatz, all maximal ideals are of the form $(x_1 - a_1, \dots, x_n - a_n)$ and hence they are k -valued points. On the other hand, if X is separated over k , then by Cancellation Theorem, all k -valued points are closed points (and in this case, we don't need k to be algebraically closed).

Remark about Hodge Structure. A pure Hodge structure of weight $k \in \mathbb{Z}$ consists of the following data

- A finitely generated abelian group $H_{\mathbb{Z}}$
- A decreasing Hodge filtration F^{\bullet} on the complex vector space $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$

such that for all p, q with $p + q = k$, the filtration induces a direct sum decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$, where $H^{p,q} = F^p \cap \overline{F^q}$.

In addition, there is an equivalent definition that a pure Hodge structure of weight k is a finitely generated abelian group $H_{\mathbb{Z}}$ together with a direct sum decomposition of its complexification

$$H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \quad (29)$$

satisfying the Hodge symmetry condition that $\overline{H^{p,q}} = H^{q,p}$, where the bar denotes complex conjugation.

Given the decomposition $H^{p,q}$, we can define the filtration as $F^p = \bigoplus_{i \geq p} H^{i,k-i}$. Conversely, given a filtration with the property $F^p \oplus \overline{F^{k-p+1}} \cong H_{\mathbb{C}}$, we can recover the decomposition as $H^{p,q} = F^p \cap \overline{F^q}$.

Define a polarization on a pure Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight k to be a non-degenerate bilinear form Q on $H_{\mathbb{Z}}$ (symmetric if k is even, alternating if k is odd) satisfying the Hodge-Riemann bilinear relations that

- Orthogonality: $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p = p'$ and $q = q'$
- Positivity: $i^{p-q} Q(v, \overline{v}) > 0$ for all non-zero $v \in H^{p,q}$

Except pure Hodge structure, there is also a context of mixed Hodge structure. Many interesting spaces in algebraic geometry do not have a pure decomposition and mixed Hodge structure is our tool to handle these cases. A mixed Hodge structure consists of:

- A finitely generated abelian group $H_{\mathbb{Z}}$
- An increasing weight filtration W_{\bullet} on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$
- A decreasing Hodge filtration F^{\bullet} on $H_{\mathbb{C}}$

such that for each integer m , the filtration F^{\bullet} induces a pure Hodge structure of weight m on $(W_m/W_{m-1}) \otimes \mathbb{C}$.