Remark

Contents

1 Rising Sea

Remark about 7.4.8. For the equivalence, consider the following commutative diagram

$$\begin{array}{ccccc} A[x_1,x_2,\cdots,x_n] & \xrightarrow{\alpha} & A/\mathfrak{p}[x_1,x_2,\cdots,x_n] & \xrightarrow{\beta} & k(\mathfrak{p})[x_1,x_2,\cdots,x_n] \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A/\mathfrak{p} & \longrightarrow & k(\mathfrak{p}) \end{array}$$

Note that α is a quation map by $\mathfrak{p}[x_1,x_2,\cdots,x_n]$ and β is a localization at $A/\mathfrak{p}\setminus\{0\}$. Thus the equivalence is clear. □

Remark about 11.3.C(a). For affine case, we can also apply Krull's Principal Ideal Theorem. But there we get $\dim(k[x_{0/i},\cdots,x_{n/i}]/(\mathfrak{p},f))\geq\dim(k[x_{0/i},\cdots,x_{n/i}]/\mathfrak{p})-1\geq 0$.

When the equality fails, we can find some prime ideal containing f so that H meets X. However, when the equality holds, it is possible that $(\mathfrak{p},f)=(1)$ inducing $H\cap X=\varnothing$. □

Remark about 13.3.L. We say two closed subschemes transversally intersect if generators of their ideal sheaf form a regular sequence in the stalk. Hence two closed subschemes intersect at point p with multiplicity at least 2 if and only if generators of their ideal sheaf do not form a regular sequence in the stalk at p.

For the case in 13.3.L, assume that line L is cut out by $n-1$ linear equations which are linearly independent, denoted by $f_1,\cdots,f_{n-1}$. For convenience, we may assume that $p=[1,a_1,\cdots,a_n]\in D_+(x_0)$ and $f,f_i\in k[x_{1/0},\cdots,x_{n/0}]$. Then $\frac{f_1}{1},\cdots,\frac{f_{n-1}}{1}$ automatically form a regular sequence in $\mathcal{O}_{\mathbb{P}^n_k,p}$. As X meets L at p with multiplicity at least 2, we get $\frac{f}{1}$ is a zero divisor in $\mathcal{O}_{\mathbb{P}^n_k,p}/(\frac{f_1}{1},\cdots,\frac{f_{n-1}}{1})$.

Note that p is a regular point, we have $\mathcal{O}_{\mathbb{P}^n_k,p}$ is a regular local ring and so is quotient one. Since regular local ring is an integral domain, we get $\frac{f}{1}\in(\frac{f_1}{1},\cdots,\frac{f_{n-1}}{1})$. Assume that $\frac{f}{1}=\frac{\sum_i g_i f_i}{s}$ so that $sf=\sum_i g_i f_i$. Multiplying the tangent space T_pX by $s(p)\neq 0$, the evaluation of s at p, we have that

$$\begin{aligned}\sum_j s(p)\frac{\partial f}{x_j}(p)(x_j-a_j) &= \sum_j(s(p)\frac{\partial f}{x_j}(p)-f(p)\frac{\partial s}{x_j}(p))(x_j-a_j)\\ &=\sum_j\frac{\partial sf}{x_j}(p)(x_j-a_j)\\ &=\sum_j\frac{\partial\sum_i g_i f_i}{x_j}(p)(x_j-a_j)\\ &=\sum_i\sum_j g_i(p)\frac{\partial f_i}{x_j}(p)(x_j-a_j)\end{aligned}\tag{1}$$

since all f_i are linear equations, we immediately get $\sum_j\frac{\partial f_i}{x_j}(p)x_j=f_i$ and $\sum_j\frac{\partial f_i}{x_j}(p)a_j=f_i(p)=0$. Thus $s(p)T_pX=\sum_i g_i(p)f_i\in(f_1,\cdots,f_{n-1})$ i.e. line L is contained in tangent space T_pX. □

Remark about 13.6.D. Since $\pi:X\to Y$ and $\rho:Y\to Z$ are smooth of relative dimensions m and n respectively, by definition, there are affine open covering pairs $(\{V_i\},\{U_i\})$ and

$(\{\widetilde{U_j}\}, \{\widetilde{W_j}\})$ such that for all i, j, there are commutative diagrams

$$\begin{tikzcd}
V_i = D(F_i) \arrow[r, hook] \arrow[d, "\pi|_{V_i}"'] & \operatorname{Spec} B_i[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[dl] \\
U_i = \operatorname{Spec} B_i &
\end{tikzcd} \tag{2}$$

and

$$\begin{tikzcd}
\widetilde{U_j} = D(G_j) \arrow[r, hook] \arrow[d, "\rho|_{\widetilde{U_j}}"'] & \operatorname{Spec} C_j[y_1, \cdots, y_{m+t}]/(g_1, \cdots, g_t) \arrow[dl] \\
\widetilde{W_j} = \operatorname{Spec} C_j &
\end{tikzcd} \tag{3}$$

where $F_i = \det(\frac{\partial f_a}{\partial x_b})_{a,b \leq r}$ and $G_j = \det(\frac{\partial g_{a'}}{\partial y_{b'}})_{a',b' \leq t}$. Take localization of B_i, we can get affine open covering pair $(\{V_{ijk}\}, \{U_{ijk}\})$ such that U_{ijk} cover $U_i \cap \widetilde{U_j}$ as standard open subsets both in U_i and $\widetilde{U_j}$.

Assume $U_{ijk} = D(H_{ijk})$ in $\widetilde{U_j}$, then we get a commutative diagram

$$\begin{tikzcd}
U_{ijk} \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} C_j[y_1, \cdots, y_{m+t}, y_{m+t+1}]/(g_1, \cdots, g_t, y_{m+t+1}H_{ijk} - 1) \arrow[d, hook] \\
\widetilde{U_j} \arrow[r, hook] \arrow[d, "\rho|_{\widetilde{U_j}}"'] & \operatorname{Spec} C_j[y_1, \cdots, y_{m+t}]/(g_1, \cdots, g_t) \arrow[dl] \\
\widetilde{W_j} &
\end{tikzcd} \tag{4}$$

Hence if take $W_{ijk} = \widetilde{W_j}$, then $(\{U_{ijk}\}, \{W_{ijk}\})$ is also an affine open covering pair representing ρ. Moreover, there is a commutative diagram

$$\begin{tikzcd}
V_{ijk} \arrow[r, hook] \arrow[d, "\pi|_{V_{ijk}}"'] & \operatorname{Spec} B_{ijk}[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[dl] \\
U_{ijk} \arrow[r, hook] \arrow[d, "\rho|_{U_{ijk}}"'] & \operatorname{Spec} C_j[y_1, \cdots, y_{m+t}, y_{m+t+1}]/(g_1, \cdots, g_t, y_{m+t+1}H_{ijk} - 1) \arrow[dl] \\
W_{ijk} &
\end{tikzcd} \tag{5}$$

Note that $B_{ijk} = C_j[y_1, \cdots, y_{m+t}, y_{m+t+1}]/(g_1, \cdots, g_t, y_{m+t+1}H_{ijk} - 1)_{G_j}$, we get

$$\begin{tikzcd}
V_{ijk} \arrow[r, hook] \arrow[d] & \operatorname{Spec} C_j[x_1, \cdots, x_{n+r}, y_1, \cdots, y_{m+t+1}]/(f_1, \cdots, f_r, g_1, \cdots, g_m, y_{m+t+1}H_{ijk} - 1) \arrow[dl] \\
W_{ijk} &
\end{tikzcd} \tag{6}$$

so that $\pi \circ \rho$ is a smooth morphism of relative dimension $n + m$. □

Remark about 14.1.D. The transition functions of $\mathcal{E}$ can be interpreted as a commutative diagram

$$\begin{tikzcd} & \mathcal{O}_{U_i\cap U_j}^{\oplus n} \arrow[dd, "T_{ij}"] \\ \mathcal{E}\big|_{U_i\cap U_j} \arrow[ur, "\varphi_i|_{U_i\cap U_j}"] \arrow[dr, "\varphi_j|_{U_i\cap U_j}"'] & \\ & \mathcal{O}_{U_i\cap U_j}^{\oplus n} \end{tikzcd} \tag{7}$$

Applying sheaf hom functor, we get

$$\begin{tikzcd} & \mathrm{Hom}(\mathcal{O}_{U_i\cap U_j}^{\oplus n}, \mathcal{O}_{U_i\cap U_j}) \arrow[dd, "\circ T_{ji}"] \\ \mathrm{Hom}(\mathcal{E}\big|_{U_i\cap U_j}, \mathcal{O}_{U_i\cap U_j}) \arrow[ur, "\circ\varphi_i|_{U_i\cap U_j}^{-1}"] \arrow[dr, "\circ\varphi_j|_{U_i\cap U_j}^{-1}"'] & \\ & \mathrm{Hom}(\mathcal{O}_{U_i\cap U_j}^{\oplus n}, \mathcal{O}_{U_i\cap U_j}) \end{tikzcd} \tag{8}$$

Hence for basis $e_1, \cdots, e_n$ of $\mathrm{Hom}(\mathcal{O}_{U_i\cap U_j}^{\oplus n}, \mathcal{O}_{U_i\cap U_j})$, under transition map, e_s is mapping to $\sum_k a_{sk}e'_k$, where a_{sk} is the (s,k)-entry of T_{ji}. Conclude the new transition map is $T'_{ij} := (T_{ij}^{-1})^T$. □

2 Higher Dimensional Algebraic Geometry

Remark about birational equivalence class of smooth projective curves. Let X be a smooth projective curve. Assume that $f : X \dashrightarrow Y$ is a birational map with Y projective. Since the question is local, we may assume that X is affine. Also, as X is smooth, it is disjoint union of irreducible component, we may also assume X is irreducible. Take some nonempty affine open subset U containing in the definition of domain. Embedding Y into $\mathbb{P}^n$ for some n, then $f\big|_U$ is given by rational functions $f_0, \cdots, f_n$ in $\Gamma(U, \mathcal{O}_X) \subseteq K(X)$.

Suppose that there exists $p \in X$ of codimension 1 where f is not defined. Note that X is smooth, we get $\mathcal{O}_{X,p}$ is a discrete valuation ring with fraction field $K(X)$. Hence we can write $f_i = \varpi^{s_i} g_i$ where ϖ is uniformizor of $\mathcal{O}_{X,p}$ and g_i is invertible in $\mathcal{O}_{X,p}$.

Take $s = \min_i\{s_i\}$. And we have that $[f_0, \cdots, f_n] = [\varpi^{s_0-s}g_0, \cdots, \varpi^{s_n-s}g_n]$. As $s_0 - s \geq 0$ for all i, we can take affine open neighbourhood U' of p such that $g_i \in \Gamma(U', \mathcal{O}_X)$ for all i. Define morphism $g : U' \to Y$ by $\varpi^{s_0-s}g_0, \cdots, \varpi^{s_n-s}g_n$. It is clear f and g coincide on $U \cap U'$. Hence f is defined over p.

Conclude that the indeterminacy locus of f is of codimension at least 2. While X is one-dimensional, we get f is in fact a morphism. What we prove is a more general proposition that indeterminacy locus of a rational map from smooth scheme to projective scheme is of codimension at least 2.

Now assume that $f : X \dashrightarrow Y$ is a birational map between smooth projective curves. By argument above, we would get both f and f^{-1} are in fact morphisms so that X is isomorphic to Y. Conclude that the birational equivalence class of smooth projective curve is itself up to isomorphism. $\square$

3 Singularities of the Minimal Model Program

Remark about Serre's Condition. Let X be a locally noetherian scheme and $n \geq 0$.
(1)We say that X is regular in codimension n, or we say X has R_n property if for every $x \in X$, we have

$$\dim(\mathcal{O}_{X,x}) \leq n \Rightarrow \mathcal{O}_{X,x} \text{ regular} \tag{9}$$

i.e. regular in codimension n.
(2)We say that X has property S_n if for every $x \in X$, we have

$$\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min(n, \dim(\mathcal{O}_{X,x})) \tag{10}$$

Remark about relatively ample divisor. Let $f : X \to Y$ be a proper morphism of schemes, $D \subset X$ divisor. We say that D is f-ample if $\mathcal{O}_X(D)$ is f-ample i.e. for large enough m, $\mathcal{O}_X(D)^{\otimes m}$ is f-very ample.

Let $\mathcal{L}$ be an invertible sheaf on X. We say that $\mathcal{L}$ is f-very ample if the induced map $X \to \mathbb{P}(f_*\mathcal{L})$ is a closed embedding. $\square$

Remark about local complete intersection. Let $f : V \hookrightarrow X$ be a locally closed immersion factoring through open subscheme $U \subset X$. We say that V is a local complete intersection if there exists an affine open covering $\mathcal{U} = \{U_i = \operatorname{Spec} A_i\}$ of U such that for all i, ideal sheaf of $V \cap U_i$ in U_i is generated by a Koszul-regular sequence $f_1, \cdots, f_r$ in A_i.

Let A be a ring. We say that $f_1, \cdots, f_r \in A$ is an H_1-regular sequence if the Koszul complex $K_\bullet(f_1, \cdots, f_r)$ is exact at $K_1(f_1, \cdots, f_r)$. Moreover, we say that $f_1, \cdots, f_r \in A$ is a Koszul-regular sequence if the Koszul complex is exact.

Let $\varphi : M \to A$ be an A-module homomorphism. Define the Koszul complex $K_\bullet(\varphi)$ with underlying graded algebra $\wedge M$ as following

$$\cdots \longrightarrow K_n(\varphi) \xrightarrow{d_n} \cdots \longrightarrow K_1(\varphi) \xrightarrow{d_1} K_0(\varphi) = A \longrightarrow 0 \tag{11}$$

where each generator of the form $e_1 \wedge \cdots \wedge e_n$ is mapping to $\sum_i (-1)^{i+1}\varphi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$ under d_n. In particular, if take $M = A^{\oplus r}$ and $\varphi = (\varphi_i : A \xrightarrow{\cdot f_i} A)_i$ where $f_i \in A$, then $K_\bullet(\varphi)$ is also called the Koszul complex of sequence $f_1, \cdots, f_r$.

The definition above is from Stacks Project. It seems that sometimes we would use the other definition where Koszul-regular condition is simply replaced by regular condition. $\square$

Remark about well-definedness of definition of canonical sheaf. Recall definition of canonical sheaf in *Singularities of the Minimal Model Program* that for an open immersion $i : X^0 \hookrightarrow X$ and a locally closed immersion $\ell : X^0 \hookrightarrow \mathbb{P}^N_B$ satisfying that

- $X \setminus X^0$ is of codimension at least 2 in X
- ℓ is a regular embedding

we define

$$\omega_{X^0/B} := \ell^*(\omega_{\mathbb{P}^N_B/B} \otimes \overset{-1}{\det}(\mathcal{I}/\mathcal{I}^2)) \tag{12}$$

and $\omega_{X/B} := i_*\omega_{X^0/B}$, where $\omega_{\mathbb{P}^N_B/B}$ is the canonical sheaf defined by differentials and $\mathcal{I}$ is the ideal sheaf corresponding to $\overline{\ell(X^0)}$.

Here we need to show that this definition is independent on the choice of ℓ. Assume there are two immersions $\ell_1 : X^0 \hookrightarrow \mathbb{P}^{N_1}_B$ and $\ell_2 : X^0 \hookrightarrow \mathbb{P}^{N_2}_B$. Consider $Y := \mathbb{P}^{N_1}_B \times_B \mathbb{P}^{N_2}_B$ with two projections $p : Y \to \mathbb{P}^{N_1}_B$ and $q : Y \to \mathbb{P}^{N_2}_B$ and locally closed immersion $j : X^0 \hookrightarrow Y$ given by $\ell_1 \times \ell_2$. We are supposed to show that $\ell_1^*(\omega_{\mathbb{P}^{N_1}_B} \otimes \det^{-1}(\mathcal{I}_1/\mathcal{I}_1^2))$ and $\ell_2^*(\omega_{\mathbb{P}^{N_2}_B} \otimes \det^{-1}(\mathcal{I}_2/\mathcal{I}_2^2))$ are same.

Note that there is an exact sequence of Kahler differentials

$$0 \longrightarrow p^*\Omega_{\mathbb{P}^{N_1}_B/B} \longrightarrow \Omega_{Y/B} \longrightarrow \Omega_{Y/\mathbb{P}^{N_1}_B} \longrightarrow 0 \tag{13}$$

By taking highest exterior power, we get $\omega_{Y/B} \cong \omega_{Y/\mathbb{P}^{N_1}_B} \otimes p^*\omega_{\mathbb{P}^{N_1}_B/B}$. In addition, there is an exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{X^0/Y} \longrightarrow \mathcal{N}_{X^0/\mathbb{P}^{N_1}_B} \longrightarrow j^*\mathcal{N}_{Y/\mathbb{P}^{N_1}_B} \longrightarrow 0 \tag{14}$$

Again by taking highest exterior power, we get $\det \mathcal{N}_{X^0/\mathbb{P}^{N_1}_B} \cong \det \mathcal{N}_{X^0/Y} \otimes j^*\mathcal{N}_{Y/\mathbb{P}^{N_1}_B}$. Then we have that

$$\begin{aligned}
\ell_1^*(\omega_{\mathbb{P}^{N_1}_B} \otimes \overset{-1}{\det}(\mathcal{I}_1/\mathcal{I}_1^2)) &\cong j^*p^*\omega_{\mathbb{P}^{N_1}_B} \otimes \det \mathcal{N}_{X^0/\mathbb{P}^{N_1}_B} \\
&\cong j^*p^*\omega_{\mathbb{P}^{N_1}_B} \otimes \det j^*\mathcal{N}_{Y/\mathbb{P}^{N_1}_B} \otimes \det \mathcal{N}_{X^0/Y} \\
&\cong j^*p^*\omega_{\mathbb{P}^{N_1}_B} \otimes j^* \det \Omega_{Y/\mathbb{P}^{N_1}_B} \otimes \det \mathcal{N}_{X^0/Y} \\
&\cong j^*p^*\omega_{\mathbb{P}^{N_1}_B} \otimes j^*\omega_{Y/\mathbb{P}^{N_1}_B} \otimes \det \mathcal{N}_{X^0/N} \\
&\cong j^*\omega_{Y/B} \otimes \det \mathcal{N}_{X^0/B}
\end{aligned} \tag{15}$$

By symmetry, we are done! □

Remark about etale topology. Let X be a scheme. Define an etale covering of X to be a family of etale morphisms $\{\varphi_i : U_i \to X\}$ such that $X = \cup\varphi_i(U_i)$. With this "topology", we can say what an etale sheaf is.

An etale sheaf $\mathcal{F}$ of sets on X is the data

- for each etale morphism $\varphi : U \to X$, there is a set $\mathcal{F}(U)$
- for each pair U, U' and morphism $U \to U'$, there is a restriction map $\rho^{U'}_U : \mathcal{F}(U') \to \mathcal{F}(U)$

where $\rho^U_U = id$ in case of $\mathrm{id}_U : U \to U$ and $\rho^{U''}_{U'} \circ \rho^{U'}_U = \rho^{U''}_U$ when we have $U \to U' \to U''$. In addition, for every etale covering $\{\varphi_i : U_i \to U\}$, the following diagram is a equalizer

$$\mathcal{F}(U) \xrightarrow{(\rho^U_{U_i})_i} \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \tag{16}$$

Similarly, we can define etale sheaf of abelian groups, rings and so on. □

4 Birational Geometry of Algebraic Varieties

Remark about Theorem 0.1. By Grothendieck's Theorem in *An Introduction to Homological Algebra*, Rotman, there is a spectral sequence with

$$E_2^{p,q} = (R^p\Gamma(Y,\cdot))(R^q f_*)\mathcal{F} \underset{p}{\Rightarrow} R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} \tag{17}$$

Hence it suffices to show that $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$. Take injective resolution of $\mathcal{F}$ in $Mod(X)$

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \tag{18}$$

Acting by functor $\Gamma(Y, f_*\cdot)$, get complex

$$0 \longrightarrow \Gamma(Y, f_*\mathcal{F}) \longrightarrow \Gamma(Y, f_*I^0) \longrightarrow \Gamma(Y, f_*I^1) \longrightarrow \cdots \tag{19}$$

Note that for quasi-coherent $\mathcal{F}$, we have that $f_*\mathcal{F} = \widetilde{H^0(X, \mathcal{F})}$. Hence we get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, I^0) \longrightarrow \Gamma(X, I^1) \longrightarrow \cdots \tag{20}$$

which is just the complex $\Gamma(X, I^*)$ so that $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$. □

Remark about transversal intersection. Let X be a smooth variety, D_1, D_2 prime divisors. We say that D_1 and D_2 intersect transversally at point $x \in D_1 \cap D_2$ if local equations defining D_1 and D_2 around x form a regular sequence in $\mathcal{O}_{X,x}$. □

5 Note on Derived Categories and Derived Functors

Remark about Triangulated Category. A triangulated category consists of three parts of datum

- An additive category $\mathscr{A}$
- An automorphism $\mathcal{T} : \mathscr{A} \to \mathscr{A}$
- A class of $\mathcal{T}$-triangle in $\mathscr{A}$ satisfying following axioms, whose elements are called exact (or distinguished) triangle

$(TR0)$: Any triangle isomorphic to exact triangle is also exact triangle.
$(TR1)$: Every morphism $f : A \to B$ in $\mathscr{A}$ is the base of some exact triangle $A \xrightarrow{f} B \to C \to \mathcal{T}A$. And for all object $A \in \mathscr{A}$, $A \xrightarrow{\mathrm{id}_A} A \to 0 \to \mathcal{T}A$ is exact triangle.
$(TR2)$: For each exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A$, its rotations are also exact triangles

$$B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A \xrightarrow{\mathcal{T}f} \mathcal{T}B \text{ and } \mathcal{T}^{-1}C \xrightarrow{\mathcal{T}^{-1}h} A \xrightarrow{f} B \xrightarrow{g} C \tag{21}$$

$(TR3)$: Given two exact triangles $A \to B \to C \to \mathcal{T}A$ and $A' \to B' \to C' \to \mathcal{T}A'$, every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{22}$$

would extend to a morphism of triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \mathcal{T}A' \end{tikzcd} \tag{23}$$

$(TR4)$: If there are three exact triangles

$$\begin{aligned} &X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow \mathcal{T}X \\ &Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow \mathcal{T}Y \\ &X \xrightarrow{g\circ f} Z \longrightarrow Y' \longrightarrow \mathcal{T}X \end{aligned} \tag{24}$$

Then there is an exact triangle $Z' \to Y' \to X' \to \mathcal{T}Z'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d] & \mathcal{T}X \arrow[d, "\mathrm{id}_{\mathcal{T}X}"] \\ X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & \mathcal{T}X \arrow[d, "\mathcal{T}f"] \\ Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"] & \mathcal{T}Y \arrow[d] \\ Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \mathcal{T}Z' \end{tikzcd} \tag{25}$$

People always prefer another interpretation of the above diagram

$$\begin{tikzcd} & Y' \arrow[dr, dashed] \arrow[ddl] & \\ Z' \arrow[ur, dashed] \arrow[d] & & X' \arrow[ll] \arrow[ddl] \\ X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[u] \arrow[uul] \\ & Y \arrow[ur, "g"'] \arrow[uul] & \end{tikzcd} \tag{26}$$

This is why axiom $(TR4)$ is called octahedral axiom. □

6 Stable Reduction

Remark about Scheme over Specturm of a DVR. Let R be a DVR, $f : V \to \operatorname{Spec} R$ a morphism. As there are only two points (0) and $\mathfrak{m}$ in $\operatorname{Spec} R$, we call their fibers general (or say generic) fiber and closed (or say special) fiber respectively.

Denote the two fibers by W and X respectively. For a Cartier divisor D on V, if $\operatorname{Supp}(D)$ is contained in X, then we say it is a vertical divisor and otherwise a horizontal divisor.

As a generalization, when base is replaced by disc Δ or a smooth curve, we can also define generic fiber and special fiber to be the fibers at the generic point and 0 respectively. However, in these cases, special fiber is not the unique closed fiber. And for a Cartier divisor D on V, we say it is a horizontal divisor if any component of its support is not contained in any closed fiber. □

Remark about Ramification. Let $f : X \to Y$ be a finite morphism of smooth algebraic curves over algebraically closed field k. Then for each closed point $P \in X$ with $f(P) = Q \in Y$, there is an inclusion $f^\sharp : \mathcal{O}_{Y,Q} \hookrightarrow \mathcal{O}_{X,P}$ of DVRs. Pick a uniformizor π at Q, then $f^\sharp(\pi) = s \cdot \varpi^{e_P}$, where $s \in \mathcal{O}_{X,P}^\times$ and $e_P \geq 1$ called the ramification index.

We say f is unramified at P if $e_P = 1$ or ramified at P if $e_P > 1$. And for a ramified point P, there is further distinction when $\operatorname{Char} k = p > 0$. We say f is tamely ramified at P if e_P is not divisible by p and the residue field extension is separable, otherwise we say f is wildly ramified at P. Obviously, when $\operatorname{Char} k = 0$, all ramified points are tame.

There is also a definition of ramification for arbitrary morphism. Let $f : X \to Y$ be a morphism of schemes. We say f is unramified at $x \in X$ if

- f is locally of finite type.
- the residue field extension $k(x)/k(f(x))$ is separable.
- $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$.

Equivalently, $\Omega_{X/Y}$ vanishes in a neighbourhood of x. □

Remark about Riemann-Hurwitz Formula. Let $f; X \to Y$ be a finite morphism of smooth algebraic curves over algebraically closed field k. When $\operatorname{Char} k = 0$ or f is tamely ramified, there is a Riemann-Hurwitz formula

$$\chi(X) = \deg f \cdot \chi(Y) - \sum_{P \in X}(e_P - 1) \tag{27}$$

where χ denotes the topological Euler characteristic $2-2g$. Equivalently, in terms of canonical divisors,

$$K_X \sim f^* K_Y + R \tag{28}$$

where $R = \sum_{P\in X}(e_P - 1)P$ is the ramification divisor. In addition, when f is wildly ramified, we still have a Riemann-Hurwitz formula where e_P are replaced by the different exponents d_P determined by higher ramification groups. □

7 Others

Remark about Hodge structure. A pure Hodge structure of weight $k \in \mathbb{Z}$ consists of the following data

- A finitely generated abelian group $H_\mathbb{Z}$
- A decreasing Hodge filtration $F^\bullet$ on the complex vector space $H_\mathbb{C} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$

such that for all p, q with $p + q = k$, the filtration induces a direct sum decomposition $H_\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$, where $H^{p,q} = F^p \cap \overline{F^q}$.

In addition, there is an equivalent definition that a pure Hodge structure of weight k is a finitely generated abelian group $H_\mathbb{Z}$ together with a direct sum decomposition of its complexification

$$H_\mathbb{C} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} = \oplus_{p+q=k} H^{p,q} \tag{29}$$

satisfying the Hodge symmetry condition that $\overline{H^{p,q}} = H^{q,p}$, where the bar denotes complex conjugation.

Given the decomposition $H^{p,q}$, we can define the filtration as $F^p = \oplus_{i\geq p} H^{i,k-i}$. Conversely, given a filtration with the property $F^p \oplus \overline{F^{k-p+1}} \cong H_{\mathbb{C}}$, we can recover the decomposition as $H^{p,q} = F^p \cap \overline{F^q}$.

Define a polarization on a pure Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight k to be a non-degenerate bilinear form Q on $H_{\mathbb{Z}}$ (symmetric if k is even, alternating if k is odd) satisfying the Hodge-Riemann bilinear relations that

- Orthogonality: $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p = p'$ and $q = q'$
- Positivity: $i^{p-q} Q(v, \overline{v}) > 0$ for all non-zero $v \in H^{p,q}$

Except pure Hodge structure, there is also a context of mixed Hodge structure. Many interesting spaces in algebraic geometry do not have a pure decomposition and mixed Hodge structure is our tool to handle these cases. A mixed Hodge structure consists of:

- A finitely generated abelian group $H_{\mathbb{Z}}$
- An increasing weight filtration $W_\bullet$ on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$
- A decreasing Hodge filtration $F^\bullet$ on $H_{\mathbb{C}}$

such that for each integer m, the filtration $F^\bullet$ induces a pure Hodge structure of weight m on $(W_m/W_{m-1}) \otimes \mathbb{C}$. □