

Remark

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# 1 Rising Sea

**Remark about 7.4.8.** For the equivalence, consider the following commutative diagram

$$\begin{array}{ccccc}
 A[x_1, x_2, \dots, x_n] & \xrightarrow{\alpha} & A/\mathfrak{p}[x_1, x_2, \dots, x_n] & \xrightarrow{\beta} & k(\mathfrak{p})[x_1, x_2, \dots, x_n] \\
 \uparrow & & \uparrow & & \uparrow \\
 A & \longrightarrow & A/\mathfrak{p} & \longrightarrow & k(\mathfrak{p})
 \end{array}$$

Note that  $\alpha$  is a quation map by  $\mathfrak{p}[x_1, x_2, \dots, x_n]$  and  $\beta$  is a localization at  $A/\mathfrak{p} \setminus \{0\}$ . Thus the equivalence is clear.  $\square$

**Remark about 11.3.C(a).** For affine case, we can also apply Krull's Principal Ideal Theorem. But there we get  $\dim(k[x_{0/i}, \dots, x_{n/i}]/(\mathfrak{p}, f)) \geq \dim(k[x_{0/i}, \dots, x_{n/i}]/\mathfrak{p}) - 1 \geq 0$ .

When the equality fails, we can find some prime ideal containing  $f$  so that  $H$  meets  $X$ . However, when the equality holds, it is possible that  $(\mathfrak{p}, f) = (1)$  inducing  $H \cap X = \emptyset$ .  $\square$

**Remark about 13.3.L.** We say two closed subschemes transversally intersect if generators of their ideal sheaf form a regular sequence in the stalk. Hence two closed subschemes intersect at point  $p$  with multiplicity at least 2 if and only if generators of their ideal sheaf do not form a regular sequence in the stalk at  $p$ .

For the case in 13.3.L, assume that line  $L$  is cut out by  $n - 1$  linear equations which are linearly independent, denoted by  $f_1, \dots, f_{n-1}$ . For convenience, we may assume that  $p = [1, a_1, \dots, a_n] \in D_+(x_0)$  and  $f, f_i \in k[x_{1/0}, \dots, x_{n/0}]$ . Then  $\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1}$  automatically form a regular sequence in  $\mathcal{O}_{\mathbb{P}_k^n, p}$ . As  $X$  meets  $L$  at  $p$  with multiplicity at least 2, we get  $\frac{f}{1}$  is a zero divisor in  $\mathcal{O}_{\mathbb{P}_k^n, p}/(\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1})$ .

Note that  $p$  is a regular point, we have  $\mathcal{O}_{\mathbb{P}_k^n, p}$  is a regular local ring and so is quotient one. Since regular local ring is an integral domain, we get  $\frac{f}{1} \in (\frac{f_1}{1}, \dots, \frac{f_{n-1}}{1})$ . Assume that  $\frac{f}{1} = \frac{\sum_i g_i f_i}{s}$  so that  $sf = \sum_i g_i f_i$ . Multiplying the tangent space  $T_p X$  by  $s(p) \neq 0$ , the evaluation of  $s$  at  $p$ , we have that

$$\begin{aligned}
 \sum_j s(p) \frac{\partial f}{\partial x_j}(p)(x_j - a_j) &= \sum_j (s(p) \frac{\partial f}{\partial x_j}(p) - f(p) \frac{\partial s}{\partial x_j}(p))(x_j - a_j) \\
 &= \sum_j \frac{\partial s f}{\partial x_j}(p)(x_j - a_j) \\
 &= \sum_j \frac{\partial \sum_i g_i f_i}{\partial x_j}(p)(x_j - a_j) \\
 &= \sum_i \sum_j g_i(p) \frac{\partial f_i}{\partial x_j}(p)(x_j - a_j)
 \end{aligned} \tag{1}$$

since all  $f_i$  are linear equations, we immediately get  $\sum_j \frac{\partial f_i}{\partial x_j}(p)x_j = f_i$  and  $\sum_j \frac{\partial f_i}{\partial x_j}(p)a_j = f_i(p) = 0$ . Thus  $s(p)T_p X = \sum_i g_i(p)f_i \in (f_1, \dots, f_{n-1})$  i.e. line  $L$  is contained in tangent space  $T_p X$ .  $\square$

**Remark about 13.6.D.** Since  $\pi : X \rightarrow Y$  and  $\rho : Y \rightarrow Z$  are smooth of relative dimensions  $m$  and  $n$  respectively, by definition, there are affine open covering pairs  $(\{V_i\}, \{U_i\})$  and

$(\{\widetilde{U}_j\}, \{\widetilde{W}_j\})$  such that for all  $i, j$ , there are commutative diagrams

$$\begin{array}{ccc} V_i = D(F_i) & \hookrightarrow & \text{Spec } B_i[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r) \\ \pi|_{V_i} \downarrow & \swarrow & \\ U_i = \text{Spec } B_i & & \end{array} \quad (2)$$

and

$$\begin{array}{ccc} \widetilde{U}_j = D(G_j) & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}]/(g_1, \dots, g_t) \\ \rho|_{\widetilde{U}_j} \downarrow & \swarrow & \\ \widetilde{W}_j = \text{Spec } C_j & & \end{array} \quad (3)$$

where  $F_i = \det(\frac{\partial f_a}{\partial x_b})_{a,b \leq r}$  and  $G_j = \det(\frac{\partial g_{a'}}{\partial y_{b'}})_{a',b' \leq t}$ . Take localization of  $B_i$ , we can get affine open covering pair  $(\{V_{ijk}\}, \{U_{ijk}\})$  such that  $U_{ijk}$  cover  $U_i \cap \widetilde{U}_j$  as standard open subsets both in  $U_i$  and  $\widetilde{U}_j$ .

Assume  $U_{ijk} = D(H_{ijk})$  in  $\widetilde{U}_j$ , then we get a commutative diagram

$$\begin{array}{ccc} U_{ijk} & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1) \\ \downarrow & & \downarrow \\ \widetilde{U}_j & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}]/(g_1, \dots, g_t) \\ \rho|_{\widetilde{U}_j} \downarrow & \swarrow & \\ \widetilde{W}_j & & \end{array} \quad (4)$$

Hence if take  $W_{ijk} = \widetilde{W}_j$ , then  $(\{U_{ijk}\}, \{W_{ijk}\})$  is also an affine open covering pair representing  $\rho$ . Moreover, there is a commutative diagram

$$\begin{array}{ccc} V_{ijk} & \hookrightarrow & \text{Spec } B_{ijk}[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r) \\ \pi|_{V_{ijk}} \downarrow & \swarrow & \\ U_{ijk} & \hookrightarrow & \text{Spec } C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1) \\ \rho|_{U_{ijk}} \downarrow & \swarrow & \\ W_{ijk} & & \end{array} \quad (5)$$

Note that  $B_{ijk} = C_j[y_1, \dots, y_{m+t}, y_{m+t+1}]/(g_1, \dots, g_t, y_{m+t+1}H_{ijk} - 1)_{G_j}$ , we get

$$\begin{array}{ccc} V_{ijk} & \hookrightarrow & \text{Spec } C_j[x_1, \dots, x_{n+r}, y_1, \dots, y_{m+t+1}]/(f_1, \dots, f_r, g_1, \dots, g_m, y_{m+t+1}H_{ijk} - 1) \\ \downarrow & \swarrow & \\ W_{ijk} & & \end{array} \quad (6)$$

so that  $\pi \circ \rho$  is a smooth morphism of relative dimension  $n + m$ .  $\square$

**Remark about 14.1.D.** The transition functions of  $\mathcal{E}$  can be interpreted as a commutative

diagram

$$\begin{array}{ccc}
 & \mathcal{O}_{U_i \cap U_j}^{\oplus n} & \\
 \varphi_i|_{U_i \cap U_j} \nearrow & \downarrow T_{ij} & \\
 \mathcal{E}|_{U_i \cap U_j} & & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \\
 \varphi_j|_{U_i \cap U_j} \searrow & & 
 \end{array} \quad (7)$$

Applying sheaf hom functor, we get

$$\begin{array}{ccc}
 & \mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j}) & \\
 \circ \varphi_i|_{U_i \cap U_j}^{-1} \nearrow & \downarrow \circ T_{ji} & \\
 \mathrm{Hom}(\mathcal{E}|_{U_i \cap U_j}, \mathcal{O}_{U_i \cap U_j}) & & \mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j}) \\
 \circ \varphi_j|_{U_i \cap U_j}^{-1} \searrow & & 
 \end{array} \quad (8)$$

Hence for basis  $e_1, \dots, e_n$  of  $\mathrm{Hom}(\mathcal{O}_{U_i \cap U_j}^{\oplus n}, \mathcal{O}_{U_i \cap U_j})$ , under transition map,  $e_s$  is mapping to  $\sum_k a_{sk} e'_k$ , where  $a_{sk}$  is the  $(s, k)$ -entry of  $T_{ji}$ . Conclude the new transition map is  $T'_{ij} := (T_{ij}^{-1})^T$ .  $\square$

## 2 Higher Dimensional Algebraic Geometry

**Remark about birational equivalence class of smooth projective curves.** Let  $X$  be a smooth projective curve. Assume that  $f : X \dashrightarrow Y$  is a birational map with  $Y$  projective. Since the question is local, we may assume that  $X$  is affine. Also, as  $X$  is smooth, it is disjoint union of irreducible component, we may also assume  $X$  is irreducible. Take some nonempty affine open subset  $U$  containing in the definition of domain. Embedding  $Y$  into  $\mathbb{P}^n$  for some  $n$ , then  $f|_U$  is given by rational functions  $f_0, \dots, f_n$  in  $\Gamma(U, \mathcal{O}_X) \subseteq K(X)$ .

Suppose that there exists  $p \in X$  of codimension 1 where  $f$  is not defined. Note that  $X$  is smooth, we get  $\mathcal{O}_{X,p}$  is a discrete valuation ring with fraction field  $K(X)$ . Hence we can write  $f_i = \varpi^{s_i} g_i$  where  $\varpi$  is uniformizer of  $\mathcal{O}_{X,p}$  and  $g_i$  is invertible in  $\mathcal{O}_{X,p}$ .

Take  $s = \min_i \{s_i\}$ . And we have that  $[f_0, \dots, f_n] = [\varpi^{s_0-s} g_0, \dots, \varpi^{s_n-s} g_n]$ . As  $s_0 - s \geq 0$  for all  $i$ , we can take affine open neighbourhood  $U'$  of  $p$  such that  $g_i \in \Gamma(U', \mathcal{O}_X)$  for all  $i$ . Define morphism  $g : U' \rightarrow Y$  by  $\varpi^{s_0-s} g_0, \dots, \varpi^{s_n-s} g_n$ . It is clear  $f$  and  $g$  coincide on  $U \cap U'$ . Hence  $f$  is defined over  $p$ .

Conclude that the indeterminacy locus of  $f$  is of codimension at least 2. While  $X$  is one-dimensional, we get  $f$  is in fact a morphism. What we prove is a more general proposition that indeterminacy locus of a rational map from smooth scheme to projective scheme is of codimension at least 2.

Now assume that  $f : X \dashrightarrow Y$  is a birational map between smooth projective curves. By argument above, we would get both  $f$  and  $f^{-1}$  are in fact morphisms so that  $X$  is isomorphic

to  $Y$ . Conclude that the birational equivalence class of smooth projective curve is itself up to isomorphism.  $\square$

### 3 Singularities of the Minimal Model Program

**Remark about Serre's Condition.** Let  $X$  be a locally noetherian scheme and  $n \geq 0$ .

(1) We say that  $X$  is regular in codimension  $n$ , or we say  $X$  has  $R_n$  property if for every  $x \in X$ , we have

$$\dim(\mathcal{O}_{X,x}) \leq n \Rightarrow \mathcal{O}_{X,x} \text{ regular} \quad (9)$$

i.e. regular in codimension  $n$ .

(2) We say that  $X$  has property  $S_n$  if for every  $x \in X$ , we have

$$\text{depth}(\mathcal{O}_{X,x}) \geq \min(n, \dim(\mathcal{O}_{X,x})) \quad (10)$$

**Remark about relatively ample divisor.** Let  $f : X \rightarrow Y$  be a proper morphism of schemes,  $D \subset X$  divisor. We say that  $D$  is  $f$ -ample if  $\mathcal{O}_X(D)$  is  $f$ -ample i.e. for large enough  $m$ ,  $\mathcal{O}_X(D)^{\otimes m}$  is  $f$ -very ample.

Let  $\mathcal{L}$  be an invertible sheaf on  $X$ . We say that  $\mathcal{L}$  is  $f$ -very ample if the induced map  $X \rightarrow \mathbb{P}(f_*\mathcal{L})$  is a closed embedding.  $\square$

**Remark about local complete intersection.** Let  $f : V \hookrightarrow X$  be a locally closed immersion factoring through open subscheme  $U \subset X$ . We say that  $V$  is a local complete intersection if there exists an affine open covering  $\mathcal{U} = \{U_i = \text{Spec } A_i\}$  of  $U$  such that for all  $i$ , ideal sheaf of  $V \cap U_i$  in  $U_i$  is generated by a Koszul-regular sequence  $f_1, \dots, f_r$  in  $A_i$ .

Let  $A$  be a ring. We say that  $f_1, \dots, f_r \in A$  is an  $H_1$ -regular sequence if the Koszul complex  $K_\bullet(f_1, \dots, f_r)$  is exact at  $K_1(f_1, \dots, f_r)$ . Moreover, we say that  $f_1, \dots, f_r \in A$  is a Koszul-regular sequence if the Koszul complex is exact.

Let  $\varphi : M \rightarrow A$  be an  $A$ -module homomorphism. Define the Koszul complex  $K_\bullet(\varphi)$  with underlying graded algebra  $\wedge M$  as following

$$\cdots \longrightarrow K_n(\varphi) \xrightarrow{d_n} \cdots \longrightarrow K_1(\varphi) \xrightarrow{d_1} K_0(\varphi) = A \longrightarrow 0 \quad (11)$$

where each generator of the form  $e_1 \wedge \cdots \wedge e_n$  is mapping to  $\sum_i (-1)^{i+1} \varphi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$  under  $d_n$ . In particular, if take  $M = A^{\oplus r}$  and  $\varphi = (\varphi_i : A \xrightarrow{f_i} A)_i$  where  $f_i \in A$ , then  $K_\bullet(\varphi)$  is also called the Koszul complex of sequence  $f_1, \dots, f_r$ .

The definition above is from Stacks Project. It seems that sometimes we would use the other definition where Koszul-regular condition is simply replaced by regular condition.  $\square$

**Remark about well-definedness of definition of canonical sheaf.** Recall definition of canonical sheaf in *Singularities of the Minimal Model Program* that for an open immersion  $i : X^0 \hookrightarrow X$  and a locally closed immersion  $\ell : X^0 \hookrightarrow \mathbb{P}_B^N$  satisfying that

- $X \setminus X^0$  is of codimension at least 2 in  $X$
- $\ell$  is a regular embedding

we define

$$\omega_{X^0/B} := \ell^*(\omega_{\mathbb{P}_B^N/B} \otimes \det^{-1}(\mathcal{I}/\mathcal{I}^2)) \quad (12)$$

and  $\omega_{X/B} := i_*\omega_{X^0/B}$ , where  $\omega_{\mathbb{P}_B^N/B}$  is the canonical sheaf defined by differentials and  $\mathcal{I}$  is the ideal sheaf corresponding to  $\overline{\ell(X^0)}$ .

Here we need to show that this definition is independent on the choice of  $\ell$ . Assume there are two immersions  $\ell_1 : X^0 \hookrightarrow \mathbb{P}_B^{N_1}$  and  $\ell_2 : X^0 \hookrightarrow \mathbb{P}_B^{N_2}$ . Consider  $Y := \mathbb{P}_B^{N_1} \times_B \mathbb{P}_B^{N_2}$  with two projections  $p : Y \rightarrow \mathbb{P}_B^{N_1}$  and  $q : Y \rightarrow \mathbb{P}_B^{N_2}$  and locally closed immersion  $j : X^0 \hookrightarrow Y$  given by  $\ell_1 \times \ell_2$ . We are supposed to show that  $\ell_1^*(\omega_{\mathbb{P}_B^{N_1}} \otimes \det^{-1}(\mathcal{I}_1/\mathcal{I}_1^2))$  and  $\ell_2^*(\omega_{\mathbb{P}_B^{N_2}} \otimes \det^{-1}(\mathcal{I}_2/\mathcal{I}_2^2))$  are same.

Note that there is an exact sequence of Kahler differentials

$$0 \longrightarrow p^*\Omega_{\mathbb{P}_B^{N_1}/B} \longrightarrow \Omega_{Y/B} \longrightarrow \Omega_{Y/\mathbb{P}_B^{N_1}} \longrightarrow 0 \quad (13)$$

By taking highest exterior power, we get  $\omega_{Y/B} \cong \omega_{Y/\mathbb{P}_B^{N_1}} \otimes p^*\omega_{\mathbb{P}_B^{N_1}/B}$ . In addition, there is an exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{X^0/Y} \longrightarrow \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \longrightarrow j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}} \longrightarrow 0 \quad (14)$$

Again by taking highest exterior power, we get  $\det \mathcal{N}_{X^0/Y} \cong \det \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \otimes j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}}$ . Then we have that

$$\begin{aligned} \ell_1^*(\omega_{\mathbb{P}_B^{N_1}} \otimes \det^{-1}(\mathcal{I}_1/\mathcal{I}_1^2)) &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/\mathbb{P}_B^{N_1}} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes \det j^*\mathcal{N}_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes j^*\det \Omega_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*p^*\omega_{\mathbb{P}_B^{N_1}} \otimes j^*\omega_{Y/\mathbb{P}_B^{N_1}} \otimes \det \mathcal{N}_{X^0/Y} \\ &\cong j^*\omega_{Y/B} \otimes \det \mathcal{N}_{X^0/B} \end{aligned} \quad (15)$$

By symmetry, we are done!  $\square$

**Remark about etale topology.** Let  $X$  be a scheme. Define an etale covering of  $X$  to be a family of etale morphisms  $\{\varphi_i : U_i \rightarrow X\}$  such that  $X = \cup \varphi_i(U_i)$ . With this "topology", we can say what an etale sheaf is.

An etale sheaf  $\mathcal{F}$  of sets on  $X$  is the data

- for each etale morphism  $\varphi : U \rightarrow X$ , there is a set  $\mathcal{F}(U)$
- for each pair  $U, U'$  and morphism  $U \rightarrow U'$ , there is a restriction map  $\rho_U^{U'} : \mathcal{F}(U') \rightarrow \mathcal{F}(U)$

where  $\rho_U^U = id$  in case of  $id_U : U \rightarrow U$  and  $\rho_U^{U''} \circ \rho_U^{U'} = \rho_U^{U''}$  when we have  $U \rightarrow U' \rightarrow U''$ . In addition, for every etale covering  $\{\varphi_i : U_i \rightarrow U\}$ , the following diagram is a equalizer

$$\mathcal{F}(U) \xrightarrow{(\rho_{U_i}^U)_i} \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \quad (16)$$

Similarly, we can define etale sheaf of abelian groups, rings and so on.  $\square$

## 4 Birational Geometry of Algebraic Varieties

**Remark about Theorem 0.1.** By Grothendieck's Theorem in *An Introduction to Homological Algebra*, Rotman, there is a spectral sequence with

$$E_2^{p,q} = (R^p\Gamma(Y, \cdot))(R^q f_*)\mathcal{F} \Rightarrow_p R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} \quad (17)$$

Hence it suffices to show that  $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$ . Take injective resolution of  $\mathcal{F}$  in  $\text{Mod}(X)$

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \quad (18)$$

Acting by functor  $\Gamma(Y, f_*\cdot)$ , get complex

$$0 \longrightarrow \Gamma(Y, f_*\mathcal{F}) \longrightarrow \Gamma(Y, f_*I^0) \longrightarrow \Gamma(Y, f_*I^1) \longrightarrow \dots \quad (19)$$

Note that for quasi-coherent  $\mathcal{F}$ , we have that  $f_*\mathcal{F} = \widetilde{H^0(X, \mathcal{F})}$ . Hence we get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, I^0) \longrightarrow \Gamma(X, I^1) \longrightarrow \dots \quad (20)$$

which is just the complex  $\Gamma(X, I^*)$  so that  $R^{p+q}(\Gamma(Y, f_*\cdot))\mathcal{F} = H^{p+q}(X, \mathcal{F})$ .  $\square$

**Remark about transversal intersection.** Let  $X$  be a smooth variety,  $D_1, D_2$  prime divisors. We say that  $D_1$  and  $D_2$  intersect transversally at point  $x \in D_1 \cap D_2$  if local equations defining  $D_1$  and  $D_2$  around  $x$  form a regular sequence in  $\mathcal{O}_{X,x}$ .  $\square$

## 5 Note on Derived Categories and Derived Functors

**Remark about Triangulated Category.** A triangulated category consists of three parts of datum

- An additive category  $\mathcal{A}$
- An automorphism  $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$
- A class of  $\mathcal{T}$ -triangle in  $\mathcal{A}$  satisfying following axioms, whose elements are called exact (or distinguished) triangle

(TR0): Any triangle isomorphic to exact triangle is also exact triangle.

(TR1): Every morphism  $f : A \rightarrow B$  in  $\mathcal{A}$  is the base of some exact triangle  $A \xrightarrow{f} B \rightarrow C \rightarrow \mathcal{T}A$ . And for all object  $A \in \mathcal{A}$ ,  $A \xrightarrow{\text{id}_A} A \rightarrow 0 \rightarrow \mathcal{T}A$  is exact triangle.

(TR2): For each exact triangle  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A$ , its rotations are also exact triangles

$$B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A \xrightarrow{\mathcal{T}f} \mathcal{T}B \text{ and } \mathcal{T}^{-1}C \xrightarrow{\mathcal{T}^{-1}h} A \xrightarrow{f} B \xrightarrow{g} C \quad (21)$$

(TR3): Given two exact triangles  $A \rightarrow B \rightarrow C \rightarrow \mathcal{T}A$  and  $A' \rightarrow B' \rightarrow C' \rightarrow \mathcal{T}A'$ , every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \quad (22)$$



would extend to a morphism of triangles

$$\begin{array}{ccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \mathcal{T} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \mathcal{T}A'
 \end{array} \tag{23}$$

(TR4): If there are three exact triangles

$$\begin{array}{lclcl}
 X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \mathcal{T}X \\
 Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & \mathcal{T}Y \\
 X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & \mathcal{T}X
 \end{array} \tag{24}$$

Then there is an exact triangle  $Z' \rightarrow Y' \rightarrow X' \rightarrow \mathcal{T}Z'$  such that the following diagram commutes

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \mathcal{T}X \\
 \downarrow \text{id}_X & & \downarrow g & & \downarrow & & \downarrow \text{id}_{\mathcal{T}X} \\
 X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & \mathcal{T}X \\
 \downarrow f & & \downarrow \text{id}_Z & & \downarrow & & \downarrow \mathcal{T}f \\
 Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & \mathcal{T}Y \\
 \downarrow & & \downarrow & & \downarrow \text{id}_{X'} & & \downarrow \\
 Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \mathcal{T}Z'
 \end{array} \tag{25}$$

People always prefer another interpretation of the above diagram

This is why axiom (TR4) is called octahedral axiom.  $\square$

## 6 Stable Reduction

**Remark about Scheme over Spectrum of a DVR.** Let  $R$  be a DVR,  $f : V \rightarrow \text{Spec } R$  a morphism. As there are only two points  $(0)$  and  $\mathfrak{m}$  in  $\text{Spec } R$ , we call their fibers general (or say generic) fiber and closed (or say special) fiber respectively.

Denote the two fibers by  $W$  and  $X$  respectively. For a Cartier divisor  $D$  on  $V$ , if  $\text{Supp}(D)$  is contained in  $X$ , then we say it is a vertical divisor and otherwise a horizontal divisor.

As a generalization, when base is replaced by disc  $\Delta$  or a smooth curve, we can also define generic fiber and special fiber to be the fibers at the generic point and  $0$  respectively. However, in these cases, special fiber is not the unique closed fiber. And for a Cartier divisor  $D$  on  $V$ , we say it is a horizontal divisor if any component of its support is not contained in any closed fiber.  $\square$

## 7 Others

**Remark about Hodge structure.** A pure Hodge structure of weight  $k \in \mathbb{Z}$  consists of the following data

- A finitely generated abelian group  $H_{\mathbb{Z}}$
- A decreasing Hodge filtration  $F^{\bullet}$  on the complex vector space  $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$

such that for all  $p, q$  with  $p + q = k$ , the filtration induces a direct sum decomposition  $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ , where  $H^{p,q} = F^p \cap \overline{F^q}$ .

In addition, there is an equivalent definition that a pure Hodge structure of weight  $k$  is a finitely generated abelian group  $H_{\mathbb{Z}}$  together with a direct sum decomposition of its complexification

$$H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \quad (27)$$

satisfying the Hodge symmetry condition that  $\overline{H^{p,q}} = H^{q,p}$ , where the bar denotes complex conjugation.

Given the decomposition  $H^{p,q}$ , we can define the filtration as  $F^p = \bigoplus_{i \geq p} H^{i,k-i}$ . Conversely, given a filtration with the property  $F^p \oplus \overline{F^{k-p+1}} \cong H_{\mathbb{C}}$ , we can recover the decomposition as  $H^{p,q} = F^p \cap \overline{F^q}$ .

Define a polarization on a pure Hodge structure  $(H_{\mathbb{Z}}, H^{p,q})$  of weight  $k$  to be a non-degenerate bilinear form  $Q$  on  $H_{\mathbb{Z}}$  (symmetric if  $k$  is even, alternating if  $k$  is odd) satisfying the Hodge-Riemann bilinear relations that

- Orthogonality:  $Q(H^{p,q}, H^{p',q'}) = 0$  unless  $p = p'$  and  $q = q'$
- Positivity:  $i^{p-q} Q(v, \overline{v}) > 0$  for all non-zero  $v \in H^{p,q}$

Except pure Hodge structure, there is also a context of mixed Hodge structure. Many interesting spaces in algebraic geometry do not have a pure decomposition and mixed Hodge structure is our tool to handle these cases. A mixed Hodge structure consists of:

- A finitely generated abelian group  $H_{\mathbb{Z}}$
- An increasing weight filtration  $W_{\bullet}$  on  $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$
- A decreasing Hodge filtration  $F^{\bullet}$  on  $H_{\mathbb{C}}$

such that for each integer  $m$ , the filtration  $F^{\bullet}$  induces a pure Hodge structure of weight  $m$  on  $(W_m/W_{m-1}) \otimes \mathbb{C}$ .  $\square$