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Hypergeometric orthogonal polynomials and their q-analogues

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Hypergeometric orthogonal polynomials

In this chapter we deal with all families of hypergeometric orthogonal polynomials appearing in the Askey scheme on page **??**. For each family of orthogonal polynomials we state the most important properties such as a representation as a hypergeometric function, orthogonality relation(s), the three-term recurrence relation, the second-order differential or difference equation, the forward shift (or degree lowering) and backward shift (or degree raising) operator, a Rodrigues-type formula and some generating functions. In each case we use the notation which seems to be most common in the literature. Moreover, in each case we mention the connection between various families by stating the appropriate limit relations. See also [**?**] for an algebraic approach of this Askey scheme and [**?**] for a view from asymptotic analysis. For notations the reader is referred to chapter **??**.

9.1 Wilson

Hypergeometric representation

$$\begin{aligned}&\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}\\&={}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\a+b,a+c,a+d\end{matrix};1\right).\end{aligned} \tag{9.1.1}$$

Orthogonality relation

If $Re(a,b,c,d)>0$ and non-real parameters occur in conjugate pairs, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2\\&\qquad\times W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)\,dx\\&\quad=\frac{\Gamma(n+a+b)\cdots\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}(n+a+b+c+d-1)_n n!\,\delta_{mn},\end{aligned}\tag{9.1.2}$$

where

$$\begin{aligned}&\Gamma(n+a+b)\cdots\Gamma(n+c+d)\\&=\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d).\end{aligned}$$

If $a<0$ and $a+b$, $a+c$, $a+d$ are positive or a pair of complex conjugates occur with positive real parts, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2\\&\qquad\times W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)\,dx\\&\quad+\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)}\\&\qquad\times\sum_{\substack{k=0,1,2\ldots\\a+k<0}}\frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k k!}\\&\qquad\qquad\times W_m(-(a+k)^2;a,b,c,d)W_n(-(a+k)^2;a,b,c,d)\\&=\frac{\Gamma(n+a+b)\cdots\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}(n+a+b+c+d-1)_n n!\,\delta_{mn}.\end{aligned}\tag{9.1.3}$$

Recurrence relation

$$-\left(a^2+x^2\right)\tilde{W}_n(x^2)=A_n\tilde{W}_{n+1}(x^2)-(A_n+C_n)\tilde{W}_n(x^2)+C_n\tilde{W}_{n-1}(x^2),\tag{9.1.4}$$

where

$$\tilde{W}_n(x^2):=\tilde{W}_n(x^2;a,b,c,d)=\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}$$

and

$$\begin{cases}A_n=\dfrac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}\\[2ex]C_n=\dfrac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}.\end{cases}$$

Normalized recurrence relation

$$xp_n(x)=p_{n+1}(x)+(A_n+C_n-a^2)p_n(x)+A_{n-1}C_np_{n-1}(x),\tag{9.1.5}$$

where

$$W_n(x^2;a,b,c,d)=(-1)^n(n+a+b+c+d-1)_np_n(x^2).$$

Difference equation

$$\begin{aligned} & n(n+a+b+c+d-1)y(x) \\ & = B(x)y(x+i) - [B(x)+D(x)]\,y(x) + D(x)y(x-i), \end{aligned} \tag{9.1.6}$$

where

$$y(x) = W_n(x^2;a,b,c,d)$$

and

$$\begin{cases} B(x) = \dfrac{(a-ix)(b-ix)(c-ix)(d-ix)}{2ix(2ix-1)} \\ D(x) = \dfrac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} & W_n((x+\tfrac{1}{2}i)^2;a,b,c,d) - W_n((x-\tfrac{1}{2}i)^2;a,b,c,d) \\ & = -2inx(n+a+b+c+d-1)W_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}) \end{aligned} \tag{9.1.7}$$

or equivalently

$$\begin{aligned} & \frac{\delta W_n(x^2;a,b,c,d)}{\delta x^2} \\ & = -n(n+a+b+c+d-1)W_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}). \end{aligned} \tag{9.1.8}$$

Backward shift operator

$$\begin{aligned}
&(a-\tfrac{1}{2}-ix)(b-\tfrac{1}{2}-ix)(c-\tfrac{1}{2}-ix)(d-\tfrac{1}{2}-ix)W_n((x+\tfrac{1}{2}i)^2;a,b,c,d)\\
&\quad-(a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)(c-\tfrac{1}{2}+ix)(d-\tfrac{1}{2}+ix)W_n((x-\tfrac{1}{2}i)^2;a,b,c,d)\\
&=-2ixW_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})
\end{aligned}\tag{9.1.9}$$

or equivalently

$$\begin{aligned}
&\frac{\delta\left[\omega(x;a,b,c,d)W_n(x^2;a,b,c,d)\right]}{\delta x^2}\\
&=\omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})W_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}),
\end{aligned}\tag{9.1.10}$$

where

$$\omega(x;a,b,c,d):=\frac{1}{2ix}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2.$$

Rodrigues-type formula

$$\begin{aligned}
&\omega(x;a,b,c,d)W_n(x^2;a,b,c,d)\\
&=\left(\frac{\delta}{\delta x^2}\right)^n\left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n,d+\tfrac{1}{2}n)\right].
\end{aligned}\tag{9.1.11}$$

Generating functions

$${}_2F_1\left(\begin{matrix}a+ix,b+ix\\a+b\end{matrix};t\right){}_2F_1\left(\begin{matrix}c-ix,d-ix\\c+d\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+b)_n(c+d)_nn!}.\tag{9.1.12}$$

$${}_2F_1\left(\begin{matrix}a+ix,c+ix\\a+c\end{matrix};t\right){}_2F_1\left(\begin{matrix}b-ix,d-ix\\b+d\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+c)_n(b+d)_nn!}.\tag{9.1.13}$$

$${}_2F_1\left(\begin{matrix}a+ix,d+ix\\a+d\end{matrix};t\right){}_2F_1\left(\begin{matrix}b-ix,c-ix\\b+c\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+d)_n(b+c)_nn!}.\tag{9.1.14}$$

$$\begin{aligned}
&(1-t)^{1-a-b-c-d}\\
&\quad\times{}_4F_3\left(\begin{matrix}\tfrac{1}{2}(a+b+c+d-1),\tfrac{1}{2}(a+b+c+d),a+ix,a-ix\\a+b,a+c,a+d\end{matrix};-\frac{4t}{(1-t)^2}\right)\\
&=\sum_{n=0}^{\infty}\frac{(a+b+c+d-1)_n}{(a+b)_n(a+c)_n(a+d)_nn!}W_n(x^2;a,b,c,d)t^n.
\end{aligned}\tag{9.1.15}$$

Limit relations

Wilson → Continuous dual Hahn

The continuous dual Hahn polynomials given by (9.3.1) can be found from the Wilson polynomials by dividing by $(a+d)_n$ and letting $d\to\infty$:

$$\lim_{d\to\infty}\frac{W_n(x^2;a,b,c,d)}{(a+d)_n}=S_n(x^2;a,b,c).\tag{9.1.16}$$

Wilson $\rightarrow$ Continuous Hahn

The continuous Hahn polynomials given by (9.4.1) are obtained from the Wilson polynomials by the substitutions $a \rightarrow a - it$, $b \rightarrow b - it$, $c \rightarrow c + it$, $d \rightarrow d + it$ and $x \rightarrow x + t$ and the limit $t \rightarrow \infty$ in the following way:

$$\lim_{t\to\infty} \frac{W_n((x+t)^2; a-it, b-it, c+it, d+it)}{(-2t)^n n!} = p_n(x; a, b, c, d). \tag{9.1.17}$$

Wilson $\rightarrow$ Jacobi

The Jacobi polynomials given by (9.8.1) can be found from the Wilson polynomials by substituting $a = b = \frac{1}{2}(\alpha+1)$, $c = \frac{1}{2}(\beta+1) + it$, $d = \frac{1}{2}(\beta+1) - it$ and $x \rightarrow t\sqrt{\frac{1}{2}(1-x)}$ in the definition (9.1.1) of the Wilson polynomials and taking the limit $t \rightarrow \infty$. In fact we have

$$\begin{aligned}&\lim_{t\to\infty} \frac{W_n(\frac{1}{2}(1-x)t^2; \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+1), \frac{1}{2}(\beta+1)+it, \frac{1}{2}(\beta+1)-it)}{t^{2n} n!} \\ &= P_n^{(\alpha,\beta)}(x).\end{aligned} \tag{9.1.18}$$

Remarks

Note that for $k < n$ we have

$$\frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b)_k(a+c)_k(a+d)_k} = (a+b+k)_{n-k}(a+c+k)_{n-k}(a+d+k)_{n-k},$$

which implies that the Wilson polynomials defined by (9.1.1) can also be seen as polynomials in the parameters a, b, c and d.
If we set

$$\begin{aligned} a &= \tfrac{1}{2}(\gamma+\delta+1), \\ b &= \tfrac{1}{2}(2\alpha-\gamma-\delta+1), \\ c &= \tfrac{1}{2}(2\beta-\gamma+\delta+1), \\ d &= \tfrac{1}{2}(\gamma-\delta+1), \end{aligned}$$

and

$$ix \rightarrow x + \tfrac{1}{2}(\gamma+\delta+1)$$

in

$$\tilde{W}_n(x^2; a, b, c, d) = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} \tag{9.1.19}$$

given by (9.1.1) and take with N a nonnegative integer, we obtain the Racah polynomials given by (9.2.1).

Symmetry

The Wilson polynomial $W_n(y; a, b, c, d)$ is symmetric in a, b, c, d.
This follows from the orthogonality relation (9.1.2) together with the value of its coefficient of y^n given in (9.1.5b). Alternatively, combine (9.1.1) with [AAR, Theorem 3.1.1].
As a consequence, it is sufficient to give generating function (9.1.12). Then the generating functions (9.1.13), (9.1.14) will follow by symmetry in the parameters.

Hypergeometric representation

In addition to (9.1.1) we have (see [513, (2.2)]):

$$W_n(x^2;a,b,c,d) = \frac{(a-ix)_n(b-ix)_n(c-ix)_n(d-ix)_n}{(-2ix)_n} \times {}_7F_6\left(\begin{array}{c} 2ix-n, ix-\frac{1}{2}n+1, a+ix, b+ix, c+ix, d+ix, -n \\ ix-\frac{1}{2}n, 1-n-a+ix, 1-n-b+ix, 1-n-c+ix, 1-n-d+ix \end{array}; 1\right). \quad (9.1.20)$$

The symmetry in a,b,c,d is clear from (9.1.20).

Special value

$$W_n(-a^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n\,, \quad (9.1.21)$$

and similarly for arguments $-b^2$, $-c^2$ and $-d^2$ by symmetry of W_n in a,b,c,d.

Uniqueness of orthogonality measure

Under the assumptions on a,b,c,d for (9.1.2) or (9.1.3) the orthogonality measure is unique up to constant factor.

For the proof assume without loss of generality (by the symmetry in a,b,c,d) that $\mathrm{Re}\, a \geq 0$. Write the right-hand side of (9.1.2) or (9.1.3) as $h_n\delta_{m,n}$. Observe from (9.1.2) and (9.1.21) that

$$\frac{|W_n(-a^2;a,b,c,d)|^2}{h_n} = O(n^{4\mathrm{Re}\,a-1}) \quad \text{as } n \to \infty.$$

Therefore (**??**) holds, from which the uniqueness of the orthogonality measure follows.

By a similar, but necessarily more complicated argument Ismail et al. [281, Section 3] proved the uniqueness of orthogonality measure for associated Wilson polynomials.

References

[**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**].

9.2 Racah

Hypergeometric representation

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{array}{c} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{array}; 1\right), \quad n = 0,1,2,\ldots,N, \quad (9.2.1)$$

where

$$\lambda(x) = x(x+\gamma+\delta+1)$$

and

$$\alpha+1 = -N \quad \text{or} \quad \beta+\delta+1 = -N \quad \text{or} \quad \gamma+1 = -N$$

with N a nonnegative integer.

Orthogonality relation

$$\sum_{x=0}^{N}\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x x!}$$
$$\times R_m(\lambda(x))R_n(\lambda(x))$$
$$= M\frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n n!}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\,\delta_{mn}, \tag{9.2.2}$$

where

$$R_n(\lambda(x)) := R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$M = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if} \quad \alpha+1=-N \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if} \quad \gamma+1=-N. \end{cases}$$

Recurrence relation

$$\lambda(x)R_n(\lambda(x)) = A_nR_{n+1}(\lambda(x)) - (A_n+C_n)R_n(\lambda(x)) + C_nR_{n-1}(\lambda(x)), \tag{9.2.3}$$

where

$$R_n(\lambda(x)) := R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$\begin{cases} A_n = \dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ C_n = \dfrac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \end{cases}$$

hence

$$A_n = \begin{cases} \dfrac{(n+\beta-N)(n+\beta+\delta+1)(n+\gamma+1)(n-N)}{(2n+\beta-N)(2n+\beta-N+1)} & \text{if} \quad \alpha+1=-N \\ \dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\gamma+1)(n-N)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n-N)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} & \text{if} \quad \gamma+1=-N \end{cases}$$

and

$$C_n = \begin{cases} \dfrac{n(n+\beta)(n+\beta-\gamma-N-1)(n-\delta-N-1)}{(2n+\beta-N-1)(2n+\beta-N)} & \text{if} \quad \alpha+1=-N \\ \dfrac{n(n+\alpha+\beta+N+1)(n+\alpha+\beta-\gamma)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{n(n+\alpha+\beta+N+1)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} & \text{if} \quad \gamma+1=-N. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) - (A_n + C_n)p_n(x) + A_{n-1}C_n p_{n-1}(x), \tag{9.2.4}$$

where

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n} p_n(\lambda(x)).$$

Difference equation

$$n(n+\alpha+\beta+1)y(x) = B(x)y(x+1) - [B(x)+D(x)]\,y(x) + D(x)y(x-1), \tag{9.2.5}$$

where

$$y(x) = R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$\begin{cases} B(x) = \dfrac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \\ D(x) = \dfrac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta) - R_n(\lambda(x);\alpha,\beta,\gamma,\delta) \\ &= \frac{n(n+\alpha+\beta+1)}{(\alpha+1)(\beta+\delta+1)(\gamma+1)} \\ &\quad \times (2x+\gamma+\delta+2)R_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma+1,\delta) \end{aligned} \tag{9.2.6}$$

or equivalently

$$\begin{aligned} &\frac{\Delta R_n(\lambda(x);\alpha,\beta,\gamma,\delta)}{\Delta\lambda(x)} \\ &= \frac{n(n+\alpha+\beta+1)}{(\alpha+1)(\beta+\delta+1)(\gamma+1)} R_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma+1,\delta). \end{aligned} \tag{9.2.7}$$

Backward shift operator

$$\begin{aligned} &(x+\alpha)(x+\beta+\delta)(x+\gamma)(x+\gamma+\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta) \\ &\qquad - x(x-\beta+\gamma)(x-\alpha+\gamma+\delta)(x+\delta)R_n(\lambda(x-1);\alpha,\beta,\gamma,\delta) \\ &= \alpha\gamma(\beta+\delta)(2x+\gamma+\delta)R_{n+1}(\lambda(x);\alpha-1,\beta-1,\gamma-1,\delta) \end{aligned} \tag{9.2.8}$$

or equivalently

$$\begin{aligned} &\frac{\nabla[\omega(x;\alpha,\beta,\gamma,\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)]}{\nabla\lambda(x)} \\ &= \frac{1}{\gamma+\delta}\omega(x;\alpha-1,\beta-1,\gamma-1,\delta)R_{n+1}(\lambda(x);\alpha-1,\beta-1,\gamma-1,\delta), \end{aligned} \tag{9.2.9}$$

where

$$\omega(x;\alpha,\beta,\gamma,\delta) = \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x x!}.$$

Rodrigues-type formula

$$\omega(x;\alpha,\beta,\gamma,\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$
$$=(\gamma+\delta+1)_n\,(\nabla_\lambda)^n\left[\omega(x;\alpha+n,\beta+n,\gamma+n,\delta)\right], \tag{9.2.10}$$

where

$$\nabla_\lambda:=\frac{\nabla}{\nabla\lambda(x)}.$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$${}_2F_1\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right){}_2F_1\left(\begin{matrix}x+\beta+\delta+1,x+\gamma+1\\ \beta+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(\beta+\delta+1)_n(\gamma+1)_n}{(\beta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n, \tag{9.2.11}$$

$${}_2F_1\left(\begin{matrix}-x,-x+\beta-\gamma\\ \beta+\delta+1\end{matrix};t\right){}_2F_1\left(\begin{matrix}x+\alpha+1,x+\gamma+1\\ \alpha-\delta+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(\alpha+1)_n(\gamma+1)_n}{(\alpha-\delta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n, \tag{9.2.12}$$

$${}_2F_1\left(\begin{matrix}-x,-x-\delta\\ \gamma+1\end{matrix};t\right){}_2F_1\left(\begin{matrix}x+\alpha+1,x+\beta+\delta+1\\ \alpha+\beta-\gamma+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(\alpha+1)_n(\beta+\delta+1)_n}{(\alpha+\beta-\gamma+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n, \tag{9.2.13}$$

$$\left[(1-t)^{-\alpha-\beta-1}\right.$$
$$\left.\times\,{}_4F_3\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2),-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};-\frac{4t}{(1-t)^2}\right)\right]_N$$
$$=\sum_{n=0}^{N}\frac{(\alpha+\beta+1)_n}{n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n. \tag{9.2.14}$$

Limit relations

Racah → Hahn

The Hahn polynomials given by (9.5.1) can be obtained from the Racah polynomials by taking $\gamma+1=-N$ and letting $\delta\to\infty$:

$$\lim_{\delta\to\infty}R_n(\lambda(x);\alpha,\beta,-N-1,\delta)=Q_n(x;\alpha,\beta,N). \tag{9.2.15}$$

The Hahn polynomials given by (9.5.1) can also be obtained from the Racah polynomials by taking $\delta=-\beta-N-1$ and letting $\gamma\to\infty$:

$$\lim_{\gamma\to\infty}R_n(\lambda(x);\alpha,\beta,\gamma,-\beta-N-1)=Q_n(x;\alpha,\beta,N). \tag{9.2.16}$$

Another way to do this is to take $\alpha+1=-N$ and $\beta\to\beta+\gamma+N+1$ and then take the limit $\delta\to\infty$. In that case we obtain the Hahn polynomials given by (9.5.1) in the following way:

$$\lim_{\delta\to\infty}R_n(\lambda(x);-N-1,\beta+\gamma+N+1,\gamma,\delta)=Q_n(x;\gamma,\beta,N). \tag{9.2.17}$$

Racah → Dual Hahn

The dual Hahn polynomials given by (9.6.1) are obtained from the Racah polynomials if we take $\alpha+1=-N$ and let $\beta\to\infty$:

$$\lim_{\beta\to\infty} R_n(\lambda(x);-N-1,\beta,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N). \tag{9.2.18}$$

The dual Hahn polynomials given by (9.6.1) are also obtained from the Racah polynomials if we take $\beta=-\delta-N-1$ and let $\alpha\to\infty$:

$$\lim_{\alpha\to\infty} R_n(\lambda(x);\alpha,-\delta-N-1,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N). \tag{9.2.19}$$

Finally, the dual Hahn polynomials given by (9.6.1) are also obtained from the Racah polynomials if we take $\gamma+1=-N$ and $\delta\to\alpha+\delta+N+1$ and take the limit $\beta\to\infty$:

$$\lim_{\beta\to\infty} R_n(\lambda(x);\alpha,\beta,-N-1,\alpha+\delta+N+1)=R_n(\lambda(x);\alpha,\delta,N). \tag{9.2.20}$$

Remark

If we set $\alpha = a+b-1$, $\beta = c+d-1$, $\gamma = a+d-1$, $\delta = a-d$ and $x \to -a+ix$ in the definition (9.2.1) of the Racah polynomials we obtain the Wilson polynomials given by (9.1.1):

$$\begin{aligned}&R_n(\lambda(-a+ix);a+b-1,c+d-1,a+d-1,a-d)\\&=\tilde{W}_n(x^2;a,b,c,d)=\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}.\end{aligned}$$

Racah in terms of Wilson

In the Remark on p.196 Racah polynomials are expressed in terms of Wilson polynomials. This can be equivalently written as

$$\begin{aligned}&R_n\big(x(x-N+\delta);\alpha,\beta,-N-1,\delta\big)\\&=\frac{W_n\big(-(x+\frac{1}{2}(\delta-N))^2;\frac{1}{2}(\delta-N),\alpha+1-\frac{1}{2}(\delta-N),\beta+\frac{1}{2}(\delta+N)+1,-\frac{1}{2}(\delta+N)\big)}{(\alpha+1)_n(\beta+\delta+1)_n(-N)_n}.\end{aligned} \tag{9.2.21}$$

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.3 Continuous dual Hahn

Hypergeometric representation

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n}={}_3F_2\left(\begin{matrix}-n,a+ix,a-ix\\a+b,a+c\end{matrix};1\right). \tag{9.3.1}$$

Orthogonality relation

If a, b and c are positive except possibly for a pair of complex conjugates with positive real parts, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2;a,b,c)S_n(x^2;a,b,c)\,dx\\&=\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)n!\,\delta_{mn}.\end{aligned} \tag{9.3.2}$$

If $a<0$ and $a+b$, $a+c$ are positive or a pair of complex conjugates with positive real parts, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2;a,b,c)S_n(x^2;a,b,c)\,dx\\&\quad+\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b-a)\Gamma(c-a)}{\Gamma(-2a)}\\&\quad\times\sum_{\substack{k=0,1,2\ldots\\a+k<0}}\frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k}{(a)_k(a-b+1)_k(a-c+1)_kk!}(-1)^k\\&\qquad\times S_m(-(a+k)^2;a,b,c)S_n(-(a+k)^2;a,b,c)\\&=\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)n!\,\delta_{mn}.\end{aligned} \tag{9.3.3}$$

Recurrence relation

$$-\left(a^2+x^2\right)\tilde{S}_n(x^2)=A_n\tilde{S}_{n+1}(x^2)-(A_n+C_n)\,\tilde{S}_n(x^2)+C_n\tilde{S}_{n-1}(x^2), \tag{9.3.4}$$

where

$$\tilde{S}_n(x^2):=\tilde{S}_n(x^2;a,b,c)=\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n}$$

and

$$\begin{cases} A_n=(n+a+b)(n+a+c) \\ C_n=n(n+b+c-1). \end{cases}$$

Normalized recurrence relation

$$xp_n(x)=p_{n+1}(x)+(A_n+C_n-a^2)p_n(x)+A_{n-1}C_np_{n-1}(x), \tag{9.3.5}$$

where

$$S_n(x^2;a,b,c)=(-1)^np_n(x^2).$$

Difference equation

$$ny(x)=B(x)y(x+i)-[B(x)+D(x)]\,y(x)+D(x)y(x-i), \tag{9.3.6}$$

where

$$y(x)=S_n(x^2;a,b,c)$$

and

$$\begin{cases} B(x)=\dfrac{(a-ix)(b-ix)(c-ix)}{2ix(2ix-1)} \\ D(x)=\dfrac{(a+ix)(b+ix)(c+ix)}{2ix(2ix+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &S_n((x+\tfrac{1}{2}i)^2;a,b,c)-S_n((x-\tfrac{1}{2}i)^2;a,b,c) \\ &=-2inxS_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2}) \end{aligned} \tag{9.3.7}$$

or equivalently

$$\frac{\delta S_n(x^2;a,b,c)}{\delta x^2}=-nS_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2}). \tag{9.3.8}$$

Backward shift operator

$$\begin{aligned} &(a-\tfrac{1}{2}-ix)(b-\tfrac{1}{2}-ix)(c-\tfrac{1}{2}-ix)S_n((x+\tfrac{1}{2}i)^2;a,b,c) \\ &\quad-(a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)(c-\tfrac{1}{2}+ix)S_n((x-\tfrac{1}{2}i)^2;a,b,c) \\ &=-2ixS_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2}) \end{aligned} \tag{9.3.9}$$

or equivalently

$$\begin{aligned} &\frac{\delta\left[\omega(x;a,b,c)S_n(x^2;a,b,c)\right]}{\delta x^2} \\ &=\omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2})S_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2}), \end{aligned} \tag{9.3.10}$$

where

$$\omega(x;a,b,c)=\frac{1}{2ix}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2.$$

Rodrigues-type formula

$$\omega(x;a,b,c)S_n(x^2;a,b,c)=\left(\frac{\delta}{\delta x^2}\right)^n\left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n)\right]. \tag{9.3.11}$$

Generating functions

$$(1-t)^{-c+ix}\,{}_2F_1\left(\begin{matrix}a+ix,b+ix\\a+b\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+b)_n n!}t^n. \tag{9.3.12}$$

$$(1-t)^{-b+ix}\,{}_2F_1\left(\begin{matrix}a+ix,c+ix\\a+c\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+c)_n n!}t^n. \tag{9.3.13}$$

$$(1-t)^{-a+ix}\,{}_2F_1\left(\begin{matrix}b+ix,c+ix\\b+c\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(b+c)_n n!}t^n. \tag{9.3.14}$$

$$\mathrm{e}^t\,{}_2F_2\left(\begin{matrix}a+ix,a-ix\\a+b,a+c\end{matrix};-t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n n!}t^n. \tag{9.3.15}$$

$$\begin{aligned}&(1-t)^{-\gamma}\,{}_3F_2\left(\begin{matrix}\gamma,a+ix,a-ix\\a+b,a+c\end{matrix};\frac{t}{t-1}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n n!}t^n,\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.3.16}$$

Limit relations

Wilson → Continuous dual Hahn

The continuous dual Hahn polynomials can be found from the Wilson polynomials given by (9.1.1) by dividing by $(a+d)_n$ and letting $d\to\infty$:

$$\lim_{d\to\infty}\frac{W_n(x^2;a,b,c,d)}{(a+d)_n}=S_n(x^2;a,b,c).$$

Continuous dual Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials given by (9.7.1) can be obtained from the continuous dual Hahn polynomials by the substitutions $x\to x-t$, $a=\lambda+it$, $b=\lambda-it$ and $c=t\cot\phi$ and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{S_n((x-t)^2;\lambda+it,\lambda-it,t\cot\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\sin\phi)^n}. \tag{9.3.17}$$

Remark

Since we have for $k<n$

$$\frac{(a+b)_n(a+c)_n}{(a+b)_k(a+c)_k}=(a+b+k)_{n-k}(a+c+k)_{n-k},$$

the continuous dual Hahn polynomials defined by (9.3.1) can also be seen as polynomials in the parameters a, b and c.

Symmetry

The continuous dual Hahn polynomial $S_n(y;a,b,c)$ is symmetric in a,b,c.
This follows from the orthogonality relation (9.3.2) together with the value of its coefficient of y^n given in (9.3.5b). Alternatively, combine (9.3.1) with [AAR, Corollary 3.3.5].
As a consequence, it is sufficient to give generating function (9.3.12). Then the generating functions (9.3.13), (9.3.14) will follow by symmetry in the parameters.

Special value

$$S_n(-a^2;a,b,c) = (a+b)_n(a+c)_n\,, \tag{9.3.18}$$

and similarly for arguments $-b^2$ and $-c^2$ by symmetry of S_n in a,b,c.

Uniqueness of orthogonality measure

Under the assumptions on a,b,c for (9.3.2) or (9.3.3) the orthogonality measure is unique up to constant factor.

For the proof assume without loss of generality (by the symmetry in a,b,c,d) that $\mathrm{Re}\,a \geq 0$. Write the right-hand side of (9.3.2) or (9.3.3) as $h_n\delta_{m,n}$. Observe from (9.3.2) and (9.3.18) that

$$\frac{|S_n(-a^2;a,b,c)|^2}{h_n} = O(n^{2\mathrm{Re}\,a-1}) \quad \text{as } n \to \infty.$$

Therefore (**??**) holds, from which the uniqueness of the orthogonality measure follows.

References

[**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**].

9.4 Continuous Hahn

Hypergeometric representation

$$\begin{aligned} &p_n(x;a,b,c,d) \\ &= i^n \frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix}; 1\right). \end{aligned} \tag{9.4.1}$$

Orthogonality relation

If $Re(a,b,c,d) > 0$, $c = \bar{a}$ and $d = \bar{b}$, then

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)p_m(x;a,b,c,d)p_n(x;a,b,c,d)\,dx \\ &= \frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)n!}\,\delta_{mn}. \end{aligned} \tag{9.4.2}$$

Recurrence relation

$$(a+ix)\tilde{p}_n(x) = A_n\tilde{p}_{n+1}(x) - (A_n+C_n)\,\tilde{p}_n(x) + C_n\tilde{p}_{n-1}(x), \tag{9.4.3}$$

where

$$\tilde{p}_n(x) := \tilde{p}_n(x;a,b,c,d) = \frac{n!}{i^n(a+c)_n(a+d)_n}p_n(x;a,b,c,d)$$

and

$$\begin{cases} A_n = -\dfrac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \\ C_n = \dfrac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + i(A_n + C_n + a)p_n(x) - A_{n-1}C_n p_{n-1}(x), \tag{9.4.4}$$

where

$$p_n(x;a,b,c,d) = \frac{(n+a+b+c+d-1)_n}{n!} p_n(x).$$

Difference equation

$$\begin{aligned} & n(n+a+b+c+d-1)y(x) \\ & = B(x)y(x+i) - [B(x)+D(x)]\, y(x) + D(x)y(x-i), \end{aligned} \tag{9.4.5}$$

where

$$y(x) = p_n(x;a,b,c,d)$$

and

$$\begin{cases} B(x) = (c-ix)(d-ix) \\ \\ D(x) = (a+ix)(b+ix). \end{cases}$$

Forward shift operator

$$\begin{aligned} & p_n(x+\tfrac{1}{2}i;a,b,c,d) - p_n(x-\tfrac{1}{2}i;a,b,c,d) \\ & = i(n+a+b+c+d-1)p_{n-1}(x;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}) \end{aligned} \tag{9.4.6}$$

or equivalently

$$\frac{\delta p_n(x;a,b,c,d)}{\delta x} = (n+a+b+c+d-1)p_{n-1}(x;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}). \tag{9.4.7}$$

Backward shift operator

$$\begin{aligned} & (c-\tfrac{1}{2}-ix)(d-\tfrac{1}{2}-ix)p_n(x+\tfrac{1}{2}i;a,b,c,d) \\ & \quad - (a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)p_n(x-\tfrac{1}{2}i;a,b,c,d) \\ & = \frac{n+1}{i} p_{n+1}(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}) \end{aligned} \tag{9.4.8}$$

or equivalently

$$\begin{aligned} & \frac{\delta\,[\omega(x;a,b,c,d)p_n(x;a,b,c,d)]}{\delta x} \\ & = -(n+1)\omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}) \\ & \quad \times p_{n+1}(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}), \end{aligned} \tag{9.4.9}$$

where

$$\omega(x;a,b,c,d) = \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix).$$

Rodrigues-type formula

$$\begin{aligned} & \omega(x;a,b,c,d)p_n(x;a,b,c,d) \\ & = \frac{(-1)^n}{n!}\left(\frac{\delta}{\delta x}\right)^n \left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n,d+\tfrac{1}{2}n)\right]. \end{aligned} \tag{9.4.10}$$

Generating functions

$$ {}_1F_1\left(\begin{matrix} a+ix \\ a+c \end{matrix};-it\right) {}_1F_1\left(\begin{matrix} d-ix \\ b+d \end{matrix};it\right) = \sum_{n=0}^{\infty} \frac{p_n(x;a,b,c,d)}{(a+c)_n(b+d)_n} t^n. \tag{9.4.11} $$

$$ {}_1F_1\left(\begin{matrix} a+ix \\ a+d \end{matrix};-it\right) {}_1F_1\left(\begin{matrix} c-ix \\ b+c \end{matrix};it\right) = \sum_{n=0}^{\infty} \frac{p_n(x;a,b,c,d)}{(a+d)_n(b+c)_n} t^n. \tag{9.4.12} $$

$$ (1-t)^{1-a-b-c-d} {}_3F_2\left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix \\ a+c, a+d \end{matrix}; -\frac{4t}{(1-t)^2}\right) $$
$$ = \sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n}{(a+c)_n(a+d)_n i^n} p_n(x;a,b,c,d) t^n. \tag{9.4.13} $$

Limit relations

Wilson → Continuous Hahn

The continuous Hahn polynomials are obtained from the Wilson polynomials given by (9.1.1) by the substitution $a \to a-it$, $b \to b-it$, $c \to c+it$, $d \to d+it$ and $x \to x+t$ and the limit $t \to \infty$ in the following way:

$$ \lim_{t\to\infty} \frac{W_n((x+t)^2; a-it, b-it, c+it, d+it)}{(-2t)^n n!} = p_n(x;a,b,c,d). $$

Continuous Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials given by (9.7.1) can be obtained from the continuous Hahn polynomials by setting $x \to x+t$, $a = \lambda - it$, $c = \lambda + it$ and $b = d = t\tan\phi$ and taking the limit $t \to \infty$:

$$ \lim_{t\to\infty} \frac{p_n(x+t; \lambda-it, t\tan\phi, \lambda+it, t\tan\phi)}{t^n} = \frac{P_n^{(\lambda)}(x;\phi)}{(\cos\phi)^n}. \tag{9.4.14} $$

Continuous Hahn → Jacobi

The Jacobi polynomials given by (9.8.1) follow from the continuous Hahn polynomials by the substitution $x \to \frac{1}{2}xt$, $a = \frac{1}{2}(\alpha+1-it)$, $b = \frac{1}{2}(\beta+1+it)$, $c = \frac{1}{2}(\alpha+1+it)$ and $d = \frac{1}{2}(\beta+1-it)$, division by t^n and the limit $t \to \infty$:

$$ \lim_{t\to\infty} \frac{p_n(\frac{1}{2}xt; \frac{1}{2}(\alpha+1-it), \frac{1}{2}(\beta+1+it), \frac{1}{2}(\alpha+1+it), \frac{1}{2}(\beta+1-it))}{t^n} $$
$$ = P_n^{(\alpha,\beta)}(x). \tag{9.4.15} $$

Continuous Hahn → Pseudo Jacobi

The pseudo Jacobi polynomials given by (9.9.1) follow from the continuous Hahn polynomials by the substitution $x \to xt$, $a = \frac{1}{2}(-N+i\nu-2t)$, $b = \frac{1}{2}(-N-i\nu+2t)$, $c = \frac{1}{2}(-N-i\nu-2t)$ and $d = \frac{1}{2}(-N+i\nu+2t)$, division by t^n and the limit $t \to \infty$:

$$ \lim_{t\to\infty} \frac{p_n(xt; \frac{1}{2}(-N+i\nu-2t), \frac{1}{2}(-N-i\nu+2t), \frac{1}{2}(-N+i\nu-2t), \frac{1}{2}(-N-i\nu+2t))}{t^n} $$
$$ = \frac{(n-2N-1)_n}{n!} P_n(x;\nu,N). \tag{9.4.16} $$

Remark

Since we have for $k<n$

$$\frac{(a+b)_n(a+c)_n}{(a+b)_k(a+c)_k}=(a+b+k)_{n-k}(a+c+k)_{n-k},$$

the continuous Hahn polynomials defined by (9.4.1) can also be seen as polynomials in the parameters a, b and c.

Orthogonality relation and symmetry

The orthogonality relation (9.4.2) holds under the more general assumption that $\mathrm{Re}\,(a,b,c,d)>0$ and $(c,d)=(\overline{a},\overline{b})$ or $(\overline{b},\overline{a})$.
Thus, under these assumptions, the continuous Hahn polynomial $p_n(x;a,b,c,d)$ is symmetric in a,b and in c,d. This follows from the orthogonality relation (9.4.2) together with the value of its coefficient of x^n given in (9.4.4b).
As a consequence, it is sufficient to give generating function (9.4.11). Then the generating function (9.4.12) will follow by symmetry in the parameters.

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.4.4) behaves as $O(n^2)$ as $n\to\infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

Special cases

In the following special case there is a reduction to Meixner-Pollaczek:

$$p_n(x;a,a+\tfrac{1}{2},a,a+\tfrac{1}{2})=\frac{(2a)_n(2a+\frac{1}{2})_n}{(4a)_n}P_n^{(2a)}(2x;\tfrac{1}{2}\pi). \tag{9.4.17}$$

See [342, (2.6)] (note that in [342, (2.3)] the Meixner-Pollaczek polynonmials are defined different from (9.7.1), without a constant factor in front).

For $0<a<1$ the continuous Hahn polynomials $p_n(x;a,1-a,a,1-a)$ are orthogonal on $(-\infty,\infty)$ with respect to the weight function $\big(\cosh(2\pi x)-\cos(2\pi a)\big)^{-1}$ (by straightforward computation from (9.4.2)). For $a=\frac{1}{4}$ the two special cases coincide: Meixner-Pollaczek with weight function $\big(\cosh(2\pi x)\big)^{-1}$.

References

[**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**].

9.5 Hahn

Hypergeometric representation

$$Q_n(x;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,-N\end{matrix};1\right),\quad n=0,1,2,\ldots,N. \tag{9.5.1}$$

Orthogonality relation

For $\alpha>-1$ and $\beta>-1$, or for $\alpha<-N$ and $\beta<-N$, we have

$$\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_m(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)$$
$$=\frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}\,\delta_{mn}. \tag{9.5.2}$$

Recurrence relation

$$-xQ_n(x) = A_n Q_{n+1}(x) - (A_n + C_n)\, Q_n(x) + C_n Q_{n-1}(x), \tag{9.5.3}$$

where

$$Q_n(x) := Q_n(x;\alpha,\beta,N)$$

and

$$\begin{cases} A_n = \dfrac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\[2ex] C_n = \dfrac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n)\, p_n(x) + A_{n-1} C_n p_{n-1}(x), \tag{9.5.4}$$

where

$$Q_n(x;\alpha,\beta,N) = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N)_n} p_n(x).$$

Difference equation

$$n(n+\alpha+\beta+1)y(x) = B(x)y(x+1) - [B(x)+D(x)]\, y(x) + D(x)y(x-1), \tag{9.5.5}$$

where

$$y(x) = Q_n(x;\alpha,\beta,N)$$

and

$$\begin{cases} B(x) = (x+\alpha+1)(x-N) \\[2ex] D(x) = x(x-\beta-N-1). \end{cases}$$

Forward shift operator

$$\begin{aligned} &Q_n(x+1;\alpha,\beta,N) - Q_n(x;\alpha,\beta,N) \\ &= -\frac{n(n+\alpha+\beta+1)}{(\alpha+1)N} Q_{n-1}(x;\alpha+1,\beta+1,N-1) \end{aligned} \tag{9.5.6}$$

or equivalently

$$\Delta Q_n(x;\alpha,\beta,N) = -\frac{n(n+\alpha+\beta+1)}{(\alpha+1)N} Q_{n-1}(x;\alpha+1,\beta+1,N-1). \tag{9.5.7}$$

Backward shift operator

$$\begin{aligned} &(x+\alpha)(N+1-x)Q_n(x;\alpha,\beta,N) - x(\beta+N+1-x)Q_n(x-1;\alpha,\beta,N) \\ &= \alpha(N+1)Q_{n+1}(x;\alpha-1,\beta-1,N+1) \end{aligned} \tag{9.5.8}$$

or equivalently

$$\begin{aligned} &\nabla\left[\omega(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)\right] \\ &= \frac{N+1}{\beta}\omega(x;\alpha-1,\beta-1,N+1)Q_{n+1}(x;\alpha-1,\beta-1,N+1), \end{aligned} \tag{9.5.9}$$

where

$$\omega(x;\alpha,\beta,N) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}.$$

Rodrigues-type formula

$$\begin{aligned}&\omega(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)\\&=\frac{(-1)^n(\beta+1)_n}{(-N)_n}\nabla^n\left[\omega(x;\alpha+n,\beta+n,N-n)\right].\end{aligned}\tag{9.5.10}$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$${}_1F_1\left(\begin{matrix}-x\\\alpha+1\end{matrix};-t\right)\,{}_1F_1\left(\begin{matrix}x-N\\\beta+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{(\beta+1)_n n!}Q_n(x;\alpha,\beta,N)t^n.\tag{9.5.11}$$

$$\begin{aligned}&{}_2F_0\left(\begin{matrix}-x,-x+\beta+N+1\\-\end{matrix};-t\right)\,{}_2F_0\left(\begin{matrix}x-N,x+\alpha+1\\-\end{matrix};t\right)\\&=\sum_{n=0}^{N}\frac{(-N)_n(\alpha+1)_n}{n!}Q_n(x;\alpha,\beta,N)t^n.\end{aligned}\tag{9.5.12}$$

$$\begin{aligned}&\left[(1-t)^{-\alpha-\beta-1}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2),-x\\\alpha+1,-N\end{matrix};-\frac{4t}{(1-t)^2}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\alpha+\beta+1)_n}{n!}Q_n(x;\alpha,\beta,N)t^n.\end{aligned}\tag{9.5.13}$$

Limit relations

Racah → Hahn

If we take $\gamma+1=-N$ and let $\delta\to\infty$ in the definition (9.2.1) of the Racah polynomials, we obtain the Hahn polynomials. Hence

$$\lim_{\delta\to\infty}R_n(\lambda(x);\alpha,\beta,-N-1,\delta)=Q_n(x;\alpha,\beta,N).$$

And if we take $\delta=-\beta-N-1$ and let $\gamma\to\infty$ in the definition (9.2.1) of the Racah polynomials, we also obtain the Hahn polynomials:

$$\lim_{\gamma\to\infty}R_n(\lambda(x);\alpha,\beta,\gamma,-\beta-N-1)=Q_n(x;\alpha,\beta,N).$$

Another way to do this is to take $\alpha+1=-N$ and $\beta\to\beta+\gamma+N+1$ in the definition (9.2.1) of the Racah polynomials and then take the limit $\delta\to\infty$. In that case we obtain the Hahn polynomials in the following way:

$$\lim_{\delta\to\infty}R_n(\lambda(x);-N-1,\beta+\gamma+N+1,\gamma,\delta)=Q_n(x;\gamma,\beta,N).$$

Hahn → Jacobi

To find the Jacobi polynomials given by (9.8.1) from the Hahn polynomials we take $x\to Nx$ and let $N\to\infty$. In fact we have

$$\lim_{N\to\infty}Q_n(Nx;\alpha,\beta,N)=\frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.\tag{9.5.14}$$

Hahn → Meixner

The Meixner polynomials given by (9.10.1) can be obtained from the Hahn polynomials by taking $\alpha = b-1$, $\beta = N(1-c)c^{-1}$ and letting $N \to \infty$:

$$\lim_{N\to\infty} Q_n(x;b-1,N(1-c)c^{-1},N) = M_n(x;b,c). \tag{9.5.15}$$

Hahn → Krawtchouk

The Krawtchouk polynomials given by (9.11.1) are obtained from the Hahn polynomials if we take $\alpha = pt$ and $\beta = (1-p)t$ and let $t \to \infty$:

$$\lim_{t\to\infty} Q_n(x;pt,(1-p)t,N) = K_n(x;p,N). \tag{9.5.16}$$

Remark

If we interchange the role of x and n in (9.5.1) we obtain the dual Hahn polynomials given by (9.6.1).
Since

$$Q_n(x;\alpha,\beta,N) = R_x(\lambda(n);\alpha,\beta,N)$$

we obtain the dual orthogonality relation for the Hahn polynomials from the orthogonality relation (9.6.2) of the dual Hahn polynomials:

$$\sum_{n=0}^{N} \frac{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!} Q_n(x;\alpha,\beta,N)Q_n(y;\alpha,\beta,N)$$
$$= \frac{\delta_{xy}}{\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}}, \quad x,y \in \{0,1,2,\ldots,N\}.$$

Special values

$$Q_n(0;\alpha,\beta,N) = 1, \quad Q_n(N;\alpha,\beta,N) = \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}. \tag{9.5.17}$$

Use (9.5.1) and compare with (9.8.1) and (9.8.22).

From (9.5.3) and (**??**) it follows that

$$Q_{2n}(N;\alpha,\alpha,2N) = \frac{(\frac{1}{2})_n(N+\alpha+1)_n}{(-N+\frac{1}{2})_n(\alpha+1)_n}. \tag{9.5.18}$$

From (9.5.1) and [DLMF, (15.4.24)] it follows that

$$Q_N(x;\alpha,\beta,N) = \frac{(-N-\beta)_x}{(\alpha+1)_x} \qquad (x = 0,1,\ldots,N). \tag{9.5.19}$$

Symmetries

By the orthogonality relation (9.5.2):

$$\frac{Q_n(N-x;\alpha,\beta,N)}{Q_n(N;\alpha,\beta,N)} = Q_n(x;\beta,\alpha,N), \tag{9.5.20}$$

It follows from (9.6.21) and (9.5.22) that

$$\frac{Q_{N-n}(x;\alpha,\beta,N)}{Q_N(x;\alpha,\beta,N)} = Q_n(x;-N-\beta-1,-N-\alpha-1,N) \qquad (x = 0,1,\ldots,N). \tag{9.5.21}$$

Duality

The Remark on p.208 gives the duality between Hahn and dual Hahn polynomials:

$$Q_n(x;\alpha,\beta,N) = R_x(n(n+\alpha+\beta+1);\alpha,\beta,N) \quad (n,x \in \{0,1,\ldots N\}). \tag{9.5.22}$$

References

[?], [?].

9.6 Dual Hahn

Hypergeometric representation

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{array}{c} -n,-x,x+\gamma+\delta+1 \\ \gamma+1,-N \end{array}; 1\right), \quad n = 0,1,2,\ldots,N, \tag{9.6.1}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1).$$

Orthogonality relation

For $\gamma > -1$ and $\delta > -1$, or for $\gamma < -N$ and $\delta < -N$, we have

$$\sum_{x=0}^{N} \frac{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_xN!}{(-1)^x(x+\gamma+\delta+1)_{N+1}(\delta+1)_xx!} R_m(\lambda(x);\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)$$
$$= \frac{\delta_{mn}}{\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}}. \tag{9.6.2}$$

Recurrence relation

$$\lambda(x)R_n(\lambda(x)) = A_nR_{n+1}(\lambda(x)) - (A_n+C_n)R_n(\lambda(x)) + C_nR_{n-1}(\lambda(x)), \tag{9.6.3}$$

where

$$R_n(\lambda(x)) := R_n(\lambda(x);\gamma,\delta,N)$$

and

$$\begin{cases} A_n = (n+\gamma+1)(n-N) \\ \\ C_n = n(n-\delta-N-1). \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) - (A_n+C_n)p_n(x) + A_{n-1}C_np_{n-1}(x), \tag{9.6.4}$$

where

$$R_n(\lambda(x);\gamma,\delta,N) = \frac{1}{(\gamma+1)_n(-N)_n}p_n(\lambda(x)).$$

Difference equation

$$-ny(x)=B(x)y(x+1)-[B(x)+D(x)]\,y(x)+D(x)y(x-1), \tag{9.6.5}$$

where

$$y(x)=R_n(\lambda(x);\gamma,\delta,N)$$

and

$$\begin{cases} B(x)=\dfrac{(x+\gamma+1)(x+\gamma+\delta+1)(N-x)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \\ D(x)=\dfrac{x(x+\gamma+\delta+N+1)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &R_n(\lambda(x+1);\gamma,\delta,N)-R_n(\lambda(x);\gamma,\delta,N) \\ &\quad=-\frac{n(2x+\gamma+\delta+2)}{(\gamma+1)N}R_{n-1}(\lambda(x);\gamma+1,\delta,N-1) \end{aligned} \tag{9.6.6}$$

or equivalently

$$\frac{\Delta R_n(\lambda(x);\gamma,\delta,N)}{\Delta\lambda(x)}=-\frac{n}{(\gamma+1)N}R_{n-1}(\lambda(x);\gamma+1,\delta,N-1). \tag{9.6.7}$$

Backward shift operator

$$\begin{aligned} &(x+\gamma)(x+\gamma+\delta)(N+1-x)R_n(\lambda(x);\gamma,\delta,N) \\ &\qquad-x(x+\gamma+\delta+N+1)(x+\delta)R_n(\lambda(x-1);\gamma,\delta,N) \\ &\quad=\gamma(N+1)(2x+\gamma+\delta)R_{n+1}(\lambda(x);\gamma-1,\delta,N+1) \end{aligned} \tag{9.6.8}$$

or equivalently

$$\begin{aligned} &\frac{\nabla\left[\omega(x;\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)\right]}{\nabla\lambda(x)} \\ &=\frac{1}{\gamma+\delta}\omega(x;\gamma-1,\delta,N+1)R_{n+1}(\lambda(x);\gamma-1,\delta,N+1), \end{aligned} \tag{9.6.9}$$

where

$$\omega(x;\gamma,\delta,N)=\frac{(-1)^x(\gamma+1)_x(\gamma+\delta+1)_x(-N)_x}{(\gamma+\delta+N+2)_x(\delta+1)_x x!}.$$

Rodrigues-type formula

$$\omega(x;\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)=(\gamma+\delta+1)_n\left(\nabla_\lambda\right)^n\left[\omega(x;\gamma+n,\delta,N-n)\right], \tag{9.6.10}$$

where

$$\nabla_\lambda:=\frac{\nabla}{\nabla\lambda(x)}.$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$$(1-t)^{N-x}\,{}_2F_1\left(\begin{matrix}-x,-x-\delta\\ \gamma+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{n!}R_n(\lambda(x);\gamma,\delta,N)t^n. \tag{9.6.11}$$

$$\begin{aligned}&(1-t)^x\,{}_2F_1\left(\begin{matrix}x-N,x+\gamma+1\\ -\delta-N\end{matrix};t\right)\\&=\sum_{n=0}^{N}\frac{(\gamma+1)_n(-N)_n}{(-\delta-N)_n n!}R_n(\lambda(x);\gamma,\delta,N)t^n.\end{aligned} \tag{9.6.12}$$

$$\left[e^t\,{}_2F_2\left(\begin{matrix}-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix};-t\right)\right]_N=\sum_{n=0}^{N}\frac{R_n(\lambda(x);\gamma,\delta,N)}{n!}t^n. \tag{9.6.13}$$

$$\begin{aligned}&\left[(1-t)^{-\varepsilon}\,{}_3F_2\left(\begin{matrix}\varepsilon,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix};\frac{t}{t-1}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\varepsilon)_n}{n!}R_n(\lambda(x);\gamma,\delta,N)t^n,\quad \varepsilon\text{ arbitrary.}\end{aligned} \tag{9.6.14}$$

Limit relations

Racah → Dual Hahn

If we take $\alpha+1=-N$ and let $\beta\to\infty$ in the definition (9.2.1) of the Racah polynomials, then we obtain the dual Hahn polynomials:

$$\lim_{\beta\to\infty}R_n(\lambda(x);-N-1,\beta,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N).$$

And if we take $\beta=-\delta-N-1$ and let $\alpha\to\infty$ in the definition (9.2.1) of the Racah polynomials, then we also obtain the dual Hahn polynomials:

$$\lim_{\alpha\to\infty}R_n(\lambda(x);\alpha,-\delta-N-1,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N).$$

Finally, if we take $\gamma+1=-N$ and $\delta\to\alpha+\delta+N+1$ in the definition (9.2.1) of the Racah polynomials and take the limit $\beta\to\infty$ we find the dual Hahn polynomials in the following way:

$$\lim_{\beta\to\infty}R_n(\lambda(x);\alpha,\beta,-N-1,\alpha+\delta+N+1)=R_n(\lambda(x);\alpha,\delta,N).$$

Dual Hahn → Meixner

The Meixner polynomials given by (9.10.1) are obtained from the dual Hahn polynomials if we take $\gamma=\beta-1$ and $\delta=N(1-c)c^{-1}$ and let $N\to\infty$:

$$\lim_{N\to\infty}R_n(\lambda(x);\beta-1,N(1-c)c^{-1},N)=M_n(x;\beta,c). \tag{9.6.15}$$

Dual Hahn → Krawtchouk

The Krawtchouk polynomials given by (9.11.1) can be obtained from the dual Hahn polynomials by setting $\gamma = pt$, $\delta = (1-p)t$ and letting $t \to \infty$:

$$\lim_{t\to\infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N). \tag{9.6.16}$$

Remark

If we interchange the role of x and n in the definition (9.6.1) of the dual Hahn polynomials we obtain the Hahn polynomials given by (9.5.1).
Since

$$R_n(\lambda(x); \gamma, \delta, N) = Q_x(n; \gamma, \delta, N)$$

we obtain the dual orthogonality relation for the dual Hahn polynomials from the orthogonality relation (9.5.2) for the Hahn polynomials:

$$\begin{aligned}&\sum_{n=0}^{N} \binom{\gamma+n}{n}\binom{\delta+N-n}{N-n} R_n(\lambda(x);\gamma,\delta,N) R_n(\lambda(y);\gamma,\delta,N)\\ &= \frac{(-1)^x (x+\gamma+\delta+1)_{N+1}(\delta+1)_x x!}{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_x N!}\delta_{xy}, \quad x,y \in \{0,1,2,\ldots,N\}.\end{aligned}$$

Special values

By (9.5.19) and (9.5.22) we have

$$R_n(N(N+\gamma+\delta+1);\gamma,\delta,N) = \frac{(-N-\delta)_n}{(\gamma+1)_n}. \tag{9.6.17}$$

It follows from (9.5.17) and (9.5.22) that

$$R_N(x(x+\gamma+\delta+1);\gamma,\delta,N) = \frac{(-1)^x(\delta+1)_x}{(\gamma+1)_x} \qquad (x = 0,1,\ldots,N). \tag{9.6.18}$$

Symmetries

Write the weight in (9.6.2) as

$$w_x(\alpha,\beta,N) := N!\,\frac{2x+\gamma+\delta+1}{(x+\gamma+\delta+1)_{N+1}}\,\frac{(\gamma+1)_x}{(\delta+1)_x}\binom{N}{x}. \tag{9.6.19}$$

Then

$$(\delta+1)_N\, w_{N-x}(\gamma,\delta,N) = (-\gamma-N)_N\, w_x(-\delta-N-1,-\gamma-N-1,N). \tag{9.6.20}$$

Hence, by (9.6.2),

$$\frac{R_n((N-x)(N-x+\gamma+\delta+1);\gamma,\delta,N)}{R_n(N(N+\gamma+\delta+1);\gamma,\delta,N)} = R_n(x(x-2N-\gamma-\delta-1);-N-\delta-1,-N-\gamma-1,N). \tag{9.6.21}$$

Alternatively, (9.6.21) follows from (9.6.1) and [DLMF, (16.4.11)].
It follows from (9.5.20) and (9.5.22) that

$$\frac{R_{N-n}(x(x+\gamma+\delta+1);\gamma,\delta,N)}{R_N(x(x+\gamma+\delta+1);\gamma,\delta,N)} = R_n(x(x+\gamma+\delta+1);\delta,\gamma,N) \qquad (x = 0,1,\ldots,N). \tag{9.6.22}$$

Re: (9.6.11).

The generating function (9.6.11) can be written in a more conceptual way as

$$(1-t)^x \, {}_2F_1\left(\begin{matrix} x-N, x+\gamma+1 \\ -\delta-N \end{matrix}; t\right) = \frac{N!}{(\delta+1)_N} \sum_{n=0}^{N} \omega_n R_n(\lambda(x); \gamma, \delta, N) \, t^n, \tag{9.6.23}$$

where

$$\omega_n := \binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}, \tag{9.6.24}$$

i.e., the denominator on the right-hand side of (9.6.2). By the duality between Hahn polynomials and dual Hahn polynomials (see (9.5.22)) the above generating function can be rewritten in terms of Hahn polynomials:

$$(1-t)^n \, {}_2F_1\left(\begin{matrix} n-N, n+\alpha+1 \\ -\beta-N \end{matrix}; t\right) = \frac{N!}{(\beta+1)_N} \sum_{x=0}^{N} w_x Q_n(x; \alpha, \beta, N) \, t^x, \tag{9.6.25}$$

where

$$w_x := \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}, \tag{9.6.26}$$

i.e., the weight occurring in the orthogonality relation (9.5.2) for Hahn polynomials.

Re: (9.6.15).

There should be a closing bracket before the equality sign.

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.7 Meixner-Pollaczek

Hypergeometric representation

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} \, {}_2F_1\left(\begin{matrix} -n, \lambda+ix \\ 2\lambda \end{matrix}; 1-e^{-2i\phi}\right). \tag{9.7.1}$$

Orthogonality relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{(2\phi-\pi)x} |\Gamma(\lambda+ix)|^2 P_m^{(\lambda)}(x; \phi) P_n^{(\lambda)}(x; \phi) \, dx = \frac{\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda} n!} \delta_{mn}, \tag{9.7.2}$$

Recurrence relation

$$\begin{aligned} &(n+1) P_{n+1}^{(\lambda)}(x; \phi) - 2\left[x\sin\phi + (n+\lambda)\cos\phi\right] P_n^{(\lambda)}(x; \phi) \\ &\quad + (n+2\lambda-1) P_{n-1}^{(\lambda)}(x; \phi) = 0. \end{aligned} \tag{9.7.3}$$

Normalized recurrence relation

$$x p_n(x) = p_{n+1}(x) - \left(\frac{n+\lambda}{\tan\phi}\right) p_n(x) + \frac{n(n+2\lambda-1)}{4\sin^2\phi} p_{n-1}(x), \tag{9.7.4}$$

where

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\sin\phi)^n}{n!} p_n(x).$$

Difference equation

$$\begin{aligned}&\mathrm{e}^{i\phi}(\lambda - ix)y(x+i) + 2i\left[x\cos\phi - (n+\lambda)\sin\phi\right]y(x)\\&\quad - \mathrm{e}^{-i\phi}(\lambda + ix)y(x-i) = 0, \quad y(x) = P_n^{(\lambda)}(x;\phi).\end{aligned} \tag{9.7.5}$$

Forward shift operator

$$P_n^{(\lambda)}(x+\tfrac{1}{2}i;\phi) - P_n^{(\lambda)}(x-\tfrac{1}{2}i;\phi) = (e^{i\phi} - e^{-i\phi})P_{n-1}^{(\lambda+\frac{1}{2})}(x;\phi) \tag{9.7.6}$$

or equivalently

$$\frac{\delta P_n^{(\lambda)}(x;\phi)}{\delta x} = 2\sin\phi\, P_{n-1}^{(\lambda+\frac{1}{2})}(x;\phi). \tag{9.7.7}$$

Backward shift operator

$$\begin{aligned}&\mathrm{e}^{i\phi}(\lambda - \tfrac{1}{2} - ix)P_n^{(\lambda)}(x+\tfrac{1}{2}i;\phi) + \mathrm{e}^{-i\phi}(\lambda - \tfrac{1}{2} + ix)P_n^{(\lambda)}(x-\tfrac{1}{2}i;\phi)\\&= (n+1)P_{n+1}^{(\lambda-\frac{1}{2})}(x;\phi)\end{aligned} \tag{9.7.8}$$

or equivalently

$$\frac{\delta\left[\omega(x;\lambda,\phi)P_n^{(\lambda)}(x;\phi)\right]}{\delta x} = -(n+1)\omega(x;\lambda-\tfrac{1}{2},\phi)P_{n+1}^{(\lambda-\frac{1}{2})}(x;\phi), \tag{9.7.9}$$

where

$$\omega(x;\lambda,\phi) = \Gamma(\lambda+ix)\Gamma(\lambda-ix)\mathrm{e}^{(2\phi-\pi)x}.$$

Rodrigues-type formula

$$\omega(x;\lambda,\phi)P_n^{(\lambda)}(x;\phi) = \frac{(-1)^n}{n!}\left(\frac{\delta}{\delta x}\right)^n\left[\omega(x;\lambda+\tfrac{1}{2}n,\phi)\right]. \tag{9.7.10}$$

Generating functions

$$(1-e^{i\phi}t)^{-\lambda+ix}(1-e^{-i\phi}t)^{-\lambda-ix}=\sum_{n=0}^{\infty}P_n^{(\lambda)}(x;\phi)t^n. \tag{9.7.11}$$

$$\mathrm{e}^t\,{}_1F_1\left(\begin{matrix}\lambda+ix\\2\lambda\end{matrix};(e^{-2i\phi}-1)t\right)=\sum_{n=0}^{\infty}\frac{P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n e^{in\phi}}t^n. \tag{9.7.12}$$

$$\begin{aligned}&(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,\lambda+ix\\2\lambda\end{matrix};\frac{(1-\mathrm{e}^{-2i\phi})t}{t-1}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2\lambda)_n}\frac{P_n^{(\lambda)}(x;\phi)}{\mathrm{e}^{in\phi}}t^n,\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.7.13}$$

Limit relations

Continuous dual Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials can be obtained from the continuous dual Hahn polynomials given by (9.3.1) by the substitutions $x\to x-t$, $a=\lambda+it$, $b=\lambda-it$ and $c=t\cot\phi$ and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{S_n((x-t)^2;\lambda+it,\lambda-it,t\cot\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\sin\phi)^n}.$$

Continuous Hahn → Meixner-Pollaczek

By setting $x\to x+t$, $a=\lambda-it$, $c=\lambda+it$ and $b=d=t\tan\phi$ in the definition (9.4.1) of the continuous Hahn polynomials and taking the limit $t\to\infty$ we obtain the Meixner-Pollaczek polynomials:

$$\lim_{t\to\infty}\frac{p_n(x+t;\lambda-it,t\tan\phi,\lambda+it,t\tan\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\cos\phi)^n}.$$

Meixner-Pollaczek → Laguerre

The Laguerre polynomials given by (9.12.1) can be obtained from the Meixner-Pollaczek polynomials by the substitution $\lambda = \frac{1}{2}(\alpha+1)$, $x \to -\frac{1}{2}\phi^{-1}x$ and the limit $\phi \to 0$:

$$\lim_{\phi\to 0} P_n^{(\frac{1}{2}\alpha+\frac{1}{2})}(-\tfrac{1}{2}\phi^{-1}x;\phi) = L_n^{(\alpha)}(x). \tag{9.7.14}$$

Meixner-Pollaczek → Hermite

The Hermite polynomials given by (9.15.1) are obtained from the Meixner-Pollaczek polynomials if we substitute $x \to (\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi)$ and then let $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}n} P_n^{(\lambda)}((\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi);\phi) = \frac{H_n(x)}{n!}. \tag{9.7.15}$$

Remark

Since we have for $k < n$

$$\frac{(2\lambda)_n}{(2\lambda)_k} = (2\lambda+k)_{n-k},$$

the Meixner-Pollaczek polynomials defined by (9.7.1) can also be seen as polynomials in the parameter λ.

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.7.4) behaves as $O(n^2)$ as $n \to \infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

References

[?], [?].

9.8 Jacobi

Hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right). \tag{9.8.1}$$

Orthogonality relation

For $\alpha > -1$ and $\beta > -1$ we have

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,dx$$
$$= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\,\delta_{mn}. \tag{9.8.2}$$

For $\alpha+\beta < -2N-1$, $\beta > -1$ and $m,n \in \{0,1,2,\ldots,N\}$ we also have

$$\int_{1}^{\infty} (x+1)^{\alpha}(x-1)^{\beta} P_m^{(\alpha,\beta)}(-x) P_n^{(\alpha,\beta)}(-x)\,dx$$
$$= -\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(-n-\alpha-\beta)\Gamma(n+\alpha+\beta+1)}{\Gamma(-n-\alpha)n!}\,\delta_{mn}. \tag{9.8.3}$$

Recurrence relation

$$\begin{aligned}
xP_n^{(\alpha,\beta)}(x) &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x) \\
&\quad + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x) \\
&\quad + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned} \tag{9.8.4}$$

Normalized recurrence relation

$$\begin{aligned}
xp_n(x) &= p_{n+1}(x) + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}p_n(x) \\
&\quad + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}p_{n-1}(x)
\end{aligned} \tag{9.8.5}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!}p_n(x).$$

Differential equation

$$\begin{aligned}
&(1-x^2)y''(x) + [\beta-\alpha-(\alpha+\beta+2)x]\,y'(x) \\
&\quad + n(n+\alpha+\beta+1)y(x) = 0, \quad y(x) = P_n^{(\alpha,\beta)}(x).
\end{aligned} \tag{9.8.6}$$

Forward shift operator

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{9.8.7}$$

Backward shift operator

$$\begin{aligned}
&(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) + [(\beta-\alpha)-(\alpha+\beta)x]\,P_n^{(\alpha,\beta)}(x) \\
&= -2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(x)
\end{aligned} \tag{9.8.8}$$

or equivalently

$$\begin{aligned}
&\frac{d}{dx}\left[(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)\right] \\
&= -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1}P_{n+1}^{(\alpha-1,\beta-1)}(x).
\end{aligned} \tag{9.8.9}$$

Rodrigues-type formula

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{9.8.10}$$

Generating functions

$$\frac{2^{\alpha+\beta}}{R(1+R-t)^{\alpha}(1+R+t)^{\beta}}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n,\quad R=\sqrt{1-2xt+t^2}. \tag{9.8.11}$$

$$\begin{aligned}&{}_0F_1\left(\begin{array}{c}-\\ \alpha+1\end{array};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{array}{c}-\\ \beta+1\end{array};\frac{(x+1)t}{2}\right)\\ &=\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)}{(\alpha+1)_n(\beta+1)_n}t^n.\end{aligned} \tag{9.8.12}$$

$$\begin{aligned}&(1-t)^{-\alpha-\beta-1}{}_2F_1\left(\begin{array}{c}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2)\\ \alpha+1\end{array};\frac{2(x-1)t}{(1-t)^2}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x)t^n.\end{aligned} \tag{9.8.13}$$

$$\begin{aligned}&(1+t)^{-\alpha-\beta-1}{}_2F_1\left(\begin{array}{c}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2)\\ \beta+1\end{array};\frac{2(x+1)t}{(1+t)^2}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(x)t^n.\end{aligned} \tag{9.8.14}$$

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\gamma,\alpha+\beta+1-\gamma\\ \alpha+1\end{array};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{array}{c}\gamma,\alpha+\beta+1-\gamma\\ \beta+1\end{array};\frac{1-R+t}{2}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\alpha+\beta+1-\gamma)_n}{(\alpha+1)_n(\beta+1)_n}P_n^{(\alpha,\beta)}(x)t^n,\quad R=\sqrt{1-2xt+t^2}\end{aligned} \tag{9.8.15}$$

with γ arbitrary.

Limit relations

Wilson → Jacobi

The Jacobi polynomials can be found from the Wilson polynomials given by (9.1.1) by substituting $a=b=\frac{1}{2}(\alpha+1)$, $c=\frac{1}{2}(\beta+1)+it$, $d=\frac{1}{2}(\beta+1)-it$ and $x\to t\sqrt{\frac{1}{2}(1-x)}$ in the definition (9.1.1) of the Wilson polynomials and taking the limit $t\to\infty$. In fact we have

$$\lim_{t\to\infty}\frac{W_n(\frac{1}{2}(1-x)t^2;\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta+1)+it,\frac{1}{2}(\beta+1)-it)}{t^{2n}n!}=P_n^{(\alpha,\beta)}(x).$$

Continuous Hahn → Jacobi

The Jacobi polynomials follow from the continuous Hahn polynomials given by (9.4.1) by using the substitution $x\to\frac{1}{2}xt$, $a=\frac{1}{2}(\alpha+1-it)$, $b=\frac{1}{2}(\beta+1+it)$, $c=\frac{1}{2}(\alpha+1+it)$ and $d=\frac{1}{2}(\beta+1-it)$ in (9.4.1), division by t^n and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{p_n(\frac{1}{2}xt;\frac{1}{2}(\alpha+1-it),\frac{1}{2}(\beta+1+it),\frac{1}{2}(\alpha+1+it),\frac{1}{2}(\beta+1-it))}{t^n}=P_n^{(\alpha,\beta)}(x).$$

Hahn → Jacobi

To find the Jacobi polynomials from the Hahn polynomials given by (9.5.1) we take $x \to Nx$ in (9.5.1) and let $N \to \infty$. In fact we have

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

Jacobi → Laguerre

The Laguerre polynomials given by (9.12.1) can be obtained from the Jacobi polynomials by setting $x \to 1-2\beta^{-1}x$ and then the limit $\beta \to \infty$:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x) = L_n^{(\alpha)}(x). \tag{9.8.16}$$

Jacobi → Bessel

The Bessel polynomials given by (9.13.1) are obtained from the Jacobi polynomials if we take $\beta = a-\alpha$ and let $\alpha \to -\infty$:

$$\lim_{\alpha\to-\infty} \frac{P_n^{(\alpha,a-\alpha)}(1+\alpha x)}{P_n^{(\alpha,a-\alpha)}(1)} = y_n(x;a). \tag{9.8.17}$$

Jacobi → Hermite

The Hermite polynomials given by (9.15.1) follow from the Jacobi polynomials by taking $\beta = \alpha$ and letting $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}}x) = \frac{H_n(x)}{2^n n!}. \tag{9.8.18}$$

Remarks

The definition (9.8.1) of the Jacobi polynomials can also be written as:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!}(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k}\left(\frac{1-x}{2}\right)^k.$$

In this way the Jacobi polynomials can also be seen as polynomials in the parameters α and β. Therefore they can be defined for all α and β. Then we have the following connection with the Meixner polynomials given by (9.10.1):

$$\frac{(\beta)_n}{n!} M_n(x;\beta,c) = P_n^{(\beta-1,-n-\beta-x)}((2-c)c^{-1}).$$

The Jacobi polynomials are related to the pseudo Jacobi polynomials defined by (9.9.1) in the following way:

$$P_n(x;\nu,N) = \frac{(-2i)^n n!}{(n-2N-1)_n} P_n^{(-N-1+i\nu,-N-1-i\nu)}(ix).$$

The Jacobi polynomials are also related to the Gegenbauer (or ultraspherical) polynomials given by (9.8.35) by the quadratic transformations:

$$C_{2n}^{(\lambda)}(x) = \frac{(\lambda)_n}{(\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

and

$$C_{2n+1}^{(\lambda)}(x) = \frac{(\lambda)_{n+1}}{(\frac{1}{2})_{n+1}} x P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1).$$

Orthogonality relation

Write the right-hand side of (9.8.2) as $h_n\,\delta_{m,n}$. Then

$$\begin{aligned}
&\frac{h_n}{h_0}=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_n\,n!}\,,\quad h_0=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,,\\
&\frac{h_n}{h_0\,(P_n^{(\alpha,\beta)}(1))^2}=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\beta+1)_n\,n!}{(\alpha+1)_n\,(\alpha+\beta+2)_n}\,.
\end{aligned}\tag{9.8.19}$$

In (9.8.3) the numerator factor $\Gamma(n+\alpha+\beta+1)$ in the last line should be $\Gamma(\beta+1)$. When thus corrected, (9.8.3) can be rewritten as:

$$\begin{aligned}
&\int_1^\infty P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)\,(x-1)^\alpha(x+1)^\beta\,dx=h_n\,\delta_{m,n}\,,\\
&\qquad\qquad -1-\beta>\alpha>-1,\quad m,n<-\tfrac12(\alpha+\beta+1),\\
&\frac{h_n}{h_0}=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_n\,n!}\,,\quad h_0=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(-\alpha-\beta-1)}{\Gamma(-\beta)}\,.
\end{aligned}\tag{9.8.20}$$

Symmetry

$$P_n^{(\alpha,\beta)}(-x)=(-1)^n\,P_n^{(\beta,\alpha)}(x).\tag{9.8.21}$$

Use (9.8.2) and (9.8.5b) or see [DLMF, Table 18.6.1].

Special values

$$P_n^{(\alpha,\beta)}(1)=\frac{(\alpha+1)_n}{n!}\,,\quad P_n^{(\alpha,\beta)}(-1)=\frac{(-1)^n(\beta+1)_n}{n!}\,,\quad \frac{P_n^{(\alpha,\beta)}(-1)}{P_n^{(\alpha,\beta)}(1)}=\frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}\,.\tag{9.8.22}$$

Use (9.8.1) and (9.8.21) or see [DLMF, Table 18.6.1].

Generating functions

Formula (9.8.15) was first obtained by Brafman [109].

Bilateral generating functions

For $0\le r<1$ and $x,y\in[-1,1]$ we have in terms of F_4 (see (**??**)):

$$\begin{aligned}
\sum_{n=0}^\infty \frac{(\alpha+\beta+1)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)&=\frac{1}{(1+r)^{\alpha+\beta+1}}\\
\times F_4\Big(\tfrac12(\alpha+\beta+1),\tfrac12(\alpha+\beta+2);\alpha+1,\beta+1;&\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big),
\end{aligned}\tag{9.8.23}$$

$$\begin{aligned}
\sum_{n=0}^\infty \frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}\,\frac{(\alpha+\beta+2)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)&=\frac{1-r}{(1+r)^{\alpha+\beta+2}}\\
\times F_4\Big(\tfrac12(\alpha+\beta+2),\tfrac12(\alpha+\beta+3);\alpha+1,\beta+1;&\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big).
\end{aligned}\tag{9.8.24}$$

Formulas (9.8.23) and (9.8.24) were first given by Bailey [91, (2.1), (2.3)]. See Stanton [485] for a shorter proof. (However, in the second line of [485, (1)] z and Z should be interchanged.) As observed in Bailey [91, p.10], (9.8.24) follows from (9.8.23) by applying the operator $r^{-\frac12(\alpha+\beta-1)}\,\frac{d}{dr}\circ r^{\frac12(\alpha+\beta+1)}$ to both sides of (9.8.23). In view of (9.8.19), formula (9.8.24) is the Poisson kernel for Jacobi polynomials. The right-hand side of (9.8.24) makes clear that this kernel is positive. See also the discussion in Askey [46, following (2.32)].

Quadratic transformations

$$\frac{C_{2n}^{(\alpha+\frac{1}{2})}(x)}{C_{2n}^{(\alpha+\frac{1}{2})}(1)}=\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}=\frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}\,, \tag{9.8.25}$$

$$\frac{C_{2n+1}^{(\alpha+\frac{1}{2})}(x)}{C_{2n+1}^{(\alpha+\frac{1}{2})}(1)}=\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)}=\frac{x\,P_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)}\,. \tag{9.8.26}$$

See p.221, Remarks, last two formulas together with (9.8.22) and (9.8.53). Or see [DLMF, (18.7.13), (18.7.14)].

Differentiation formulas

Each differentiation formula is given in two equivalent forms.

$$\begin{gathered}\frac{d}{dx}\left((1-x)^\alpha P_n^{(\alpha,\beta)}(x)\right)=-(n+\alpha)\,(1-x)^{\alpha-1}P_n^{(\alpha-1,\beta+1)}(x),\\ \left((1-x)\frac{d}{dx}-\alpha\right)P_n^{(\alpha,\beta)}(x)=-(n+\alpha)\,P_n^{(\alpha-1,\beta+1)}(x).\end{gathered} \tag{9.8.27}$$

$$\begin{gathered}\frac{d}{dx}\left((1+x)^\beta P_n^{(\alpha,\beta)}(x)\right)=(n+\beta)\,(1+x)^{\beta-1}P_n^{(\alpha+1,\beta-1)}(x),\\ \left((1+x)\frac{d}{dx}+\beta\right)P_n^{(\alpha,\beta)}(x)=(n+\beta)\,P_n^{(\alpha+1,\beta-1)}(x).\end{gathered} \tag{9.8.28}$$

Formulas (9.8.27) and (9.8.28) follow from [DLMF, (15.5.4), (15.5.6)] together with (9.8.1). They also follow from each other by (9.8.21).

Generalized Gegenbauer polynomials

These are defined by

$$S_{2m}^{(\alpha,\beta)}(x):=\text{const.}\,P_m^{(\alpha,\beta)}(2x^2-1),\qquad S_{2m+1}^{(\alpha,\beta)}(x):=\text{const.}\,x\,P_m^{(\alpha,\beta+1)}(2x^2-1) \tag{9.8.29}$$

in the notation of [146, p.156] (see also [**?**]), while [**?**, Section 1.5.2] has $C_n^{(\lambda,\mu)}(x)=\text{const.}$ $\times\,S_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(x)$. For $\alpha,\beta>-1$ we have the orthogonality relation

$$\int_{-1}^1 S_m^{(\alpha,\beta)}(x)\,S_n^{(\alpha,\beta)}(x)\,|x|^{2\beta+1}(1-x^2)^\alpha\,dx=0\qquad(m\neq n). \tag{9.8.30}$$

For $\beta=\alpha-1$ generalized Gegenbauer polynomials are limit cases of continuous q-ultraspherical polynomials, see (**??**).

If we define the *Dunkl operator* T_μ by

$$(T_\mu f)(x):=f'(x)+\mu\,\frac{f(x)-f(-x)}{x} \tag{9.8.31}$$

and if we choose the constants in (9.8.29) as

$$S_{2m}^{(\alpha,\beta)}(x)=\frac{(\alpha+\beta+1)_m}{(\beta+1)_m}\,P_m^{(\alpha,\beta)}(2x^2-1),\quad S_{2m+1}^{(\alpha,\beta)}(x)=\frac{(\alpha+\beta+1)_{m+1}}{(\beta+1)_{m+1}}\,x\,P_m^{(\alpha,\beta+1)}(2x^2-1) \tag{9.8.32}$$

then (see [**?**, (1.6)])

$$T_{\beta+\frac{1}{2}}S_n^{(\alpha,\beta)}=2(\alpha+\beta+1)\,S_{n-1}^{(\alpha+1,\beta)}. \tag{9.8.33}$$

Formula (9.8.33) with (9.8.32) substituted gives rise to two differentiation formulas involving Jacobi polynomials which are equivalent to (9.8.7) and (9.8.28).

Composition of (9.8.33) with itself gives

$$T^2_{\beta+\frac{1}{2}} S_n^{(\alpha,\beta)} = 4(\alpha+\beta+1)(\alpha+\beta+2)\, S_{n-2}^{(\alpha+2,\beta)},$$

which is equivalent to the composition of (9.8.7) and (9.8.28):

$$\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(2x^2-1) = 4(n+\alpha+\beta+1)(n+\beta)\, P_{n-1}^{(\alpha+2,\beta)}(2x^2-1). \quad (9.8.34)$$

Formula (9.8.34) was also given in [322, (2.4)].

References

[?], [?].

Special cases

9.8.1 Gegenbauer / Ultraspherical

Hypergeometric representation

The Gegenbauer (or ultraspherical) polynomials are Jacobi polynomials with $\alpha=\beta=\lambda-\frac{1}{2}$ and another normalization:

$$\begin{aligned} C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) \\ &= \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda+\frac{1}{2} \end{matrix}; \frac{1-x}{2}\right), \quad \lambda \neq 0. \end{aligned} \quad (9.8.35)$$

Orthogonality relation

$$\begin{aligned} &\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x)\, dx \\ &= \frac{\pi\Gamma(n+2\lambda)2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2 (n+\lambda) n!}\,\delta_{mn}, \quad \lambda > -\frac{1}{2} \quad \lambda \neq 0. \end{aligned} \quad (9.8.36)$$

Recurrence relation

$$2(n+\lambda)xC_n^{(\lambda)}(x) = (n+1)C_{n+1}^{(\lambda)}(x) + (n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \quad (9.8.37)$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n(n+2\lambda-1)}{4(n+\lambda-1)(n+\lambda)} p_{n-1}(x), \quad (9.8.38)$$

where

$$C_n^{(\lambda)}(x) = \frac{2^n(\lambda)_n}{n!} p_n(x).$$

Differential equation

$$(1-x^2)y''(x)-(2\lambda+1)xy'(x)+n(n+2\lambda)y(x)=0, \quad y(x)=C_n^{(\lambda)}(x). \tag{9.8.39}$$

Forward shift operator

$$\frac{d}{dx}C_n^{(\lambda)}(x)=2\lambda C_{n-1}^{(\lambda+1)}(x). \tag{9.8.40}$$

Backward shift operator

$$(1-x^2)\frac{d}{dx}C_n^{(\lambda)}(x)+(1-2\lambda)xC_n^{(\lambda)}(x)=-\frac{(n+1)(2\lambda+n-1)}{2(\lambda-1)}C_{n+1}^{(\lambda-1)}(x) \tag{9.8.41}$$

or equivalently

$$\begin{aligned}&\frac{d}{dx}\left[(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)\right]\\&=-\frac{(n+1)(2\lambda+n-1)}{2(\lambda-1)}(1-x^2)^{\lambda-\frac{3}{2}}C_{n+1}^{(\lambda-1)}(x).\end{aligned} \tag{9.8.42}$$

Rodrigues-type formula

$$(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)=\frac{(2\lambda)_n(-1)^n}{(\lambda+\frac{1}{2})_n 2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x^2)^{\lambda+n-\frac{1}{2}}\right]. \tag{9.8.43}$$

Generating functions

$$(1-2xt+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n. \tag{9.8.44}$$

$$R^{-1}\left(\frac{1+R-xt}{2}\right)^{\frac{1}{2}-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n}C_n^{(\lambda)}(x)t^n, \quad R=\sqrt{1-2xt+t^2}. \tag{9.8.45}$$

$${}_0F_1\left(\begin{matrix}-\\ \lambda+\frac{1}{2}\end{matrix};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{matrix}-\\ \lambda+\frac{1}{2}\end{matrix};\frac{(x+1)t}{2}\right)=\sum_{n=0}^{\infty}\frac{C_n^{(\lambda)}(x)}{(2\lambda)_n(\lambda+\frac{1}{2})_n}t^n. \tag{9.8.46}$$

$$\mathrm{e}^{xt}\,{}_0F_1\left(\begin{matrix}-\\ \lambda+\frac{1}{2}\end{matrix};\frac{(x^2-1)t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{C_n^{(\lambda)}(x)}{(2\lambda)_n}t^n. \tag{9.8.47}$$

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}\gamma,2\lambda-\gamma\\ \lambda+\frac{1}{2}\end{matrix};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{matrix}\gamma,2\lambda-\gamma\\ \lambda+\frac{1}{2}\end{matrix};\frac{1-R+t}{2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(2\lambda-\gamma)_n}{(2\lambda)_n(\lambda+\frac{1}{2})_n}C_n^{(\lambda)}(x)t^n, \quad R=\sqrt{1-2xt+t^2}, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.48}$$

$$\begin{aligned}&(1-xt)^{-\gamma}{}_2F_1\left(\begin{matrix}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \lambda+\frac{1}{2}\end{matrix};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2\lambda)_n}C_n^{(\lambda)}(x)t^n, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.49}$$

Limit relation

Gegenbauer / Ultraspherical → Hermite

The Hermite polynomials given by (9.15.1) follow from the Gegenbauer (or ultraspherical) polynomials by taking $\lambda = \alpha + \frac{1}{2}$ and letting $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} C_n^{(\alpha+\frac{1}{2})}(\alpha^{-\frac{1}{2}}x) = \frac{H_n(x)}{n!}. \tag{9.8.50}$$

Remarks

The case $\lambda = 0$ needs another normalization. In that case we have the Chebyshev polynomials of the first kind described in the next subsection.
The Gegenbauer (or ultraspherical) polynomials are related to the Jacobi polynomials given by (9.8.1) by the quadratic transformations:

$$C_{2n}^{(\lambda)}(x) = \frac{(\lambda)_n}{(\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

and

$$C_{2n+1}^{(\lambda)}(x) = \frac{(\lambda)_{n+1}}{(\frac{1}{2})_{n+1}} x P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1).$$

Notation

Here the Gegenbauer polynomial is denoted by C_n^λ instead of $C_n^{(\lambda)}$.

Orthogonality relation

Write the right-hand side of (9.8.20) as $h_n\,\delta_{m,n}$. Then

$$\frac{h_n}{h_0} = \frac{\lambda}{\lambda+n}\,\frac{(2\lambda)_n}{n!}, \quad h_0 = \frac{\pi^{\frac{1}{2}}\,\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}, \quad \frac{h_n}{h_0\,(C_n^\lambda(1))^2} = \frac{\lambda}{\lambda+n}\,\frac{n!}{(2\lambda)_n}. \tag{9.8.51}$$

Hypergeometric representation

Beside (9.8.19) we have also

$$C_n^\lambda(x) = \sum_{\ell=0}^{\lfloor n/2\rfloor} \frac{(-1)^\ell(\lambda)_{n-\ell}}{\ell!\,(n-2\ell)!}(2x)^{n-2\ell} = (2x)^n\,\frac{(\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}\\ 1-\lambda-n\end{matrix}; \frac{1}{x^2}\right). \tag{9.8.52}$$

See [DLMF, (18.5.10)].

Special value

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}. \tag{9.8.53}$$

Use (9.8.19) or see [DLMF, Table 18.6.1].

Expression in terms of Jacobi

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \frac{P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)}{P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(1)}, \qquad C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \tag{9.8.54}$$

Re: (9.8.21)

By iteration of recurrence relation (9.8.21):

$$x^2C_n^\lambda(x)=\frac{(n+1)(n+2)}{4(n+\lambda)(n+\lambda+1)}C_{n+2}^\lambda(x)+\frac{n^2+2n\lambda+\lambda-1}{2(n+\lambda-1)(n+\lambda+1)}C_n^\lambda(x)\\ +\frac{(n+2\lambda-1)(n+2\lambda-2)}{4(n+\lambda)(n+\lambda-1)}C_{n-2}^\lambda(x). \quad (9.8.55)$$

Bilateral generating functions

$$\sum_{n=0}^\infty\frac{n!}{(2\lambda)_n}r^nC_n^\lambda(x)C_n^\lambda(y)=\frac{1}{(1-2rxy+r^2)^\lambda}\,{}_2F_1\left(\begin{matrix}\frac12\lambda,\frac12(\lambda+1)\\ \lambda+\frac12\end{matrix};\frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right)\\ (r\in(-1,1),\ x,y\in[-1,1]). \quad (9.8.56)$$

For the proof put $\beta:=\alpha$ in (9.8.23), then use (**??**) and (9.8.54). The Poisson kernel for Gegenbauer polynomials can be derived in a similar way from (9.8.24), or alternatively by applying the operator $r^{-\lambda+1}\frac{d}{dr}\circ r^\lambda$ to both sides of (9.8.56):

$$\sum_{n=0}^\infty\frac{\lambda+n}{\lambda}\frac{n!}{(2\lambda)_n}r^nC_n^\lambda(x)C_n^\lambda(y)=\frac{1-r^2}{(1-2rxy+r^2)^{\lambda+1}}\\ \times{}_2F_1\left(\begin{matrix}\frac12(\lambda+1),\frac12(\lambda+2)\\ \lambda+\frac12\end{matrix};\frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right)\qquad(r\in(-1,1),\ x,y\in[-1,1]). \quad (9.8.57)$$

Formula (9.8.57) was obtained by Gasper & Rahman [234, (4.4)] as a limit case of their formula for the Poisson kernel for continuous q-ultraspherical polynomials.

Trigonometric expansions

By [DLMF, (18.5.11), (15.8.1)]:

$$C_n^\lambda(\cos\theta)=\sum_{k=0}^n\frac{(\lambda)_k(\lambda)_{n-k}}{k!\,(n-k)!}e^{i(n-2k)\theta}=e^{in\theta}\frac{(\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\lambda\\ 1-\lambda-n\end{matrix};e^{-2i\theta}\right) \quad (9.8.58)$$

$$=\frac{(\lambda)_n}{2^\lambda n!}e^{-\frac12 i\lambda\pi}e^{i(n+\lambda)\theta}(\sin\theta)^{-\lambda}\,{}_2F_1\left(\begin{matrix}\lambda,1-\lambda\\ 1-\lambda-n\end{matrix};\frac{ie^{-i\theta}}{2\sin\theta}\right) \quad (9.8.59)$$

$$=\frac{(\lambda)_n}{n!}\sum_{k=0}^\infty\frac{(\lambda)_k(1-\lambda)_k}{(1-\lambda-n)_kk!}\frac{\cos((n-k+\lambda)\theta+\frac12(k-\lambda)\pi)}{(2\sin\theta)^{k+\lambda}}. \quad (9.8.60)$$

In (9.8.59) and (9.8.60) we require that $\frac16\pi<\theta<\frac56\pi$. Then the convergence is absolute for $\lambda>\frac12$ and conditional for $0<\lambda\le\frac12$.

By [DLMF, (14.13.1), (14.3.21), (15.8.1)]]:

$$C_n^\lambda(\cos\theta)=\frac{2\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{1-2\lambda}\sum_{k=0}^\infty\frac{(1-\lambda)_k(n+1)_k}{(n+\lambda+1)_k k!}\sin\big((2k+n+1)\theta\big) \tag{9.8.61}$$

$$=\frac{2\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{1-2\lambda}\,\mathrm{Im}\bigg(e^{i(n+1)\theta}\,{}_2F_1\left({1-\lambda,n+1 \atop n+\lambda+1};e^{2i\theta}\right)\bigg)$$

$$=\frac{2^\lambda\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{-\lambda}\,\mathrm{Re}\bigg(e^{-\frac12 i\lambda\pi}e^{i(n+\lambda)\theta}\,{}_2F_1\left({\lambda,1-\lambda \atop 1+\lambda+n};\frac{e^{i\theta}}{2i\sin\theta}\right)\bigg)$$

$$=\frac{2^{2\lambda}\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\sum_{k=0}^\infty\frac{(\lambda)_k(1-\lambda)_k}{(1+\lambda+n)_k k!}\,\frac{\cos((n+k+\lambda)\theta-\frac12(k+\lambda)\pi)}{(2\sin\theta)^{k+\lambda}}. \tag{9.8.62}$$

We require that $0<\theta<\pi$ in (9.8.61) and $\frac16\pi<\theta<\frac56\pi$ in (9.8.62) The convergence is absolute for $\lambda>\frac12$ and conditional for $0<\lambda\le\frac12$. For $\lambda\in\mathbb{Z}_{>0}$ the above series terminate after the term with $k=\lambda-1$. Formulas (9.8.61) and (9.8.62) are also given in [Sz, (4.9.22), (4.9.25)].

Fourier transform

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac12)\Gamma(\frac12)}\int_{-1}^1\frac{C_n^\lambda(y)}{C_n^\lambda(1)}(1-y^2)^{\lambda-\frac12}e^{ixy}\,dy=i^n\,2^\lambda\,\Gamma(\lambda+1)\,x^{-\lambda}\,J_{\lambda+n}(x). \tag{9.8.63}$$

See [DLMF, (18.17.17) and (18.17.18)].

Laplace transforms

$$\frac{2}{n!\,\Gamma(\lambda)}\int_0^\infty H_n(tx)\,t^{n+2\lambda-1}\,e^{-t^2}\,dt=C_n^\lambda(x). \tag{9.8.64}$$

See Nielsen [**?**, p.48, (4) with p.47, (1) and p.28, (10)] (1918) or Feldheim [**?**, (28)] (1942).

$$\frac{2}{\Gamma(\lambda+\frac12)}\int_0^1\frac{C_n^\lambda(t)}{C_n^\lambda(1)}(1-t^2)^{\lambda-\frac12}\,t^{-1}\,(x/t)^{n+2\lambda+1}\,e^{-x^2/t^2}\,dt=2^{-n}H_n(x)\,e^{-x^2}\quad(\lambda>-\tfrac12). \tag{9.8.65}$$

Use Askey & Fitch [**?**, (3.29)] for $\alpha=\pm\frac12$ together with (9.8.21), (9.8.25), (9.8.26), (9.14.17) and (9.14.18).

Addition formula

(see [AAR, (9.8.5′)]])

$$R_n^{(\alpha,\alpha)}\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)=\sum_{k=0}^n\frac{(-1)^k(-n)_k\,(n+2\alpha+1)_k}{2^{2k}((\alpha+1)_k)^2}$$

$$\times(1-x^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-y^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(y)\,\omega_k^{(\alpha-\frac12,\alpha-\frac12)}\,R_k^{(\alpha-\frac12,\alpha-\frac12)}(t), \tag{9.8.66}$$

where

$$R_n^{(\alpha,\beta)}(x):=P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1),\quad \omega_n^{(\alpha,\beta)}:=\frac{\int_{-1}^1(1-x)^\alpha(1+x)^\beta\,dx}{\int_{-1}^1(R_n^{(\alpha,\beta)}(x))^2\,(1-x)^\alpha(1+x)^\beta\,dx}.$$

References

[**?**], [**?**].

9.8.2 Chebyshev

Hypergeometric representation

The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = -\frac{1}{2}$:

$$T_n(x) = \frac{P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)}{P_n^{(-\frac{1}{2},-\frac{1}{2})}(1)} = {}_2F_1\left(\begin{matrix}-n,n\\ \frac{1}{2}\end{matrix}; \frac{1-x}{2}\right) \tag{9.8.67}$$

and the Chebyshev polynomials of the second kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = \frac{1}{2}$:

$$U_n(x) = (n+1)\frac{P_n^{(\frac{1}{2},\frac{1}{2})}(x)}{P_n^{(\frac{1}{2},\frac{1}{2})}(1)} = (n+1)\,{}_2F_1\left(\begin{matrix}-n,n+2\\ \frac{3}{2}\end{matrix}; \frac{1-x}{2}\right). \tag{9.8.68}$$

Orthogonality relation

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_m(x)T_n(x)\,dx = \begin{cases}\dfrac{\pi}{2}\,\delta_{mn},\ n\neq 0\\[2ex] \pi\,\delta_{mn},\ \ n=0.\end{cases} \tag{9.8.69}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_m(x)U_n(x)\,dx = \frac{\pi}{2}\,\delta_{mn}. \tag{9.8.70}$$

Recurrence relations

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x), \quad T_0(x) = 1 \quad \text{and} \quad T_1(x) = x. \tag{9.8.71}$$

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_0(x) = 1 \quad \text{and} \quad U_1(x) = 2x. \tag{9.8.72}$$

Normalized recurrence relations

$$xp_n(x) = p_{n+1}(x) + \frac{1}{4}p_{n-1}(x), \tag{9.8.73}$$

where

$$T_1(x) = p_1(x) = x \quad \text{and} \quad T_n(x) = 2^n p_n(x), \quad n \neq 1.$$

$$xp_n(x) = p_{n+1}(x) + \frac{1}{4}p_{n-1}(x), \tag{9.8.74}$$

where

$$U_n(x) = 2^n p_n(x).$$

Differential equations

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0, \quad y(x) = T_n(x). \tag{9.8.75}$$

$$(1-x^2)y''(x) - 3xy'(x) + n(n+2)y(x) = 0, \quad y(x) = U_n(x). \tag{9.8.76}$$

Forward shift operator

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \tag{9.8.77}$$

Backward shift operator

$$(1-x^2)\frac{d}{dx}U_n(x)-xU_n(x)=-(n+1)T_{n+1}(x) \tag{9.8.78}$$

or equivalently

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}}U_n(x)\right]=-(n+1)\left(1-x^2\right)^{-\frac{1}{2}}T_{n+1}(x). \tag{9.8.79}$$

Rodrigues-type formulas

$$(1-x^2)^{-\frac{1}{2}}T_n(x)=\frac{(-1)^n}{(\frac{1}{2})_n 2^n}\left(\frac{d}{dx}\right)^n\left[(1-x^2)^{n-\frac{1}{2}}\right]. \tag{9.8.80}$$

$$(1-x^2)^{\frac{1}{2}}U_n(x)=\frac{(n+1)(-1)^n}{(\frac{3}{2})_n 2^n}\left(\frac{d}{dx}\right)^n\left[(1-x^2)^{n+\frac{1}{2}}\right]. \tag{9.8.81}$$

Generating functions

$$\frac{1-xt}{1-2xt+t^2}=\sum_{n=0}^{\infty}T_n(x)t^n. \tag{9.8.82}$$

$$R^{-1}\sqrt{\frac{1}{2}(1+R-xt)}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}T_n(x)t^n, \quad R=\sqrt{1-2xt+t^2}. \tag{9.8.83}$$

$${}_0F_1\left(\begin{array}{c}-\\ \frac{1}{2}\end{array};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{array}{c}-\\ \frac{1}{2}\end{array};\frac{(x+1)t}{2}\right)=\sum_{n=0}^{\infty}\frac{T_n(x)}{\left(\frac{1}{2}\right)_n n!}t^n. \tag{9.8.84}$$

$$\mathrm{e}^{xt}\,{}_0F_1\left(\begin{array}{c}-\\ \frac{1}{2}\end{array};\frac{(x^2-1)t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n. \tag{9.8.85}$$

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\gamma,-\gamma\\ \frac{1}{2}\end{array};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{array}{c}\gamma,-\gamma\\ \frac{1}{2}\end{array};\frac{1-R+t}{2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(-\gamma)_n}{\left(\frac{1}{2}\right)_n n!}T_n(x)t^n, \quad R=\sqrt{1-2xt+t^2}, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.86}$$

$$\begin{aligned}&(1-xt)^{-\gamma}{}_2F_1\left(\begin{array}{c}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \frac{1}{2}\end{array};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}T_n(x)t^n, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.87}$$

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n. \tag{9.8.88}$$

$$\frac{1}{R\sqrt{\frac{1}{2}(1+R-xt)}}=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(n+1)!}U_n(x)t^n, \quad R=\sqrt{1-2xt+t^2}. \tag{9.8.89}$$

$${}_0F_1\left(\begin{array}{c}-\\ \frac{3}{2}\end{array};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{array}{c}-\\ \frac{3}{2}\end{array};\frac{(x+1)t}{2}\right)=\sum_{n=0}^{\infty}\frac{U_n(x)}{\left(\frac{3}{2}\right)_n (n+1)!}t^n. \tag{9.8.90}$$

$$e^{xt}\,{}_0F_1\left(\begin{array}{c}-\\ \frac{3}{2}\end{array};\frac{(x^2-1)t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{U_n(x)}{(n+1)!}t^n. \tag{9.8.91}$$

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\gamma,2-\gamma\\ \frac{3}{2}\end{array};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{array}{c}\gamma,2-\gamma\\ \frac{3}{2}\end{array};\frac{1-R+t}{2}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\gamma)_n(2-\gamma)_n}{\left(\frac{3}{2}\right)_n(n+1)!}U_n(x)t^n,\quad R=\sqrt{1-2xt+t^2},\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.8.92}$$

$$\begin{aligned}&(1-xt)^{-\gamma}{}_2F_1\left(\begin{array}{c}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \frac{3}{2}\end{array};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(n+1)!}U_n(x)t^n,\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.8.93}$$

Remarks

The Chebyshev polynomials can also be written as:

$$T_n(x)=\cos(n\theta),\quad x=\cos\theta$$

and

$$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\quad x=\cos\theta.$$

Further we have

$$U_n(x)=C_n^{(1)}(x)$$

where $C_n^{(\lambda)}(x)$ denotes the Gegenbauer (or ultraspherical) polynomial given by (9.8.35) in the preceding subsection. In addition to the Chebyshev polynomials T_n of the first kind (9.8.35) and U_n of the second kind (9.8.36),

$$T_n(x):=\frac{P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)}{P_n^{(-\frac{1}{2},-\frac{1}{2})}(1)}=\cos(n\theta),\quad x=\cos\theta, \tag{9.8.94}$$

$$U_n(x):=(n+1)\,\frac{P_n^{(\frac{1}{2},\frac{1}{2})}(x)}{P_n^{(\frac{1}{2},\frac{1}{2})}(1)}=\frac{\sin((n+1)\theta)}{\sin\theta},\quad x=\cos\theta, \tag{9.8.95}$$

we have Chebyshev polynomials V_n *of the third kind* and W_n *of the fourth kind*,

$$V_n(x):=\frac{P_n^{(-\frac{1}{2},\frac{1}{2})}(x)}{P_n^{(-\frac{1}{2},\frac{1}{2})}(1)}=\frac{\cos((n+\frac{1}{2})\theta)}{\cos(\frac{1}{2}\theta)},\quad x=\cos\theta, \tag{9.8.96}$$

$$W_n(x):=(2n+1)\,\frac{P_n^{(\frac{1}{2},-\frac{1}{2})}(x)}{P_n^{(\frac{1}{2},-\frac{1}{2})}(1)}=\frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)},\quad x=\cos\theta, \tag{9.8.97}$$

see [**?**, Section 1.2.3]. Then there is the symmetry

$$V_n(-x)=(-1)^nW_n(x). \tag{9.8.98}$$

The names of Chebyshev polynomials of the third and fourth kind and the notation $V_n(x)$ are due to Gautschi [**?**]. The notation $W_n(x)$ was first used by Mason [**?**]. Names and notations for Chebyshev polynomials of the third and fourth kind are interchanged in [AAR, Remark 2.5.3] and [DLMF, Table 18.3.1].

References

[?], [?].

9.8.3 Legendre / Spherical

Hypergeometric representation

The Legendre (or spherical) polynomials are Jacobi polynomials with $\alpha = \beta = 0$:

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix}; \frac{1-x}{2}\right). \tag{9.8.99}$$

Orthogonality relation

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{mn}. \tag{9.8.100}$$

Recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x). \tag{9.8.101}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n^2}{(2n-1)(2n+1)} p_{n-1}(x), \tag{9.8.102}$$

where

$$P_n(x) = \binom{2n}{n} \frac{1}{2^n} p_n(x).$$

Differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0, \quad y(x) = P_n(x). \tag{9.8.103}$$

Rodrigues-type formula

$$P_n(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n \left[(1-x^2)^n\right]. \tag{9.8.104}$$

Generating functions

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{9.8.105}$$

$${}_0F_1\left(\begin{matrix} - \\ 1 \end{matrix}; \frac{(x-1)t}{2}\right) {}_0F_1\left(\begin{matrix} - \\ 1 \end{matrix}; \frac{(x+1)t}{2}\right) = \sum_{n=0}^{\infty} \frac{P_n(x)}{(n!)^2} t^n. \tag{9.8.106}$$

$$\mathrm{e}^{xt}\, {}_0F_1\left(\begin{matrix} - \\ 1 \end{matrix}; \frac{(x^2-1)t^2}{4}\right) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n. \tag{9.8.107}$$

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} \gamma, 1-\gamma \\ 1 \end{matrix}; \frac{1-R-t}{2}\right) {}_2F_1\left(\begin{matrix} \gamma, 1-\gamma \\ 1 \end{matrix}; \frac{1-R+t}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_n (1-\gamma)_n}{(n!)^2} P_n(x) t^n, \quad R = \sqrt{1-2xt+t^2}, \quad \gamma \text{ arbitrary}. \end{aligned} \tag{9.8.108}$$

$$(1-xt)^{-\gamma}{}_2F_1\left(\begin{array}{c}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ 1\end{array};\frac{(x^2-1)t^2}{(1-xt)^2}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}P_n(x)t^n,\quad \gamma \text{ arbitrary.} \tag{9.8.109}$$

References

[?], [?].

9.9 Pseudo Jacobi

Hypergeometric representation

$$\begin{aligned} P_n(x;\nu,N) &= \frac{(-2i)^n(-N+i\nu)_n}{(n-2N-1)_n}\,{}_2F_1\left(\begin{matrix}-n,n-2N-1\\ -N+i\nu\end{matrix};\frac{1-ix}{2}\right) \\ &= (x+i)^n\,{}_2F_1\left(\begin{matrix}-n,N+1-n-i\nu\\ 2N+2-2n\end{matrix};\frac{2}{1-ix}\right), \quad n=0,1,2,\ldots,N. \end{aligned} \tag{9.9.1}$$

Orthogonality relation

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}P_m(x;\nu,N)P_n(x;\nu,N)\,dx \\ &= \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)\,|\Gamma(N+1-n+i\nu)|^2}\,\delta_{mn}. \end{aligned} \tag{9.9.2}$$

Recurrence relation

$$\begin{aligned} xP_n(x;\nu,N) &= P_{n+1}(x;\nu,N)+\frac{(N+1)\nu}{(n-N-1)(n-N)}P_n(x;\nu,N) \\ &\quad -\frac{n(n-2N-2)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)} \\ &\quad \times(n-N-1-i\nu)(n-N-1+i\nu)P_{n-1}(x;\nu,N). \end{aligned} \tag{9.9.3}$$

Normalized recurrence relation

$$\begin{aligned} xp_n(x) &= p_{n+1}(x)+\frac{(N+1)\nu}{(n-N-1)(n-N)}p_n(x) \\ &\quad -\frac{n(n-2N-2)(n-N-1-i\nu)(n-N-1+i\nu)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)}p_{n-1}(x), \end{aligned} \tag{9.9.4}$$

where

$$P_n(x;\nu,N)=p_n(x).$$

Differential equation

$$(1+x^2)y''(x)+2\,(\nu-Nx)\,y'(x)-n(n-2N-1)y(x)=0, \tag{9.9.5}$$

where

$$y(x)=P_n(x;\nu,N).$$

Forward shift operator

$$\frac{d}{dx}P_n(x;\nu,N)=nP_{n-1}(x;\nu,N-1). \tag{9.9.6}$$

Backward shift operator

$$
\begin{aligned}
&(1+x^2)\frac{d}{dx}P_n(x;\nu,N)+2\left[\nu-(N+1)x\right]P_n(x;\nu,N)\\
&=(n-2N-2)P_{n+1}(x;\nu,N+1) \qquad (9.9.7)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\frac{d}{dx}\left[(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}P_n(x;\nu,N)\right]\\
&=(n-2N-2)(1+x^2)^{-N-2}\mathrm{e}^{2\nu\arctan x}P_{n+1}(x;\nu,N+1). \qquad (9.9.8)
\end{aligned}
$$

Rodrigues-type formula

$$
P_n(x;\nu,N)=\frac{(1+x^2)^{N+1}e^{-2\nu\arctan x}}{(n-2N-1)_n}\left(\frac{d}{dx}\right)^n\left[(1+x^2)^{n-N-1}e^{2\nu\arctan x}\right]. \qquad (9.9.9)
$$

Generating function

$$
\begin{aligned}
&\left[{}_0F_1\left(\begin{array}{c}-\\-N+i\nu\end{array};(x+i)t\right){}_0F_1\left(\begin{array}{c}-\\-N-i\nu\end{array};(x-i)t\right)\right]_N\\
&=\sum_{n=0}^{N}\frac{(n-2N-1)_n}{(-N+i\nu)_n(-N-i\nu)_n n!}P_n(x;\nu,N)t^n. \qquad (9.9.10)
\end{aligned}
$$

Limit relation

Continuous Hahn → Pseudo Jacobi

The pseudo Jacobi polynomials follow from the continuous Hahn polynomials given by (9.4.1) by the substitutions $x\to xt$, $a=\frac{1}{2}(-N+i\nu-2t)$, $b=\frac{1}{2}(-N-i\nu+2t)$, $c=\frac{1}{2}(-N-i\nu-2t)$ and $d=\frac{1}{2}(-N+i\nu+2t)$, division by t^n and the limit $t\to\infty$:

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{p_n(xt;\frac{1}{2}(-N+i\nu-2t),\frac{1}{2}(-N-i\nu+2t),\frac{1}{2}(-N+i\nu-2t),\frac{1}{2}(-N-i\nu+2t))}{t^n}\\
&=\frac{(n-2N-1)_n}{n!}P_n(x;\nu,N).
\end{aligned}
$$

Remarks

Since we have for $k<n$

$$
\frac{(-N+i\nu)_n}{(-N+i\nu)_k}=(-N+i\nu+k)_{n-k},
$$

the pseudo Jacobi polynomials given by (9.9.1) can also be seen as polynomials in the parameter ν.

The weight function for the pseudo Jacobi polynomials can be written as

$$
(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}=(1+ix)^{-N-1-i\nu}(1-ix)^{-N-1+i\nu}.
$$

The pseudo Jacobi polynomials are related to the Jacobi polynomials defined by (9.8.1) in the following way:

$$
P_n(x;\nu,N)=\frac{(-2i)^n n!}{(n-2N-1)_n}P_n^{(-N-1+i\nu,-N-1-i\nu)}(ix).
$$

If we set $x\to\nu x$ in the definition (9.9.1) of the pseudo Jacobi polynomials and take the limit $\nu\to\infty$ we obtain a special case of the Bessel polynomials given by (9.13.1) in the following way:

$$
\lim_{\nu\to\infty}\frac{P_n(\nu x;\nu,N)}{\nu^n}=\frac{2^n}{(n-2N-1)_n}y_n(x;-2N-2).
$$

References

[?], [?], [?].

9.10 Meixner

Hypergeometric representation

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1-\frac{1}{c}\right). \tag{9.10.1}$$

Orthogonality relation

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_m(x;\beta,c) M_n(x;\beta,c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^{\beta}}\, \delta_{mn} \tag{9.10.2}$$

Recurrence relation

$$\begin{aligned}(c-1)xM_n(x;\beta,c) &= c(n+\beta)M_{n+1}(x;\beta,c) \\ &\quad - [n+(n+\beta)c]\, M_n(x;\beta,c) + nM_{n-1}(x;\beta,c).\end{aligned} \tag{9.10.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n+(n+\beta)c}{1-c} p_n(x) + \frac{n(n+\beta-1)c}{(1-c)^2} p_{n-1}(x), \tag{9.10.4}$$

where

$$M_n(x;\beta,c) = \frac{1}{(\beta)_n}\left(\frac{c-1}{c}\right)^n p_n(x).$$

Difference equation

$$n(c-1)y(x) = c(x+\beta)y(x+1) - [x+(x+\beta)c]\, y(x) + xy(x-1), \tag{9.10.5}$$

where

$$y(x) = M_n(x;\beta,c).$$

Forward shift operator

$$M_n(x+1;\beta,c) - M_n(x;\beta,c) = \frac{n}{\beta}\left(\frac{c-1}{c}\right) M_{n-1}(x;\beta+1,c) \tag{9.10.6}$$

or equivalently

$$\Delta M_n(x;\beta,c) = \frac{n}{\beta}\left(\frac{c-1}{c}\right) M_{n-1}(x;\beta+1,c). \tag{9.10.7}$$

Backward shift operator

$$c(\beta+x-1)M_n(x;\beta,c) - xM_n(x-1;\beta,c) = c(\beta-1)M_{n+1}(x;\beta-1,c) \tag{9.10.8}$$

or equivalently

$$\nabla\left[\frac{(\beta)_x c^x}{x!} M_n(x;\beta,c)\right] = \frac{(\beta-1)_x c^x}{x!} M_{n+1}(x;\beta-1,c). \tag{9.10.9}$$

Rodrigues-type formula

$$\frac{(\beta)_x c^x}{x!} M_n(x;\beta,c) = \nabla^n \left[\frac{(\beta+n)_x c^x}{x!}\right]. \tag{9.10.10}$$

Generating functions

$$\left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x;\beta,c)t^n. \tag{9.10.11}$$

$$\mathrm{e}^t \, {}_1F_1\left(\begin{matrix} -x \\ \beta \end{matrix}; \left(\frac{1-c}{c}\right)t\right) = \sum_{n=0}^{\infty} \frac{M_n(x;\beta,c)}{n!} t^n. \tag{9.10.12}$$

$$(1-t)^{-\gamma} \, {}_2F_1\left(\begin{matrix} \gamma, -x \\ \beta \end{matrix}; \frac{(1-c)t}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} M_n(x;\beta,c)t^n, \quad \gamma \text{ arbitrary}. \tag{9.10.13}$$

Limit relations

Hahn → Meixner

If we take $\alpha = b-1$, $\beta = N(1-c)c^{-1}$ in the definition (9.5.1) of the Hahn polynomials and let $N \to \infty$ we find the Meixner polynomials:

$$\lim_{N\to\infty} Q_n(x;b-1,N(1-c)c^{-1},N) = M_n(x;b,c).$$

Dual Hahn → Meixner

To obtain the Meixner polynomials from the dual Hahn polynomials we have to take $\gamma = \beta - 1$ and $\delta = N(1-c)c^{-1}$ in the definition (9.6.1) of the dual Hahn polynomials and let $N \to \infty$:

$$\lim_{N\to\infty} R_n(\lambda(x);\beta-1,N(1-c)c^{-1},N) = M_n(x;\beta,c).$$

Meixner → Laguerre

The Laguerre polynomials given by (9.12.1) are obtained from the Meixner polynomials if we take $\beta = \alpha+1$ and $x \to (1-c)^{-1}x$ and let $c \to 1$:

$$\lim_{c\to 1} M_n((1-c)^{-1}x;\alpha+1,c) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}. \tag{9.10.14}$$

Meixner → Charlier

The Charlier polynomials given by (9.14.1) are obtained from the Meixner polynomials if we take $c = (a+\beta)^{-1}a$ and let $\beta \to \infty$:

$$\lim_{\beta\to\infty} M_n(x;\beta,(a+\beta)^{-1}a) = C_n(x;a). \tag{9.10.15}$$

Remarks

The Meixner polynomials are related to the Jacobi polynomials given by (9.8.1) in the following way:

$$\frac{(\beta)_n}{n!}M_n(x;\beta,c) = P_n^{(\beta-1,-n-\beta-x)}((2-c)c^{-1}).$$

The Meixner polynomials are also related to the Krawtchouk polynomials given by (9.11.1) in the following way:

$$K_n(x;p,N) = M_n(x;-N,(p-1)^{-1}p).$$

In this section in [KLS] the pseudo Jacobi polynomial $P_n(x;\nu,N)$ in (9.9.1) is considered for $N\in\mathbb{Z}_{\geq 0}$ and $n=0,1,\ldots,n$. However, we can more generally take $-\frac{1}{2}<N\in\mathbb{R}$ (so here I overrule my convention formulated in the beginning of this paper), N_0 integer such that $N-\frac{1}{2}\leq N_0<N+\frac{1}{2}$, and $n=0,1,\ldots,N_0$ (see [382, §5, case A.4]). The orthogonality relation (9.9.2) is valid for $m,n=0,1,\ldots,N_0$.

History

These polynomials were first obtained by Routh [**?**] in 1885, and later, independently, by Romanovski [463] in 1929.

Limit relation:

Pseudo big q-Jacobi $\longrightarrow$ Pseudo Jacobi
See also (**??**).

References

See also [Ism, §20.1], [51], [384], [**?**], [**?**], [**?**].

References

[**?**], [**?**].

9.11 Krawtchouk

Hypergeometric representation

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};\frac{1}{p}\right),\quad n=0,1,2,\ldots,N. \tag{9.11.1}$$

Orthogonality relation

$$\begin{aligned}&\sum_{x=0}^{N}\binom{N}{x}p^x(1-p)^{N-x}K_m(x;p,N)K_n(x;p,N)\\&=\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{mn},\quad 0<p<1.\end{aligned} \tag{9.11.2}$$

Recurrence relation

$$
\begin{aligned}
-xK_n(x;p,N) = p(N-n)K_{n+1}(x;p,N)& \\
-\left[p(N-n)+n(1-p)\right]K_n(x;p,N)& \\
+n(1-p)K_{n-1}(x;p,N).&
\end{aligned}
\tag{9.11.3}
$$

Normalized recurrence relation

$$
\begin{aligned}
xp_n(x) = p_{n+1}(x) + \left[p(N-n)+n(1-p)\right]p_n(x)& \\
+np(1-p)(N+1-n)p_{n-1}(x),&
\end{aligned}
\tag{9.11.4}
$$

where

$$K_n(x;p,N) = \frac{1}{(-N)_n p^n} p_n(x).$$

Difference equation

$$
\begin{aligned}
-ny(x) = p(N-x)y(x+1)& \\
-\left[p(N-x)+x(1-p)\right]y(x)+x(1-p)y(x-1),&
\end{aligned}
\tag{9.11.5}
$$

where

$$y(x) = K_n(x;p,N).$$

Forward shift operator

$$K_n(x+1;p,N) - K_n(x;p,N) = -\frac{n}{Np}K_{n-1}(x;p,N-1) \tag{9.11.6}$$

or equivalently

$$\Delta K_n(x;p,N) = -\frac{n}{Np}K_{n-1}(x;p,N-1). \tag{9.11.7}$$

Backward shift operator

$$
\begin{aligned}
&(N+1-x)K_n(x;p,N) - x\left(\frac{1-p}{p}\right)K_n(x-1;p,N) \\
&= (N+1)K_{n+1}(x;p,N+1)
\end{aligned}
\tag{9.11.8}
$$

or equivalently

$$\nabla\left[\binom{N}{x}\left(\frac{p}{1-p}\right)^x K_n(x;p,N)\right] = \binom{N+1}{x}\left(\frac{p}{1-p}\right)^x K_{n+1}(x;p,N+1). \tag{9.11.9}$$

Rodrigues-type formula

$$\binom{N}{x}\left(\frac{p}{1-p}\right)^x K_n(x;p,N) = \nabla^n\left[\binom{N-n}{x}\left(\frac{p}{1-p}\right)^x\right]. \tag{9.11.10}$$

Generating functions

For $x = 0, 1, 2, \ldots, N$ we have

$$\left(1 - \frac{(1-p)}{p}t\right)^x (1+t)^{N-x} = \sum_{n=0}^{N} \binom{N}{n} K_n(x;p,N)t^n. \tag{9.11.11}$$

$$\left[e^t \, {}_1F_1\left(\begin{matrix} -x \\ -N \end{matrix}; -\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{K_n(x;p,N)}{n!} t^n. \tag{9.11.12}$$

$$\left[(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix} \gamma, -x \\ -N \end{matrix}; \frac{t}{p(t-1)}\right)\right]_N$$
$$= \sum_{n=0}^{N} \frac{(\gamma)_n}{n!} K_n(x;p,N)t^n, \quad \gamma \text{ arbitrary}. \tag{9.11.13}$$

Limit relations

Hahn → Krawtchouk

If we take $\alpha = pt$ and $\beta = (1-p)t$ in the definition (9.5.1) of the Hahn polynomials and let $t \to \infty$ we obtain the Krawtchouk polynomials:

$$\lim_{t\to\infty} Q_n(x; pt, (1-p)t, N) = K_n(x; p, N).$$

Dual Hahn → Krawtchouk

The Krawtchouk polynomials follow from the dual Hahn polynomials given by (9.6.1) if we set $\gamma = pt$, $\delta = (1-p)t$ and let $t \to \infty$:

$$\lim_{t\to\infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N).$$

Krawtchouk → Charlier

The Charlier polynomials given by (9.14.1) can be found from the Krawtchouk polynomials by taking $p = N^{-1}a$ and letting $N \to \infty$:

$$\lim_{N\to\infty} K_n(x; N^{-1}a, N) = C_n(x; a). \tag{9.11.14}$$

Krawtchouk → Hermite

The Hermite polynomials given by (9.15.1) follow from the Krawtchouk polynomials by setting $x \to pN + x\sqrt{2p(1-p)N}$ and then letting $N \to \infty$:

$$\lim_{N\to\infty} \sqrt{\binom{N}{n}} K_n(pN + x\sqrt{2p(1-p)N}; p, N) = \frac{(-1)^n H_n(x)}{\sqrt{2^n n! \left(\frac{p}{1-p}\right)^n}}. \tag{9.11.15}$$

Remarks

The Krawtchouk polynomials are self-dual, which means that

$$K_n(x;p,N)=K_x(n;p,N),\quad n,x\in\{0,1,2,\ldots,N\}.$$

By using this relation we easily obtain the so-called dual orthogonality relation from the orthogonality relation (9.11.2):

$$\sum_{n=0}^{N}\binom{N}{n}p^n(1-p)^{N-n}K_n(x;p,N)K_n(y;p,N)=\frac{\left(\frac{1-p}{p}\right)^x}{\binom{N}{x}}\delta_{xy},$$

where $0<p<1$ and $x,y\in\{0,1,2,\ldots,N\}$.
The Krawtchouk polynomials are related to the Meixner polynomials given by (9.10.1) in the following way:

$$K_n(x;p,N)=M_n(x;-N,(p-1)^{-1}p).$$

History

In 1934 Meixner [406] (see (1.1) and case IV on pp. 10, 11 and 12) gave the orthogonality measure for the polynomials P_n given by the generating function

$$e^{xu(t)}\,f(t)=\sum_{n=0}^{\infty}P_n(x)\,\frac{t^n}{n!},$$

where

$$e^{u(t)}=\left(\frac{1-\beta t}{1-\alpha t}\right)^{\frac{1}{\alpha-\beta}},\quad f(t)=\frac{(1-\beta t)^{\frac{k_2}{\beta(\alpha-\beta)}}}{(1-\alpha t)^{\frac{k_2}{\alpha(\alpha-\beta)}}}\quad (k_2<0;\ \alpha>\beta>0\ \text{ or }\ \alpha<\beta<0).$$

Then P_n can be expressed as a Meixner polynomial:

$$P_n(x)=(-k_2(\alpha\beta)^{-1})_n\beta^n M_n\left(-\frac{x+k_2\alpha^{-1}}{\alpha-\beta},-k_2(\alpha\beta)^{-1},\beta\alpha^{-1}\right).$$

In 1938 Gottlieb [**?**, §2] introduces polynomials l_n “of Laguerre type” which turn out to be special Meixner polynomials: $l_n(x)=e^{-n\lambda}M_n(x;1,e^{-\lambda})$.

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.10.4) behaves as $O(n^2)$ as $n\to\infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

References

[**?**], [**?**].

9.12 Laguerre

Hypergeometric representation

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right).\tag{9.12.1}$$

Orthogonality relation

$$\int_0^\infty e^{-x}x^\alpha L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{mn}, \quad \alpha > -1. \tag{9.12.2}$$

Recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0. \tag{9.12.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (2n+\alpha+1)p_n(x) + n(n+\alpha)p_{n-1}(x), \tag{9.12.4}$$

where

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}p_n(x).$$

Differential equation

$$xy''(x) + (\alpha+1-x)y'(x) + ny(x) = 0, \quad y(x) = L_n^{(\alpha)}(x). \tag{9.12.5}$$

Forward shift operator

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{9.12.6}$$

Backward shift operator

$$x\frac{d}{dx}L_n^{(\alpha)}(x) + (\alpha - x)L_n^{(\alpha)}(x) = (n+1)L_{n+1}^{(\alpha-1)}(x) \tag{9.12.7}$$

or equivalently

$$\frac{d}{dx}\left[e^{-x}x^{\alpha}L_n^{(\alpha)}(x)\right] = (n+1)e^{-x}x^{\alpha-1}L_{n+1}^{(\alpha-1)}(x). \tag{9.12.8}$$

Rodrigues-type formula

$$\mathrm{e}^{-x}x^{\alpha}L_n^{(\alpha)}(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n\left[e^{-x}x^{n+\alpha}\right]. \tag{9.12.9}$$

Generating functions

$$(1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n. \tag{9.12.10}$$

$$\mathrm{e}^t\,{}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -xt\right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n}t^n. \tag{9.12.11}$$

$$(1-t)^{-\gamma}\,{}_1F_1\left(\begin{matrix} \gamma \\ \alpha+1 \end{matrix}; \frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\alpha+1)_n}L_n^{(\alpha)}(x)t^n, \quad \gamma \text{ arbitrary}. \tag{9.12.12}$$

Limit relations

Meixner-Pollaczek → Laguerre

The Laguerre polynomials can be obtained from the Meixner-Pollaczek polynomials given by (9.7.1) by the substitution $\lambda = \frac{1}{2}(\alpha+1)$, $x \to -\frac{1}{2}\phi^{-1}x$ and the limit $\phi \to 0$:

$$\lim_{\phi\to 0} P_n^{(\frac{1}{2}\alpha+\frac{1}{2})}(-\tfrac{1}{2}\phi^{-1}x;\phi) = L_n^{(\alpha)}(x).$$

Jacobi → Laguerre

The Laguerre polynomials are obtained from the Jacobi polynomials given by (9.8.1) if we set $x \to 1-2\beta^{-1}x$ and then take the limit $\beta \to \infty$:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x) = L_n^{(\alpha)}(x).$$

Meixner → Laguerre

If we take $\beta = \alpha+1$ and $x \to (1-c)^{-1}x$ in the definition (9.10.1) of the Meixner polynomials and let $c \to 1$ we obtain the Laguerre polynomials:

$$\lim_{c\to 1} M_n((1-c)^{-1}x;\alpha+1,c) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$

Laguerre → Hermite

The Hermite polynomials given by (9.15.1) can be obtained from the Laguerre polynomials by taking the limit $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}((2\alpha)^{\frac{1}{2}}x+\alpha) = \frac{(-1)^n}{n!}H_n(x). \tag{9.12.13}$$

Remarks

The definition (9.12.1) of the Laguerre polynomials can also be written as:

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}(\alpha+k+1)_{n-k}x^k.$$

In this way the Laguerre polynomials can also be seen as polynomials in the parameter α. Therefore they can be defined for all α.
The Laguerre polynomials are related to the Bessel polynomials given by (9.13.1) in the following way:

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!}y_n(2x^{-1};-2n-\alpha-1).$$

The Laguerre polynomials are related to the Charlier polynomials given by (9.14.1) in the following way:

$$\frac{(-a)^n}{n!}C_n(x;a) = L_n^{(x-n)}(a).$$

The Laguerre polynomials and the Hermite polynomials given by (9.15.1) are also connected by the following quadratic transformations:

$$H_{2n}(x) = (-1)^n n!\, 2^{2n} L_n^{(-\frac{1}{2})}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n n!\, 2^{2n+1} x L_n^{(\frac{1}{2})}(x^2).$$

In combinatorics the Laguerre polynomials with $\alpha = 0$ are often called Rook polynomials.

Special values

By (9.11.1) and the binomial formula:

$$K_n(0;p,N) = 1, \qquad K_n(N;p,N) = (1-p^{-1})^n. \tag{9.12.14}$$

The self-duality (p.240, Remarks, first formula)

$$K_n(x;p,N) = K_x(n;p,N) \qquad (n,x \in \{0,1,\ldots,N\}) \tag{9.12.15}$$

combined with (9.12.14) yields:

$$K_N(x;p,N) = (1-p^{-1})^x \qquad (x \in \{0,1,\ldots,N\}). \tag{9.12.16}$$

Symmetry

By the orthogonality relation (9.11.2):

$$\frac{K_n(N-x;p,N)}{K_n(N;p,N)} = K_n(x;1-p,N). \tag{9.12.17}$$

By (9.12.17) and (9.12.15) we have also

$$\frac{K_{N-n}(x;p,N)}{K_N(x;p,N)} = K_n(x;1-p,N) \qquad (n,x \in \{0,1,\ldots,N\}), \tag{9.12.18}$$

and, by (9.12.18), (9.12.17) and (9.12.14),

$$K_{N-n}(N-x;p,N) = \left(\frac{p}{p-1}\right)^{n+x-N} K_n(x;p,N) \qquad (n,x \in \{0,1,\ldots,N\}). \tag{9.12.19}$$

A particular case of (9.12.17) is:

$$K_n(N-x;\tfrac{1}{2},N) = (-1)^n K_n(x;\tfrac{1}{2},N). \tag{9.12.20}$$

Hence

$$K_{2m+1}(N;\tfrac{1}{2},2N) = 0. \tag{9.12.21}$$

From (9.11.11):

$$K_{2m}(N;\tfrac{1}{2},2N) = \frac{(\frac{1}{2})_m}{(-N+\frac{1}{2})_m}\,. \tag{9.12.22}$$

Quadratic transformations

$$K_{2m}(x+N;\tfrac{1}{2},2N) = \frac{(\frac{1}{2})_m}{(-N+\frac{1}{2})_m} R_m(x^2;-\tfrac{1}{2},-\tfrac{1}{2},N), \tag{9.12.23}$$

$$K_{2m+1}(x+N;\tfrac{1}{2},2N) = -\frac{(\frac{3}{2})_m}{N(-N+\frac{1}{2})_m} x R_m(x^2-1;\tfrac{1}{2},\tfrac{1}{2},N-1), \tag{9.12.24}$$

$$K_{2m}(x+N+1;\tfrac{1}{2},2N+1) = \frac{(\frac{1}{2})_m}{(-N-\frac{1}{2})_m} R_m(x(x+1);-\tfrac{1}{2},\tfrac{1}{2},N), \tag{9.12.25}$$

$$K_{2m+1}(x+N+1;\tfrac{1}{2},2N+1) = \frac{(\frac{3}{2})_m}{(-N-\frac{1}{2})_{m+1}} (x+\tfrac{1}{2}) R_m(x(x+1);\tfrac{1}{2},-\tfrac{1}{2},N), \tag{9.12.26}$$

where R_m is a dual Hahn polynomial (9.6.1). For the proofs use (9.6.2), (9.11.2), (9.6.4) and (9.11.4).

Generating functions

$$\sum_{x=0}^{N} \binom{N}{x} K_m(x;p,N) K_n(x;q,N) z^x$$
$$= \left(\frac{p-z+pz}{p}\right)^m \left(\frac{q-z+qz}{q}\right)^n (1+z)^{N-m-n} K_m\left(n;-\frac{(p-z+pz)(q-z+qz)}{z},N\right). \tag{9.12.27}$$

This follows immediately from Rosengren [**?**, (3.5)], which goes back to Meixner [**?**].

References

[?], [?].

9.13 Bessel

Hypergeometric representation

$$\begin{aligned} y_n(x;a) &= {}_2F_0\left(\begin{matrix} -n, n+a+1 \\ - \end{matrix}; -\frac{x}{2}\right) \\ &= (n+a+1)_n \left(\frac{x}{2}\right)^n {}_1F_1\left(\begin{matrix} -n \\ -2n-a \end{matrix}; \frac{2}{x}\right), \quad n = 0,1,2,\ldots,N. \end{aligned} \tag{9.13.1}$$

Orthogonality relation

$$\begin{aligned} &\int_0^\infty x^a \mathrm{e}^{-\frac{2}{x}} y_m(x;a) y_n(x;a)\, dx \\ &= -\frac{2^{a+1}}{2n+a+1}\Gamma(-n-a)n!\,\delta_{mn}, \quad a < -2N-1. \end{aligned} \tag{9.13.2}$$

Recurrence relation

$$\begin{aligned} &2(n+a+1)(2n+a)y_{n+1}(x;a) \\ &= (2n+a+1)\left[2a+(2n+a)(2n+a+2)x\right] y_n(x;a) \\ &\quad + 2n(2n+a+2)y_{n-1}(x;a). \end{aligned} \tag{9.13.3}$$

Normalized recurrence relation

$$\begin{aligned} xp_n(x) = p_{n+1}(x) &- \frac{2a}{(2n+a)(2n+a+2)} p_n(x) \\ &- \frac{4n(n+a)}{(2n+a-1)(2n+a)^2(2n+a+1)} p_{n-1}(x), \end{aligned} \tag{9.13.4}$$

where

$$y_n(x;a) = \frac{(n+a+1)_n}{2^n} p_n(x).$$

Differential equation

$$x^2 y''(x) + [(a+2)x+2]\, y'(x) - n(n+a+1)y(x) = 0, \quad y(x) = y_n(x;a). \tag{9.13.5}$$

Forward shift operator

$$\frac{d}{dx} y_n(x;a) = \frac{n(n+a+1)}{2} y_{n-1}(x;a+2). \tag{9.13.6}$$

Backward shift operator

$$x^2\frac{d}{dx}y_n(x;a)+(ax+2)y_n(x;a)=2y_{n+1}(x;a-2) \tag{9.13.7}$$

or equivalently

$$\frac{d}{dx}\left[x^a e^{-\frac{2}{x}}y_n(x;a)\right]=2x^{a-2}e^{-\frac{2}{x}}y_{n+1}(x;a-2). \tag{9.13.8}$$

Rodrigues-type formula

$$y_n(x;a)=2^{-n}x^{-a}e^{\frac{2}{x}}D^n\left(x^{2n+a}e^{-\frac{2}{x}}\right). \tag{9.13.9}$$

Generating function

$$(1-2xt)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-2xt}}\right)^a\exp\left(\frac{2t}{1+\sqrt{1-2xt}}\right)=\sum_{n=0}^{\infty}y_n(x;a)\frac{t^n}{n!}. \tag{9.13.10}$$

Limit relation

Jacobi → Bessel

If we take $\beta=a-\alpha$ in the definition (9.8.1) of the Jacobi polynomials and let $\alpha\to-\infty$ we find the Bessel polynomials:

$$\lim_{\alpha\to-\infty}\frac{P_n^{(\alpha,a-\alpha)}(1+\alpha x)}{P_n^{(\alpha,a-\alpha)}(1)}=y_n(x;a).$$

Remarks

The following notations are also used for the Bessel polynomials:

$$y_n(x;a,b)=y_n(2b^{-1}x;a) \quad \text{and} \quad \theta_n(x;a,b)=x^n y_n(x^{-1};a,b).$$

However, the Bessel polynomials essentially depend on only one parameter.
The Bessel polynomials are related to the Laguerre polynomials given by (9.12.1) in the following way:

$$L_n^{(\alpha)}(x)=\frac{(-x)^n}{n!}y_n(2x^{-1};-2n-\alpha-1).$$

The special case $a=-2N-2$ of the Bessel polynomials can be obtained from the pseudo Jacobi polynomials by setting $x\to\nu x$ in the definition (9.9.1) of the pseudo Jacobi polynomials and taking the limit $\nu\to\infty$ in the following way:

$$\lim_{\nu\to\infty}\frac{P_n(\nu x;\nu,N)}{\nu^n}=\frac{2^n}{(n-2N-1)_n}y_n(x;-2N-2).$$

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.14 Charlier

Hypergeometric representation

$$C_n(x;a)={}_2F_0\left(\begin{matrix}-n,-x\\ -\end{matrix};-\frac{1}{a}\right). \tag{9.14.1}$$

Orthogonality relation

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x;a) C_n(x;a) = a^{-n} e^a n! \, \delta_{mn}, \quad a > 0. \tag{9.14.2}$$

Recurrence relation

$$-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a). \tag{9.14.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (n+a)p_n(x) + nap_{n-1}(x), \tag{9.14.4}$$

where

$$C_n(x;a) = \left(-\frac{1}{a}\right)^n p_n(x).$$

Difference equation

$$-ny(x) = ay(x+1) - (x+a)y(x) + xy(x-1), \quad y(x) = C_n(x;a). \tag{9.14.5}$$

Forward shift operator

$$C_n(x+1;a) - C_n(x;a) = -\frac{n}{a} C_{n-1}(x;a) \tag{9.14.6}$$

or equivalently

$$\Delta C_n(x;a) = -\frac{n}{a} C_{n-1}(x;a). \tag{9.14.7}$$

Backward shift operator

$$C_n(x;a) - \frac{x}{a} C_n(x-1;a) = C_{n+1}(x;a) \tag{9.14.8}$$

or equivalently

$$\nabla \left[\frac{a^x}{x!} C_n(x;a)\right] = \frac{a^x}{x!} C_{n+1}(x;a). \tag{9.14.9}$$

Rodrigues-type formula

$$\frac{a^x}{x!} C_n(x;a) = \nabla^n \left[\frac{a^x}{x!}\right]. \tag{9.14.10}$$

Generating function

$$\mathrm{e}^t \left(1 - \frac{t}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n. \tag{9.14.11}$$

Limit relations

Meixner → Charlier

If we take $c = (a+\beta)^{-1}a$ in the definition (9.10.1) of the Meixner polynomials and let $\beta \to \infty$ we find the Charlier polynomials:

$$\lim_{\beta \to \infty} M_n(x; \beta, (a+\beta)^{-1}a) = C_n(x;a).$$

Krawtchouk → Charlier

The Charlier polynomials can be found from the Krawtchouk polynomials given by (9.11.1) by taking $p = N^{-1}a$ and letting $N \to \infty$:

$$\lim_{N\to\infty} K_n(x; N^{-1}a, N) = C_n(x;a).$$

Charlier → Hermite

The Hermite polynomials given by (9.15.1) are obtained from the Charlier polynomials if we set $x \to (2a)^{1/2}x + a$ and let $a \to \infty$. In fact we have

$$\lim_{a\to\infty} (2a)^{\frac{1}{2}n} C_n((2a)^{\frac{1}{2}}x + a; a) = (-1)^n H_n(x). \tag{9.14.12}$$

Remark

The Charlier polynomials are related to the Laguerre polynomials given by (9.12.1) in the following way:

$$\frac{(-a)^n}{n!} C_n(x;a) = L_n^{(x-n)}(a).$$

Notation

Here the Laguerre polynomial is denoted by L_n^α instead of $L_n^{(\alpha)}$.

Hypergeometric representation

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right) \tag{9.14.13}$$

$$= \frac{(-x)^n}{n!}\,{}_2F_0\left(\begin{matrix}-n,-n-\alpha\\ -\end{matrix};-\frac{1}{x}\right) \tag{9.14.14}$$

$$= \frac{(-x)^n}{n!} C_n(n+\alpha;x), \tag{9.14.15}$$

where C_n in (9.14.15) is a Charlier polynomial. Formula (9.14.13) is (9.12.1). Then (9.14.14) follows by reversal of summation. Finally (9.14.15) follows by (9.14.14) and (9.15.15). It is also the remark on top of p.244 in [KLS], and it is essentially [416, (2.7.10)].

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.12.4) behaves as $O(n^2)$ as $n \to \infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

Special value

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}. \tag{9.14.16}$$

Use (9.12.1) or see [DLMF, 18.6.1)].

Quadratic transformations

$$H_{2n}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{-1/2}(x^2), \tag{9.14.17}$$

$$H_{2n+1}(x) = (-1)^n\, 2^{2n+1}\, n!\, x L_n^{1/2}(x^2). \tag{9.14.18}$$

See p.244, Remarks, last two formulas. Or see [DLMF, (18.7.19), (18.7.20)].

Fourier transform

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty \frac{L_n^\alpha(y)}{L_n^\alpha(0)}\,e^{-y}\,y^\alpha\,e^{ixy}\,dy = i^n\,\frac{y^n}{(iy+1)^{n+\alpha+1}}\,, \tag{9.14.19}$$

see [DLMF, (18.17.34)].

Differentiation formulas

Each differentiation formula is given in two equivalent forms.

$$\frac{d}{dx}\left(x^\alpha L_n^\alpha(x)\right) = (n+\alpha)\,x^{\alpha-1}L_n^{\alpha-1}(x), \qquad \left(x\frac{d}{dx}+\alpha\right)L_n^\alpha(x) = (n+\alpha)\,L_n^{\alpha-1}(x). \tag{9.14.20}$$

$$\frac{d}{dx}\left(e^{-x}L_n^\alpha(x)\right) = -e^{-x}L_n^{\alpha+1}(x), \qquad \left(\frac{d}{dx}-1\right)L_n^\alpha(x) = -L_n^{\alpha+1}(x). \tag{9.14.21}$$

Formulas (9.14.20) and (9.14.21) follow from [DLMF, (13.3.18), (13.3.20)] together with (9.12.1).

Generalized Hermite polynomials

See [146, p.156], [**?**, Section 1.5.1]. These are defined by

$$H_{2m}^\mu(x) := \mathrm{const.}\,L_m^{\mu-\frac12}(x^2), \qquad H_{2m+1}^\mu(x) := \mathrm{const.}\,x\,L_m^{\mu+\frac12}(x^2). \tag{9.14.22}$$

Then for $\mu > -\frac12$ we have orthogonality relation

$$\int_{-\infty}^\infty H_m^\mu(x)\,H_n^\mu(x)\,|x|^{2\mu}e^{-x^2}\,dx = 0 \qquad (m\neq n). \tag{9.14.23}$$

Let the Dunkl operator T_μ be defined by (9.8.31). If we choose the constants in (9.14.22) as

$$H_{2m}^\mu(x) = \frac{(-1)^m(2m)!}{(\mu+\frac12)_m}\,L_m^{\mu-\frac12}(x^2), \qquad H_{2m+1}^\mu(x) = \frac{(-1)^m(2m+1)!}{(\mu+\frac12)_{m+1}}\,x\,L_m^{\mu+\frac12}(x^2) \tag{9.14.24}$$

then (see [**?**, (1.6)])

$$T_\mu H_n^\mu = 2n\,H_{n-1}^\mu. \tag{9.14.25}$$

Formula (9.14.25) with (9.14.24) substituted gives rise to two differentiation formulas involving Laguerre polynomials which are equivalent to (9.12.6) and (9.14.20).

Composition of (9.14.25) with itself gives

$$T_\mu^2 H_n^\mu = 4n(n-1)\,H_{n-2}^\mu,$$

which is equivalent to the composition of (9.12.6) and (9.14.20):

$$\left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\,\frac{d}{dx}\right)L_n^\alpha(x^2) = -4(n+\alpha)\,L_{n-1}^\alpha(x^2). \tag{9.14.26}$$

References

[**?**], [**?**].

9.15 Hermite

Hypergeometric representation

$$H_n(x) = (2x)^n\,{}_2F_0\left(\begin{array}{c}-n/2,-(n-1)/2\\ -\end{array};-\frac{1}{x^2}\right). \tag{9.15.1}$$

Orthogonality relation

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = 2^n n!\,\delta_{mn}. \tag{9.15.2}$$

Recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{9.15.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n}{2}p_{n-1}(x), \tag{9.15.4}$$

where

$$H_n(x) = 2^n p_n(x).$$

Differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad y(x) = H_n(x). \tag{9.15.5}$$

Forward shift operator

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x). \tag{9.15.6}$$

Backward shift operator

$$\frac{d}{dx}H_n(x) - 2xH_n(x) = -H_{n+1}(x) \tag{9.15.7}$$

or equivalently

$$\frac{d}{dx}\left[e^{-x^2}H_n(x)\right] = -e^{-x^2}H_{n+1}(x). \tag{9.15.8}$$

Rodrigues-type formula

$$\mathrm{e}^{-x^2}H_n(x) = (-1)^n\left(\frac{d}{dx}\right)^n\left[e^{-x^2}\right]. \tag{9.15.9}$$

Generating functions

$$\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{9.15.10}$$

$$\begin{cases} e^t\cos(2x\sqrt{t}) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}H_{2n}(x)t^n \\ \dfrac{e^t}{\sqrt{t}}\sin(2x\sqrt{t}) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}H_{2n+1}(x)t^n. \end{cases} \tag{9.15.11}$$

$$\begin{cases} e^{-t^2}\cosh(2xt) = \displaystyle\sum_{n=0}^{\infty}\frac{H_{2n}(x)}{(2n)!}t^{2n} \\ e^{-t^2}\sinh(2xt) = \displaystyle\sum_{n=0}^{\infty}\frac{H_{2n+1}(x)}{(2n+1)!}t^{2n+1}. \end{cases} \tag{9.15.12}$$

$$\begin{cases} (1+t^2)^{-\gamma}\,{}_1F_1\left(\begin{matrix}\gamma\\ \frac{1}{2}\end{matrix};\frac{x^2t^2}{1+t^2}\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2n)!}H_{2n}(x)t^{2n}\\ \frac{xt}{\sqrt{1+t^2}}\,{}_1F_1\left(\begin{matrix}\gamma+\frac{1}{2}\\ \frac{3}{2}\end{matrix};\frac{x^2t^2}{1+t^2}\right)=\sum_{n=0}^{\infty}\frac{(\gamma+\frac{1}{2})_n}{(2n+1)!}H_{2n+1}(x)t^{2n+1}\end{cases}\tag{9.15.13}$$

with γ arbitrary.

$$\frac{1+2xt+4t^2}{(1+4t^2)^{\frac{3}{2}}}\exp\left(\frac{4x^2t^2}{1+4t^2}\right)=\sum_{n=0}^{\infty}\frac{H_n(x)}{\lfloor n/2\rfloor\,!}t^n,\tag{9.15.14}$$

where $\lfloor n/2\rfloor$ denotes the largest integer smaller than or equal to $n/2$.

Limit relations

Meixner-Pollaczek → Hermite

If we take $x\to(\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi)$ in the definition (9.7.1) of the Meixner-Pollaczek polynomials and then let $\lambda\to\infty$ we obtain the Hermite polynomials:

$$\lim_{\lambda\to\infty}\lambda^{-\frac{1}{2}n}P_n^{(\lambda)}((\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi);\phi)=\frac{H_n(x)}{n!}.$$

Jacobi → Hermite

The Hermite polynomials follow from the Jacobi polynomials given by (9.8.1) by taking $\beta=\alpha$ and letting $\alpha\to\infty$ in the following way:

$$\lim_{\alpha\to\infty}\alpha^{-\frac{1}{2}n}P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}}x)=\frac{H_n(x)}{2^n n!}.$$

Gegenbauer / Ultraspherical → Hermite

The Hermite polynomials follow from the Gegenbauer (or ultraspherical) polynomials given by (9.8.35) by taking $\lambda=\alpha+\frac{1}{2}$ and letting $\alpha\to\infty$ in the following way:

$$\lim_{\alpha\to\infty}\alpha^{-\frac{1}{2}n}C_n^{(\alpha+\frac{1}{2})}(\alpha^{-\frac{1}{2}}x)=\frac{H_n(x)}{n!}.$$

Krawtchouk → Hermite

The Hermite polynomials follow from the Krawtchouk polynomials given by (9.11.1) by setting $x\to pN+x\sqrt{2p(1-p)N}$ and then letting $N\to\infty$:

$$\lim_{N\to\infty}\sqrt{\binom{N}{n}}K_n(pN+x\sqrt{2p(1-p)N};p,N)=\frac{(-1)^nH_n(x)}{\sqrt{2^n n!\left(\frac{p}{1-p}\right)^n}}.$$

Laguerre → Hermite

The Hermite polynomials can be obtained from the Laguerre polynomials given by (9.12.1) by taking the limit $\alpha\to\infty$ in the following way:

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha}\right)^{\frac{1}{2}n}L_n^{(\alpha)}((2\alpha)^{\frac{1}{2}}x+\alpha)=\frac{(-1)^n}{n!}H_n(x).$$

Charlier → Hermite

If we set $x \to (2a)^{1/2}x + a$ in the definition (9.14.1) of the Charlier polynomials and let $a \to \infty$ we find the Hermite polynomials. In fact we have

$$\lim_{a\to\infty} (2a)^{\frac{1}{2}n} C_n((2a)^{\frac{1}{2}}x + a; a) = (-1)^n H_n(x).$$

Remarks

The Hermite polynomials can also be written as:

$$\frac{H_n(x)}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!},$$

where $\lfloor n/2 \rfloor$ denotes the largest integer smaller than or equal to $n/2$.
The Hermite polynomials and the Laguerre polynomials given by (9.12.1) are also connected by the following quadratic transformations:

$$H_{2n}(x) = (-1)^n n!\, 2^{2n} L_n^{(-\frac{1}{2})}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n n!\, 2^{2n+1} x L_n^{(\frac{1}{2})}(x^2).$$

Hypergeometric representation

$$C_n(x;a) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} ; -\frac{1}{a}\right) \tag{9.15.15}$$

$$= \frac{(-x)_n}{a^n}\, {}_1F_1\left(\begin{matrix} -n \\ x-n+1 \end{matrix} ; a\right) \tag{9.15.16}$$

$$= \frac{n!}{(-a)^n} L_n^{x-n}(a), \tag{9.15.17}$$

where $L_n^{\alpha}(x)$ is a Laguerre polynomial. Formula (9.15.15) is (9.14.1). Then (9.15.16) follows by reversal of the summation. Finally (9.15.17) follows by (9.15.16) and (9.12.1). It is also the Remark on p.249 of [KLS], and it was earlier given in [416, (2.7.10)].

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.14.4) behaves as $O(n)$ as $n \to \infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

References

Uniqueness of orthogonality measure

The coefficient of $p_{n-1}(x)$ in (9.15.4) behaves as $O(n)$ as $n \to \infty$. Hence (**??**) holds, by which the orthogonality measure is unique.

Fourier transforms

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H_n(y)\,e^{-\frac{1}{2}y^2}\,e^{ixy}\,dy = i^n\,H_n(x)\,e^{-\frac{1}{2}x^2}, \tag{9.15.18}$$

see [AAR, (6.1.15) and Exercise 6.11].

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(y)\,e^{-y^2}\,e^{ixy}\,dy = i^n\,x^n\,e^{-\frac{1}{4}x^2}, \tag{9.15.19}$$

see [DLMF, (18.17.35)].

$$\frac{i^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty} y^n\,e^{-\frac{1}{4}y^2}\,e^{-ixy}\,dy = H_n(x)\,e^{-x^2}, \tag{9.15.20}$$

see [AAR, (6.1.4)]. [?], [?].