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Hypergeometric orthogonal polynomials and their q-analogues

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9

Hypergeometric orthogonal polynomials

In this chapter we deal with all families of hypergeometric orthogonal polynomials appearing in the Askey scheme on page **??**. For each family of orthogonal polynomials we state the most important properties such as a representation as a hypergeometric function, orthogonality relation(s), the three-term recurrence relation, the second-order differential or difference equation, the forward shift (or degree lowering) and backward shift (or degree raising) operator, a Rodrigues-type formula and some generating functions. In each case we use the notation which seems to be most common in the literature. Moreover, in each case we mention the connection between various families by stating the appropriate limit relations. See also [**?**] for an algebraic approach of this Askey scheme and [**?**] for a view from asymptotic analysis. For notations the reader is referred to chapter **??**.

9.1 Wilson

Hypergeometric representation

$$\begin{aligned}&\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}\\&={}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\a+b,a+c,a+d\end{matrix};1\right).\end{aligned} \tag{9.1.1}$$

Orthogonality relation

If $Re(a,b,c,d) > 0$ and non-real parameters occur in conjugate pairs, then

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 \times W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)\,dx = \frac{\Gamma(n+a+b)\cdots\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}(n+a+b+c+d-1)_n n!\,\delta_{mn}, \tag{9.1.2}$$

where

$$\Gamma(n+a+b)\cdots\Gamma(n+c+d) = \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d).$$

If $a<0$ and $a+b$, $a+c$, $a+d$ are positive or a pair of complex conjugates occur with positive real parts, then

$$\begin{aligned}\frac{1}{2\pi}\int_0^\infty &\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 \times W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)\,dx \\ &+ \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)} \\ &\times \sum_{\substack{k=0,1,2\ldots \\ a+k<0}} \frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k k!} \\ &\times W_m(-(a+k)^2;a,b,c,d)W_n(-(a+k)^2;a,b,c,d) \\ =& \frac{\Gamma(n+a+b)\cdots\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}(n+a+b+c+d-1)_n n!\,\delta_{mn}.\end{aligned} \tag{9.1.3}$$

Recurrence relation

$$-\left(a^2+x^2\right)\tilde{W}_n(x^2) = A_n\tilde{W}_{n+1}(x^2) - (A_n+C_n)\tilde{W}_n(x^2) + C_n\tilde{W}_{n-1}(x^2), \tag{9.1.4}$$

where

$$\tilde{W}_n(x^2) := \tilde{W}_n(x^2;a,b,c,d) = \frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}$$

and

$$\begin{cases} A_n = \dfrac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \\ C_n = \dfrac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (A_n+C_n-a^2)p_n(x) + A_{n-1}C_n p_{n-1}(x), \tag{9.1.5}$$

where

$$W_n(x^2;a,b,c,d) = (-1)^n(n+a+b+c+d-1)_n p_n(x^2).$$

Difference equation

$$\begin{aligned} &n(n+a+b+c+d-1)y(x) \\ &= B(x)y(x+i) - [B(x)+D(x)]\,y(x) + D(x)y(x-i), \end{aligned} \tag{9.1.6}$$

where

$$y(x) = W_n(x^2;a,b,c,d)$$

and

$$\begin{cases} B(x) = \dfrac{(a-ix)(b-ix)(c-ix)(d-ix)}{2ix(2ix-1)} \\ D(x) = \dfrac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &W_n((x+\tfrac{1}{2}i)^2;a,b,c,d) - W_n((x-\tfrac{1}{2}i)^2;a,b,c,d) \\ &= -2inx(n+a+b+c+d-1)W_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}) \end{aligned} \tag{9.1.7}$$

or equivalently

$$\begin{aligned} &\frac{\delta W_n(x^2;a,b,c,d)}{\delta x^2} \\ &= -n(n+a+b+c+d-1)W_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}). \end{aligned} \tag{9.1.8}$$

Backward shift operator

$$\begin{aligned}&(a-\tfrac{1}{2}-ix)(b-\tfrac{1}{2}-ix)(c-\tfrac{1}{2}-ix)(d-\tfrac{1}{2}-ix)W_n((x+\tfrac{1}{2}i)^2;a,b,c,d)\\&\quad-(a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)(c-\tfrac{1}{2}+ix)(d-\tfrac{1}{2}+ix)W_n((x-\tfrac{1}{2}i)^2;a,b,c,d)\\&=-2ixW_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})\end{aligned}\tag{9.1.9}$$

or equivalently

$$\begin{aligned}&\frac{\delta\left[\omega(x;a,b,c,d)W_n(x^2;a,b,c,d)\right]}{\delta x^2}\\&=\omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})W_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}),\end{aligned}\tag{9.1.10}$$

where

$$\omega(x;a,b,c,d):=\frac{1}{2ix}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2.$$

Rodrigues-type formula

$$\begin{aligned}&\omega(x;a,b,c,d)W_n(x^2;a,b,c,d)\\&=\left(\frac{\delta}{\delta x^2}\right)^n\left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n,d+\tfrac{1}{2}n)\right].\end{aligned}\tag{9.1.11}$$

Generating functions

$${}_2F_1\left(\begin{array}{c}a+ix,b+ix\\a+b\end{array};t\right){}_2F_1\left(\begin{array}{c}c-ix,d-ix\\c+d\end{array};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+b)_n(c+d)_nn!}.\tag{9.1.12}$$

$${}_2F_1\left(\begin{array}{c}a+ix,c+ix\\a+c\end{array};t\right){}_2F_1\left(\begin{array}{c}b-ix,d-ix\\b+d\end{array};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+c)_n(b+d)_nn!}.\tag{9.1.13}$$

$${}_2F_1\left(\begin{array}{c}a+ix,d+ix\\a+d\end{array};t\right){}_2F_1\left(\begin{array}{c}b-ix,c-ix\\b+c\end{array};t\right)=\sum_{n=0}^{\infty}\frac{W_n(x^2;a,b,c,d)t^n}{(a+d)_n(b+c)_nn!}.\tag{9.1.14}$$

$$\begin{aligned}&(1-t)^{1-a-b-c-d}\\&\quad\times{}_4F_3\left(\begin{array}{c}\tfrac{1}{2}(a+b+c+d-1),\tfrac{1}{2}(a+b+c+d),a+ix,a-ix\\a+b,a+c,a+d\end{array};-\frac{4t}{(1-t)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(a+b+c+d-1)_n}{(a+b)_n(a+c)_n(a+d)_nn!}W_n(x^2;a,b,c,d)t^n.\end{aligned}\tag{9.1.15}$$

Limit relations

Wilson → Continuous dual Hahn

The continuous dual Hahn polynomials given by (9.3.1) can be found from the Wilson polynomials by dividing by $(a+d)_n$ and letting $d\to\infty$:

$$\lim_{d\to\infty}\frac{W_n(x^2;a,b,c,d)}{(a+d)_n}=S_n(x^2;a,b,c).\tag{9.1.16}$$

Wilson → Continuous Hahn

The continuous Hahn polynomials given by (9.4.1) are obtained from the Wilson polynomials by the substitutions $a \to a - it$, $b \to b - it$, $c \to c + it$, $d \to d + it$ and $x \to x + t$ and the limit $t \to \infty$ in the following way:

$$\lim_{t\to\infty} \frac{W_n((x+t)^2; a-it, b-it, c+it, d+it)}{(-2t)^n n!} = p_n(x; a, b, c, d). \tag{9.1.17}$$

Wilson → Jacobi

The Jacobi polynomials given by (9.8.1) can be found from the Wilson polynomials by substituting $a = b = \frac{1}{2}(\alpha+1)$, $c = \frac{1}{2}(\beta+1) + it$, $d = \frac{1}{2}(\beta+1) - it$ and $x \to t\sqrt{\frac{1}{2}(1-x)}$ in the definition (9.1.1) of the Wilson polynomials and taking the limit $t \to \infty$. In fact we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{W_n(\frac{1}{2}(1-x)t^2; \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+1), \frac{1}{2}(\beta+1)+it, \frac{1}{2}(\beta+1)-it)}{t^{2n} n!} \\ &= P_n^{(\alpha,\beta)}(x). \end{aligned} \tag{9.1.18}$$

Remarks

Note that for $k < n$ we have

$$\frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b)_k(a+c)_k(a+d)_k} = (a+b+k)_{n-k}(a+c+k)_{n-k}(a+d+k)_{n-k},$$

which implies that the Wilson polynomials defined by (9.1.1) can also be seen as polynomials in the parameters a, b, c and d.
If we set

$$\begin{aligned} a &= \tfrac{1}{2}(\gamma+\delta+1), \\ b &= \tfrac{1}{2}(2\alpha-\gamma-\delta+1), \\ c &= \tfrac{1}{2}(2\beta-\gamma+\delta+1), \\ d &= \tfrac{1}{2}(\gamma-\delta+1), \end{aligned}$$

and

$$ix \to x + \tfrac{1}{2}(\gamma+\delta+1)$$

in

$$\tilde{W}_n(x^2; a, b, c, d) = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} \tag{9.1.19}$$

given by (9.1.1) and take with N a nonnegative integer, we obtain the Racah polynomials given by (9.2.1).

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.2 Racah

Hypergeometric representation

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};1\right),\quad n=0,1,2,\ldots,N, \tag{9.2.1}$$

where

$$\lambda(x)=x(x+\gamma+\delta+1)$$

and

$$\alpha+1=-N \quad \text{or} \quad \beta+\delta+1=-N \quad \text{or} \quad \gamma+1=-N$$

with N a nonnegative integer.

Orthogonality relation

$$\sum_{x=0}^{N}\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x x!} \times R_m(\lambda(x))R_n(\lambda(x)) = M\frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n n!}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\delta_{mn}, \tag{9.2.2}$$

where

$$R_n(\lambda(x)):=R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$M=\begin{cases}\dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if} \quad \alpha+1=-N\\[2ex] \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if} \quad \beta+\delta+1=-N\\[2ex] \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if} \quad \gamma+1=-N.\end{cases}$$

Recurrence relation

$$\lambda(x)R_n(\lambda(x))=A_nR_{n+1}(\lambda(x))-(A_n+C_n)R_n(\lambda(x))+C_nR_{n-1}(\lambda(x)), \tag{9.2.3}$$

where

$$R_n(\lambda(x)):=R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$\begin{cases}A_n=\dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\\[2ex] C_n=\dfrac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},\end{cases}$$

hence

$$A_n = \begin{cases} \dfrac{(n+\beta-N)(n+\beta+\delta+1)(n+\gamma+1)(n-N)}{(2n+\beta-N)(2n+\beta-N+1)} & \text{if} \quad \alpha+1=-N \\ \dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\gamma+1)(n-N)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n-N)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} & \text{if} \quad \gamma+1=-N \end{cases}$$

and

$$C_n = \begin{cases} \dfrac{n(n+\beta)(n+\beta-\gamma-N-1)(n-\delta-N-1)}{(2n+\beta-N-1)(2n+\beta-N)} & \text{if} \quad \alpha+1=-N \\ \dfrac{n(n+\alpha+\beta+N+1)(n+\alpha+\beta-\gamma)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{n(n+\alpha+\beta+N+1)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} & \text{if} \quad \gamma+1=-N. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) - (A_n + C_n)p_n(x) + A_{n-1}C_n p_{n-1}(x), \tag{9.2.4}$$

where

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n} p_n(\lambda(x)).$$

Difference equation

$$n(n+\alpha+\beta+1)y(x) = B(x)y(x+1) - [B(x)+D(x)]\,y(x) + D(x)y(x-1), \tag{9.2.5}$$

where

$$y(x) = R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$\begin{cases} B(x) = \dfrac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \\ D(x) = \dfrac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta) - R_n(\lambda(x);\alpha,\beta,\gamma,\delta) \\ &\quad = \frac{n(n+\alpha+\beta+1)}{(\alpha+1)(\beta+\delta+1)(\gamma+1)} \\ &\qquad \times (2x+\gamma+\delta+2)R_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma+1,\delta) \end{aligned} \tag{9.2.6}$$

or equivalently

$$\begin{aligned} &\frac{\Delta R_n(\lambda(x);\alpha,\beta,\gamma,\delta)}{\Delta\lambda(x)} \\ &= \frac{n(n+\alpha+\beta+1)}{(\alpha+1)(\beta+\delta+1)(\gamma+1)} R_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma+1,\delta). \end{aligned} \tag{9.2.7}$$

Backward shift operator

$$\begin{aligned}
&(x+\alpha)(x+\beta+\delta)(x+\gamma)(x+\gamma+\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\\
&\qquad -x(x-\beta+\gamma)(x-\alpha+\gamma+\delta)(x+\delta)R_n(\lambda(x-1);\alpha,\beta,\gamma,\delta)\\
&=\alpha\gamma(\beta+\delta)(2x+\gamma+\delta)R_{n+1}(\lambda(x);\alpha-1,\beta-1,\gamma-1,\delta)
\end{aligned}\tag{9.2.8}$$

or equivalently

$$\begin{aligned}
&\frac{\nabla\left[\omega(x;\alpha,\beta,\gamma,\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\right]}{\nabla\lambda(x)}\\
&=\frac{1}{\gamma+\delta}\omega(x;\alpha-1,\beta-1,\gamma-1,\delta)R_{n+1}(\lambda(x);\alpha-1,\beta-1,\gamma-1,\delta),
\end{aligned}\tag{9.2.9}$$

where

$$\omega(x;\alpha,\beta,\gamma,\delta)=\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x x!}.$$

Rodrigues-type formula

$$\begin{aligned}
&\omega(x;\alpha,\beta,\gamma,\delta)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\\
&=(\gamma+\delta+1)_n\left(\nabla_\lambda\right)^n\left[\omega(x;\alpha+n,\beta+n,\gamma+n,\delta)\right],
\end{aligned}\tag{9.2.10}$$

where

$$\nabla_\lambda:=\frac{\nabla}{\nabla\lambda(x)}.$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$${}_2F_1\left(\begin{array}{c}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{array};t\right){}_2F_1\left(\begin{array}{c}x+\beta+\delta+1,x+\gamma+1\\ \beta+1\end{array};t\right)=\sum_{n=0}^{N}\frac{(\beta+\delta+1)_n(\gamma+1)_n}{(\beta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n,\tag{9.2.11}$$

$${}_2F_1\left(\begin{array}{c}-x,-x+\beta-\gamma\\ \beta+\delta+1\end{array};t\right){}_2F_1\left(\begin{array}{c}x+\alpha+1,x+\gamma+1\\ \alpha-\delta+1\end{array};t\right)=\sum_{n=0}^{N}\frac{(\alpha+1)_n(\gamma+1)_n}{(\alpha-\delta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n,\tag{9.2.12}$$

$${}_2F_1\left(\begin{array}{c}-x,-x-\delta\\ \gamma+1\end{array};t\right){}_2F_1\left(\begin{array}{c}x+\alpha+1,x+\beta+\delta+1\\ \alpha+\beta-\gamma+1\end{array};t\right)=\sum_{n=0}^{N}\frac{(\alpha+1)_n(\beta+\delta+1)_n}{(\alpha+\beta-\gamma+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n,\tag{9.2.13}$$

$$\begin{aligned}
&\Bigg[(1-t)^{-\alpha-\beta-1}\\
&\qquad\times{}_4F_3\left(\begin{array}{c}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2),-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{array};-\frac{4t}{(1-t)^2}\right)\Bigg]_N\\
&=\sum_{n=0}^{N}\frac{(\alpha+\beta+1)_n}{n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n.
\end{aligned}\tag{9.2.14}$$

Limit relations

Racah → Hahn

The Hahn polynomials given by (9.5.1) can be obtained from the Racah polynomials by taking $\gamma+1=-N$ and letting $\delta\to\infty$:

$$\lim_{\delta\to\infty} R_n(\lambda(x);\alpha,\beta,-N-1,\delta)=Q_n(x;\alpha,\beta,N). \tag{9.2.15}$$

The Hahn polynomials given by (9.5.1) can also be obtained from the Racah polynomials by taking $\delta=-\beta-N-1$ and letting $\gamma\to\infty$:

$$\lim_{\gamma\to\infty} R_n(\lambda(x);\alpha,\beta,\gamma,-\beta-N-1)=Q_n(x;\alpha,\beta,N). \tag{9.2.16}$$

Another way to do this is to take $\alpha+1=-N$ and $\beta\to\beta+\gamma+N+1$ and then take the limit $\delta\to\infty$. In that case we obtain the Hahn polynomials given by (9.5.1) in the following way:

$$\lim_{\delta\to\infty} R_n(\lambda(x);-N-1,\beta+\gamma+N+1,\gamma,\delta)=Q_n(x;\gamma,\beta,N). \tag{9.2.17}$$

Racah → Dual Hahn

The dual Hahn polynomials given by (9.6.1) are obtained from the Racah polynomials if we take $\alpha+1=-N$ and let $\beta\to\infty$:

$$\lim_{\beta\to\infty} R_n(\lambda(x);-N-1,\beta,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N). \tag{9.2.18}$$

The dual Hahn polynomials given by (9.6.1) are also obtained from the Racah polynomials if we take $\beta=-\delta-N-1$ and let $\alpha\to\infty$:

$$\lim_{\alpha\to\infty} R_n(\lambda(x);\alpha,-\delta-N-1,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N). \tag{9.2.19}$$

Finally, the dual Hahn polynomials given by (9.6.1) are also obtained from the Racah polynomials if we take $\gamma+1=-N$ and $\delta\to\alpha+\delta+N+1$ and take the limit $\beta\to\infty$:

$$\lim_{\beta\to\infty} R_n(\lambda(x);\alpha,\beta,-N-1,\alpha+\delta+N+1)=R_n(\lambda(x);\alpha,\delta,N). \tag{9.2.20}$$

Remark

If we set $\alpha = a+b-1$, $\beta = c+d-1$, $\gamma = a+d-1$, $\delta = a-d$ and $x \to -a+ix$ in the definition (9.2.1) of the Racah polynomials we obtain the Wilson polynomials given by (9.1.1):

$$R_n(\lambda(-a+ix); a+b-1, c+d-1, a+d-1, a-d)$$
$$= \tilde{W}_n(x^2; a,b,c,d) = \frac{W_n(x^2; a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n}.$$

References

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9.3 Continuous dual Hahn

Hypergeometric representation

$$\frac{S_n(x^2; a,b,c)}{(a+b)_n(a+c)_n} = {}_3F_2\left(\begin{array}{c} -n, a+ix, a-ix \\ a+b, a+c \end{array}; 1\right). \tag{9.3.1}$$

Orthogonality relation

If a, b and c are positive except possibly for a pair of complex conjugates with positive real parts, then

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2; a,b,c) S_n(x^2; a,b,c)\, dx$$
$$= \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c) n!\, \delta_{mn}. \tag{9.3.2}$$

If $a < 0$ and $a+b$, $a+c$ are positive or a pair of complex conjugates with positive real parts, then

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2; a,b,c) S_n(x^2; a,b,c)\, dx$$
$$+ \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b-a)\Gamma(c-a)}{\Gamma(-2a)}$$
$$\times \sum_{\substack{k=0,1,2\ldots \\ a+k<0}} \frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k}{(a)_k(a-b+1)_k(a-c+1)_k k!}(-1)^k$$
$$\times S_m(-(a+k)^2; a,b,c) S_n(-(a+k)^2; a,b,c)$$
$$= \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c) n!\, \delta_{mn}. \tag{9.3.3}$$

Recurrence relation

$$-\left(a^2+x^2\right)\tilde{S}_n(x^2) = A_n \tilde{S}_{n+1}(x^2) - (A_n + C_n)\tilde{S}_n(x^2) + C_n \tilde{S}_{n-1}(x^2), \tag{9.3.4}$$

where

$$\tilde{S}_n(x^2) := \tilde{S}_n(x^2; a,b,c) = \frac{S_n(x^2; a,b,c)}{(a+b)_n(a+c)_n}$$

and

$$\begin{cases} A_n = (n+a+b)(n+a+c) \\ C_n = n(n+b+c-1). \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n - a^2)p_n(x) + A_{n-1}C_n p_{n-1}(x), \tag{9.3.5}$$

where

$$S_n(x^2;a,b,c) = (-1)^n p_n(x^2).$$

Difference equation

$$ny(x) = B(x)y(x+i) - [B(x) + D(x)]\,y(x) + D(x)y(x-i), \tag{9.3.6}$$

where

$$y(x) = S_n(x^2;a,b,c)$$

and

$$\begin{cases} B(x) = \dfrac{(a-ix)(b-ix)(c-ix)}{2ix(2ix-1)} \\ D(x) = \dfrac{(a+ix)(b+ix)(c+ix)}{2ix(2ix+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned} &S_n((x+\tfrac{1}{2}i)^2;a,b,c) - S_n((x-\tfrac{1}{2}i)^2;a,b,c) \\ &= -2inxS_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2}) \end{aligned} \tag{9.3.7}$$

or equivalently

$$\frac{\delta S_n(x^2;a,b,c)}{\delta x^2} = -nS_{n-1}(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2}). \tag{9.3.8}$$

Backward shift operator

$$\begin{aligned} &(a-\tfrac{1}{2}-ix)(b-\tfrac{1}{2}-ix)(c-\tfrac{1}{2}-ix)S_n((x+\tfrac{1}{2}i)^2;a,b,c) \\ &\quad - (a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)(c-\tfrac{1}{2}+ix)S_n((x-\tfrac{1}{2}i)^2;a,b,c) \\ &= -2ixS_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2}) \end{aligned} \tag{9.3.9}$$

or equivalently

$$\begin{aligned} &\frac{\delta\left[\omega(x;a,b,c)S_n(x^2;a,b,c)\right]}{\delta x^2} \\ &= \omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2})S_{n+1}(x^2;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2}), \end{aligned} \tag{9.3.10}$$

where

$$\omega(x;a,b,c) = \frac{1}{2ix}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2.$$

Rodrigues-type formula

$$\omega(x;a,b,c)S_n(x^2;a,b,c) = \left(\frac{\delta}{\delta x^2}\right)^n \left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n)\right]. \tag{9.3.11}$$

Generating functions

$$(1-t)^{-c+ix}\,{}_2F_1\left(\begin{matrix}a+ix,b+ix\\a+b\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+b)_n n!}t^n. \tag{9.3.12}$$

$$(1-t)^{-b+ix}\,{}_2F_1\left(\begin{matrix}a+ix,c+ix\\a+c\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+c)_n n!}t^n. \tag{9.3.13}$$

$$(1-t)^{-a+ix}\,{}_2F_1\left(\begin{matrix}b+ix,c+ix\\b+c\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(b+c)_n n!}t^n. \tag{9.3.14}$$

$$\mathrm{e}^t\,{}_2F_2\left(\begin{matrix}a+ix,a-ix\\a+b,a+c\end{matrix};-t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n n!}t^n. \tag{9.3.15}$$

$$\begin{aligned}&(1-t)^{-\gamma}\,{}_3F_2\left(\begin{matrix}\gamma,a+ix,a-ix\\a+b,a+c\end{matrix};\frac{t}{t-1}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n n!}t^n,\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.3.16}$$

Limit relations

Wilson → Continuous dual Hahn

The continuous dual Hahn polynomials can be found from the Wilson polynomials given by (9.1.1) by dividing by $(a+d)_n$ and letting $d\to\infty$:

$$\lim_{d\to\infty}\frac{W_n(x^2;a,b,c,d)}{(a+d)_n}=S_n(x^2;a,b,c).$$

Continuous dual Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials given by (9.7.1) can be obtained from the continuous dual Hahn polynomials by the substitutions $x\to x-t$, $a=\lambda+it$, $b=\lambda-it$ and $c=t\cot\phi$ and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{S_n((x-t)^2;\lambda+it,\lambda-it,t\cot\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\sin\phi)^n}. \tag{9.3.17}$$

Remark

Since we have for $k<n$

$$\frac{(a+b)_n(a+c)_n}{(a+b)_k(a+c)_k}=(a+b+k)_{n-k}(a+c+k)_{n-k},$$

the continuous dual Hahn polynomials defined by (9.3.1) can also be seen as polynomials in the parameters a, b and c.

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.4 Continuous Hahn

Hypergeometric representation

$$p_n(x;a,b,c,d) = i^n \frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix\\ a+c, a+d\end{matrix};1\right). \tag{9.4.1}$$

Orthogonality relation

If $Re(a,b,c,d) > 0$, $c = \bar{a}$ and $d = \bar{b}$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)p_m(x;a,b,c,d)p_n(x;a,b,c,d)\,dx = \frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)n!}\,\delta_{mn}. \tag{9.4.2}$$

Recurrence relation

$$(a+ix)\tilde{p}_n(x) = A_n\tilde{p}_{n+1}(x) - (A_n+C_n)\,\tilde{p}_n(x) + C_n\tilde{p}_{n-1}(x), \tag{9.4.3}$$

where

$$\tilde{p}_n(x) := \tilde{p}_n(x;a,b,c,d) = \frac{n!}{i^n(a+c)_n(a+d)_n}p_n(x;a,b,c,d)$$

and

$$\begin{cases} A_n = -\dfrac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \\ C_n = \dfrac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + i(A_n+C_n+a)p_n(x) - A_{n-1}C_np_{n-1}(x), \tag{9.4.4}$$

where

$$p_n(x;a,b,c,d) = \frac{(n+a+b+c+d-1)_n}{n!}p_n(x).$$

Difference equation

$$n(n+a+b+c+d-1)y(x) = B(x)y(x+i) - [B(x)+D(x)]\,y(x) + D(x)y(x-i), \tag{9.4.5}$$

where

$$y(x) = p_n(x;a,b,c,d)$$

and

$$\begin{cases} B(x) = (c-ix)(d-ix) \\ D(x) = (a+ix)(b+ix). \end{cases}$$

Forward shift operator

$$\begin{aligned}&p_n(x+\tfrac{1}{2}i;a,b,c,d)-p_n(x-\tfrac{1}{2}i;a,b,c,d)\\&=i(n+a+b+c+d-1)p_{n-1}(x;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2})\end{aligned} \tag{9.4.6}$$

or equivalently

$$\frac{\delta p_n(x;a,b,c,d)}{\delta x}=(n+a+b+c+d-1)p_{n-1}(x;a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}). \tag{9.4.7}$$

Backward shift operator

$$\begin{aligned}&(c-\tfrac{1}{2}-ix)(d-\tfrac{1}{2}-ix)p_n(x+\tfrac{1}{2}i;a,b,c,d)\\&\quad-(a-\tfrac{1}{2}+ix)(b-\tfrac{1}{2}+ix)p_n(x-\tfrac{1}{2}i;a,b,c,d)\\&=\frac{n+1}{i}p_{n+1}(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})\end{aligned} \tag{9.4.8}$$

or equivalently

$$\begin{aligned}&\frac{\delta\left[\omega(x;a,b,c,d)p_n(x;a,b,c,d)\right]}{\delta x}\\&=-(n+1)\omega(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2})\\&\quad\times p_{n+1}(x;a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}),\end{aligned} \tag{9.4.9}$$

where

$$\omega(x;a,b,c,d)=\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix).$$

Rodrigues-type formula

$$\begin{aligned}&\omega(x;a,b,c,d)p_n(x;a,b,c,d)\\&=\frac{(-1)^n}{n!}\left(\frac{\delta}{\delta x}\right)^n\left[\omega(x;a+\tfrac{1}{2}n,b+\tfrac{1}{2}n,c+\tfrac{1}{2}n,d+\tfrac{1}{2}n)\right].\end{aligned} \tag{9.4.10}$$

Generating functions

$${}_1F_1\left(\begin{matrix}a+ix\\a+c\end{matrix};-it\right){}_1F_1\left(\begin{matrix}d-ix\\b+d\end{matrix};it\right)=\sum_{n=0}^{\infty}\frac{p_n(x;a,b,c,d)}{(a+c)_n(b+d)_n}t^n. \tag{9.4.11}$$

$${}_1F_1\left(\begin{matrix}a+ix\\a+d\end{matrix};-it\right){}_1F_1\left(\begin{matrix}c-ix\\b+c\end{matrix};it\right)=\sum_{n=0}^{\infty}\frac{p_n(x;a,b,c,d)}{(a+d)_n(b+c)_n}t^n. \tag{9.4.12}$$

$$\begin{aligned}&(1-t)^{1-a-b-c-d}{}_3F_2\left(\begin{matrix}\frac{1}{2}(a+b+c+d-1),\frac{1}{2}(a+b+c+d),a+ix\\a+c,a+d\end{matrix};-\frac{4t}{(1-t)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(a+b+c+d-1)_n}{(a+c)_n(a+d)_ni^n}p_n(x;a,b,c,d)t^n.\end{aligned} \tag{9.4.13}$$

Limit relations

Wilson → Continuous Hahn

The continuous Hahn polynomials are obtained from the Wilson polynomials given by (9.1.1) by the substitution $a \to a - it$, $b \to b - it$, $c \to c + it$, $d \to d + it$ and $x \to x + t$ and the limit $t \to \infty$ in the following way:

$$\lim_{t\to\infty} \frac{W_n((x+t)^2; a-it, b-it, c+it, d+it)}{(-2t)^n n!} = p_n(x; a, b, c, d).$$

Continuous Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials given by (9.7.1) can be obtained from the continuous Hahn polynomials by setting $x \to x + t$, $a = \lambda - it$, $c = \lambda + it$ and $b = d = t\tan\phi$ and taking the limit $t \to \infty$:

$$\lim_{t\to\infty} \frac{p_n(x+t; \lambda - it, t\tan\phi, \lambda + it, t\tan\phi)}{t^n} = \frac{P_n^{(\lambda)}(x;\phi)}{(\cos\phi)^n}. \tag{9.4.14}$$

Continuous Hahn → Jacobi

The Jacobi polynomials given by (9.8.1) follow from the continuous Hahn polynomials by the substitution $x \to \frac{1}{2}xt$, $a = \frac{1}{2}(\alpha + 1 - it)$, $b = \frac{1}{2}(\beta + 1 + it)$, $c = \frac{1}{2}(\alpha + 1 + it)$ and $d = \frac{1}{2}(\beta + 1 - it)$, division by t^n and the limit $t \to \infty$:

$$\begin{aligned}&\lim_{t\to\infty} \frac{p_n(\frac{1}{2}xt; \frac{1}{2}(\alpha+1-it), \frac{1}{2}(\beta+1+it), \frac{1}{2}(\alpha+1+it), \frac{1}{2}(\beta+1-it))}{t^n}\\ &= P_n^{(\alpha,\beta)}(x).\end{aligned} \tag{9.4.15}$$

Continuous Hahn → Pseudo Jacobi

The pseudo Jacobi polynomials given by (9.9.1) follow from the continuous Hahn polynomials by the substitution $x \to xt$, $a = \frac{1}{2}(-N + i\nu - 2t)$, $b = \frac{1}{2}(-N - i\nu + 2t)$, $c = \frac{1}{2}(-N - i\nu - 2t)$ and $d = \frac{1}{2}(-N + i\nu + 2t)$, division by t^n and the limit $t \to \infty$:

$$\begin{aligned}&\lim_{t\to\infty} \frac{p_n(xt; \frac{1}{2}(-N+i\nu-2t), \frac{1}{2}(-N-i\nu+2t), \frac{1}{2}(-N+i\nu-2t), \frac{1}{2}(-N-i\nu+2t))}{t^n}\\ &= \frac{(n-2N-1)_n}{n!} P_n(x;\nu,N).\end{aligned} \tag{9.4.16}$$

Remark

Since we have for $k < n$

$$\frac{(a+b)_n(a+c)_n}{(a+b)_k(a+c)_k} = (a+b+k)_{n-k}(a+c+k)_{n-k},$$

the continuous Hahn polynomials defined by (9.4.1) can also be seen as polynomials in the parameters a, b and c.

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.5 Hahn

Hypergeometric representation

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1, -N\end{matrix};1\right), \quad n=0,1,2,\ldots,N. \tag{9.5.1}$$

Orthogonality relation

For $\alpha > -1$ and $\beta > -1$, or for $\alpha < -N$ and $\beta < -N$, we have

$$\begin{aligned}&\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_m(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)\\ &= \frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}\,\delta_{mn}.\end{aligned} \tag{9.5.2}$$

Recurrence relation

$$-xQ_n(x) = A_nQ_{n+1}(x) - (A_n+C_n)\,Q_n(x) + C_nQ_{n-1}(x), \tag{9.5.3}$$

where

$$Q_n(x) := Q_n(x;\alpha,\beta,N)$$

and

$$\begin{cases} A_n = \dfrac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\\ C_n = \dfrac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.\end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (A_n+C_n)\,p_n(x) + A_{n-1}C_np_{n-1}(x), \tag{9.5.4}$$

where

$$Q_n(x;\alpha,\beta,N) = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N)_n}p_n(x).$$

Difference equation

$$n(n+\alpha+\beta+1)y(x) = B(x)y(x+1) - [B(x)+D(x)]\,y(x) + D(x)y(x-1), \tag{9.5.5}$$

where

$$y(x) = Q_n(x;\alpha,\beta,N)$$

and

$$\begin{cases} B(x) = (x+\alpha+1)(x-N)\\ D(x) = x(x-\beta-N-1).\end{cases}$$

Forward shift operator

$$\begin{aligned}&Q_n(x+1;\alpha,\beta,N) - Q_n(x;\alpha,\beta,N)\\ &= -\frac{n(n+\alpha+\beta+1)}{(\alpha+1)N}Q_{n-1}(x;\alpha+1,\beta+1,N-1)\end{aligned} \tag{9.5.6}$$

or equivalently

$$\Delta Q_n(x;\alpha,\beta,N) = -\frac{n(n+\alpha+\beta+1)}{(\alpha+1)N}Q_{n-1}(x;\alpha+1,\beta+1,N-1). \tag{9.5.7}$$

Backward shift operator

$$\begin{aligned}&(x+\alpha)(N+1-x)Q_n(x;\alpha,\beta,N)-x(\beta+N+1-x)Q_n(x-1;\alpha,\beta,N)\\&=\alpha(N+1)Q_{n+1}(x;\alpha-1,\beta-1,N+1)\end{aligned}\tag{9.5.8}$$

or equivalently

$$\begin{aligned}&\nabla\left[\omega(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)\right]\\&=\frac{N+1}{\beta}\omega(x;\alpha-1,\beta-1,N+1)Q_{n+1}(x;\alpha-1,\beta-1,N+1),\end{aligned}\tag{9.5.9}$$

where

$$\omega(x;\alpha,\beta,N)=\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}.$$

Rodrigues-type formula

$$\begin{aligned}&\omega(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N)\\&=\frac{(-1)^n(\beta+1)_n}{(-N)_n}\nabla^n\left[\omega(x;\alpha+n,\beta+n,N-n)\right].\end{aligned}\tag{9.5.10}$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$${}_1F_1\left(\begin{matrix}-x\\\alpha+1\end{matrix};-t\right){}_1F_1\left(\begin{matrix}x-N\\\beta+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{(\beta+1)_n n!}Q_n(x;\alpha,\beta,N)t^n.\tag{9.5.11}$$

$$\begin{aligned}&{}_2F_0\left(\begin{matrix}-x,-x+\beta+N+1\\-\end{matrix};-t\right){}_2F_0\left(\begin{matrix}x-N,x+\alpha+1\\-\end{matrix};t\right)\\&=\sum_{n=0}^{N}\frac{(-N)_n(\alpha+1)_n}{n!}Q_n(x;\alpha,\beta,N)t^n.\end{aligned}\tag{9.5.12}$$

$$\begin{aligned}&\left[(1-t)^{-\alpha-\beta-1}{}_3F_2\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2),-x\\\alpha+1,-N\end{matrix};-\frac{4t}{(1-t)^2}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\alpha+\beta+1)_n}{n!}Q_n(x;\alpha,\beta,N)t^n.\end{aligned}\tag{9.5.13}$$

Limit relations

Racah → Hahn

If we take $\gamma+1=-N$ and let $\delta\to\infty$ in the definition (9.2.1) of the Racah polynomials, we obtain the Hahn polynomials. Hence

$$\lim_{\delta\to\infty}R_n(\lambda(x);\alpha,\beta,-N-1,\delta)=Q_n(x;\alpha,\beta,N).$$

And if we take $\delta=-\beta-N-1$ and let $\gamma\to\infty$ in the definition (9.2.1) of the Racah polynomials, we also obtain the Hahn polynomials:

$$\lim_{\gamma\to\infty} R_n(\lambda(x);\alpha,\beta,\gamma,-\beta-N-1) = Q_n(x;\alpha,\beta,N).$$

Another way to do this is to take $\alpha+1=-N$ and $\beta \to \beta+\gamma+N+1$ in the definition (9.2.1) of the Racah polynomials and then take the limit $\delta \to \infty$. In that case we obtain the Hahn polynomials in the following way:

$$\lim_{\delta\to\infty} R_n(\lambda(x);-N-1,\beta+\gamma+N+1,\gamma,\delta) = Q_n(x;\gamma,\beta,N).$$

Hahn → Jacobi

To find the Jacobi polynomials given by (9.8.1) from the Hahn polynomials we take $x \to Nx$ and let $N \to \infty$. In fact we have

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}. \tag{9.5.14}$$

Hahn → Meixner

The Meixner polynomials given by (9.10.1) can be obtained from the Hahn polynomials by taking $\alpha = b-1$, $\beta = N(1-c)c^{-1}$ and letting $N \to \infty$:

$$\lim_{N\to\infty} Q_n(x;b-1,N(1-c)c^{-1},N) = M_n(x;b,c). \tag{9.5.15}$$

Hahn → Krawtchouk

The Krawtchouk polynomials given by (9.11.1) are obtained from the Hahn polynomials if we take $\alpha = pt$ and $\beta = (1-p)t$ and let $t \to \infty$:

$$\lim_{t\to\infty} Q_n(x;pt,(1-p)t,N) = K_n(x;p,N). \tag{9.5.16}$$

Remark

If we interchange the role of x and n in (9.5.1) we obtain the dual Hahn polynomials given by (9.6.1).
Since

$$Q_n(x;\alpha,\beta,N) = R_x(\lambda(n);\alpha,\beta,N)$$

we obtain the dual orthogonality relation for the Hahn polynomials from the orthogonality relation (9.6.2) of the dual Hahn polynomials:

$$\sum_{n=0}^{N} \frac{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!} Q_n(x;\alpha,\beta,N)Q_n(y;\alpha,\beta,N)$$
$$= \frac{\delta_{xy}}{\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}}, \quad x,y \in \{0,1,2,\ldots,N\}.$$

References

[?], [?].

9.6 Dual Hahn

Hypergeometric representation

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-n,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix};1\right), \quad n=0,1,2,\ldots,N, \tag{9.6.1}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1).$$

Orthogonality relation

For $\gamma > -1$ and $\delta > -1$, or for $\gamma < -N$ and $\delta < -N$, we have

$$\begin{aligned}&\sum_{x=0}^{N}\frac{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_xN!}{(-1)^x(x+\gamma+\delta+1)_{N+1}(\delta+1)_xx!}R_m(\lambda(x);\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)\\ &= \frac{\delta_{mn}}{\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}}.\end{aligned} \tag{9.6.2}$$

Recurrence relation

$$\lambda(x)R_n(\lambda(x)) = A_nR_{n+1}(\lambda(x)) - (A_n+C_n)R_n(\lambda(x)) + C_nR_{n-1}(\lambda(x)), \tag{9.6.3}$$

where

$$R_n(\lambda(x)) := R_n(\lambda(x);\gamma,\delta,N)$$

and

$$\begin{cases} A_n = (n+\gamma+1)(n-N) \\ C_n = n(n-\delta-N-1). \end{cases}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) - (A_n+C_n)p_n(x) + A_{n-1}C_np_{n-1}(x), \tag{9.6.4}$$

where

$$R_n(\lambda(x);\gamma,\delta,N) = \frac{1}{(\gamma+1)_n(-N)_n}p_n(\lambda(x)).$$

Difference equation

$$-ny(x) = B(x)y(x+1) - [B(x)+D(x)]\,y(x) + D(x)y(x-1), \tag{9.6.5}$$

where

$$y(x) = R_n(\lambda(x);\gamma,\delta,N)$$

and

$$\begin{cases} B(x) = \dfrac{(x+\gamma+1)(x+\gamma+\delta+1)(N-x)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \\ D(x) = \dfrac{x(x+\gamma+\delta+N+1)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}. \end{cases}$$

Forward shift operator

$$\begin{aligned}&R_n(\lambda(x+1);\gamma,\delta,N)-R_n(\lambda(x);\gamma,\delta,N)\\&=-\frac{n(2x+\gamma+\delta+2)}{(\gamma+1)N}R_{n-1}(\lambda(x);\gamma+1,\delta,N-1)\end{aligned}\tag{9.6.6}$$

or equivalently

$$\frac{\Delta R_n(\lambda(x);\gamma,\delta,N)}{\Delta\lambda(x)}=-\frac{n}{(\gamma+1)N}R_{n-1}(\lambda(x);\gamma+1,\delta,N-1).\tag{9.6.7}$$

Backward shift operator

$$\begin{aligned}&(x+\gamma)(x+\gamma+\delta)(N+1-x)R_n(\lambda(x);\gamma,\delta,N)\\&\qquad-x(x+\gamma+\delta+N+1)(x+\delta)R_n(\lambda(x-1);\gamma,\delta,N)\\&=\gamma(N+1)(2x+\gamma+\delta)R_{n+1}(\lambda(x);\gamma-1,\delta,N+1)\end{aligned}\tag{9.6.8}$$

or equivalently

$$\begin{aligned}&\frac{\nabla\left[\omega(x;\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)\right]}{\nabla\lambda(x)}\\&=\frac{1}{\gamma+\delta}\omega(x;\gamma-1,\delta,N+1)R_{n+1}(\lambda(x);\gamma-1,\delta,N+1),\end{aligned}\tag{9.6.9}$$

where

$$\omega(x;\gamma,\delta,N)=\frac{(-1)^x(\gamma+1)_x(\gamma+\delta+1)_x(-N)_x}{(\gamma+\delta+N+2)_x(\delta+1)_x x!}.$$

Rodrigues-type formula

$$\omega(x;\gamma,\delta,N)R_n(\lambda(x);\gamma,\delta,N)=(\gamma+\delta+1)_n\left(\nabla_\lambda\right)^n\left[\omega(x;\gamma+n,\delta,N-n)\right], \tag{9.6.10}$$

where

$$\nabla_\lambda:=\frac{\nabla}{\nabla\lambda(x)}.$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$$(1-t)^{N-x}\,{}_2F_1\left(\begin{array}{c}-x,-x-\delta\\ \gamma+1\end{array};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{n!}R_n(\lambda(x);\gamma,\delta,N)t^n. \tag{9.6.11}$$

$$\begin{aligned}&(1-t)^x\,{}_2F_1\left(\begin{array}{c}x-N,x+\gamma+1\\ -\delta-N\end{array};t\right)\\&=\sum_{n=0}^{N}\frac{(\gamma+1)_n(-N)_n}{(-\delta-N)_n n!}R_n(\lambda(x);\gamma,\delta,N)t^n.\end{aligned} \tag{9.6.12}$$

$$\left[e^t\,{}_2F_2\left(\begin{array}{c}-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{array};-t\right)\right]_N=\sum_{n=0}^{N}\frac{R_n(\lambda(x);\gamma,\delta,N)}{n!}t^n. \tag{9.6.13}$$

$$\begin{aligned}&\left[(1-t)^{-\varepsilon}\,{}_3F_2\left(\begin{array}{c}\varepsilon,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{array};\frac{t}{t-1}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\varepsilon)_n}{n!}R_n(\lambda(x);\gamma,\delta,N)t^n,\quad \varepsilon \text{ arbitrary.}\end{aligned} \tag{9.6.14}$$

Limit relations

Racah → Dual Hahn

If we take $\alpha+1=-N$ and let $\beta\to\infty$ in the definition (9.2.1) of the Racah polynomials, then we obtain the dual Hahn polynomials:

$$\lim_{\beta\to\infty}R_n(\lambda(x);-N-1,\beta,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N).$$

And if we take $\beta=-\delta-N-1$ and let $\alpha\to\infty$ in the definition (9.2.1) of the Racah polynomials, then we also obtain the dual Hahn polynomials:

$$\lim_{\alpha\to\infty}R_n(\lambda(x);\alpha,-\delta-N-1,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta,N).$$

Finally, if we take $\gamma+1=-N$ and $\delta\to\alpha+\delta+N+1$ in the definition (9.2.1) of the Racah polynomials and take the limit $\beta\to\infty$ we find the dual Hahn polynomials in the following way:

$$\lim_{\beta\to\infty}R_n(\lambda(x);\alpha,\beta,-N-1,\alpha+\delta+N+1)=R_n(\lambda(x);\alpha,\delta,N).$$

Dual Hahn → Meixner

The Meixner polynomials given by (9.10.1) are obtained from the dual Hahn polynomials if we take $\gamma=\beta-1$ and $\delta=N(1-c)c^{-1}$ and let $N\to\infty$:

$$\lim_{N\to\infty}R_n(\lambda(x);\beta-1,N(1-c)c^{-1},N)=M_n(x;\beta,c). \tag{9.6.15}$$

Dual Hahn → Krawtchouk

The Krawtchouk polynomials given by (9.11.1) can be obtained from the dual Hahn polynomials by setting $\gamma = pt$, $\delta = (1-p)t$ and letting $t \to \infty$:

$$\lim_{t\to\infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N). \tag{9.6.16}$$

Remark

If we interchange the role of x and n in the definition (9.6.1) of the dual Hahn polynomials we obtain the Hahn polynomials given by (9.5.1).
Since

$$R_n(\lambda(x); \gamma, \delta, N) = Q_x(n; \gamma, \delta, N)$$

we obtain the dual orthogonality relation for the dual Hahn polynomials from the orthogonality relation (9.5.2) for the Hahn polynomials:

$$\sum_{n=0}^{N} \binom{\gamma+n}{n}\binom{\delta+N-n}{N-n} R_n(\lambda(x); \gamma, \delta, N) R_n(\lambda(y); \gamma, \delta, N)$$
$$= \frac{(-1)^x (x+\gamma+\delta+1)_{N+1} (\delta+1)_x x!}{(2x+\gamma+\delta+1)(\gamma+1)_x (-N)_x N!} \delta_{xy}, \quad x, y \in \{0, 1, 2, \ldots, N\}.$$

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.7 Meixner-Pollaczek

Hypergeometric representation

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix} -n, \lambda+ix \\ 2\lambda \end{matrix}; 1-e^{-2i\phi}\right). \tag{9.7.1}$$

Orthogonality relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{(2\phi-\pi)x} |\Gamma(\lambda+ix)|^2 P_m^{(\lambda)}(x; \phi) P_n^{(\lambda)}(x; \phi)\, dx = \frac{\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda} n!} \delta_{mn}, \tag{9.7.2}$$

Recurrence relation

$$(n+1) P_{n+1}^{(\lambda)}(x; \phi) - 2\left[x\sin\phi + (n+\lambda)\cos\phi\right] P_n^{(\lambda)}(x; \phi)$$
$$+ (n+2\lambda-1) P_{n-1}^{(\lambda)}(x; \phi) = 0. \tag{9.7.3}$$

Normalized recurrence relation

$$x p_n(x) = p_{n+1}(x) - \left(\frac{n+\lambda}{\tan\phi}\right) p_n(x) + \frac{n(n+2\lambda-1)}{4\sin^2\phi} p_{n-1}(x), \tag{9.7.4}$$

where

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\sin\phi)^n}{n!} p_n(x).$$

Difference equation

$$\begin{aligned} &\mathrm{e}^{i\phi}(\lambda - ix)y(x+i) + 2i\left[x\cos\phi - (n+\lambda)\sin\phi\right]y(x) \\ &\quad - \mathrm{e}^{-i\phi}(\lambda + ix)y(x-i) = 0, \quad y(x) = P_n^{(\lambda)}(x;\phi). \end{aligned} \tag{9.7.5}$$

Forward shift operator

$$P_n^{(\lambda)}(x+\tfrac{1}{2}i;\phi) - P_n^{(\lambda)}(x-\tfrac{1}{2}i;\phi) = (e^{i\phi} - e^{-i\phi})P_{n-1}^{(\lambda+\frac{1}{2})}(x;\phi) \tag{9.7.6}$$

or equivalently

$$\frac{\delta P_n^{(\lambda)}(x;\phi)}{\delta x} = 2\sin\phi\, P_{n-1}^{(\lambda+\frac{1}{2})}(x;\phi). \tag{9.7.7}$$

Backward shift operator

$$\begin{aligned} &\mathrm{e}^{i\phi}(\lambda - \tfrac{1}{2} - ix)P_n^{(\lambda)}(x+\tfrac{1}{2}i;\phi) + \mathrm{e}^{-i\phi}(\lambda - \tfrac{1}{2} + ix)P_n^{(\lambda)}(x-\tfrac{1}{2}i;\phi) \\ &= (n+1)P_{n+1}^{(\lambda-\frac{1}{2})}(x;\phi) \end{aligned} \tag{9.7.8}$$

or equivalently

$$\frac{\delta\left[\omega(x;\lambda,\phi)P_n^{(\lambda)}(x;\phi)\right]}{\delta x} = -(n+1)\omega(x;\lambda-\tfrac{1}{2},\phi)P_{n+1}^{(\lambda-\frac{1}{2})}(x;\phi), \tag{9.7.9}$$

where

$$\omega(x;\lambda,\phi) = \Gamma(\lambda+ix)\Gamma(\lambda-ix)\mathrm{e}^{(2\phi-\pi)x}.$$

Rodrigues-type formula

$$\omega(x;\lambda,\phi)P_n^{(\lambda)}(x;\phi) = \frac{(-1)^n}{n!}\left(\frac{\delta}{\delta x}\right)^n\left[\omega(x;\lambda+\tfrac{1}{2}n,\phi)\right]. \tag{9.7.10}$$

Generating functions

$$(1-e^{i\phi}t)^{-\lambda+ix}(1-e^{-i\phi}t)^{-\lambda-ix}=\sum_{n=0}^{\infty}P_n^{(\lambda)}(x;\phi)t^n. \tag{9.7.11}$$

$$\mathrm{e}^t\,{}_1F_1\left(\begin{array}{c}\lambda+ix\\2\lambda\end{array};(e^{-2i\phi}-1)t\right)=\sum_{n=0}^{\infty}\frac{P_n^{(\lambda)}(x;\phi)}{(2\lambda)_n e^{in\phi}}t^n. \tag{9.7.12}$$

$$\begin{aligned}&(1-t)^{-\gamma}{}_2F_1\left(\begin{array}{c}\gamma,\lambda+ix\\2\lambda\end{array};\frac{(1-\mathrm{e}^{-2i\phi})t}{t-1}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2\lambda)_n}\frac{P_n^{(\lambda)}(x;\phi)}{\mathrm{e}^{in\phi}}t^n,\quad \gamma\text{ arbitrary.}\end{aligned} \tag{9.7.13}$$

Limit relations

Continuous dual Hahn → Meixner-Pollaczek

The Meixner-Pollaczek polynomials can be obtained from the continuous dual Hahn polynomials given by (9.3.1) by the substitutions $x\to x-t$, $a=\lambda+it$, $b=\lambda-it$ and $c=t\cot\phi$ and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{S_n((x-t)^2;\lambda+it,\lambda-it,t\cot\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\sin\phi)^n}.$$

Continuous Hahn → Meixner-Pollaczek

By setting $x\to x+t$, $a=\lambda-it$, $c=\lambda+it$ and $b=d=t\tan\phi$ in the definition (9.4.1) of the continuous Hahn polynomials and taking the limit $t\to\infty$ we obtain the Meixner-Pollaczek polynomials:

$$\lim_{t\to\infty}\frac{p_n(x+t;\lambda-it,t\tan\phi,\lambda+it,t\tan\phi)}{t^n n!}=\frac{P_n^{(\lambda)}(x;\phi)}{(\cos\phi)^n}.$$

Meixner-Pollaczek → Laguerre

The Laguerre polynomials given by (9.12.1) can be obtained from the Meixner-Pollaczek polynomials by the substitution $\lambda = \frac{1}{2}(\alpha+1)$, $x \to -\frac{1}{2}\phi^{-1}x$ and the limit $\phi \to 0$:

$$\lim_{\phi\to 0} P_n^{(\frac{1}{2}\alpha+\frac{1}{2})}(-\tfrac{1}{2}\phi^{-1}x;\phi) = L_n^{(\alpha)}(x). \tag{9.7.14}$$

Meixner-Pollaczek → Hermite

The Hermite polynomials given by (9.15.1) are obtained from the Meixner-Pollaczek polynomials if we substitute $x \to (\sin\phi)^{-1}(x\sqrt{\lambda} - \lambda\cos\phi)$ and then let $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}n} P_n^{(\lambda)}((\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi);\phi) = \frac{H_n(x)}{n!}. \tag{9.7.15}$$

Remark

Since we have for $k < n$

$$\frac{(2\lambda)_n}{(2\lambda)_k} = (2\lambda+k)_{n-k},$$

the Meixner-Pollaczek polynomials defined by (9.7.1) can also be seen as polynomials in the parameter λ.

References

[?], [?].

9.8 Jacobi

Hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right). \tag{9.8.1}$$

Orthogonality relation

For $\alpha > -1$ and $\beta > -1$ we have

$$\begin{aligned}&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\,dx\\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\,\delta_{mn}.\end{aligned} \tag{9.8.2}$$

For $\alpha+\beta < -2N-1$, $\beta > -1$ and $m,n \in \{0,1,2,\ldots,N\}$ we also have

$$\begin{aligned}&\int_{1}^{\infty}(x+1)^{\alpha}(x-1)^{\beta}P_m^{(\alpha,\beta)}(-x)P_n^{(\alpha,\beta)}(-x)\,dx\\ &= -\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(-n-\alpha-\beta)\Gamma(n+\alpha+\beta+1)}{\Gamma(-n-\alpha)n!}\,\delta_{mn}.\end{aligned} \tag{9.8.3}$$

Recurrence relation

$$\begin{aligned}
xP_n^{(\alpha,\beta)}(x) &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x)\\
&\quad + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x). \qquad (9.8.4)
\end{aligned}$$

Normalized recurrence relation

$$\begin{aligned}
xp_n(x) &= p_{n+1}(x) + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}p_n(x)\\
&\quad + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}p_{n-1}(x) \qquad (9.8.5)
\end{aligned}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!}p_n(x).$$

Differential equation

$$\begin{aligned}
&(1-x^2)y''(x) + [\beta-\alpha-(\alpha+\beta+2)x]\,y'(x)\\
&\quad + n(n+\alpha+\beta+1)y(x) = 0, \quad y(x) = P_n^{(\alpha,\beta)}(x). \qquad (9.8.6)
\end{aligned}$$

Forward shift operator

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \qquad (9.8.7)$$

Backward shift operator

$$\begin{aligned}
&(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) + [(\beta-\alpha)-(\alpha+\beta)x]\,P_n^{(\alpha,\beta)}(x)\\
&= -2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(x) \qquad (9.8.8)
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\frac{d}{dx}\left[(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\right]\\
&= -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1}P_{n+1}^{(\alpha-1,\beta-1)}(x). \qquad (9.8.9)
\end{aligned}$$

Rodrigues-type formula

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \qquad (9.8.10)$$

Generating functions

$$\frac{2^{\alpha+\beta}}{R(1+R-t)^{\alpha}(1+R+t)^{\beta}}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n,\quad R=\sqrt{1-2xt+t^2}. \tag{9.8.11}$$

$$\begin{aligned}&{}_0F_1\left(\begin{array}{c}-\\ \alpha+1\end{array};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{array}{c}-\\ \beta+1\end{array};\frac{(x+1)t}{2}\right)\\&=\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)}{(\alpha+1)_n(\beta+1)_n}t^n.\end{aligned} \tag{9.8.12}$$

$$\begin{aligned}&(1-t)^{-\alpha-\beta-1}{}_2F_1\left(\begin{array}{c}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2)\\ \alpha+1\end{array};\frac{2(x-1)t}{(1-t)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x)t^n.\end{aligned} \tag{9.8.13}$$

$$\begin{aligned}&(1+t)^{-\alpha-\beta-1}{}_2F_1\left(\begin{array}{c}\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2)\\ \beta+1\end{array};\frac{2(x+1)t}{(1+t)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(x)t^n.\end{aligned} \tag{9.8.14}$$

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\gamma,\alpha+\beta+1-\gamma\\ \alpha+1\end{array};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{array}{c}\gamma,\alpha+\beta+1-\gamma\\ \beta+1\end{array};\frac{1-R+t}{2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\alpha+\beta+1-\gamma)_n}{(\alpha+1)_n(\beta+1)_n}P_n^{(\alpha,\beta)}(x)t^n,\quad R=\sqrt{1-2xt+t^2}\end{aligned} \tag{9.8.15}$$

with γ arbitrary.

Limit relations

Wilson → Jacobi

The Jacobi polynomials can be found from the Wilson polynomials given by (9.1.1) by substituting $a=b=\frac{1}{2}(\alpha+1)$, $c=\frac{1}{2}(\beta+1)+it$, $d=\frac{1}{2}(\beta+1)-it$ and $x\rightarrow t\sqrt{\frac{1}{2}(1-x)}$ in the definition (9.1.1) of the Wilson polynomials and taking the limit $t\rightarrow\infty$. In fact we have

$$\lim_{t\rightarrow\infty}\frac{W_n(\frac{1}{2}(1-x)t^2;\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta+1)+it,\frac{1}{2}(\beta+1)-it)}{t^{2n}n!}=P_n^{(\alpha,\beta)}(x).$$

Continuous Hahn → Jacobi

The Jacobi polynomials follow from the continuous Hahn polynomials given by (9.4.1) by using the substitution $x\rightarrow\frac{1}{2}xt$, $a=\frac{1}{2}(\alpha+1-it)$, $b=\frac{1}{2}(\beta+1+it)$, $c=\frac{1}{2}(\alpha+1+it)$ and $d=\frac{1}{2}(\beta+1-it)$ in (9.4.1), division by t^n and the limit $t\rightarrow\infty$:

$$\lim_{t\rightarrow\infty}\frac{p_n(\frac{1}{2}xt;\frac{1}{2}(\alpha+1-it),\frac{1}{2}(\beta+1+it),\frac{1}{2}(\alpha+1+it),\frac{1}{2}(\beta+1-it))}{t^n}=P_n^{(\alpha,\beta)}(x).$$

Hahn → Jacobi

To find the Jacobi polynomials from the Hahn polynomials given by (9.5.1) we take $x \to Nx$ in (9.5.1) and let $N \to \infty$. In fact we have

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

Jacobi → Laguerre

The Laguerre polynomials given by (9.12.1) can be obtained from the Jacobi polynomials by setting $x \to 1-2\beta^{-1}x$ and then the limit $\beta \to \infty$:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x) = L_n^{(\alpha)}(x). \tag{9.8.16}$$

Jacobi → Bessel

The Bessel polynomials given by (9.13.1) are obtained from the Jacobi polynomials if we take $\beta = a-\alpha$ and let $\alpha \to -\infty$:

$$\lim_{\alpha\to-\infty} \frac{P_n^{(\alpha,a-\alpha)}(1+\alpha x)}{P_n^{(\alpha,a-\alpha)}(1)} = y_n(x;a). \tag{9.8.17}$$

Jacobi → Hermite

The Hermite polynomials given by (9.15.1) follow from the Jacobi polynomials by taking $\beta = \alpha$ and letting $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}}x) = \frac{H_n(x)}{2^n n!}. \tag{9.8.18}$$

Remarks

The definition (9.8.1) of the Jacobi polynomials can also be written as:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!}(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k}\left(\frac{1-x}{2}\right)^k.$$

In this way the Jacobi polynomials can also be seen as polynomials in the parameters α and β. Therefore they can be defined for all α and β. Then we have the following connection with the Meixner polynomials given by (9.10.1):

$$\frac{(\beta)_n}{n!}M_n(x;\beta,c) = P_n^{(\beta-1,-n-\beta-x)}((2-c)c^{-1}).$$

The Jacobi polynomials are related to the pseudo Jacobi polynomials defined by (9.9.1) in the following way:

$$P_n(x;\nu,N) = \frac{(-2i)^n n!}{(n-2N-1)_n} P_n^{(-N-1+i\nu,-N-1-i\nu)}(ix).$$

The Jacobi polynomials are also related to the Gegenbauer (or ultraspherical) polynomials given by (9.8.19) by the quadratic transformations:

$$C_{2n}^{(\lambda)}(x) = \frac{(\lambda)_n}{(\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

and

$$C_{2n+1}^{(\lambda)}(x) = \frac{(\lambda)_{n+1}}{(\frac{1}{2})_{n+1}} x P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1).$$

References

[?], [?].

Special cases

9.8.1 Gegenbauer / Ultraspherical

Hypergeometric representation

The Gegenbauer (or ultraspherical) polynomials are Jacobi polynomials with $\alpha=\beta=\lambda-\frac{1}{2}$ and another normalization:

$$\begin{aligned} C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) \\ &= \frac{(2\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix};\frac{1-x}{2}\right), \quad \lambda\neq 0. \end{aligned} \tag{9.8.19}$$

Orthogonality relation

$$\begin{aligned} &\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_m^{(\lambda)}(x)C_n^{(\lambda)}(x)\,dx \\ &= \frac{\pi\Gamma(n+2\lambda)2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2(n+\lambda)n!}\,\delta_{mn}, \quad \lambda>-\frac{1}{2} \quad \lambda\neq 0. \end{aligned} \tag{9.8.20}$$

Recurrence relation

$$2(n+\lambda)xC_n^{(\lambda)}(x) = (n+1)C_{n+1}^{(\lambda)}(x)+(n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \tag{9.8.21}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x)+\frac{n(n+2\lambda-1)}{4(n+\lambda-1)(n+\lambda)}p_{n-1}(x), \tag{9.8.22}$$

where

$$C_n^{(\lambda)}(x) = \frac{2^n(\lambda)_n}{n!}p_n(x).$$

Differential equation

$$(1-x^2)y''(x)-(2\lambda+1)xy'(x)+n(n+2\lambda)y(x)=0, \quad y(x)=C_n^{(\lambda)}(x). \tag{9.8.23}$$

Forward shift operator

$$\frac{d}{dx}C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x). \tag{9.8.24}$$

Backward shift operator

$$(1-x^2)\frac{d}{dx}C_n^{(\lambda)}(x)+(1-2\lambda)xC_n^{(\lambda)}(x)=-\frac{(n+1)(2\lambda+n-1)}{2(\lambda-1)}C_{n+1}^{(\lambda-1)}(x) \tag{9.8.25}$$

or equivalently

$$\begin{aligned}&\frac{d}{dx}\left[(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)\right]\\&=-\frac{(n+1)(2\lambda+n-1)}{2(\lambda-1)}(1-x^2)^{\lambda-\frac{3}{2}}C_{n+1}^{(\lambda-1)}(x).\end{aligned} \tag{9.8.26}$$

Rodrigues-type formula

$$(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)=\frac{(2\lambda)_n(-1)^n}{(\lambda+\frac{1}{2})_n 2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x^2)^{\lambda+n-\frac{1}{2}}\right]. \tag{9.8.27}$$

Generating functions

$$(1-2xt+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n. \tag{9.8.28}$$

$$R^{-1}\left(\frac{1+R-xt}{2}\right)^{\frac{1}{2}-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n}C_n^{(\lambda)}(x)t^n, \quad R=\sqrt{1-2xt+t^2}. \tag{9.8.29}$$

$${}_0F_1\left(\begin{array}{c}-\\ \lambda+\frac{1}{2}\end{array};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{array}{c}-\\ \lambda+\frac{1}{2}\end{array};\frac{(x+1)t}{2}\right)=\sum_{n=0}^{\infty}\frac{C_n^{(\lambda)}(x)}{(2\lambda)_n(\lambda+\frac{1}{2})_n}t^n. \tag{9.8.30}$$

$$\mathrm{e}^{xt}\,{}_0F_1\left(\begin{array}{c}-\\ \lambda+\frac{1}{2}\end{array};\frac{(x^2-1)t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{C_n^{(\lambda)}(x)}{(2\lambda)_n}t^n. \tag{9.8.31}$$

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\gamma,2\lambda-\gamma\\ \lambda+\frac{1}{2}\end{array};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{array}{c}\gamma,2\lambda-\gamma\\ \lambda+\frac{1}{2}\end{array};\frac{1-R+t}{2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(2\lambda-\gamma)_n}{(2\lambda)_n(\lambda+\frac{1}{2})_n}C_n^{(\lambda)}(x)t^n, \quad R=\sqrt{1-2xt+t^2}, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.32}$$

$$\begin{aligned}&(1-xt)^{-\gamma}{}_2F_1\left(\begin{array}{c}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \lambda+\frac{1}{2}\end{array};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2\lambda)_n}C_n^{(\lambda)}(x)t^n, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.33}$$

Limit relation

Gegenbauer / Ultraspherical → Hermite

The Hermite polynomials given by (9.15.1) follow from the Gegenbauer (or ultraspherical) polynomials by taking $\lambda=\alpha+\frac{1}{2}$ and letting $\alpha\to\infty$ in the following way:

$$\lim_{\alpha\to\infty}\alpha^{-\frac{1}{2}n}C_n^{(\alpha+\frac{1}{2})}(\alpha^{-\frac{1}{2}}x)=\frac{H_n(x)}{n!}. \tag{9.8.34}$$

Remarks

The case $\lambda = 0$ needs another normalization. In that case we have the Chebyshev polynomials of the first kind described in the next subsection.
The Gegenbauer (or ultraspherical) polynomials are related to the Jacobi polynomials given by (9.8.1) by the quadratic transformations:

$$C_{2n}^{(\lambda)}(x) = \frac{(\lambda)_n}{\left(\frac{1}{2}\right)_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

and

$$C_{2n+1}^{(\lambda)}(x) = \frac{(\lambda)_{n+1}}{\left(\frac{1}{2}\right)_{n+1}} x P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1).$$

References

[?], [?].

9.8.2 Chebyshev

Hypergeometric representation

The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = -\frac{1}{2}$:

$$T_n(x) = \frac{P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)}{P_n^{(-\frac{1}{2},-\frac{1}{2})}(1)} = {}_2F_1\left(\begin{matrix}-n,n\\ \frac{1}{2}\end{matrix};\frac{1-x}{2}\right) \tag{9.8.35}$$

and the Chebyshev polynomials of the second kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = \frac{1}{2}$:

$$U_n(x) = (n+1)\frac{P_n^{(\frac{1}{2},\frac{1}{2})}(x)}{P_n^{(\frac{1}{2},\frac{1}{2})}(1)} = (n+1)\,{}_2F_1\left(\begin{matrix}-n,n+2\\ \frac{3}{2}\end{matrix};\frac{1-x}{2}\right). \tag{9.8.36}$$

Orthogonality relation

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_m(x)T_n(x)\,dx = \begin{cases}\dfrac{\pi}{2}\,\delta_{mn},\ n\neq 0\\[2ex] \pi\,\delta_{mn},\ n=0.\end{cases} \tag{9.8.37}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_m(x)U_n(x)\,dx = \frac{\pi}{2}\,\delta_{mn}. \tag{9.8.38}$$

Recurrence relations

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x), \quad T_0(x) = 1 \quad \text{and} \quad T_1(x) = x. \tag{9.8.39}$$

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_0(x) = 1 \quad \text{and} \quad U_1(x) = 2x. \tag{9.8.40}$$

Normalized recurrence relations

$$xp_n(x) = p_{n+1}(x) + \frac{1}{4}p_{n-1}(x), \tag{9.8.41}$$

where

$$T_1(x) = p_1(x) = x \quad \text{and} \quad T_n(x) = 2^n p_n(x), \quad n \neq 1.$$

$$xp_n(x) = p_{n+1}(x) + \frac{1}{4}p_{n-1}(x), \tag{9.8.42}$$

where

$$U_n(x) = 2^n p_n(x).$$

Differential equations

$$(1-x^2)y''(x) - xy'(x) + n^2 y(x) = 0, \quad y(x) = T_n(x). \tag{9.8.43}$$

$$(1-x^2)y''(x) - 3xy'(x) + n(n+2)y(x) = 0, \quad y(x) = U_n(x). \tag{9.8.44}$$

Forward shift operator

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \tag{9.8.45}$$

Backward shift operator

$$(1-x^2)\frac{d}{dx}U_n(x) - xU_n(x) = -(n+1)T_{n+1}(x) \tag{9.8.46}$$

or equivalently

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}} U_n(x)\right] = -(n+1)\left(1-x^2\right)^{-\frac{1}{2}} T_{n+1}(x). \tag{9.8.47}$$

Rodrigues-type formulas

$$(1-x^2)^{-\frac{1}{2}} T_n(x) = \frac{(-1)^n}{(\frac{1}{2})_n 2^n}\left(\frac{d}{dx}\right)^n \left[(1-x^2)^{n-\frac{1}{2}}\right]. \tag{9.8.48}$$

$$(1-x^2)^{\frac{1}{2}} U_n(x) = \frac{(n+1)(-1)^n}{(\frac{3}{2})_n 2^n}\left(\frac{d}{dx}\right)^n \left[(1-x^2)^{n+\frac{1}{2}}\right]. \tag{9.8.49}$$

Generating functions

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n. \tag{9.8.50}$$

$$R^{-1}\sqrt{\frac{1}{2}(1+R-xt)} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} T_n(x)t^n, \quad R = \sqrt{1-2xt+t^2}. \tag{9.8.51}$$

$${}_0F_1\left(\begin{array}{c} - \\ \frac{1}{2} \end{array}; \frac{(x-1)t}{2}\right) {}_0F_1\left(\begin{array}{c} - \\ \frac{1}{2} \end{array}; \frac{(x+1)t}{2}\right) = \sum_{n=0}^{\infty} \frac{T_n(x)}{\left(\frac{1}{2}\right)_n n!} t^n. \tag{9.8.52}$$

$$\mathrm{e}^{xt}\, {}_0F_1\left(\begin{array}{c} - \\ \frac{1}{2} \end{array}; \frac{(x^2-1)t^2}{4}\right) = \sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n. \tag{9.8.53}$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\gamma,-\gamma\\ \frac{1}{2}\end{matrix};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{matrix}\gamma,-\gamma\\ \frac{1}{2}\end{matrix};\frac{1+R+t}{2}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(-\gamma)_n}{\left(\frac{1}{2}\right)_n n!}T_n(x)t^n,\quad R=\sqrt{1-2xt+t^2},\quad \gamma\text{ arbitrary.}
\end{aligned}\tag{9.8.54}
$$

$$
\begin{aligned}
&(1-xt)^{-\gamma}{}_2F_1\left(\begin{matrix}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \frac{1}{2}\end{matrix};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}T_n(x)t^n,\quad \gamma\text{ arbitrary.}
\end{aligned}\tag{9.8.55}
$$

$$
\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n.\tag{9.8.56}
$$

$$
\frac{1}{R\sqrt{\frac{1}{2}(1+R-xt)}}=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(n+1)!}U_n(x)t^n,\quad R=\sqrt{1-2xt+t^2}.\tag{9.8.57}
$$

$$
{}_0F_1\left(\begin{matrix}-\\ \frac{3}{2}\end{matrix};\frac{(x-1)t}{2}\right){}_0F_1\left(\begin{matrix}-\\ \frac{3}{2}\end{matrix};\frac{(x+1)t}{2}\right)=\sum_{n=0}^{\infty}\frac{U_n(x)}{\left(\frac{3}{2}\right)_n(n+1)!}t^n.\tag{9.8.58}
$$

$$
\mathrm{e}^{xt}\,{}_0F_1\left(\begin{matrix}-\\ \frac{3}{2}\end{matrix};\frac{(x^2-1)t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{U_n(x)}{(n+1)!}t^n.\tag{9.8.59}
$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\gamma,2-\gamma\\ \frac{3}{2}\end{matrix};\frac{1-R-t}{2}\right){}_2F_1\left(\begin{matrix}\gamma,2-\gamma\\ \frac{3}{2}\end{matrix};\frac{1-R+t}{2}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n(2-\gamma)_n}{\left(\frac{3}{2}\right)_n(n+1)!}U_n(x)t^n,\quad R=\sqrt{1-2xt+t^2},\quad \gamma\text{ arbitrary.}
\end{aligned}\tag{9.8.60}
$$

$$
\begin{aligned}
&(1-xt)^{-\gamma}{}_2F_1\left(\begin{matrix}\frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2}\\ \frac{3}{2}\end{matrix};\frac{(x^2-1)t^2}{(1-xt)^2}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(n+1)!}U_n(x)t^n,\quad \gamma\text{ arbitrary.}
\end{aligned}\tag{9.8.61}
$$

Remarks

The Chebyshev polynomials can also be written as:

$$
T_n(x)=\cos(n\theta),\quad x=\cos\theta
$$

and

$$
U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\quad x=\cos\theta.
$$

Further we have

$$
U_n(x)=C_n^{(1)}(x)
$$

where $C_n^{(\lambda)}(x)$ denotes the Gegenbauer (or ultraspherical) polynomial given by (9.8.19) in the preceding subsection.

References

[?], [?].

9.8.3 Legendre / Spherical

Hypergeometric representation

The Legendre (or spherical) polynomials are Jacobi polynomials with $\alpha = \beta = 0$:

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left(\begin{matrix}-n, n+1\\ 1\end{matrix}; \frac{1-x}{2}\right). \tag{9.8.62}$$

Orthogonality relation

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\,\delta_{mn}. \tag{9.8.63}$$

Recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x). \tag{9.8.64}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n^2}{(2n-1)(2n+1)}p_{n-1}(x), \tag{9.8.65}$$

where

$$P_n(x) = \binom{2n}{n}\frac{1}{2^n}p_n(x).$$

Differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0, \quad y(x) = P_n(x). \tag{9.8.66}$$

Rodrigues-type formula

$$P_n(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x^2)^n\right]. \tag{9.8.67}$$

Generating functions

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{9.8.68}$$

$${}_0F_1\left(\begin{matrix}-\\ 1\end{matrix}; \frac{(x-1)t}{2}\right){}_0F_1\left(\begin{matrix}-\\ 1\end{matrix}; \frac{(x+1)t}{2}\right) = \sum_{n=0}^{\infty}\frac{P_n(x)}{(n!)^2}t^n. \tag{9.8.69}$$

$$\mathrm{e}^{xt}\,{}_0F_1\left(\begin{matrix}-\\ 1\end{matrix}; \frac{(x^2-1)t^2}{4}\right) = \sum_{n=0}^{\infty}\frac{P_n(x)}{n!}t^n. \tag{9.8.70}$$

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}\gamma, 1-\gamma\\ 1\end{matrix}; \frac{1-R-t}{2}\right){}_2F_1\left(\begin{matrix}\gamma, 1-\gamma\\ 1\end{matrix}; \frac{1-R+t}{2}\right)\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_n(1-\gamma)_n}{(n!)^2}P_n(x)t^n, \quad R = \sqrt{1-2xt+t^2}, \quad \gamma \text{ arbitrary}.\end{aligned} \tag{9.8.71}$$

$$(1-xt)^{-\gamma}{}_2F_1\left(\begin{array}{c}\frac{1}{2}\gamma, \frac{1}{2}\gamma+\frac{1}{2}\\ 1\end{array}; \frac{(x^2-1)t^2}{(1-xt)^2}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}P_n(x)t^n, \quad \gamma \text{ arbitrary.} \tag{9.8.72}$$

References

[?], **[?]**.

9.9 Pseudo Jacobi

Hypergeometric representation

$$P_n(x;\nu,N) = \frac{(-2i)^n(-N+i\nu)_n}{(n-2N-1)_n}\,{}_2F_1\left(\begin{array}{c}-n, n-2N-1\\ -N+i\nu\end{array};\frac{1-ix}{2}\right) \tag{9.9.1}$$
$$= (x+i)^n\,{}_2F_1\left(\begin{array}{c}-n, N+1-n-i\nu\\ 2N+2-2n\end{array};\frac{2}{1-ix}\right),\quad n=0,1,2,\ldots,N.$$

Orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}P_m(x;\nu,N)P_n(x;\nu,N)\,dx$$
$$= \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)\,|\Gamma(N+1-n+i\nu)|^2}\,\delta_{mn}. \tag{9.9.2}$$

Recurrence relation

$$xP_n(x;\nu,N) = P_{n+1}(x;\nu,N) + \frac{(N+1)\nu}{(n-N-1)(n-N)}P_n(x;\nu,N)$$
$$-\frac{n(n-2N-2)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)}$$
$$\times(n-N-1-i\nu)(n-N-1+i\nu)P_{n-1}(x;\nu,N). \tag{9.9.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{(N+1)\nu}{(n-N-1)(n-N)}p_n(x)$$
$$-\frac{n(n-2N-2)(n-N-1-i\nu)(n-N-1+i\nu)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)}p_{n-1}(x), \tag{9.9.4}$$

where

$$P_n(x;\nu,N) = p_n(x).$$

Differential equation

$$(1+x^2)y''(x) + 2(\nu - Nx)\,y'(x) - n(n-2N-1)y(x) = 0, \tag{9.9.5}$$

where

$$y(x) = P_n(x;\nu,N).$$

Forward shift operator

$$\frac{d}{dx}P_n(x;\nu,N) = nP_{n-1}(x;\nu,N-1). \tag{9.9.6}$$

Backward shift operator

$$(1+x^2)\frac{d}{dx}P_n(x;\nu,N)+2\left[\nu-(N+1)x\right]P_n(x;\nu,N)$$
$$=(n-2N-2)P_{n+1}(x;\nu,N+1) \tag{9.9.7}$$

or equivalently

$$\frac{d}{dx}\left[(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}P_n(x;\nu,N)\right]$$
$$=(n-2N-2)(1+x^2)^{-N-2}\mathrm{e}^{2\nu\arctan x}P_{n+1}(x;\nu,N+1). \tag{9.9.8}$$

Rodrigues-type formula

$$P_n(x;\nu,N)=\frac{(1+x^2)^{N+1}e^{-2\nu\arctan x}}{(n-2N-1)_n}\left(\frac{d}{dx}\right)^n\left[(1+x^2)^{n-N-1}e^{2\nu\arctan x}\right]. \tag{9.9.9}$$

Generating function

$$\left[{}_0F_1\left(\begin{array}{c}-\\-N+i\nu\end{array};(x+i)t\right){}_0F_1\left(\begin{array}{c}-\\-N-i\nu\end{array};(x-i)t\right)\right]_N$$
$$=\sum_{n=0}^{N}\frac{(n-2N-1)_n}{(-N+i\nu)_n(-N-i\nu)_n n!}P_n(x;\nu,N)t^n. \tag{9.9.10}$$

Limit relation

Continuous Hahn → Pseudo Jacobi

The pseudo Jacobi polynomials follow from the continuous Hahn polynomials given by (9.4.1) by the substitutions $x\to xt$, $a=\frac{1}{2}(-N+i\nu-2t)$, $b=\frac{1}{2}(-N-i\nu+2t)$, $c=\frac{1}{2}(-N-i\nu-2t)$ and $d=\frac{1}{2}(-N+i\nu+2t)$, division by t^n and the limit $t\to\infty$:

$$\lim_{t\to\infty}\frac{p_n(xt;\frac{1}{2}(-N+i\nu-2t),\frac{1}{2}(-N-i\nu+2t),\frac{1}{2}(-N+i\nu-2t),\frac{1}{2}(-N-i\nu+2t))}{t^n}$$
$$=\frac{(n-2N-1)_n}{n!}P_n(x;\nu,N).$$

Remarks

Since we have for $k<n$

$$\frac{(-N+i\nu)_n}{(-N+i\nu)_k}=(-N+i\nu+k)_{n-k},$$

the pseudo Jacobi polynomials given by (9.9.1) can also be seen as polynomials in the parameter ν.

The weight function for the pseudo Jacobi polynomials can be written as

$$(1+x^2)^{-N-1}\mathrm{e}^{2\nu\arctan x}=(1+ix)^{-N-1-i\nu}(1-ix)^{-N-1+i\nu}.$$

The pseudo Jacobi polynomials are related to the Jacobi polynomials defined by (9.8.1) in the following way:

$$P_n(x;\nu,N)=\frac{(-2i)^n n!}{(n-2N-1)_n}P_n^{(-N-1+i\nu,-N-1-i\nu)}(ix).$$

If we set $x\to\nu x$ in the definition (9.9.1) of the pseudo Jacobi polynomials and take the limit $\nu\to\infty$ we obtain a special case of the Bessel polynomials given by (9.13.1) in the following way:

$$\lim_{\nu\to\infty}\frac{P_n(\nu x;\nu,N)}{\nu^n}=\frac{2^n}{(n-2N-1)_n}y_n(x;-2N-2).$$

References

[?], [?], [?].

9.10 Meixner

Hypergeometric representation

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{c}\right). \tag{9.10.1}$$

Orthogonality relation

$$\sum_{x=0}^{\infty}\frac{(\beta)_x}{x!}c^x M_m(x;\beta,c)M_n(x;\beta,c) = \frac{c^{-n}n!}{(\beta)_n(1-c)^\beta}\,\delta_{mn} \tag{9.10.2}$$

Recurrence relation

$$\begin{aligned}(c-1)xM_n(x;\beta,c) &= c(n+\beta)M_{n+1}(x;\beta,c)\\ &\quad -[n+(n+\beta)c]\,M_n(x;\beta,c)+nM_{n-1}(x;\beta,c).\end{aligned} \tag{9.10.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x)+\frac{n+(n+\beta)c}{1-c}p_n(x)+\frac{n(n+\beta-1)c}{(1-c)^2}p_{n-1}(x), \tag{9.10.4}$$

where

$$M_n(x;\beta,c) = \frac{1}{(\beta)_n}\left(\frac{c-1}{c}\right)^n p_n(x).$$

Difference equation

$$n(c-1)y(x) = c(x+\beta)y(x+1)-[x+(x+\beta)c]\,y(x)+xy(x-1), \tag{9.10.5}$$

where

$$y(x) = M_n(x;\beta,c).$$

Forward shift operator

$$M_n(x+1;\beta,c)-M_n(x;\beta,c) = \frac{n}{\beta}\left(\frac{c-1}{c}\right)M_{n-1}(x;\beta+1,c) \tag{9.10.6}$$

or equivalently

$$\Delta M_n(x;\beta,c) = \frac{n}{\beta}\left(\frac{c-1}{c}\right)M_{n-1}(x;\beta+1,c). \tag{9.10.7}$$

Backward shift operator

$$c(\beta+x-1)M_n(x;\beta,c)-xM_n(x-1;\beta,c) = c(\beta-1)M_{n+1}(x;\beta-1,c) \tag{9.10.8}$$

or equivalently

$$\nabla\left[\frac{(\beta)_x c^x}{x!}M_n(x;\beta,c)\right] = \frac{(\beta-1)_x c^x}{x!}M_{n+1}(x;\beta-1,c). \tag{9.10.9}$$

Rodrigues-type formula

$$\frac{(\beta)_x c^x}{x!} M_n(x;\beta,c) = \nabla^n \left[\frac{(\beta+n)_x c^x}{x!}\right]. \tag{9.10.10}$$

Generating functions

$$\left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x;\beta,c) t^n. \tag{9.10.11}$$

$$\mathrm{e}^t \, {}_1F_1\left(\begin{matrix}-x\\ \beta\end{matrix}; \left(\frac{1-c}{c}\right)t\right) = \sum_{n=0}^{\infty} \frac{M_n(x;\beta,c)}{n!} t^n. \tag{9.10.12}$$

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma, -x\\ \beta\end{matrix}; \frac{(1-c)t}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} M_n(x;\beta,c) t^n, \quad \gamma \text{ arbitrary}. \tag{9.10.13}$$

Limit relations

Hahn → Meixner

If we take $\alpha = b-1$, $\beta = N(1-c)c^{-1}$ in the definition (9.5.1) of the Hahn polynomials and let $N \to \infty$ we find the Meixner polynomials:

$$\lim_{N\to\infty} Q_n(x;b-1,N(1-c)c^{-1},N) = M_n(x;b,c).$$

Dual Hahn → Meixner

To obtain the Meixner polynomials from the dual Hahn polynomials we have to take $\gamma = \beta - 1$ and $\delta = N(1-c)c^{-1}$ in the definition (9.6.1) of the dual Hahn polynomials and let $N \to \infty$:

$$\lim_{N\to\infty} R_n(\lambda(x);\beta-1,N(1-c)c^{-1},N) = M_n(x;\beta,c).$$

Meixner → Laguerre

The Laguerre polynomials given by (9.12.1) are obtained from the Meixner polynomials if we take $\beta = \alpha+1$ and $x \to (1-c)^{-1}x$ and let $c \to 1$:

$$\lim_{c\to 1} M_n((1-c)^{-1}x;\alpha+1,c) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}. \tag{9.10.14}$$

Meixner → Charlier

The Charlier polynomials given by (9.14.1) are obtained from the Meixner polynomials if we take $c = (a+\beta)^{-1}a$ and let $\beta \to \infty$:

$$\lim_{\beta\to\infty} M_n(x;\beta,(a+\beta)^{-1}a) = C_n(x;a). \tag{9.10.15}$$

Remarks

The Meixner polynomials are related to the Jacobi polynomials given by (9.8.1) in the following way:

$$\frac{(\beta)_n}{n!} M_n(x;\beta,c) = P_n^{(\beta-1,-n-\beta-x)}((2-c)c^{-1}).$$

The Meixner polynomials are also related to the Krawtchouk polynomials given by (9.11.1) in the following way:

$$K_n(x;p,N) = M_n(x;-N,(p-1)^{-1}p).$$

References

[?], [?].

9.11 Krawtchouk

Hypergeometric representation

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};\frac{1}{p}\right), \quad n=0,1,2,\ldots,N. \tag{9.11.1}$$

Orthogonality relation

$$\begin{aligned}&\sum_{x=0}^{N}\binom{N}{x}p^x(1-p)^{N-x}K_m(x;p,N)K_n(x;p,N)\\&=\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{mn}, \quad 0<p<1.\end{aligned} \tag{9.11.2}$$

Recurrence relation

$$\begin{aligned}-xK_n(x;p,N) = {}& p(N-n)K_{n+1}(x;p,N)\\&-[p(N-n)+n(1-p)]\,K_n(x;p,N)\\&+n(1-p)K_{n-1}(x;p,N).\end{aligned} \tag{9.11.3}$$

Normalized recurrence relation

$$\begin{aligned}xp_n(x) = {}& p_{n+1}(x)+[p(N-n)+n(1-p)]\,p_n(x)\\&+np(1-p)(N+1-n)p_{n-1}(x),\end{aligned} \tag{9.11.4}$$

where

$$K_n(x;p,N) = \frac{1}{(-N)_n p^n}p_n(x).$$

Difference equation

$$\begin{aligned}-ny(x) = {}& p(N-x)y(x+1)\\&-[p(N-x)+x(1-p)]\,y(x)+x(1-p)y(x-1),\end{aligned} \tag{9.11.5}$$

where

$$y(x) = K_n(x;p,N).$$

Forward shift operator

$$K_n(x+1;p,N)-K_n(x;p,N) = -\frac{n}{Np}K_{n-1}(x;p,N-1) \tag{9.11.6}$$

or equivalently

$$\Delta K_n(x;p,N) = -\frac{n}{Np}K_{n-1}(x;p,N-1). \tag{9.11.7}$$

Backward shift operator

$$\begin{aligned}&(N+1-x)K_n(x;p,N)-x\left(\frac{1-p}{p}\right)K_n(x-1;p,N)\\&=(N+1)K_{n+1}(x;p,N+1)\end{aligned}\tag{9.11.8}$$

or equivalently

$$\nabla\left[\binom{N}{x}\left(\frac{p}{1-p}\right)^xK_n(x;p,N)\right]=\binom{N+1}{x}\left(\frac{p}{1-p}\right)^xK_{n+1}(x;p,N+1).\tag{9.11.9}$$

Rodrigues-type formula

$$\binom{N}{x}\left(\frac{p}{1-p}\right)^xK_n(x;p,N)=\nabla^n\left[\binom{N-n}{x}\left(\frac{p}{1-p}\right)^x\right].\tag{9.11.10}$$

Generating functions

For $x=0,1,2,\ldots,N$ we have

$$\left(1-\frac{(1-p)}{p}t\right)^x(1+t)^{N-x}=\sum_{n=0}^{N}\binom{N}{n}K_n(x;p,N)t^n.\tag{9.11.11}$$

$$\left[e^t\,{}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N=\sum_{n=0}^{N}\frac{K_n(x;p,N)}{n!}t^n.\tag{9.11.12}$$

$$\begin{aligned}&\left[(1-t)^{-\gamma}\,{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\gamma)_n}{n!}K_n(x;p,N)t^n,\quad\gamma\text{ arbitrary.}\end{aligned}\tag{9.11.13}$$

Limit relations

Hahn → Krawtchouk

If we take $\alpha=pt$ and $\beta=(1-p)t$ in the definition (9.5.1) of the Hahn polynomials and let $t\to\infty$ we obtain the Krawtchouk polynomials:

$$\lim_{t\to\infty}Q_n(x;pt,(1-p)t,N)=K_n(x;p,N).$$

Dual Hahn → Krawtchouk

The Krawtchouk polynomials follow from the dual Hahn polynomials given by (9.6.1) if we set $\gamma=pt$, $\delta=(1-p)t$ and let $t\to\infty$:

$$\lim_{t\to\infty}R_n(\lambda(x);pt,(1-p)t,N)=K_n(x;p,N).$$

Krawtchouk → Charlier

The Charlier polynomials given by (9.14.1) can be found from the Krawtchouk polynomials by taking $p=N^{-1}a$ and letting $N\to\infty$:

$$\lim_{N\to\infty}K_n(x;N^{-1}a,N)=C_n(x;a).\tag{9.11.14}$$

Krawtchouk → Hermite

The Hermite polynomials given by (9.15.1) follow from the Krawtchouk polynomials by setting $x \to pN + x\sqrt{2p(1-p)N}$ and then letting $N \to \infty$:

$$\lim_{N\to\infty} \sqrt{\binom{N}{n}} K_n(pN + x\sqrt{2p(1-p)N}; p, N) = \frac{(-1)^n H_n(x)}{\sqrt{2^n n! \left(\frac{p}{1-p}\right)^n}}. \tag{9.11.15}$$

Remarks

The Krawtchouk polynomials are self-dual, which means that

$$K_n(x;p,N) = K_x(n;p,N), \quad n,x \in \{0,1,2,\dots,N\}.$$

By using this relation we easily obtain the so-called dual orthogonality relation from the orthogonality relation (9.11.2):

$$\sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} K_n(x;p,N) K_n(y;p,N) = \frac{\left(\frac{1-p}{p}\right)^x}{\binom{N}{x}} \delta_{xy},$$

where $0 < p < 1$ and $x, y \in \{0,1,2,\dots,N\}$.
The Krawtchouk polynomials are related to the Meixner polynomials given by (9.10.1) in the following way:

$$K_n(x;p,N) = M_n(x; -N, (p-1)^{-1}p).$$

References

[?], [?].

9.12 Laguerre

Hypergeometric representation

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right). \tag{9.12.1}$$

Orthogonality relation

$$\int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn}, \quad \alpha > -1. \tag{9.12.2}$$

Recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0. \tag{9.12.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (2n+\alpha+1)p_n(x) + n(n+\alpha)p_{n-1}(x), \tag{9.12.4}$$

where

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} p_n(x).$$

Differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0, \quad y(x) = L_n^{(\alpha)}(x). \tag{9.12.5}$$

Forward shift operator

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{9.12.6}$$

Backward shift operator

$$x\frac{d}{dx}L_n^{(\alpha)}(x) + (\alpha - x)L_n^{(\alpha)}(x) = (n+1)L_{n+1}^{(\alpha-1)}(x) \tag{9.12.7}$$

or equivalently

$$\frac{d}{dx}\left[e^{-x}x^{\alpha}L_n^{(\alpha)}(x)\right] = (n+1)e^{-x}x^{\alpha-1}L_{n+1}^{(\alpha-1)}(x). \tag{9.12.8}$$

Rodrigues-type formula

$$\mathrm{e}^{-x}x^{\alpha}L_n^{(\alpha)}(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n\left[e^{-x}x^{n+\alpha}\right]. \tag{9.12.9}$$

Generating functions

$$(1-t)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n. \tag{9.12.10}$$

$$\mathrm{e}^t\,{}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -xt\right) = \sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n}t^n. \tag{9.12.11}$$

$$(1-t)^{-\gamma}\,{}_1F_1\left(\begin{matrix} \gamma \\ \alpha+1 \end{matrix}; \frac{xt}{t-1}\right) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\alpha+1)_n}L_n^{(\alpha)}(x)t^n, \quad \gamma \text{ arbitrary}. \tag{9.12.12}$$

Limit relations

Meixner-Pollaczek → Laguerre

The Laguerre polynomials can be obtained from the Meixner-Pollaczek polynomials given by (9.7.1) by the substitution $\lambda = \frac{1}{2}(\alpha+1)$, $x \to -\frac{1}{2}\phi^{-1}x$ and the limit $\phi \to 0$:

$$\lim_{\phi\to 0} P_n^{(\frac{1}{2}\alpha+\frac{1}{2})}(-\tfrac{1}{2}\phi^{-1}x;\phi) = L_n^{(\alpha)}(x).$$

Jacobi → Laguerre

The Laguerre polynomials are obtained from the Jacobi polynomials given by (9.8.1) if we set $x \to 1-2\beta^{-1}x$ and then take the limit $\beta \to \infty$:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x) = L_n^{(\alpha)}(x).$$

Meixner → Laguerre

If we take $\beta = \alpha+1$ and $x \to (1-c)^{-1}x$ in the definition (9.10.1) of the Meixner polynomials and let $c \to 1$ we obtain the Laguerre polynomials:

$$\lim_{c\to 1} M_n((1-c)^{-1}x;\alpha+1,c) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$

Laguerre → Hermite

The Hermite polynomials given by (9.15.1) can be obtained from the Laguerre polynomials by taking the limit $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}((2\alpha)^{\frac{1}{2}}x+\alpha) = \frac{(-1)^n}{n!} H_n(x). \tag{9.12.13}$$

Remarks

The definition (9.12.1) of the Laguerre polynomials can also be written as:

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!}(\alpha+k+1)_{n-k}x^k.$$

In this way the Laguerre polynomials can also be seen as polynomials in the parameter α. Therefore they can be defined for all α.
The Laguerre polynomials are related to the Bessel polynomials given by (9.13.1) in the following way:

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!} y_n(2x^{-1}; -2n-\alpha-1).$$

The Laguerre polynomials are related to the Charlier polynomials given by (9.14.1) in the following way:

$$\frac{(-a)^n}{n!} C_n(x;a) = L_n^{(x-n)}(a).$$

The Laguerre polynomials and the Hermite polynomials given by (9.15.1) are also connected by the following quadratic transformations:

$$H_{2n}(x) = (-1)^n n!\, 2^{2n} L_n^{(-\frac{1}{2})}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n n!\, 2^{2n+1} x L_n^{(\frac{1}{2})}(x^2).$$

In combinatorics the Laguerre polynomials with $\alpha = 0$ are often called Rook polynomials.

References

[?], [?].

9.13 Bessel

Hypergeometric representation

$$\begin{aligned} y_n(x;a) &= {}_2F_0\left(\begin{matrix} -n, n+a+1 \\ - \end{matrix}; -\frac{x}{2}\right) \\ &= (n+a+1)_n \left(\frac{x}{2}\right)^n {}_1F_1\left(\begin{matrix} -n \\ -2n-a \end{matrix}; \frac{2}{x}\right), \quad n = 0,1,2,\ldots,N. \end{aligned} \tag{9.13.1}$$

Orthogonality relation

$$\int_0^\infty x^a e^{-\frac{2}{x}} y_m(x;a) y_n(x;a)\, dx$$
$$= -\frac{2^{a+1}}{2n+a+1}\Gamma(-n-a) n!\, \delta_{mn}, \quad a < -2N-1. \tag{9.13.2}$$

Recurrence relation

$$\begin{aligned} &2(n+a+1)(2n+a)y_{n+1}(x;a) \\ &= (2n+a+1)\left[2a+(2n+a)(2n+a+2)x\right] y_n(x;a) \\ &\quad + 2n(2n+a+2)y_{n-1}(x;a). \end{aligned} \tag{9.13.3}$$

Normalized recurrence relation

$$\begin{aligned} xp_n(x) = p_{n+1}(x) &- \frac{2a}{(2n+a)(2n+a+2)} p_n(x) \\ &- \frac{4n(n+a)}{(2n+a-1)(2n+a)^2(2n+a+1)} p_{n-1}(x), \end{aligned} \tag{9.13.4}$$

where

$$y_n(x;a) = \frac{(n+a+1)_n}{2^n} p_n(x).$$

Differential equation

$$x^2 y''(x) + [(a+2)x+2]\, y'(x) - n(n+a+1)y(x) = 0, \quad y(x) = y_n(x;a). \tag{9.13.5}$$

Forward shift operator

$$\frac{d}{dx} y_n(x;a) = \frac{n(n+a+1)}{2} y_{n-1}(x;a+2). \tag{9.13.6}$$

Backward shift operator

$$x^2 \frac{d}{dx} y_n(x;a) + (ax+2) y_n(x;a) = 2y_{n+1}(x;a-2) \tag{9.13.7}$$

or equivalently

$$\frac{d}{dx}\left[x^a e^{-\frac{2}{x}} y_n(x;a)\right] = 2x^{a-2} e^{-\frac{2}{x}} y_{n+1}(x;a-2). \tag{9.13.8}$$

Rodrigues-type formula

$$y_n(x;a) = 2^{-n} x^{-a} e^{\frac{2}{x}} D^n \left(x^{2n+a} e^{-\frac{2}{x}}\right). \tag{9.13.9}$$

Generating function

$$(1-2xt)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-2xt}}\right)^a \exp\left(\frac{2t}{1+\sqrt{1-2xt}}\right) = \sum_{n=0}^\infty y_n(x;a) \frac{t^n}{n!}. \tag{9.13.10}$$

Limit relation

Jacobi → Bessel

If we take $\beta = a - \alpha$ in the definition (9.8.1) of the Jacobi polynomials and let $\alpha \to -\infty$ we find the Bessel polynomials:

$$\lim_{\alpha\to-\infty} \frac{P_n^{(\alpha,a-\alpha)}(1+\alpha x)}{P_n^{(\alpha,a-\alpha)}(1)} = y_n(x;a).$$

Remarks

The following notations are also used for the Bessel polynomials:

$$y_n(x;a,b) = y_n(2b^{-1}x;a) \quad \text{and} \quad \theta_n(x;a,b) = x^n y_n(x^{-1};a,b).$$

However, the Bessel polynomials essentially depend on only one parameter.
The Bessel polynomials are related to the Laguerre polynomials given by (9.12.1) in the following way:

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!} y_n(2x^{-1}; -2n-\alpha-1).$$

The special case $a = -2N-2$ of the Bessel polynomials can be obtained from the pseudo Jacobi polynomials by setting $x \to \nu x$ in the definition (9.9.1) of the pseudo Jacobi polynomials and taking the limit $\nu \to \infty$ in the following way:

$$\lim_{\nu\to\infty} \frac{P_n(\nu x;\nu,N)}{\nu^n} = \frac{2^n}{(n-2N-1)_n} y_n(x;-2N-2).$$

References

[?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?].

9.14 Charlier

Hypergeometric representation

$$C_n(x;a) = {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array}; -\frac{1}{a}\right). \tag{9.14.1}$$

Orthogonality relation

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x;a) C_n(x;a) = a^{-n} e^a n!\, \delta_{mn}, \quad a > 0. \tag{9.14.2}$$

Recurrence relation

$$-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a). \tag{9.14.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + (n+a)p_n(x) + nap_{n-1}(x), \tag{9.14.4}$$

where

$$C_n(x;a) = \left(-\frac{1}{a}\right)^n p_n(x).$$

Difference equation

$$-ny(x) = ay(x+1) - (x+a)y(x) + xy(x-1), \quad y(x) = C_n(x;a). \tag{9.14.5}$$

Forward shift operator

$$C_n(x+1;a) - C_n(x;a) = -\frac{n}{a}C_{n-1}(x;a) \tag{9.14.6}$$

or equivalently

$$\Delta C_n(x;a) = -\frac{n}{a}C_{n-1}(x;a). \tag{9.14.7}$$

Backward shift operator

$$C_n(x;a) - \frac{x}{a}C_n(x-1;a) = C_{n+1}(x;a) \tag{9.14.8}$$

or equivalently

$$\nabla\left[\frac{a^x}{x!}C_n(x;a)\right] = \frac{a^x}{x!}C_{n+1}(x;a). \tag{9.14.9}$$

Rodrigues-type formula

$$\frac{a^x}{x!}C_n(x;a) = \nabla^n\left[\frac{a^x}{x!}\right]. \tag{9.14.10}$$

Generating function

$$\mathrm{e}^t\left(1-\frac{t}{a}\right)^x = \sum_{n=0}^{\infty}\frac{C_n(x;a)}{n!}t^n. \tag{9.14.11}$$

Limit relations

Meixner → Charlier

If we take $c = (a+\beta)^{-1}a$ in the definition (9.10.1) of the Meixner polynomials and let $\beta \to \infty$ we find the Charlier polynomials:

$$\lim_{\beta\to\infty} M_n(x;\beta,(a+\beta)^{-1}a) = C_n(x;a).$$

Krawtchouk → Charlier

The Charlier polynomials can be found from the Krawtchouk polynomials given by (9.11.1) by taking $p = N^{-1}a$ and letting $N \to \infty$:

$$\lim_{N\to\infty} K_n(x;N^{-1}a,N) = C_n(x;a).$$

Charlier → Hermite

The Hermite polynomials given by (9.15.1) are obtained from the Charlier polynomials if we set $x \to (2a)^{1/2}x + a$ and let $a \to \infty$. In fact we have

$$\lim_{a\to\infty}(2a)^{\frac{1}{2}n}C_n((2a)^{\frac{1}{2}}x+a;a) = (-1)^nH_n(x). \tag{9.14.12}$$

Remark

The Charlier polynomials are related to the Laguerre polynomials given by (9.12.1) in the following way:

$$\frac{(-a)^n}{n!}C_n(x;a) = L_n^{(x-n)}(a).$$

References

[?], [?].

9.15 Hermite

Hypergeometric representation

$$H_n(x) = (2x)^n \, {}_2F_0\left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix}; -\frac{1}{x^2}\right). \tag{9.15.1}$$

Orthogonality relation

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = 2^n n!\,\delta_{mn}. \tag{9.15.2}$$

Recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{9.15.3}$$

Normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{n}{2}p_{n-1}(x), \tag{9.15.4}$$

where

$$H_n(x) = 2^n p_n(x).$$

Differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad y(x) = H_n(x). \tag{9.15.5}$$

Forward shift operator

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x). \tag{9.15.6}$$

Backward shift operator

$$\frac{d}{dx}H_n(x) - 2xH_n(x) = -H_{n+1}(x) \tag{9.15.7}$$

or equivalently

$$\frac{d}{dx}\left[e^{-x^2}H_n(x)\right] = -e^{-x^2}H_{n+1}(x). \tag{9.15.8}$$

Rodrigues-type formula

$$e^{-x^2}H_n(x)=(-1)^n\left(\frac{d}{dx}\right)^n\left[e^{-x^2}\right]. \tag{9.15.9}$$

Generating functions

$$\exp\left(2xt-t^2\right)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{9.15.10}$$

$$\begin{cases} e^t\cos(2x\sqrt{t})=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}H_{2n}(x)t^n \\ \displaystyle\frac{e^t}{\sqrt{t}}\sin(2x\sqrt{t})=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}H_{2n+1}(x)t^n. \end{cases} \tag{9.15.11}$$

$$\begin{cases} e^{-t^2}\cosh(2xt)=\displaystyle\sum_{n=0}^{\infty}\frac{H_{2n}(x)}{(2n)!}t^{2n} \\ e^{-t^2}\sinh(2xt)=\displaystyle\sum_{n=0}^{\infty}\frac{H_{2n+1}(x)}{(2n+1)!}t^{2n+1}. \end{cases} \tag{9.15.12}$$

$$\begin{cases} (1+t^2)^{-\gamma}\,{}_1F_1\left(\begin{array}{c}\gamma\\ \frac{1}{2}\end{array};\frac{x^2t^2}{1+t^2}\right)=\displaystyle\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(2n)!}H_{2n}(x)t^{2n} \\ \displaystyle\frac{xt}{\sqrt{1+t^2}}\,{}_1F_1\left(\begin{array}{c}\gamma+\frac{1}{2}\\ \frac{3}{2}\end{array};\frac{x^2t^2}{1+t^2}\right)=\sum_{n=0}^{\infty}\frac{(\gamma+\frac{1}{2})_n}{(2n+1)!}H_{2n+1}(x)t^{2n+1} \end{cases} \tag{9.15.13}$$

with γ arbitrary.

$$\frac{1+2xt+4t^2}{(1+4t^2)^{\frac{3}{2}}}\exp\left(\frac{4x^2t^2}{1+4t^2}\right)=\sum_{n=0}^{\infty}\frac{H_n(x)}{\lfloor n/2\rfloor\,!}t^n, \tag{9.15.14}$$

where $\lfloor n/2\rfloor$ denotes the largest integer smaller than or equal to $n/2$.

Limit relations

Meixner-Pollaczek → Hermite

If we take $x\to(\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi)$ in the definition (9.7.1) of the Meixner-Pollaczek polynomials and then let $\lambda\to\infty$ we obtain the Hermite polynomials:

$$\lim_{\lambda\to\infty}\lambda^{-\frac{1}{2}n}P_n^{(\lambda)}((\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi);\phi)=\frac{H_n(x)}{n!}.$$

Jacobi → Hermite

The Hermite polynomials follow from the Jacobi polynomials given by (9.8.1) by taking $\beta=\alpha$ and letting $\alpha\to\infty$ in the following way:

$$\lim_{\alpha\to\infty}\alpha^{-\frac{1}{2}n}P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}}x)=\frac{H_n(x)}{2^nn!}.$$

Gegenbauer / Ultraspherical → Hermite

The Hermite polynomials follow from the Gegenbauer (or ultraspherical) polynomials given by (9.8.19) by taking $\lambda = \alpha + \frac{1}{2}$ and letting $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} C_n^{(\alpha+\frac{1}{2})}(\alpha^{-\frac{1}{2}}x) = \frac{H_n(x)}{n!}.$$

Krawtchouk → Hermite

The Hermite polynomials follow from the Krawtchouk polynomials given by (9.11.1) by setting $x \to pN + x\sqrt{2p(1-p)N}$ and then letting $N \to \infty$:

$$\lim_{N\to\infty} \sqrt{\binom{N}{n}} K_n(pN + x\sqrt{2p(1-p)N}; p, N) = \frac{(-1)^n H_n(x)}{\sqrt{2^n n! \left(\frac{p}{1-p}\right)^n}}.$$

Laguerre → Hermite

The Hermite polynomials can be obtained from the Laguerre polynomials given by (9.12.1) by taking the limit $\alpha \to \infty$ in the following way:

$$\lim_{\alpha\to\infty} \left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}((2\alpha)^{\frac{1}{2}}x + \alpha) = \frac{(-1)^n}{n!} H_n(x).$$

Charlier → Hermite

If we set $x \to (2a)^{1/2}x + a$ in the definition (9.14.1) of the Charlier polynomials and let $a \to \infty$ we find the Hermite polynomials. In fact we have

$$\lim_{a\to\infty} (2a)^{\frac{1}{2}n} C_n((2a)^{\frac{1}{2}}x + a; a) = (-1)^n H_n(x).$$

Remarks

The Hermite polynomials can also be written as:

$$\frac{H_n(x)}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!},$$

where $\lfloor n/2 \rfloor$ denotes the largest integer smaller than or equal to $n/2$.
The Hermite polynomials and the Laguerre polynomials given by (9.12.1) are also connected by the following quadratic transformations:

$$H_{2n}(x) = (-1)^n n!\, 2^{2n} L_n^{(-\frac{1}{2})}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n n!\, 2^{2n+1} x L_n^{(\frac{1}{2})}(x^2).$$

References

[?], [?].