Zeta and Related Functions

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Part I
Riemann Zeta Function

1 Definition and Expansions

1.1 Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

1.2 Other Infinite Series

$$\zeta(s) = \frac{1}{1-2^{-s}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \tag{2}$$

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{3}$$

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \tag{4}$$

$$\zeta'(s) = -\sum_{n=2}^{\infty} (\ln n) n^{-s} \tag{5}$$

$$\zeta^{(k)} @s = (-1)^k \sum_{n=2}^{\infty} (\ln n)^k n^{-s} \tag{6}$$

1.3 Representations by the Euler–Maclaurin Formula

$$\zeta(s) = \sum_{k=1}^{N} \frac{1}{k^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx \tag{7}$$

$$\zeta(s) = \sum_{k=1}^{N} \frac{1}{k^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + \sum_{k=1}^{n} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} N^{1-s-2k} - \binom{s+2n}{2n+1} \int_N^{\infty} \frac{\widetilde{B}_{2n+1}(x)}{x^{s+2n+1}}\, dx \tag{8}$$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{n} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} - \binom{s+2n}{2n+1} \int_1^{\infty} \frac{\widetilde{B}_{2n+1}(x)}{x^{s+2n+1}}\, dx \tag{9}$$

1.4 Infinite Products

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \tag{10}$$

$$\zeta(s) = \frac{(2\pi)^s e^{-s-(\gamma s/2)}}{2(s-1)\,\Gamma\bigl(\frac{1}{2}s+1\bigr)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{11}$$

2 Reflection Formulas

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\bigl(\tfrac{1}{2}\pi s\bigr)\,\Gamma(s)\,\zeta(s) \tag{12}$$

$$\zeta(s) = 2(2\pi)^{s-1} \sin\bigl(\tfrac{1}{2}\pi s\bigr)\,\Gamma(1-s)\,\zeta(1-s) \tag{13}$$

$$\xi(s) = \xi(1-s) \tag{14}$$

$$\xi(s) = \tfrac{1}{2}s(s-1)\,\Gamma\bigl(\tfrac{1}{2}s\bigr)\pi^{-s/2}\,\zeta(s) \tag{15}$$

$$(-1)^k \,\zeta^{(k)}\, @1-s = \frac{2}{(2\pi)^s} \sum_{m=0}^{k} \sum_{r=0}^{m} \binom{k}{m}\binom{m}{r} \bigl(\Re(c^{k-m}) \cos\bigl(\tfrac{1}{2}\pi s\bigr) + \Im(c^{k-m}) \sin\bigl(\tfrac{1}{2}\pi s\bigr)\bigr)\,\Gamma^{(r)}\, @s\,\zeta^{(m-r)}\, @s \tag{16}$$

3 Integral Representations

3.1 In Terms of Elementary Functions

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx \tag{17}$$

$$\zeta(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{e^x x^s}{(e^x - 1)^2}\,dx \tag{18}$$

$$\zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1}\,dx \tag{19}$$

$$\zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s+1)} \int_0^\infty \frac{e^x x^s}{(e^x + 1)^2}\,dx \tag{20}$$

$$\zeta(s) = -s \int_0^\infty \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}}\,dx \tag{21}$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \frac{x^{s-1}}{e^x}\,dx \tag{22}$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \frac{\Gamma(s+2m-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1}\right) \frac{x^{s-1}}{e^x}\,dx \tag{23}$$

$$\zeta(s) = \frac{1}{2(1-2^{-s})\,\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\sinh x}\,dx \tag{24}$$

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{(\sinh x)^2}\,dx \tag{25}$$

$$\zeta(s) = \frac{2^{s-1}}{1-2^{1-s}} \int_0^\infty \frac{\cos(s \arctan x)}{(1+x^2)^{s/2} \cosh\bigl(\frac{1}{2}\pi x\bigr)}\,dx \tag{26}$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2 \int_0^\infty \frac{\sin(s \arctan x)}{(1+x^2)^{s/2}(e^{2\pi x} - 1)}\,dx \tag{27}$$

$$\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty \frac{\sin(s \arctan x)}{(1+x^2)^{s/2}(e^{\pi x} + 1)}\,dx \tag{28}$$

3.2 In Terms of Other Functions

$$\zeta(s) = \frac{\pi^{s/2}}{s(s-1)\,\Gamma\left(\frac{1}{2}s\right)} + \frac{\pi^{s/2}}{\Gamma\left(\frac{1}{2}s\right)} \int_1^\infty \left(x^{s/2} + x^{(1-s)/2}\right) \frac{\omega(x)}{x}\, dx \tag{29}$$

$$\zeta(s) = \frac{1}{s-1} + \frac{\sin(\pi s)}{\pi} \int_0^\infty (\ln(1+x) - \psi(1+x))x^{-s}\, dx \tag{30}$$

$$\zeta(s) = \frac{1}{s-1} + \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty \left(\frac{1}{1+x} - \psi'(1+x)\right) x^{1-s}\, dx \tag{31}$$

$$\zeta(1+s) = \frac{\sin(\pi s)}{\pi} \int_0^\infty (\gamma + \psi(1+x))\, x^{-s-1}\, dx \tag{32}$$

$$\zeta(1+s) = \frac{\sin(\pi s)}{\pi s} \int_0^\infty \psi'(1+x) x^{-s}\, dx \tag{33}$$

$$\zeta(m+s) = (-1)^{m-1} \frac{\Gamma(s)\sin(\pi s)}{\pi\,\Gamma(m+s)} \int_0^\infty \psi^{(m)}\, @1 + x x^{-s}\, dx \tag{34}$$

3.3 Contour Integrals

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{z^{s-1}}{e^{-z}-1}\, dz \tag{35}$$

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i(1-2^{1-s})} \int_{-\infty}^{(0+)} \frac{z^{s-1}}{e^{-z}+1}\, dz \tag{36}$$

4 Integer Arguments

4.1 Function Values

$$\zeta(0) = -\frac{1}{2} \tag{37}$$

$$\zeta(2) = \frac{\pi^2}{6} \tag{38}$$

$$\zeta(4) = \frac{\pi^4}{90} \tag{39}$$

$$\zeta(6) = \frac{\pi^6}{945} \tag{40}$$

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}| \tag{41}$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{42}$$

$$\zeta(-2n) = 0 \tag{43}$$

$$\zeta(k+1) = \frac{1}{k!} \sum_{n_1=1}^\infty \cdots \sum_{n_k=1}^\infty \frac{1}{n_1 \cdots n_k (n_1 + \cdots + n_k)} \tag{44}$$

$$\zeta(2k+1) = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(t)\cot(\pi t)\,dt \tag{45}$$

$$\zeta(2) = \int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy \tag{46}$$

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} \tag{47}$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \tag{48}$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4\binom{2k}{k}} \tag{49}$$

4.2 Derivative Values

$$\zeta'(0) = -\tfrac{1}{2}\ln(2\pi) \tag{50}$$

$$\zeta''(0) = -\tfrac{1}{2}(\ln(2\pi))^2 + \tfrac{1}{2}\gamma^2 - \tfrac{1}{24}\pi^2 + \gamma_1 \tag{51}$$

$$(-1)^k\,\zeta^{(k)}\,@{-2n} = \frac{2(-1)^n}{(2\pi)^{2n+1}} \sum_{m=0}^{k}\sum_{r=0}^{m} \binom{k}{m}\binom{m}{r} \Im(c^{k-m})\;\Gamma^{(r)}\,@2n+1\,\zeta^{(m-r)}\,@2n+1 \tag{52}$$

$$(-1)^k\,\zeta^{(k)}\,@1-2n = \frac{2(-1)^n}{(2\pi)^{2n}} \sum_{m=0}^{k}\sum_{r=0}^{m} \binom{k}{m}\binom{m}{r} \Re(c^{k-m})\;\Gamma^{(r)}\,@2n\,\zeta^{(m-r)}\,@2n \tag{53}$$

$$\zeta'(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}\left(2n\,\zeta'(1-2n) - (\psi(2n) - \ln(2\pi))\,B_{2n}\right) \tag{54}$$

4.3 Recursion Formulas

$$\left(n+\tfrac{1}{2}\right)\zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k)\,\zeta(2n-2k) \tag{55}$$

$$\left(n+\tfrac{3}{4}\right)\zeta(4n+2) = \sum_{k=1}^{n} \zeta(2k)\,\zeta(4n+2-2k) \tag{56}$$

$$\left(n+\tfrac{1}{4}\right)\zeta(4n) + \tfrac{1}{2}(\zeta(2n))^2 \; = \sum_{k=1}^{n} \zeta(2k)\,\zeta(4n-2k) \tag{57}$$

$$\left(m+n+\tfrac{3}{2}\right)\zeta(2m+2n+2) = \left(\sum_{k=1}^{m} + \sum_{k=1}^{n}\right)\zeta(2k)\,\zeta(2m+2n+2-2k) \tag{58}$$

$$\tfrac{1}{2}(2^{2n}-1)\,\zeta(2n) = \sum_{k=1}^{n-1}(2^{2n-2k}-1)\,\zeta(2n-2k)\,\zeta(2k) \tag{59}$$

5 Sums

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1 \tag{60}$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{(k+1)!} (\zeta(s+k) - 1) = \Gamma(s-1) \tag{61}$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(s+k)\,\zeta(s+k)}{k!\,\Gamma(s) 2^{s+k}} = (1 - 2^{-s})\,\zeta(s) \tag{62}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (\zeta(nk) - 1) = \ln\left(\prod_{j=0}^{n-1} \Gamma\Big(2 - e^{(2j+1)\pi i/n}\Big)\right) \tag{63}$$

$$\sum_{k=2}^{\infty} \zeta(k) z^k = -\gamma z - z\,\psi(1-z) \tag{64}$$

$$\sum_{k=0}^{\infty} \zeta(2k) z^{2k} = -\tfrac{1}{2}\pi z \cot(\pi z) \tag{65}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k} z^k = -\gamma z + \ln\Gamma(1-z) \tag{66}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} z^{2k} = \ln\left(\frac{\pi z}{\sin(\pi z)}\right) \tag{67}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) 2^{2k}} = \frac{1}{2} - \frac{1}{2}\ln 2 \tag{68}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2) 2^{2k}} = \frac{1}{4} - \frac{7}{4\pi^2}\,\zeta(3) \tag{69}$$

6 Asymptotic Approximations

$$\zeta(\sigma + it) = \sum_{1 \le n \le x} \frac{1}{n^s} + \chi(s) \sum_{1 \le n \le y} \frac{1}{n^{1-s}} + O\big(x^{-\sigma}\big) + O\Big(y^{\sigma-1} t^{\frac{1}{2}-\sigma}\Big) \tag{70}$$

$$\zeta\big(\tfrac{1}{2} + it\big) = \sum_{n=1}^{m} \frac{1}{n^{\frac{1}{2}+it}} + \chi\left(\tfrac{1}{2} + it\right) \sum_{n=1}^{m} \frac{1}{n^{\frac{1}{2}-it}} + O\Big(t^{-1/4}\Big) \tag{71}$$

7 Zeros

7.1 Distribution

$$Z(t) = \exp(i\vartheta(t))\,\zeta\big(\tfrac{1}{2} + it\big) \tag{72}$$

7.2 Riemann–Siegel Formula

$$Z(t) = 2\sum_{n=1}^{m} \frac{\cos(\vartheta(t) - t\ln n)}{n^{1/2}} + R(t) \tag{73}$$

Part II
Related Functions

8 Hurwitz Zeta Function

8.1 Definition

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{74}$$

$$\zeta(s,1) = \zeta(s) \tag{75}$$

$$\zeta(s,a) = \zeta(s,a+1) + a^{-s} \tag{76}$$

$$\zeta(s,a) = \zeta(s,a+m) + \sum_{n=0}^{m-1} \frac{1}{(n+a)^s} \tag{77}$$

8.2 Representations by the Euler–Maclaurin Formula

$$\zeta(s,a) = \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s\int_N^{\infty} \frac{x - \lfloor x \rfloor}{(x+a)^{s+1}}\,dx \tag{78}$$

$$\zeta(s,a) = \frac{1}{a^s}\left(\frac{1}{2} + \frac{a}{s-1}\right) - s(s+1)\int_0^{\infty} \frac{\widetilde{B}_2(x)}{(x+a)^{s+2}}\,dx \tag{79}$$

$$\zeta(s,a) = \frac{1}{a^s} + \frac{1}{(1+a)^s}\left(\frac{1}{2} + \frac{1+a}{s-1}\right) + \sum_{k=1}^{n} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} \frac{1}{(1+a)^{s+2k-1}} - \binom{s+2n}{2n+1}\int_1^{\infty} \frac{\widetilde{B}_{2n+1}(x)}{(x+a)^{s+2n+1}}\,dx \tag{80}$$

8.3 Series Representations

$$\zeta\big(s, \tfrac{1}{2}a\big) = \zeta\big(s, \tfrac{1}{2}a + \tfrac{1}{2}\big) + 2^s \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s} \tag{81}$$

$$\zeta(1-s,a) = \frac{2\,\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \cos\big(\tfrac{1}{2}\pi s - 2n\pi a\big) \tag{82}$$

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!\,\Gamma(s)}\,\zeta(n+s)(1-a)^n \tag{83}$$

8.4 Special Values

$$\zeta\left(s, \tfrac{1}{2}\right) = (2^s - 1)\,\zeta(s) \tag{84}$$

$$\zeta(n+1, a) = \frac{(-1)^{n+1}\,\psi^{(n)}\,@a}{n!} \tag{85}$$

$$\zeta(0, a) = \tfrac{1}{2} - a \tag{86}$$

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1} \tag{87}$$

$$\zeta(s, ka) = k^{-s} \sum_{n=0}^{k-1} \zeta\left(s, a + \frac{n}{k}\right) \tag{88}$$

$$\zeta\left(1 - s, \frac{h}{k}\right) = \frac{2\,\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{k}\right) \zeta\left(s, \frac{r}{k}\right) \tag{89}$$

8.5 Derivatives

a-Derivative

$$\frac{\partial}{\partial a}\,\zeta(s, a) = -s\,\zeta(s+1, a) \tag{90}$$

s-Derivatives

$$\zeta'(0, a) = \ln\Gamma(a) - \tfrac{1}{2}\ln(2\pi) \tag{91}$$

$$\zeta'(s, a) = -\frac{\ln a}{a^s}\left(\frac{1}{2} + \frac{a}{s-1}\right) - \frac{a^{1-s}}{(s-1)^2} + s(s+1)\int_0^\infty \frac{\widetilde{B}_2(x)\ln(x+a)}{(x+a)^{s+2}}\,dx - (2s+1)\int_0^\infty \frac{\widetilde{B}_2(x)}{(x+a)^{s+2}}\,dx \tag{92}$$

$$\begin{aligned}(-1)^k\,\zeta^{(k)}\,@sa = {} & \frac{(\ln a)^k}{a^s}\left(\frac{1}{2} + \frac{a}{s-1}\right) + k!\,a^{1-s}\sum_{r=0}^{k-1}\frac{(\ln a)^r}{r!\,(s-1)^{k-r+1}} - s(s+1)\int_0^\infty \frac{\widetilde{B}_2(x)(\ln(x+a))^k}{(x+a)^{s+2}}\,dx \\ & + k(2s+1)\int_0^\infty \frac{\widetilde{B}_2(x)(\ln(x+a))^{k-1}}{(x+a)^{s+2}}\,dx - k(k-1)\int_0^\infty \frac{\widetilde{B}_2(x)(\ln(x+a))^{k-2}}{(x+a)^{s+2}}\,dx\end{aligned} \tag{93}$$

$$\begin{aligned}\zeta'\left(1-2n, \frac{h}{k}\right) = {} & \frac{(\psi(2n) - \ln(2\pi k))\,B_{2n}(h/k)}{2n} - \frac{(\psi(2n) - \ln(2\pi))\,B_{2n}}{2nk^{2n}} + \frac{(-1)^{n+1}\pi}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\sin\left(\frac{2\pi r h}{k}\right)\psi^{(2n-1)}\,@\frac{r}{k} \\ & + \frac{(-1)^{n+1}2\cdot(2n-1)!}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\cos\left(\frac{2\pi r h}{k}\right)\zeta'\left(2n, \frac{r}{k}\right) + \frac{\zeta'(1-2n)}{k^{2n}}\end{aligned} \tag{94}$$

$$\zeta'\left(1-2n, \tfrac{1}{2}\right) = -\frac{B_{2n}\ln 2}{n\cdot 4^n} - \frac{(2^{2n-1}-1)\,\zeta'(1-2n)}{2^{2n-1}} \tag{95}$$

$$\zeta'\left(1-2n, \tfrac{1}{3}\right) = -\frac{\pi(9^n-1)\,B_{2n}}{8n\sqrt{3}(3^{2n-1}-1)} - \frac{B_{2n}\ln 3}{4n\cdot 3^{2n-1}} - \frac{(-1)^n\,\psi^{(2n-1)}\,@\frac{1}{3}}{2\sqrt{3}(6\pi)^{2n-1}} - \frac{\left(3^{2n-1}-1\right)\zeta'(1-2n)}{2\cdot 3^{2n-1}} \tag{96}$$

$$\sum_{r=1}^{k-1}\zeta'\left(s, \frac{r}{k}\right) = (k^s - 1)\,\zeta'(s) + k^s\,\zeta(s)\ln k \tag{97}$$

8.6 Integral Representations

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx \tag{98}$$

$$\zeta(s,a) = -s \int_{-a}^\infty \frac{x - \lfloor x \rfloor - \frac{1}{2}}{(x+a)^{s+1}}\,dx \tag{99}$$

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{x^{s-1}}{e^{ax}}\,dx \tag{100}$$

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + \sum_{k=1}^{n} \frac{\Gamma(s+2k-1)}{\Gamma(s)} \frac{B_{2k}}{(2k)!} a^{-2k-s+1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} - \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} x^{2k-1} \right) x^{s-1}e^{-ax}\,dx \tag{101}$$

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin(s \arctan(x/a))}{(a^2+x^2)^{s/2}(e^{2\pi x}-1)}\,dx \tag{102}$$

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{e^{az}z^{s-1}}{1-e^z}\,dz \tag{103}$$

8.7 Further Integral Representations

$$\frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{2\cosh x}\,dx = 4^{-s}\left(\zeta\left(s, \tfrac{1}{4} + \tfrac{1}{4}a\right) - \zeta\left(s, \tfrac{3}{4} + \tfrac{1}{4}a\right)\right) \tag{104}$$

$$\int_0^a x^n\,\psi(x)\,dx = (-1)^{n-1}\,\zeta'(-n) + (-1)^n\,H_n\,\frac{B_{n+1}}{n+1} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} h(k) \frac{B_{k+1}(a)}{k+1} a^{n-k} + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \zeta'(-k,a) a^{n-k} \tag{105}$$

$$n\int_0^a \zeta'(1-n,x)\,dx = \zeta'(-n,a) - \zeta'(-n) + \frac{B_{n+1} - B_{n+1}(a)}{n(n+1)} \tag{106}$$

8.8 Further Series Representations

$$\sum_{n=0}^\infty \frac{(-1)^n}{(n+a)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{1+e^{-x}}\,dx = 2^{-s}\left(\zeta\left(s, \tfrac{1}{2}a\right) - \zeta\left(s, \tfrac{1}{2}(1+a)\right)\right) \tag{107}$$

$$\sum_{n=1}^\infty \frac{\chi(n)}{n^s} = k^{-s} \sum_{r=1}^{k} \chi(r)\,\zeta\left(s, \frac{r}{k}\right) \tag{108}$$

8.9 Sums

$$\sum_{k=1}^\infty \frac{(-1)^k}{k}\,\zeta(nk,a) = -n\ln\Gamma(a) + \ln\left(\prod_{j=0}^{n-1} \Gamma\left(a - e^{(2j+1)\pi i/n}\right) \right) \tag{109}$$

$$\sum_{k=1}^\infty \binom{n+k}{k} \zeta(n+k+1,a)z^k = \frac{(-1)^n}{n!}\left(\psi^{(n)}\,@a - \psi^{(n)}\,@a-z\right) \tag{110}$$

$$\sum_{k=2}^\infty \frac{k}{2^k}\,\zeta\left(k+1, \tfrac{3}{4}\right) = 8G \tag{111}$$

8.10 a-Asymptotic Behavior

$$\zeta(s,a+1)=\zeta(s)-s\,\zeta(s+1)a+O\big(a^2\big) \tag{112}$$

$$\zeta(s,\alpha+i\beta)\to 0 \tag{113}$$

$$\zeta(s,a)-\frac{a^{1-s}}{s-1}-\frac{1}{2}a^{-s}\sim\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\frac{\Gamma(s+2k-1)}{\Gamma(s)}a^{1-s-2k} \tag{114}$$

$$\zeta'(-1,a)-\frac{1}{12}+\frac{1}{4}a^2-\left(\frac{1}{12}-\frac{1}{2}a+\frac{1}{2}a^2\right)\ln a\sim-\sum_{k=1}^{\infty}\frac{B_{2k+2}}{(2k+2)(2k+1)2k}a^{-2k} \tag{115}$$

$$\zeta'(-2,a)-\frac{1}{12}a+\frac{1}{9}a^3-\left(\frac{1}{6}a-\frac{1}{2}a^2+\frac{1}{3}a^3\right)\ln a\sim\sum_{k=1}^{\infty}\frac{2B_{2k+2}}{(2k+2)(2k+1)2k(2k-1)}a^{-(2k-1)} \tag{116}$$

9 Polylogarithms

9.1 Dilogarithms

$$\mathrm{Li}_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2} \tag{117}$$

$$\mathrm{Li}_2(z)=-\int_0^z t^{-1}\ln(1-t)\,dt \tag{118}$$

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}(\ln(1-z))^2 \tag{119}$$

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right)=-\frac{1}{6}\pi^2-\frac{1}{2}(\ln(-z))^2 \tag{120}$$

$$\mathrm{Li}_2(z^m)=m\sum_{k=0}^{m-1}\mathrm{Li}_2\Big(ze^{2\pi ik/m}\Big) \tag{121}$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{1}{6}\pi^2-(\ln x)\ln(1-x) \tag{122}$$

$$\mathrm{Li}_2\big(e^{i\theta}\big)=\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}+i\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2} \tag{123}$$

$$\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}=\frac{\pi^2}{6}-\frac{\pi\theta}{2}+\frac{\theta^2}{4} \tag{124}$$

$$\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}=-\int_0^{\theta}\ln\big(2\sin\big(\tfrac{1}{2}x\big)\big)\,dx \tag{125}$$

9.2 Polylogarithms

$$\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s} \tag{126}$$

Integral Representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - z}\, dx \tag{127}$$

$$\mathrm{Li}_s(z) = \Gamma(1-s) \left(\ln \frac{1}{z} \right)^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n) \frac{(\ln z)^n}{n!} \tag{128}$$

$$\mathrm{Li}_s\left(e^{2\pi i a}\right) + e^{\pi i s}\, \mathrm{Li}_s\left(e^{-2\pi i a}\right) = \frac{(2\pi)^s e^{\pi i s/2}}{\Gamma(s)}\, \zeta(1-s, a) \tag{129}$$

9.3 Fermi–Dirac and Bose–Einstein Integrals

$$F_s(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^{t-x}+1}\, dt \tag{130}$$

$$G_s(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^{t-x}-1}\, dt \tag{131}$$

$$F_s(x) = -\,\mathrm{Li}_{s+1}(-e^x) \tag{132}$$

$$G_s(x) = \mathrm{Li}_{s+1}(e^x) \tag{133}$$

10 Periodic Zeta Function

$$F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s} \tag{134}$$

$$F(x,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(e^{\pi i(1-s)/2}\, \zeta(1-s,x) + e^{\pi i(s-1)/2}\, \zeta(1-s,1-x) \right) \tag{135}$$

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2}\, F(x,s) + e^{\pi i s/2}\, F(-x,s) \right) \tag{136}$$

11 Lerch's Transcendent

11.1 Definition

$$\Phi(z,s,a) \ = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s} \tag{137}$$

$$\zeta(s,a) = \Phi(1,s,a) \tag{138}$$

$$\mathrm{Li}_s(z) = z\, \Phi(z,s,1) \tag{139}$$

11.2 Properties

$$\Phi(z,s,a) = z^m\, \Phi(z,s,a+m) + \sum_{n=0}^{m-1} \frac{z^n}{(a+n)^s} \tag{140}$$

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - z e^{-x}}\, dx \tag{141}$$

$$\Phi(z,s,a) = \frac{1}{2} a^{-s} + \int_0^\infty \frac{z^x}{(a+x)^s}\, dx - 2 \int_0^\infty \frac{\sin(x \ln z - s \arctan(x/a))}{(a^2+x^2)^{s/2} (e^{2\pi x} - 1)}\, dx \tag{142}$$

12 Dirichlet L-functions

12.1 Definitions and Basic Properties

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{143}$$

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{144}$$

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k-1} \chi(r)\,\zeta\left(s, \frac{r}{k}\right) \tag{145}$$

$$L(s,\chi) = L(s,\chi_0) \prod_{p|k} \left(1 - \frac{\chi_0(p)}{p^s}\right) \tag{146}$$

$$L(1-s,\chi) = \frac{k^{s-1}\,\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} + \chi(-1) e^{\pi i s/2}\right) G(\chi)\, L(s,\overline{\chi}) \tag{147}$$

12.2 Zeros

$$L(-2n,\chi) = 0 \quad \text{if} \quad \chi(-1) = 1 \tag{148}$$

$$L(-2n-1,\chi) = 0 \quad \text{if} \quad \chi(-1) = -1 \tag{149}$$

$$L(1,\chi) \neq 0 \quad \text{if} \quad \chi \neq -1 \tag{150}$$

$$L(0,\chi) = \begin{cases} -\dfrac{1}{k} \displaystyle\sum_{r=1}^{k} r\chi(r), & \chi \neq \chi_1, \\ 0, & \chi = \chi_1. \end{cases} \tag{151}$$

Part III
Applications

13 Mathematical Applications

13.1 Distribution of Primes

$$\psi(x) = \sum_{m=1}^{\infty} \sum_{p^m \leq x} \ln p \tag{152}$$

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{x^{\rho}}{\rho} + o(1) \tag{153}$$

$$\psi(x) = x + o(x) \tag{154}$$

$$\psi(x) = x + O\left(x^{\frac{1}{2}+\epsilon}\right) \tag{155}$$

13.2 Euler Sums

$$H(s)=\sum_{n=1}^{\infty}\frac{H_n}{n^s} \tag{156}$$

$$H(s)=-\zeta'(s)+\gamma\,\zeta(s)+\frac{1}{2}\,\zeta(s+1)+\sum_{r=1}^{k}\zeta(1-2r)\,\zeta(s+2r)+\sum_{n=1}^{\infty}\frac{1}{n^s}\int_n^{\infty}\frac{\widetilde{B}_{2k+1}(x)}{x^{2k+2}}\,dx \tag{157}$$

$$H(s)=\frac{1}{2}\,\zeta(s+1)+\frac{\zeta(s)}{s-1}-\sum_{r=1}^{k}\binom{s+2r-2}{2r-1}\zeta(1-2r)\,\zeta(s+2r)-\binom{s+2k}{2k+1}\sum_{n=1}^{\infty}\frac{1}{n}\int_n^{\infty}\frac{\widetilde{B}_{2k+1}(x)}{x^{s+2k+1}}\,dx \tag{158}$$

$$H(2)=2\,\zeta(3) \tag{159}$$

$$H(3)=\frac{5}{4}\,\zeta(4) \tag{160}$$

$$H(a)=\frac{a+2}{2}\,\zeta(a+1)-\frac{1}{2}\sum_{r=1}^{a-2}\zeta(r+1)\,\zeta(a-r) \tag{161}$$

$$H(-2a)=\frac{1}{2}\,\zeta(1-2a)=-\frac{B_{2a}}{4a} \tag{162}$$

$$H(s,z)=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{m=1}^{n}\frac{1}{m^z} \tag{163}$$

$$H(s,z)+H(z,s)=\zeta(s)\,\zeta(z)+\zeta(s+z) \tag{164}$$

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\,\zeta(4) \tag{165}$$

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{rk(r+k)}=\frac{5}{4}\,\zeta(3) \tag{166}$$

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^2(r+k)}=\frac{3}{4}\,\zeta(3) \tag{167}$$

Part IV
BOF

Part V
Gamma function: Properties

14 Definitions

14.1 Gamma and Psi Functions

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}\,dt \tag{168}$$

15 Gamma function: Functional Relations

15.1 Gamma function: Recurrence

$$\Gamma(z+1) = z\,\Gamma(z) \tag{169}$$

15.2 Gamma function: Reflection

$$\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z) \tag{170}$$

Part VI

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