

The Set Theoretic Structure of Analysis

Contents

1	Introduction	2
1.1	A list of axioms	2
1.2	Some Consistency Results	4
2	Dependent Choice	4
3	Discontinuous Linear Maps and the Baire Property	8
4	Other examples of nonconstructive objects in analysis	12
5	A Strong Fubini Theorem	17
5.1	The Continuum Hypothesis and Martin's Axiom	20
5.2	Axioms of Symmetry: A_{null} and A_{\aleph_0}	21
6	A Result of Erdos	22
7	Automatic Continuity	24
8	Conclusion	24

1 Introduction

Many theorems in analysis draw heavily on set theory, although generally the more basic side of the subject, rather than any of the depths of what modern set theorists consider. Thus it should come as no surprise that many features of analysis are dependent on exactly which axioms we choose for our set theory.

In this essay I will look at various questions in analysis that are affected by these set theoretical considerations. In particular I will look at the roles of the axiom of choice (AC) and the continuum hypothesis (CH), as well as various weakenings of these statements and other statements which contradict them. In particular I will be looking at the statements ‘every subset of \mathbb{R} has the Baire Property’ (BP), which contradicts AC, and one of Freiling’s axioms of symmetry, specifically A_{null} , which contradicts CH.

In looking at these I will show how various natural constructions and examples in analysis depend on them, and will at the end discuss what the implications for this sort of result have for analysis.

I will start with a brief look at how much of analysis we can do with only dependent choice, as a prelude to looking at the Baire property and its use in demonstrating the nonconstructive nature of various objects in analysis. (i.e. showing that their nonexistence is consistent with $ZF + DC$). Most of this will consist of developing some of the basic tools of descriptive set theory in $ZF + DC$ and then only using BP quite late on in proving the specific results.

Next I shall look at the role of the continuum hypothesis in the measure theoretic structure of \mathbb{R} , and in particular discuss the connection of this with a strong form of Fubini’s Theorem, giving some axioms which alternately prove and negate it.

I shall finish off with a look at a result of Erdos giving an analytic equivalent of CH and a brief note about a result in automatic continuity theory and its connection with CH.

1.1 A list of axioms

The following are a list of the axioms I will be investigating. Many of them will be repeated in the relevant sections, but I include them here anyway for completeness.

1. AC: The Axiom of Choice

Let X be a collection of non-empty sets. There is a function $f : X \rightarrow \bigcup X$ such that $\forall x \ f(x) \in x$.

2. DC: Dependent Choice

Let X be a set and let \prec be a relation on X such that $\forall x \in X \ \exists y \in X \ x \prec y$. There exists a sequence x_n in X with $\forall n \ x_n \prec x_{n+1}$.

3. UF: The Ultrafilter Theorem

Every filter on a set is contained in some ultrafilter.

4. WUF: The Weak Ultrafilter Theorem

There is a free ultrafilter on \mathbb{N} .

5. BP: Every subset of \mathbb{R} has the Baire property.

6. LM: Every subset of \mathbb{R} is Lebesgue measurable.

7. CH: The Continuum Hypothesis

$$c = 2^{\aleph_0}$$

8. $MA(\kappa)$

Let X be a poset. X is said to be ccc if every antichain is countable. $A \subseteq X$ is said to be upwards directed if $\forall x, y \in A \exists z \in A \ x \leq z, \ y \leq z$.

For a cardinal κ , $MA(\kappa)$ is the following statement:

Let X be a non-empty ccc poset. Then for any set T of cofinal subsets of X with $|T| \leq \kappa$ there is a directed subset of X , P such that $\forall A \in T \ A \cap P \neq \emptyset$.

9. MA: Martin's Axiom

$$\forall \kappa < c \ MA(\kappa)$$

10. A_{\aleph_0}

$\forall f : \mathbb{R} \rightarrow \mathbb{R}_{\aleph_0} \exists x, y \ x \notin f(y) \ y \notin f(x)$, where \mathbb{R}_{\aleph_0} is the set of countable subsets of \mathbb{R}

11. A_{null}

The same as A_{\aleph_0} but with \mathbb{R}_{\aleph_0} replaced by \mathbb{R}_{null} , the set of null subsets of \mathbb{R} .

12. There exists an uncountable family F of entire functions such that for every $z \in \mathbb{C}$ we have $\{f(z) : f \in F\}$ countable.

There are also various theorems whose independence properties I would like to investigate, so I will give them similar acronyms:

1. HB: The Hahn Banach Theorem

Let V be a vector space over \mathbb{R} and $f : V \rightarrow \mathbb{R}$ be sub-linear. i.e. for $x, y \in V$ and $r, s \geq 0 \ f(rx + sy) \leq rf(x) + sf(y)$.

Let W be a subspace of V and $T : W \rightarrow \mathbb{R}$ satisfy $\forall x \in W \ T(x) \leq f(x)$. Then there exists a linear map $\tilde{T} : V \rightarrow \mathbb{R}$ such that $\forall x \in V \ T(x) \leq \tilde{T}(x)$ and $\tilde{T}|_W = T$.

2. SFT: Strong Fubini Theorem

Let $f : [0, 1]^2 \rightarrow \mathbb{R}$ be a bounded function such that each of the integrals $\int_0^1 \int_0^1 f(x, y) dx dy$ and $\int_0^1 \int_0^1 f(x, y) dy dx$ exist. Then they are equal.

I will precede every theorem whose proof uses more than ZF with a list of what extra axioms are being used in its proof. This should not necessarily be taken to be a statement that no proper subset of these axioms is sufficient to prove the theorem, although this will often be the case.

1.2 Some Consistency Results

We shall quote the following consistency results. Most of them are proved in [7].

1. ZF + DC + BP
2. ZF + DC + LM + BP (assuming the existence of an inaccessible cardinal)
3. ZFC + CH
4. ZFC + \neg CH + MA
5. ZFC + A_{null} (see [5])

2 Dependent Choice

At various points we will want to work with axioms that contradict the full axiom of choice. Thus it will become relevant exactly how much choice various theorems of analysis need to prove. Dependent choice will be a convenient amount to assume, as it is a relatively weak form of choice, but almost impossible to avoid in doing analysis, due to sequence picking and category arguments. In this section I'll sketch some features of the role of dependent choice in analysis and how much it lets us prove.

In many cases, both in this section and later, it will be entirely clear that a standard proof of the result only uses dependent choice. In these cases I will often omit the proof in the interests of brevity.

Theorem 2.1 (DC). *The Baire Category Theorem*

Let X be a complete metric space and let $(U_n)_{n \geq 0}$ be a sequence of open dense subsets. Then $\bigcap U_n$ is dense.

It is clear from any standard proof of the Baire Category Theorem that the only form of choice which is used is DC. See e.g. [3]

More interesting for our purposes is the following:

Theorem 2.2. *Baire Category Theorem \implies DC.*

We first need a lemma. The proof is standard and rather elementary, so I will omit it.

Lemma 2.3. *Let X_1, \dots, X_n, \dots be metric spaces. Then $X = \prod X_n$ is metrizable. Further if the X_n are complete then we can choose a complete metric for the product as well.*

We now prove the result, roughly following the method and notation of [11].

Proof. Let X be a non-empty set and \prec be a relation on X such that $\forall x \in X \exists y x \prec y$.

Give X the discrete topology and let $T = X^{\mathbb{N}}$. This is a complete metric space by the above lemma.

Define $A_k = \bigcup_{l > k} \bigcup_{x \prec y} \{t \in T : t_k = x, t_l = y\}$

Claim: A_k is open and dense.

The first part is easy. $\{t \in T : t_k = x, t_l = y\}$ is open because $\{x\}$ and $\{y\}$ are open and because of the definition of the product topology. Thus A_k is a union of open sets and so open.

Showing that A_k is dense is only marginally trickier.

Suppose $t \in T$. We wish to show that for any n we can find an element of A_k which agrees with it for the first n terms. We may assume without loss of generality that $n \geq k$. Let $x = t_k$. Then define $s \in A_k$ as follows: For the first n places, $s_i = t_i$. Then pick some y such that $x \prec y$ and for $m > n$ let $s_m = y$. Then by definition $s \in A_k$. Hence A_k is dense.

Thus by the Baire category theorem, $\exists t \in \bigcap A_k$.

We define a subsequence of t which will be our desired sequence. Let $k_1 = 1$. Having defined k_r , we define k_{r+1} as follows:

$t \in A_{k_r}$. Thus there exists $m > k_r$ with $t_{k_r} \prec t_m$. Let k_{r+1} be the least such m .

□

Note that the space we used is not separable. In fact, the Baire category theorem for separable spaces is a theorem of ZF (because you can make the choices in the usual proof of the baire category theorem explicitly, as you'll always be able to pick an element of the countable dense subset for them).

It is in general much easier to prove separable cases in weaker axiomatisations of set theory. We shall briefly establish some basic examples of theorems for which we can prove separable versions of with only ZF + DC. We will establish later that for many of these you cannot prove much stronger results in ZF + DC (In particular the full theorem cannot be proven).

Lemma 2.4 (DC). *Let X, Y be metric spaces with Y complete. Let $A \subseteq X$ be dense and $f : A \rightarrow Y$ uniformly continuous. There is a unique extension of f to a continuous $\tilde{f} : X \rightarrow Y$.*

The usual proof of this clearly only uses DC, so we will not repeat it here. In fact if you alter the definitions slightly to ones which are equivalent in ZFC but merely imply the usual ones in ZF you can make this a theorem of ZF. We will not go into this.

Theorem 2.5 (DC). *Let V be a separable normed vector space and $\rho : V \rightarrow \mathbb{R}$ a continuous sublinear functional. Let $W \subseteq V$ be a subspace and $f : W \rightarrow \mathbb{R}$ a linear functional such that $\forall x \in W \quad f(x) \leq \rho(x)$. There is an extension of f to a linear map $\tilde{f} : V \rightarrow \mathbb{R}$ such that $\forall x \in V \quad f(x) \leq \rho(x)$.*

Proof. The standard proof of Hahn-Banach gives a canonical way of extending a linear map as above from W to $\langle W \cup \{x\} \rangle$. We will assume this.

Note that because ρ is continuous, any linear functional bounded by ρ is also continuous.

Let x_1, \dots, x_n, \dots be a dense subset of V . We will define a sequence of subspaces W_n by $W_0 = W$ and $W_{n+1} = \langle W_n, x_{n+1} \rangle$. Further we shall define a linear functional $f_n : W_n \rightarrow \mathbb{R}$ with $f_n \leq \rho$ such that for $m > n$, $f_m|_{W_n} = f_n$.

If $x_{n+1} \in W_n$ then $W_{n+1} = W_n$. If $x_{n+1} \notin W_n$ then as discussed we have a canonical way of extending f_n to f_{n+1} on W_{n+1} with $f_{n+1} \leq \rho$.

Now take the union of the W_n . By the compatibility condition on the f_n we can extend them to a linear functional $g : \bigcup W_n \rightarrow \mathbb{R}$ with $g \leq \rho$. This union is dense in V because it contains $\{x_n\}$. Because g is linear and continuous it is uniformly continuous, so we can extend g to a map $g : V \rightarrow \mathbb{R}$. By the continuity of addition, multiplication and ρ it then follows that this extension is linear and $\leq \rho$. Further it is clearly an extension of f .

□

We shall see later that you cannot drop the hypothesis of separability for V without assuming more choice, even if we still assume that ρ is continuous.

Theorem 2.6 (DC). *Let X_0, \dots, X_n, \dots be compact metric spaces. Then $X = \prod X_n$ is compact in the product topology.*

Proof. The standard proof that a metric space is compact iff it is sequentially compact clearly uses only DC. Further, we know that X is metrizable. Thus it will suffice to show that it is sequentially compact.

Let $(x_n)_{n \geq 0}$ be a sequence in x . By taking a subsequence such that the k th coordinate converges, we may inductively pick a sequence of subsequences $(x_{m_r^{(k)}})_{r \geq 0}$ so that the first k coordinates of $(x_{m_r^{(k)}})_{r \geq 0}$ converge, and $(x_{m_r^{(k+1)}})_{r \geq 0}$ is a subsequence of $(x_{m_r^{(k)}})_{r \geq 0}$. Then let $t_r = m_r^{(r)}$. Ignoring the first k terms, $x_{t_r}^{(k)}$ is a subsequence of $(x_{m_r^{(k)}})_{r \geq 0}$, and so convergent. Hence every coordinate of this subsequence converges, and thus by standard properties of the product topology so does the subsequence.

□

This is useful because it will provide us with a proof of the separable case of the Banach-Alaoglu theorem, that the unit ball of the dual of a Banach space is compact in the weak* topology.

Lemma 2.7. *Let X be a normed vector space and let D be a dense subspace. Let T_n be a sequence of bounded linear functions on X with $\forall n \ ||T_n|| \leq K$ for some K . Suppose $\forall x \in D \ T_n(x) \rightarrow 0$. Then $\forall x \in X \ T_n(x) \rightarrow 0$.*

Proof. Let $x \in X$ and $\epsilon > 0$. Consider $d \in D$ with $\|d - x\| \leq \epsilon$.

$$\begin{aligned} \|T_n x\| &= \|T_n(x - d) + T_n d\| \\ &\leq \|T_n(x - d)\| + \|T_n d\| \\ &\leq K\epsilon + \|T_n d\| \\ \limsup \|T_n x\| &\leq K\epsilon \end{aligned}$$

Hence $\limsup \|T_n x\| = 0$, and so $T_n x \rightarrow 0$.

□

Theorem 2.8 (DC). *Let X be a separable normed space and B be the unit ball of X^* . Then B is compact in the weak* topology.*

Proof. Recall that the weak* topology is the topology of pointwise convergence. i.e. a net $f_\alpha \rightarrow f$ iff $\forall x \ f_\alpha(x) \rightarrow f(x)$.

We can identify this with a closed subset of the product topology $\prod_{x \in X} \overline{B}_{\mathbb{C}}(0, \|x\|)$. Thus if we were assuming AC (or even just UF) this would immediately be compact. However with only DC we will need to work harder.

Let D be a countable dense subset of X which is closed under taking $\mathbb{Q}(i)$ linear combinations. We shall establish a homeomorphism between B and a closed subset of $\prod_{x \in D} \overline{B}_{\mathbb{C}}(0, \|x\|)$, which DC proves to be compact as D is countable and all the spaces involved are metric.

There is an obvious injection from B into K . We send $f \in B$ to the sequence $f(x_n)$. Because $\|f\| \leq 1$ we have $\|f(x_n)\| \leq \|x_n\|$ and so this is indeed in K . We first wish to show that the image of this is closed.

For each $x \in D$ we also have $rx \in D$ by hypothesis ($r \in \mathbb{Q}(i)$). Because $x \rightarrow rx$ is continuous we have that the set $\{f \in K \mid f(rx) = rf(x)\}$ is closed. Similarly for $x, y \in D$ the set $\{f \in K \mid f(x+y) = f(x) + f(y)\}$ is closed. Thus the set of rational linear functions in D is closed in K . I claim that this is the image of B .

Certainly the image of B is contained in it. In order to show that they are equal we shall show that every rational linear functional on D can be extended to a linear functional on X . But this is obvious - because the norm is ≤ 1 any such rational linear function on D is uniformly continuous, so can be extended to a continuous map on X . Further it is clear that the extension of a $\mathbb{Q}(i)$ linear

map is \mathbb{C} -linear and that it has norm ≤ 1 . Let F be the image of B , and let $\phi : F \rightarrow B$ be the inverse of the bijection from B to F .

We will show ϕ is continuous. This will immediately establish that B is compact as it is then the continuous image of a compact space, but indeed the map will also be a homeomorphism because it is a bijection and B is hausdorff.

Because we know that K is metrizable, it suffices to show that if $f_n \rightarrow f$ (with $\|f_n\|, \|f\| \leq 1$) then $\phi(f_n) \rightarrow \phi(f)$ (note that this does not need that B is metrizable, which we don't know). This in turn reduces to showing that if $f_n \rightarrow 0$ (with $\|f_n\| \leq 2$) then $\phi(f_n) \rightarrow 0$.

However this is just the content of our previous lemma. If $f_n \rightarrow 0$ then $\phi(f_n)(x) \rightarrow 0$ for x in a dense subset. Hence $\forall x \phi(f_n)(x) \rightarrow 0$, which is exactly what we mean by $\phi(f_n) \rightarrow 0$ in the weak* topology. Hence ϕ is continuous and the result is proved.

□

3 Discontinuous Linear Maps and the Baire Property

Theorem 3.1 (AC). *Let V be a normed vector space. There is a discontinuous map $T : V \rightarrow \mathbb{R}$.*

Proof. Let B be a basis of V . Normalise it so that every element has norm 1. Pick some countably infinite subset b_n . Define $T(b_n) = 4^n$, $T(b) = 0$ for other B and then extend T by linearity. $x_n = 2^{-n}b_n \rightarrow 0$ but $T(x_n) = 2^n \rightarrow \infty$.

□

We can even do the above constructively in certain cases. For example let V be the set of functions $f : \mathbb{N} \rightarrow \mathbb{R}$ with finite support. Then let $|x| = \sum |x_n|$ and $T(x) = \sum 2^n x_n$.

However, after fiddling with some examples of such, a common theme becomes evident: None of the examples we can constructively produce are on a *complete* normed space.

We will show that it is consistent with ZF + DC that all linear functions defined on a Banach space are continuous. In order to do so we shall have to call in some ideas from descriptive set theory. I shall state some of the elementary results without proof, as to prove them here would be too much of a digression. See e.g. [8] for proofs.

Definition 3.2. *Let X be a topological space. $F \subseteq X$ is said to be nowhere dense if $\text{int}(\text{cl}(F)) = \emptyset$.*

A set is said to be meager if it is the countable union of nowhere dense sets.

We define a relation \sim on $P(X)$ by saying $X \sim Y$ iff $X \triangle Y = M$ for some meager set M . A set $A \subseteq X$ has the Baire property if $A \sim U$ for some open U .

BP is the statement every subset of \mathbb{R} has the Baire property.

Lemma 3.3 (DC). \sim as defined above has the following properties:

1. It is an equivalence relation.
2. If $A \sim B$ then $A^c \sim B^c$
3. If $\forall n \ A_n \sim B_n$ then $\bigcup A_n \sim \bigcup B_n$.
4. The collection of subsets of X with the Baire property is a sigma-algebra containing all the open sets of X (and thus all the Borel sets of X).

We will need some standard results about the so called Baire space, $\mathbb{N}^{\mathbb{N}}$.

Lemma 3.4. Put a metric d on $\mathbb{N}^{\mathbb{N}}$ by $d(x, x) = 0$ and if $x \neq y$ then $d(x, y) = 2^{-n}$ where n is the smallest number such that $x_n \neq y_n$.

Then:

1. d is a complete metric giving the product topology on $\mathbb{N}^{\mathbb{N}}$.
2. $\mathbb{N}^{\mathbb{N}}$ is separable.
3. $\mathbb{N}^{\mathbb{N}}$ has a basis of clopen sets.
4. Any compact subset of $\mathbb{N}^{\mathbb{N}}$ has empty interior.

Theorem 3.5 (DC). Let X be a non-empty separable completely metrizable topological space with a basis of clopen sets such that every compact subset has empty interior. X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

We shall also need the following standard result:

Lemma 3.6 (DC). Let X be a complete metric space and $A \subseteq X$ be a G_δ set. Then A is completely metrizable.

Theorem 3.7 (DC). Let X be a complete separable metric space without isolated points. There is a meager set $M \subseteq X$ such that $X \setminus M$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Proof. Let U_n be a countable basis for the topology of X . Let $D = \bigcup \overline{U_n} \setminus U_n$. D is meager.

$A = X \setminus D$ is a G_δ subset of X , so it is completely metrizable. It is separable as it is a subset of a separable metric space. $U_n \cap A$ is a countable basis of clopen sets for A .

Now let C be a countable dense subset of A . $S = A \setminus C$ is once again completely metrizable, separable, has a basis of clopen sets and has no isolated points. By

the Baire category theorem S is dense in A , and so in X . However so is S^c , as it contains C , which is a dense subset of X . Let $M = C \cup D$, so that $S = X \setminus M$.

We now claim that no compact subset of S has non-empty interior. Suppose $F \subseteq S$ is compact with non-empty interior. Then for some n we have $U_n \cap A \subseteq F$. F is compact, so closed as a subset of X . Thus $\overline{U_n \cap A} \subseteq F$. But $\overline{U_n \cap A} = \overline{U_n}$, as A is dense. So $U_n \subseteq F$. But C is dense in X so meets U_n . So there is some $x \in U_n \cap C$. Then $x \notin A$, so $x \notin F$. Contradiction.

Hence $X \setminus M$ is a non-empty complete separable metric space with a basis of clopen sets such that every compact subset has empty interior. Thus by 3.5 it is homeomorphic to $\mathbb{N}^{\mathbb{N}}$

□

Lemma 3.8. *Let X be a topological space and $A \subseteq X$ dense and comeager¹. Then $W \subseteq X$ has the Baire property in X iff $W \cap A$ does in A .*

Proof. First note that $M \subseteq X$ is nowhere dense in X iff $M \cap A$ is nowhere dense in A :

One direction is clear. If $U \subseteq cl_X(M)$ is open and non-empty then $U \cap A \subseteq cl_A(M \cap A)$ is open, and it is non-empty because A is dense in X . So if $M \cap A$ is nowhere dense in A then M is nowhere dense in X .

Suppose now that $U \cap A \subseteq cl_A(M \cap A)$. Then $U \cap A \subseteq cl_X(M)$. So $cl_X(U \cap A) \subseteq cl_X(M)$. But because U is open and A is dense, $cl_X(U \cap A) = cl_X(U)$. Thus $U \subseteq cl_X(M)$. So if M is nowhere dense in X , then $M \cap A$ is in A .

It is then trivial to extend this to M is meager in X iff $M \cap A$ is in A .

Suppose W has the Baire property in X . Then $W \triangle U = M$ for some open U and meager M . So $(W \cap A) \triangle (U \cap A) = M \cap A$. By the above $M \cap A$ is meager, so $W \cap A$ has the Baire property in A .

Conversely, suppose that $(W \cap A) \triangle (U \cap A) = M$, with M meager in A . Let $N = X \setminus A$. Then $W \triangle U \subseteq M \cup N$, which is a union of two meager sets and so itself meager. Thus W has the Baire property in X .

□

Theorem 3.9 (DC + BP). *Let X be a complete separable metric space without isolated points. Then every subset of X has the Baire property.*

Proof. By 3.7, every complete separable metric space without isolated points has a comeager subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$. First considering \mathbb{R} and applying 3.8, we have that every subset of $\mathbb{N}^{\mathbb{N}}$ has the Baire property. Now applying 3.7 to X we have a dense comeager subset of X which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Applying 3.7 again we get that every subset of X has the Baire property.

□

¹Note that dense is a redundant hypothesis if X is a Baire space.

Theorem 3.10 (DC). *Let G be a topological group which is a Baire space and let $A \subseteq X$ have the Baire property and be non-meager. Then $A^{-1}A$ contains a neighbourhood of the identity.*

Proof. Let U be non-empty and open with $A \triangle U$ meager. The map $(x, y) \rightarrow xy^{-1}$ is continuous so we may find $g \in G$, V an open neighbourhood of the identity with $gVV^{-1} \subseteq U$. So for any $h \in V$, $gV \subseteq Uh$. In particular $e \in V$, so $gV \subseteq U$ and thus $gV \subseteq Uh \cap U$.

Now, $(U \cap Uh) \triangle (A \cap Ah) \subseteq (U \triangle A) \cup ((U \triangle A)h)$. Thus $(U \cap Uh) \triangle (A \cap Ah)$ is meager.

If $A \cap Ah$ is empty then $U \cap Uh$ is meager and thus so is gV , but we know gV is non-empty, contradicting that G is Baire.

Thus $\forall h \in V \exists x \in A \cap Ah$. So $\exists x, y \in A$ with $x = yh$, so $h = y^{-1}x \in A^{-1}A$. Thus $V \subseteq A^{-1}A$.

□

Definition 3.11. *Let X, Y be topological spaces. $f : X \rightarrow Y$ is said to have the Baire Property if for every Borel set $U \subseteq Y$, $f^{-1}(U)$ has the Baire property. Equivalently if for every open U the same holds (as the open subsets generate the Borel sigma-algebra).*

Lemma 3.12 (DC). *Let G, H be topological groups with H separable and G Baire. Let $\phi : G \rightarrow H$ be a group homomorphism with ϕ having the Baire property. Then ϕ is continuous.*

Proof. It suffices to show that ϕ is continuous at e_G , the identity. Let U be an open neighbourhood of e_H . Let V be a balanced ($V = V^{-1}$) neighbourhood of e_H with $V^2 \subseteq U$.

Now, H is separable so let $\{h_n\}$ be a countable dense subset. For any $x \in G$, xV is a neighbourhood of x , so it meets $\{h_n\}$. Say $h_n \in xV$. Then because V is balanced, $x \in h_nV$. Thus $H = \bigcup h_nV$. Hence $G = \phi^{-1}(\bigcup h_nV) = \bigcup \phi^{-1}(h_nV)$.

Now, G is Baire so for some n , $\phi^{-1}(h_nV)$ is non-meager. Further it has the Baire property because ϕ does and h_nV is open.

Hence by the previous theorem we have $(\phi^{-1}(h_nV))^{-1}\phi^{-1}(h_nV)$ contains a neighbourhood of the identity. But $(\phi^{-1}(h_nV))^{-1}\phi^{-1}(h_nV) \subseteq \phi^{-1}(V^{-1}h_n^{-1}h_nV) = \phi^{-1}(V^2) \subseteq \phi^{-1}(U)$.

Hence ϕ is continuous at e_G and so continuous.

□

Theorem 3.13 (DC + BP). *Let X, Y be metrizable topological groups with X completely metrizable. Let $T : X \rightarrow Y$ be a group homomorphism. T is continuous.*

Proof. It will suffice to show that T is continuous at 0.

Suppose not. Then we can find $\epsilon > 0$, $x_n \in X$ with $x_n \rightarrow 0$ and $\|Tx_n\| \geq \epsilon$.

Let X' be the closed subgroup generated by $\{x_n\}$ and let $Y' = T(X')$. Now, X' and Y' are both separable.

Because X' is separable we know that every subset of it has the Baire property. Thus $T|_{X'}$ trivially has the Baire property. Further because it is complete we know that is a Baire space. Hence, because Y' is also separable, we may apply 3.12 and conclude that $T|_{X'}$ is continuous, contradicting the fact that $x_n \rightarrow 0$ and $Tx_n \not\rightarrow 0$.

□

Corollary 3.13.1 (DC + BP). *Let X be a Banach space and Y a normed space. Let $T : X \rightarrow Y$ be linear. T is continuous.*

Corollary 3.13.2 (DC + BP). *The groups $(\mathbb{R}^n, +)$ are not isomorphic for distinct n .*

Proof. Any homomorphism $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is \mathbb{Q} linear and continuous, thus \mathbb{R} linear. An \mathbb{R} linear map between vector spaces can only be an isomorphism when $m = n$. □

4 Other examples of nonconstructive objects in analysis

We will now show that BP is inconsistent with any reasonably general version of HB (although it is of course consistent with HB for continuous sublinear functionals on separable normed spaces). We will use similar constructions to show that ZF + DC + BP proves the non-existence of various nonconstructive objects (what Schechter calls ‘intangibles’).

Lemma 4.1 (HB). *There is a finitely additive probability measure on $P(\mathbb{N})$ which assigns measure zero to finite sets.*

Proof. We will obtain it from a linear functional on $B(\{0, 1\}^{\mathbb{N}})$.

First let $V_0 \subseteq B(\{0, 1\}^{\mathbb{N}})$ be the subspace of elements of finite support. i.e. $V_0 = \{x \in B(\{0, 1\}^{\mathbb{N}}) : \{n : x_n \neq 0\} \text{ is finite}\}$.

Let $V_1 = \text{span } V_0 \cup \{1\}$, where 1 is the sequence $(1, 1, \dots, 1, \dots)$. We define f on V_1 by $f(x) = 0$ if $x \in V_0$ and $f(1) = 1$, extending linearly.

f clearly has norm 1. Thus applying the Hahn Banach theorem we can extend it to a linear function $f : B(\{0, 1\}^{\mathbb{N}}) \rightarrow \mathbb{R}$ of norm 1.

Suppose $x \in B(\{0,1\}^{\mathbb{N}})$ with $x \geq 0$. Then $f(x) = \|x\| - f(\|x\|1 - x) \geq \|x\| - \|\|x\|1 - x\|$. But for every $n \in \mathbb{N}$ $0 \leq x_n \leq \|x\|$, so $\|x\| \geq \|x\| - x_n \geq 0$ and $\|\|x\|1 - x\| \leq \|x\|$.

Thus $f(x) \geq 0$.

Hence f is a positive functional.

Now, $\mu(U) = f(I_U)$ is a finitely additive probability measure on $P(\mathbb{N})$. For any U , $I_U \geq 0$ so $\mu(U) \geq 0$. If A, B are disjoint then $I_{A \cup B} = I_A + I_B$ so $\mu(A \cup B) = f(I_A) + f(I_B) = \mu(A) + \mu(B)$ and $\mu(\mathbb{N}) = f(1) = 1$.

□

Note that all we really used was Hahn-Banach for continuous sub-linear functionals on just about the simplest example of an inseparable normed space.

We will need the following two, somewhat technical, lemmas:

Definition 4.2. $A \subseteq \{0,1\}^{\mathbb{N}}$ is called a tail set if whenever $x \in A$ and $\{n : y_n \neq x_n\}$ is finite, $y \in A$.

Lemma 4.3. Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be a tail set. If A has the Baire property then A is either meager or comeager.

Proof. A has the Baire property, so there exists U open and M meager with $A \triangle U = M$. If A is non-meager then $U \neq \emptyset$.

U is non-empty open so it contains $N_s = \{x \in \{0,1\}^{\mathbb{N}} : s \text{ is an initial segment of } x\}$ for some $s \in \{0,1\}^{<\mathbb{N}}$. Say $l(s) = n$.

Let V_t be $N_s \setminus M$ with the first n entries of every element replaced with s . Then V_t is meager and $A \supseteq N_s \setminus V_s$. Because A is a tail set and every element of $N_s \setminus V_s$ differs from one of $N_t \setminus V_t$, $A \subseteq N_t \setminus V_t$ for every $t \in \{0,1\}^n$. So $A \supseteq \bigcup_{t \in \{0,1\}^n} N_t \setminus V_t \supseteq \bigcup_{t \in \{0,1\}^n} N_t \setminus \bigcup_{t \in \{0,1\}^n} V_t = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{t \in \{0,1\}^n} V_t$. Each V_t is meager, so $\bigcup_{t \in \{0,1\}^n} V_t$ is meager. Thus A contains a comeager set and so is comeager.

□

Definition 4.4. Let A be a set. We define $A^{<\mathbb{N}}$ to be the set of all finite sequences with terms in A .

For $x, y \in A^{<\mathbb{N}}$ we say $x \prec y$ if x is an initial segment of y .

We define a tree on A to be a set $T \subseteq A^{<\mathbb{N}}$ such that $y \in A$ and $x \prec y$ then $x \in T$.

A tree T is said to be pruned if for every $x \in T$ there is some $y \neq x$ with $y \in T$ and $x \prec y$.

Given a tree T we define $[T]$ to be the set of infinite branches of the tree. That is $[T] = \{x \in A^{\mathbb{N}} : \forall n \ x|_n \in T\}$.

Given a closed set $F \subseteq A^{\mathbb{N}}$ we define $T_F = \{x|_n : x \in F, n \in \mathbb{N}\}$. This is clearly a pruned tree.

The maps $F \rightarrow T_F$ and $T \rightarrow [T]$ are mutually inverse bijections between the pruned trees on A and the closed subsets of $A^{\mathbb{N}}$.

Lemma 4.5. For $x \in \{0, 1\}^{\leq \mathbb{N}}$ define $K(x) = \{n \in \mathbb{N} : n < l(x), x_n = 1\}$.

Let $M \subseteq \{0, 1\}^{\mathbb{N}}$ be meager. There is an uncountable set $B \subseteq \{0, 1\}^{\mathbb{N}}$ such that $B \cap M = \emptyset$ and for $x \neq y$ $K(x) \cap K(y)$ is finite.

Proof. Let $M = \bigcup Q_p$, with the Q_p nowhere dense. By passing to their closures and rearranging we may without loss of generality assume that the Q_p are closed and $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$. In a mild abuse of notation we shall also denote the pruned tree associated with Q_p by Q_p .

We will construct an injective monotone function $F : \{0, 1\}^{<\mathbb{N}} \rightarrow \{0, 1\}^{<\mathbb{N}}$ such that:

1. $l(F(x))$ only depends on $l(x)$.
2. $l(F(x)) \rightarrow \infty$ as $l(x) \rightarrow \infty$
3. If $l(x) = p$ then $F(x) \notin Q_p$
4. If $F(x)_n = F(y)_n = 1$ then $\forall m < n, F(x)_m = F(y)_m$

Having done this we can extend it to a function $F : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$, by letting $F(x)$ be the unique y such that $\forall n, F(x|_n)$ is an initial segment of y . The set B we desire is then $F(\{0, 1\}^{\mathbb{N}})$. We shall prove this after constructing F .

Our construction will proceed as follows:

$$F(\emptyset) = \emptyset$$

Having constructed $F(x)$ for all x with $l(x) = p$, we shall construct $F(x \wedge i)$ with $l(x) = p$ and $i \in \{0, 1\}$ as follows. Enumerate the sequences of length p as x_1, \dots, x_{2^p} .

Note that because Q_p is nowhere dense, for every $a \in Q_p$ there is some b such that $a \cap b \notin Q_p$. (as if not then it would contain $\{x : a \prec x\}$ which is an open set). Thus given any sequence we can extend it to a sequence not in Q_p (this extension is trivial if the sequence is already not in Q_p).

We will now build sequences $y_1^i, \dots, y_{2^p}^i$ with $i \in \{0, 1\}$. We will then define $F(x_k \wedge i) = F(x) \wedge y_k^i$.

Start each y_k^i off as 2^p blocks of two. All of these blocks except the k th are 00. The k th block is 10 if $i = 0$ and 01 if $i = 1$. This guarantees that the map will be injective, and these share no common 1s so if at this point $F(x \wedge i)$ and $F(y \wedge j)$, with $l(x) = l(y) = p$ share a common 1 in this part of the string then $x \wedge i = y \wedge j$. If they share a common 1 earlier then the condition is already satisfied, by the inductive construction.

We now need to guarantee that these are not in Q_{p+1} , and do this in a way that doesn't disrupt the common 0s property.

We now list the strings y_k^i in some order. Take the first one and extend it to some string not in Q_{p+1} . Fill all the others up with 0s to make them the same length. Now do this with the second one, etc.

The resulting strings constructed will give us our desired F .

We now show that $B = F(\{0,1\}^{\mathbb{N}})$ has the desired property.

F is injective, so certainly B is uncountable. Further, by construction every point in the image of F is not contained in any of the Q_p , thus not contained in their union.

Suppose $u = F(x) \neq F(y) = v$. Then for some n we have $u_n \neq v_n$. Thus for any $m > n$ we cannot have $u_m = v_m = 1$, as our construction would then ensure that $u_n = v_n$.

□

Theorem 4.6 (DC). *Let μ be a finitely additive probability measure on \mathbb{N} which assigns measure 0 to finite sets. Equivalently we can regard it as a function on $\{0,1\}^{\mathbb{N}}$. Then $T = \{x \in \{0,1\}^{\mathbb{N}} : \mu(x) = 0\}$ does not have the Baire property.*

Proof. Because μ vanishes on finite sets, T is a tail set - if $x \in T$ and $\{n : y_n \neq x_n\}$ is finite, then $y \in T$. Thus if it has the Baire property it is either meager or comeager.

Let $a \rightarrow a'$ denote the map sending (a_n) to $(1 - a_n)$. This is a homeomorphism. Further because μ is finitely additive we have $\mu(a) + \mu(a') = 1$. So $T' = \{x : \mu(x) = 1\}$.

Now, suppose T is comeager. Then $T^c = \{x : \mu(x) > 0\}$ and $T'^c = \{x : \mu(x) < 1\}$ are meager. But $\{0,1\}^{\mathbb{N}} = T^c \cup T'^c$, contradicting that it is a Baire space.

Thus T must be meager.

By the previous lemma we have that there is an uncountable set $B \subseteq P(\mathbb{N})$ such that $B \cap T = \emptyset$ and for any $U, V \in B$ $|U \cap V| < \infty$ (just making the obvious conversion between a subset of $\{0,1\}^{\mathbb{N}}$ and one of $P(\mathbb{N})$).

Every element of B has positive measure, and thus measure $> \frac{1}{k}$ for some k . In particular for some k we must have uncountably many elements of B with measure $> \frac{1}{k}$ (else B would be a countable union of countable sets). In particular there are more than k such sets. Say A_1, \dots, A_k . Because their intersections are finite we can remove a finite number of points from each to make them disjoint. Since μ vanishes on finite sets, this does not change the measure. Now $\mu(\bigcup A_k) = \sum \mu(A_k) > \sum \frac{1}{k} > 1$. This contradicts the fact that μ is a probability measure.

□

Corollary 4.6.1 (DC + BP). *The Hahn Banach theorem for continuous sublinear functionals is false.*

However, as we established in the previous section, a weaker form of the Hahn Banach theorem is true in $\text{ZF} + \text{DC}$. This proves that we cannot prove a much stronger version of the result in $\text{ZF} + \text{DC}$.

This theorem can be used to show the non-existence in $\text{ZF} + \text{DC} + \text{BP}$ of a surprisingly large number of objects that are ubiquitous in ZFC .

Corollary 4.6.2 (DC + BP). *Let $f \in l^{\infty*}$. There exists $y \in l^1$ such that $\forall x \ f(x) = \sum x_i y_i$. i.e. $l^{\infty*} = l^1$.*

Proof. Suppose f does not arise from l^1 in this way. We can write $f = f^+ - f^-$ for two positive linear functionals f^+, f^- . At least one of these must not arise from l^1 , so we may assume wlog that f is positive.

Denote by 1 the sequence all of whose entries are 1.

Now, let $y_i = f(e_i)$, where e_i is the element of l^∞ that is 1 in its i th entry and 0 elsewhere. $\sum_{i=1}^n y_i = f((1, \dots, 1, 0, \dots)) \leq f(1)$. Thus $y_i \in l^1$.

Let $g(x) = \sum x_i y_i$. Note that if $x \geq 0$ then $g(x) \leq f(x)$. Let $h(x) = f(x) - g(x)$. Then this is a positive linear functional which vanishes on each of the e_i and thus on any element of l^∞ which is non-zero for only finitely many entries.

h is non-zero as else $f = g$, which contradicts our hypothesis. $h(1)$ is non-zero, as if it were not then for any $x \geq 0$ we would have $h(x) \leq h(\|x\|1) = 0$, contradicting that h is non-zero. Dividing by $h(1)$ we can normalise it so that $h(1) = 1$.

Now let $\mu(A) = h(1_A)$, where 1_A is the characteristic function of $A \subseteq \mathbb{N}$. μ is a finitely additive probability measure on \mathbb{N} which vanishes on finite sets, contradicting BP.

□

Corollary 4.6.3 (DC + BP). $\neg \text{WUF}$. i.e. *there are no free ultrafilters on \mathbb{N} .*

Proof. Suppose F were a free ultrafilter on \mathbb{N} . Then $\mu(A) = 1$ if $A \in F$ and 0 otherwise is a finitely additive probability measure on \mathbb{N} which vanishes on finite sets. □

Corollary 4.6.4 (DC + BP).

The unit ball of the dual space of l^∞ is not compact in the weak topology.*

Proof. Suppose it were. Then the space of all characters on l^∞ (considered as a Banach algebra) would be compact. The space of coordinate maps ($a \rightarrow a_n$ for some n) is homeomorphic to \mathbb{N} , so is not compact. Thus there is some

multiplicative linear functional $f : l^\infty \rightarrow \mathbb{C}$ which is not one of the coordinate maps.

Note that for $A \subseteq N$ we have $I_A^2 = I_A$, so $f(I_A)^2 = f(I_A)$ and thus $f(I_A) = 0$ or 1. Further if $A \subseteq B$ then $I_B - I_A = I_{B \setminus A}$. Thus if $f(I_A) = 1$ we have $f(I_B) = 1$.

Now, if $f_{I_{\{n\}}} = 1$ we have $f(I_A) = 1$ iff $n \in A$. It is then easy to deduce from this that f is the evaluation map for the n th coordinate. Thus for all n we have $f_{I_{\{n\}}} = 0$.

Further, $f(1) = f(I_{\mathbb{N}}) = 1$. Thus we have $\mu(A) = f(I_A)$ is a finitely additive probability measure on \mathbb{N} which assigns measure zero to finite sets. This is a contradiction.

□

Corollary 4.6.5 (DC + BP). *The Banach-Alaoglu theorem is false in ZF + DC + BP.*

Thus, as promised, we have shown that ZF + DC + BP contradicts the full versions of many standard analytic theorems, despite the seperable cases being theorems of ZF + DC.

5 A Strong Fubini Theorem

Theorem 5.1 (ZF + DC). *Fubini's Theorem*

Let X, Y be finite measure spaces and $f : X \times Y \rightarrow \mathbb{R}$ bounded and measurable. Then

$$\int_Y \int_X f(x, y) dx dy = \int_X \int_Y f(x, y) dy dx$$

It is easy but long to verify that the usual proof of Fubini's Theorem carries through as normal in ZF + DC. We shall not do it here.

The question we want to ask is whether we can weaken the hypotheses on f . It is well known that you cannot remove the boundedness hypothesis, for example $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$ on $[0, 1]^2$, but can we weaken measurability to merely require that the two iterated integrals exist?

Because ZFC + DC + LM is consistent (modulo large cardinals), we know that ZF + DC + SFT is consistent, as all functions $f : [0, 1]^2 \rightarrow \mathbb{R}$ will be lebesgue measurable in ZF + DC + LM, so the measurability hypothesis becomes unnecessary. Thus we will be interested more in the case where we assume the entirety of AC.

We state SFT entirely in terms of $[0, 1]$ rather than general probability spaces. In fact this is much more general than one might think, as we have the following theorem:

Theorem 5.2 (DC). *Borel Isomorphism Theorem.*

Let X be a separable completely metrizable topological space, and let μ be an atomless borel probability measure on X . Then $(X, B(X), \mu)$ is isomorphic to $([0, 1], B([0, 1]), \lambda)$. where λ is the usual measure on $[0, 1]$.

We shall not prove this here. See e.g. [8]. This means that the version of SFT we consider is in fact very general, and includes almost all probability spaces (and thus finite measure spaces) that we are interested in, as we can break up any measure on a separable completely metrizable space into an atomic and non-atomic part, and SFT for the atomic part is elementary (the atomic part has countable support it's just the monotone convergence theorem).

The following lemma will essentially characterise how SFT can fail:

Lemma 5.3 (ZF + DC + \neg SFT). *There exists a set $A \subseteq [0, 1]^2$ with $\{x : (x, y) \in A\}$ null for almost every $y \in [0, 1]$ and $\{y : (x, y) \in A\}$ having measure 1 for almost every $x \in [0, 1]$.*

Proof. By hypothesis, there exists $f : [0, 1]^2 \rightarrow \mathbb{R}$ bounded with $\int_0^1 \int_0^1 f(x, y) dx dy \neq \int_0^1 \int_0^1 f(x, y) dy dx$ (and both of these integrals exist).

We may assume without loss of generality that $\int_0^1 \int_0^1 f(x, y) dy dx < \int_0^1 \int_0^1 f(x, y) dx dy$.

We will construct such a set as a subset of $[0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}}$. By the Borel isomorphism theorem this will then give us such a subset of $[0, 1]^2$.

Now, for the actual construction:

First pick γ with $\int_0^1 \int_0^1 f(x, y) dy dx < \gamma < \int_0^1 \int_0^1 f(x, y) dx dy$

Because the integrals are well defined, for almost every x the integral $\int f(x, y) dy$ converges. Thus for almost every $(x_n) \in [0, 1]^{\mathbb{N}}$ each of the integrals $\int f(x_n, y) dy$ converges.

Fix such an x . The functions on $[0, 1]^{\mathbb{N}}$ given by $g_i(y) = f(x_i, y_i)$ clearly form a sequence of independent random variables. Further because f is bounded, so are they. Thus we may apply the strong law of large numbers to get that

$$\{y \in [0, 1]^{\mathbb{N}} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(f(x_i, y_i) - \int f(x_i, y) dy \right) = 0\}$$

has measure 1.

Again by the law of large numbers,

$$\{x \in [0, 1]^{\mathbb{N}} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \int f(x_i, y) dy = \int f(x, y) dy dx\}$$

also has measure 1.

Thus, combining these, we get that

$$\{y \in [0, 1]^{\mathbb{N}} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i, y_i) = \int f(x, y) dy dx\}$$

again has measure 1.

Swapping the roles of x and y we get the same result for x :

$$\{x \in [0, 1]^{\mathbb{N}} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i, y_i) = \int f(x, y) dx dy\}$$

has measure 1.

Now define $A = \{(x, y) : \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i, y_i) > \gamma\}$.

The first part of the above proves that for almost every x , the set $\{y : (x, y) \in A\}$ has measure 1. The second part gives that for almost every y , the set $\{x : (x, y) \notin A\}$ has measure 1, and so $\{x : (x, y) \in A\}$ has measure 0.

This gives the desired result.

□

Theorem 5.4 (DC). *Suppose there exists a set $A \subseteq [0, 1]^2$ with $\{x : (x, y) \in A\}$ null for almost every $y \in [0, 1]$ and $\{y : (x, y) \in A\}$ having measure 1 for almost every $x \in [0, 1]$. Then if $f(x, y) = I_A$ we have*

$$\int_0^1 \int_0^1 f(x, y) dx dy \neq \int_0^1 \int_0^1 f(x, y) dy dx$$

Proof. Fix x such that $\{y : (x, y) \in A\}$ has measure 1. Then $\int_0^1 f(x, y) dy = 1$, as $f(x, y) = 1$ for a.e. y . So this integral is 1 for a.e. x , and thus $\int_0^1 \int_0^1 f(x, y) dy dx = 1$. Essentially the same proof shows that $\int_0^1 \int_0^1 f(x, y) dx dy = 0$.

□

Corollary 5.4.1 (DC). $\neg SFT \iff$ *there exists a set of the above form.*

We will use this to demonstrate that additional axioms can be added to ZFC which either prove or refute SFT.

5.1 The Continuum Hypothesis and Martin's Axiom

Theorem 5.5 (AC + CH). $\neg SFT$

Proof. We well order $[0, 1]$ with order type ω_1 . i.e. every initial segment is countable. Call this well ordering \prec .

Let $A = \{(x, y) : x \prec y\}$. Then for $x \in [0, 1]$ the set $\{y : (x, y) \in A\}$ is cocountable, so has measure 1, and for $y \in [0, 1]$ the set $\{x : (x, y) \in A\}$ is countable, so has measure 0.

□

Note that all we really used of CH is that every subset of \mathbb{R} of cardinality $< c$ is null. This actually a much weaker statement than CH. We shall establish it from Martin's Axiom instead, which is known to be consistent with $\neg CH$.

Theorem 5.6 (AC + MA(κ)). *The union of κ many null subsets of \mathbb{R} is null.*

Proof. Let $\{F_\alpha : \alpha < \kappa\}$ be a collection of null subsets of \mathbb{R} .

Fix $\epsilon > 0$. Let $C_\epsilon = \{a \subseteq \mathbb{R} : a \text{ is open and non-empty and } \mu(a) < \epsilon\}$. Order this by inclusion.

We wish to show this is ccc.

Suppose $A \subseteq C_\epsilon$ is an uncountable up-antichain. Then for some n the set $\{a \in A : \mu(a) < (1 - \frac{1}{k})\epsilon\}$ is uncountable.

Now, let $\{U_n\}$ be the set of all finite unions of rational intervals. For each $a \in A$ we can pick n with $U_n \subseteq a$ and $\mu(a \setminus U_n) < \frac{1}{k}\epsilon$.

Suppose we have done so. Then because A is uncountable there must be two distinct a, b for which we have picked the same U_n .

But then $\mu(a \cup b) \leq \mu(a) + \mu(b - U_n) < (1 - \frac{1}{k})\epsilon + \frac{1}{k}\epsilon = \epsilon$. Thus $a \cup b$ is a common upper bound for a and b in C_ϵ , contradicting that A was an antichain.

Now, fix $\alpha < \kappa$. Define $D_\alpha = \{U \in C_\epsilon : F_\alpha \subseteq U\}$. Because F_α is null, this is dense: Let $a \in C_\epsilon$. Then $\mu(a) < \epsilon$. Pick some open subset b , say b , with $\mu(b) < \epsilon - \mu(a)$ and $F_\alpha \subseteq b$. Then $a \cup b$ is an upper bound for a which is in D_α .

Thus, by MA(κ), we may conclude that there is an upwards directed set $P \subseteq C_\epsilon$ which meets each F_α . Let $V = \bigcup P$.

For each $b \in P$ and $x \in b$ there is some U_n with $x \in U_n \subseteq b$. Thus we can pick a countable subset of P , say b_n with $V = \bigcup b_n$. Now, because P is upwards

directed we can pick c_n with $\forall j \leq n \ b_n \leq c_n$ and $c_j \leq c_n$. Then $V = \bigcup c_n$ is an increasing union, so $\mu(V) = \lim_{n \rightarrow \infty} \mu(c_n) \leq \epsilon$. Hence $\mu(V) \leq \epsilon$.

Now, because P met each F_α , we have for all α , $F_\alpha \subseteq V$. Thus $\bigcup F_\alpha \subseteq V$.

Hence for all $\epsilon > 0$ there is a set V such that $\mu(V) \leq \epsilon$ and $\bigcup F_\alpha \subseteq V$. Thus $\bigcup F_\alpha$ is null.

□

Corollary 5.6.1 (AC + MA). *A union of fewer than c many null subsets of \mathbb{R} is null.*

Corollary 5.6.2 (AC + MA). *Any subset of \mathbb{R} of cardinality less than c is null.*

This then allows us to carry out our construction exactly as in the CH case, so we further have:

Corollary 5.6.3 (AC + MA). $\neg SFT$

This is related to a major area of intersection between analysis and set theory to do with so called cardinal invariants of \mathbb{R} . For example, the above can be regarded as a statement about an invariant called the additivity of lebesgue measure. A measure is κ additive if the union of κ many null sets is null. The additivity, $add(\mu)$ is the smallest κ such that μ is not κ additive. All ZFC proves about $\kappa(\mu)$ where μ is lebesgue measure is that $add(\mu)$ is regular and $\aleph_1 \leq add(\mu) \leq c$. Other than this result, we shan't really look at cardinal invariants.

5.2 Axioms of Symmetry: A_{null} and A_{\aleph_0}

Freiling's Axioms of Symmetry are to do with suggesting various plausible negations to CH . It starts with a heuristic argument to do with 'throwing darts at the real line' to get the statement A_{\aleph_0} . In ZFC this turns out to be precisely $\neg CH$:

Theorem 5.7 (ZFC). $A_{\aleph_0} \iff \neg CH$

Proof. Assume CH . Well order \mathbb{R} with order type ω_1 . Call this well ordering \prec . Then let $f(x) = \{y : y \prec x\}$. We have for all x, y $x \in f(y)$ or $y \in f(x)$, contradicting A_{\aleph_0} .

Now assume $\neg CH$. Pick a set $A \subseteq \mathbb{R}$ of size \aleph_1 . Now $\bigcup_{a \in A} f(a)$ is a union of \aleph_1 countable sets, so has size \aleph_1 . Thus we can pick some y not in it. Now $\forall x \in A \ y \notin f(x)$. $f(y)$ is countable, so $A \setminus f(y)$ is non-empty. Thus we can pick some $x \in A \setminus f(y)$. Now $x \notin f(y)$ and $y \notin f(x)$.

□

As we have established, we need some form of strong negation of CH in order to prove SFT - merely \neg CH is not enough, as $\text{MA} + \neg$ CH is consistent, but $\text{MA} \implies \neg$ SFT. Thus it is perhaps unsurprising that a strengthening of A_{\aleph_0} gives us SFT, especially given the axiom's measure theoretic definition.

Theorem 5.8 ($\text{DC} + A_{\text{null}}$). *SFT.*

Proof. By the above, it suffices to prove that there is no set $A \subseteq [0, 1]^2$ such that for almost every x , $\{y : (x, y) \in A\}$ has measure 0 and for almost every y , $\{x : (x, y) \in A\}$ has measure 1.

Suppose such an A existed.

Define $A_x = \{y : (x, y) \in A\}$ and $A^y = \{x : (x, y) \in A\}$.

The hypothesis state that for a.e. x , A_x is null. Say for $x \notin C$, with C null. Similarly let B be a null set such that whenever $y \notin B$, A^y has measure 1.

Define $f_1(x) = A_x \cup B$ if $x \notin C$ and $f_1(x) = B$ otherwise, $f_2(y) = (A^y)^c \cup C$ if $y \notin B$ and $f_2(y) = C$ otherwise and $f(t) = f_1(t) \cup f_2(t)$. Then $f : [0, 1] \rightarrow [0, 1]_{\text{null}}$.

Now, by axiom A_{null} we have there exist x, y with $x \notin f(y)$ and $y \notin f(x)$. $B \subseteq f_1(x) \subseteq f(x)$, so $y \notin B$. Similarly $x \notin C$. Because $x \notin C$ we have $f(x) \supseteq f_1(x) = A_x \cup B$. Thus $y \notin A_x$. $x \notin f(y) \supseteq (A^y)^c \cup C$. Thus $x \notin (A^y)^c$ and so $x \in A^y$.

So we have $x \in A^y$ and $y \notin A_x$. This contradicts the definitions of such.

□

Corollary 5.8.1 ($\text{AC} + A_{\text{null}}$). *There exists $A \subseteq \mathbb{R}$ with $|A| < c$ which is not null.*

Corollary 5.8.2 ($\text{AC} + A_{\text{null}}$). $\neg \text{MA}$

6 A Result of Erdos

This section is a little tangential to the main point of this essay, and I agonised over whether or not to include it. In the end I decided to simply because of the fact that the main result of this section is really very pretty. This result is proved in [4].

Wetzel originally asked the following question: Is there an uncountable family F of entire functions such that for every $z \in \mathbb{C}$ we have $\{f(z) : f \in F\}$ countable.

We will call the statement that there exists such a family W .

Lemma 6.1 (AC). $W \implies \text{CH}$

Proof. Suppose CH were false, and let F be a family as in the definition of W . By considering a subset if necessary we may let $|F| = \aleph_1$.

Now, let $f, g \in F$ with $f \neq g$. $\{z : f(z) = g(z)\}$ is countable, for if not then it would have a limit point and thus we would have $f = g$.

Consider

$$\bigcup_{f, g \in F, f \neq g} \{z : f(z) = g(z)\}$$

This is a union of \aleph_1 many sets of cardinality \aleph_0 , so has cardinality \aleph_1 . Thus there is some z not in it. Then by definition we have $f(z) \neq g(z)$ for any $f, g \in F$ with $f \neq g$. Thus the map $f \rightarrow f(z)$ is injective and $\{f(z) : f \in F\}$ is uncountable, contradicting the hypotheses on F .

□

Lemma 6.2 (AC). $CH \implies W$

Proof. Let S be a countable dense subset of \mathbb{C} (e.g. \mathbb{Q}^2). Well order \mathbb{C} as $\{z_\alpha : \alpha < \omega_1\}$.

We will construct a sequence of distinct functions $F = \{f_\alpha : \alpha < \omega_1\}$ such that if $\alpha < \beta$ then $f_\alpha(z_\beta) \in S$. This will clearly satisfy the requirements because $\{f(z_\beta) : f \in F\} \subseteq \{f_\alpha(z_\beta) : \beta \leq \alpha\} \cup S$, which is countable because initial segments of ω_1 are.

We do this by transfinite induction. Suppose we have done so for $\alpha < \beta$. The sets $\{f_\alpha : \alpha < \beta\}$ and $\{z_\alpha : \alpha < \beta\}$ are countable, so we list them as g_n and w_n respectively (no essential change needs be made for the case where they are actually finite, so we will ignore this).

We will construct an entire g so that $n, g(w_n) \in S$ and $g(w_n) \neq g_n(w_n)$. Clearly $f_\beta = g$ will then work.

We define $p_n(z) = \prod_{i=1}^n (z - w_i)$. Then $p_n(w_k) = 0$ for $k \leq n$, $p_n(w_k) \neq 0$ for $k > n$, and $|p_n(z)| \leq K_n(1 + |z|^n)$ for some K_n .

Now, we inductively pick a sequence a_n such that $|a_n| < \frac{1}{n!} K_n$ and $\sum_{k=0}^n a_k p_k(w_n) \in S$. We can do this because S is dense, so can find a point of S arbitrarily close to $\sum_{k=0}^{n-1} a_k p_k(w_n)$, and choose a_n to make the sum equal to that value. Further we can choose it so that $\sum_{k=0}^n a_k p_k(w_n) \neq g_n(w_n)$.

Now, define $g(z) = \sum a_n p_n(z)$. Because of our choice of a_n we have that this sum is locally uniformly convergent, and thus g is entire.

Now, $g(w_n) = \sum_{k=0}^n a_k p_k(w_n) \in S$ and $g(w_n) \neq g_n(w_n)$.

This completes our construction.

□

Thus we have:

Theorem 6.3 (AC). $CH \iff W$

7 Automatic Continuity

Automatic continuity is a fascinating area of modern Banach algebra research, which studies when the algebraic structure of a map between Banach algebras guarantees that it is continuous, without any further assumptions.

This theory of course needs the full of AC to make it really interesting, since as we have established in $\text{ZF} + \text{DC} + \text{BP}$ any linear map, let alone homomorphisms, whose domain is a Banach space are continuous.

There are various beautiful theorems in the theory, for example:

Theorem 7.1 (AC). *Let A, B be commutative Banach algebras with B semisimple. Every homomorphism $f : A \rightarrow B$ is continuous.*

There are many other theorems of this sort. Thus to someone not expecting it, the following might come as a surprise. Recall that NDH is the statement that every homomorphism from $C(X)$ into a Banach algebra is continuous (where X is a compact Hausdorff space).

Theorem 7.2.

1. $\text{ZFC} + \text{CH} \implies \neg \text{NDH}$
2. $\text{ZFC} + \text{MA} + \text{NDH}$ is consistent.

Unfortunately a proof of this is far beyond the scope of this essay. The first part was established in [1]. See e.g. [2] for a proof of the second.

It is also known that $\text{ZFC} + 2^{\aleph_0} = \aleph_2 + \neg \text{NDH}$ is consistent. See [12] for details.

8 Conclusion

Much of the theory we have developed here can be regarded (from an Analyst's perspective) as examining how counterexamples to certain conjectures can arise.

For example, after looking at some concrete examples one might naturally be led to conclude that all linear functions on a Banach space were continuous, or that l^1 was reflexive (i.e. $l^{\infty*} = l^1$). The Axiom of Choice gives us, nonconstructively, various counterexamples to this conjecture, and our work with the Baire property proves that we do in fact need some reasonably strong form of choice to imply the existence of these counterexamples.

In fact, much of what we looked at shows that $\text{DC} + \text{BP}$ is a very nice theory for developing a quasi-constructive analysis in, as it removes many of the worst examples of non-constructive or pathological objects. In particular the strong automatic continuity result we proved is rather nice (although, it must be said, does rather trivialise a lot of interesting areas of study in analysis).

This is of course a somewhat special type of restriction, in that analysts will generally assume the axiom of choice without worrying about it, and thus it is of limited use to know. However, the fact that there are no *explicit* counterexamples should be reassuring, because it means that we can generally assume that any such function we write down will be of the relevant type.

For the purpose of general theory, the second type of example we looked at is perhaps more relevant, as it works in ZFC. SFT is an example of something one might easily conjecture after having looked at various examples. Thus the fact that it is actually independent of ZFC may be regarded as a fact of genuine analytic interest.

This example demonstrates an equally valid way of looking at the question: What set theoretic machinery do we need to construct *examples*? Sometimes we might want a counterexample to SFT. For example in [9], Pitowsky makes use of the continuum hypothesis to get a counterexample to SFT and then goes on to use to develop an interesting physical theory based on this.

Automatic continuity is another excellent example of a question in analysis that is independent of ZFC. Automatic continuity questions are one of the major areas of study in modern Banach algebra theory, so the question of whether every homomorphism from $C(X)$ is continuous is a very natural one to ask. Thus, once again, the fact that it is independent of ZFC shows that the natural area of study for analysts can easily move outside of it.

A slight point of interest is that the continuum hypothesis tends to produce more examples (counterexamples to SFT, discontinuous homomorphisms, families of the form proposed in W). However in most cases it seems that $\neg\text{CH}$ is not strong enough to refute the existence of these examples (for example $\neg\text{CH} + \neg\text{SFT}$ is consistent, as is $\neg\text{CH} + \neg\text{NDH}$).

All of this points to the fact that the study of analysis increasingly begins to ask questions which are sensitively dependent on the set theory behind them. Thus, in conclusion, while it is not always *neccesary* to look at the detailed set theory involved with a question, studying analysis in isolation from the associated set theory proves increasingly artificial and unable to answer various interesting questions that we want to ask.

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