Analysis in the world according to David
(Until a better title arrives)

David R. MacIver

Contents

0.1 Some Notation

(This is a temporary page, and will be replaced by a proper one at a later date).

Here is some notation which I will be using throughout the book:

Firstly, the natural numbers $\mathbb{N}$ contain 0.

For a set X and some function f defined on X, $f(X) = \{f(x) : x \in X\}$.

Similarily for a function f and a set X contained in the range of f, we define $f^{-1}(X) = \{t : f(t) \in X\}$. Note that this doesn't require f to be invertible - e.g. if $f(x) = x^2$ then $f^{-1}\{1\} = \{-1, 1\}$.

If we have an equivalence relation $\sim$ then $[x]_\sim$ is the equivalence class of x with the relation $\sim$.

Given sets X and $Y \subseteq X$ we define the indicator function $I_Y : X \to 0, 1$ by $I_Y(x) = 0$ if $x \notin Y$ and $I_Y(x) = 1$ if $x \in Y$.

Chapter 1

Algebra and Geometry

At its most basic, analysis is almost entirely concerned with the links between algebra and geometry; both in using algebraic ideas to solve geometric ones, and to use geometry to solve algebraic problems. Using the motivation of geometrical ideas we go on to develop a theory of curves, continuity, limits, differentiation and integration. These concepts, as well as giving us a more detailed and deeper understanding of geometry, are both mathematically interesting in their own right and very applicable to practical problems.

This chapter will mostly consist of an informal discussion of ideas. We will use geometric notions to suggest areas worth exploring, and use this to develop the theory that will allow a more precise mathematical discussion of the subject. If you are offended by the hand waving and imprecise arguments of this chapter, I apologise, and promise that while this is a necessary preamble to the main material it will not continue beyond this chapter.

1.1 Numbers and Plane Geometry

To start with, we shall consider the geometry of the plane. In theory we could start with the axioms of euclidean geometry, but this is really more trouble than it's worth so we will not bother.

Define the set $\mathbb{E}^2$ to be the set of all the points in the plane. We would also like a set of numbers, $\mathbb{F}$ on which we will do algebra that is going to tell us about the plane.

We would like to be able to use these numbers to define some sense of distance between two points. Specifically a function $d : (\mathbb{E}^2)^2 \to \mathbb{F}$. The distance should satisfy:

1. There is a number $0 \in \mathbb{F}$ that represents the distance from a point to itself, and distinct points have non-zero distance between them. i.e. $d(x, y) = 0$

if and only if $x = y$

2. The distance between x and y is the same as the distance between y and x. i.e. $d(x, y) = d(y, x)$
3. The distance between two points is preserved by translating them both by the same amount.
4. The distance between two points is preserved by rotating them through a fixed angle around some point.
5. If a point z lies on the line between x and y then $d(x, z) \leq d(y, z)$. Further if $z \neq y$ then $d(x, y) \neq d(x, z)$. (This of course means that we want some ordering $\leq$ on our numbers. We would then write $x < y$ to mean $x \leq y$ and $x \neq y$).

You may find all these assumptions blindingly obvious - indeed, you would be right. That isn't to say however they aren't useful, as they tell us a lot about what properties our numbers need to have.

For now let us assume that all our numbers are represented as the distance between two points. This will of course mean that all our numbers are positive (i.e. greater than or equal to zero), as if $t = d(x, y)$ then x lies on the line between x and y, so $0 = d(x, x) \leq d(x, y) = t$. We will however be able to introduce negative numbers later, without any real difficulty.

Now lets see what these assumptions tell us about the ordering of our numbers.

Consider two numbers $u = d(x_1, y_1)$ and $v = d(x_2, y_2)$. Then translate x_2 and y_2 so that the translation of x_2 is x_1, and rotate so that the line between x_2 and y_2 lies along the line between x_1 and y_1. Then, intuitively, either the two lines are the same or one lies within the other. So we have either $u < v$, $v < u$ or $u = v$, by our assumption 5 (and exactly one of these holds). This also gives us that if $u \leq v$ and $v \leq u$ then $u = v$.

We also trivially have that for any number u, $u \leq u$, as if $u = d(x, y)$ then y lies on the line between x and y, so $u = d(x, y) \leq d(x, y) = u$.

Finally, if $t \leq u$ and $u \leq v$ then find w, x, y, z with $d(w, x) = t$, $d(w, y) = u$, $d(w, z) = v$) (we can do this by finding pairs of points appropriate distances apart and then translating). Rotate all the line segments so they are all along the same line. We now have the line between w and x contained in the line between w and y, which is in turn contained within the line between w and z. Thus the line between w and x is contained within the line between w and y. Hence $t = d(w, x) \leq d(w, z) = v$.

To summarise:

1. $u \leq v$
2. If $u \leq v$ and $v \leq u$ then $u = v$.
3. If $t \leq u$ and $u \leq v$ then $t \leq v$.

4. For every x, y either $x \leq y$ or $y \leq x$

A relation which satisfies the first three of those conditions is called a partial order. If it also satisfies the fourth it is called a total order (we won't meet a great variety of these yet, but they will come in useful later at various stages).

Now that we've got some of the basic notions of distance out of the way, lets consider other things we can do with numbers. For a start, we haven't actually got much in the way of an algebraic structure so far - all we have is an order. With numbers we have encountered before, e.g. the rationals, we can add them and multiply by them. How would this work on our geometry?

First of all take an infinite straight line L. If we indicate some preferred direction of the line, we can translate points along it as follows:

Denote one direction along the line to be the left, the other the right - this need not correspond to an actual left and right, it's just convenient to have a name for each of the two directions. We will consider translations to the right.

Take a number u and find a pair of points x, y such that $d(x, y) = u$. Rotate and translate the line between x and y so that it lies on L and x is to the left of y. Now, given a point z on L, slide x, y along so that x lies on z. Map z to the new position of y.

So, any number $u \in \mathbb{F}$ gives us a function $f_u : L \to L$.

If $u \neq 0$, then f_u preserves the order of points along the line (i.e. if x is to the left of y then $f_u(x)$ is to the left of $f_u(y)$), so is injective. Also it is surjective, as if you put a line of length u with the rightmost point of it at y then the leftmost point maps to it under f_u.

Also, any translation of the line corresponds to some f_u, as if you take any point on the line, find the distance it is translated by (i.e. $u = d(x, g(x))$ for a translation g), then we must have $g = f_u$.

If we compose f_u with f_v this gives us another translation, so it corresponds to f_t for some t. Intuitively, this t should be $u + v$. So we have an addition on the numbers, which we could in fact define by $f_t \cdot f_u = f_{t+u}$.

So lets see what we can deduce about this addition. First of all, composition of functions is associative. Thus so is the addition. i.e. $t + (u + v) = (t + u) + v$.

Translations commute, so we should have $u + v = v + u$.

Translating along the line preserves the ordering, so $u < v$ if and only if $u + t < u + v$. This also gives us that if $u + t = v + t$ then $u = v$.

This allows us to define $u - v$ for $u > v$ as the unique element t such that $t + v = u$. There certainly is such an element, and if we have two such elements t and t' then $t + v = t' + v$ so $t = t'$.

Proposition 1.1

If x, y are two points on the plane and z lies on the line between them then $d(x, y) = d(x, z) + d(y, z)$.

Proof:

Give the line a direction by saying x is at the left and y at the right. By definition, if we translate x by $d(x, z)$ we get z, and if we translate z by $d(z, y)$ we get y (as z is to the left of y). So the translation from x to y is same as the translation from x to z followed by the translation from z to y. Because a translation is uniquely determined by where it maps one point to we thus must have the two translations being identically equal.

So $f_{d(x,y)} = f_{d(x,z)} \cdot f_{d(z,y)} = f_{d(x,y)+d(y,z)}$. Hence $d(x, y) = d(x, z) + d(z, y)$.

QED.

As well as translations, we would also like some sense of being able to expand and contract the plane by a factor $u \in \mathbb{F}$ (A map which is either a contraction or an expansion will be called a dilation). This will correspond to multiplying numbers.

Fix a point y in the plane. We want to consider dilating the plane around this point. So, for each number u, associate a function g_u to it. We require g_u to be a bijection unless $u = 0$ (in which case we are shrinking the plane to y). We want these maps to satisfy the following:

1. There should be a number corresponding to no change. i.e. $\exists 1 \in \mathbb{F}$ such that g_1 is the identity map.
2. For any of these maps other than g_0, it should be possible to reverse it by dilating. i.e. $\forall u \in \mathbb{F},\ \exists v \in \mathbb{F},\ g_v = g_u^{-1}$
3. The composition of two dilations is itself a dilation. So for any u, v there exists a t such that $g_u \cdot g_v = g_t$.
4. For any line L passing through y, g_u preserves the order of points on the line.
5. Given a line L passing through y and a point x to the right of y, $g_u(x)$ is to the right of $g_v(x)$ if and only if $u > v$.
6. The order of the dilations shouldn't matter. So $g_u \cdot g_v = g_v \cdot g_u$.

Property 5 means that if $g_u = g_v$ then $u = v$. This means we can use dilations to get a multiplication on our numbers by defining $u \cdot v$ to be the unique number t such that $g_t = g_u \cdot g_v$.

What does our discussion of the dilation functions now tell us about multiplication?

1. Multiplication is associative, as composition of functions is associative. $\forall x, y, z \ \ x \cdot (y \cdot z) = (x \cdot y) \cdot z$

2. Multiplication is commutative, by property 6.
   $\forall x, y \quad x \cdot y = y \cdot x$
3. Multiplication has an identity, 1.
   $\forall x \quad 1 \cdot x = x$
4. Every non-zero number has a unique inverse x^{-1} - a number that multiplies with it to give one.
   $\forall x \; x^{-1} \cdot x = 1$
5. So, according to the previous 4 properties, the set of non-zero numbers forms an abelian group under multiplication.

We would like to know more about how multiplication by a number affects the ordering of numbers. Unfortunately we cannot deduce this from the properties we've already assumed. In order for us to have our intuitive notion of dilation as a scaling, we will require the following:

Let u be any number, and x, y and z points. A dilation through a factor u about z scales the distance between x and y by u. i.e. $d(g_u(x), g_u(y)) = ud(x, y)$.

So we can now deduce two more properties about multiplication:

Let u, v be numbers. Take points x, y, z such that y lies on the line between x and z, $d(x, y) = u$ and $d(x, z) = v$. Order things so that x is at the left and z is at the right.

Now dilate by t. This stretches $d(x, y)$ to $t \cdot u$ and $d(x, z)$ to $t \cdot v$, and it preserves the order of points. So we must have the new distance $t \cdot u = d(g_t(x), g_t(y)) < d(g_t(x), g_t(z)) = t \cdot v$.

This also gives us information about how multiplication and addition are related. Let $u, v \in \mathbb{F}$. Pick points x, y, z with $d(x, y) = u$ and $d(y, z) = v$. By proposition 1, $d(x, z) = u + v$. Now dilate about x by a factor t. All the lengths are stretched by a factor of t, so applying proposition 1 again we get $t \cdot (u + v) = (t \cdot u) + (t \cdot v)$.

For convenience we will now summarise all the relevant properties of multiplication, addition and ordering we have deduced, summarised as a proposition for later reference:

Proposition 1.2

1. $(\forall x) \quad x + 0 = x$
2. $(\forall x, y, z) \quad x + (y + z) = (x + y) + z$
3. $(\forall x, y) \quad x + y = y + x$
4. $(\forall x) \quad x \cdot 1 = x$
5. $(\forall x, y, z) \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$
6. $(\forall x, y) \quad x \cdot y = y \cdot x$

7. $(\forall x, y, z)$ if $x + z = y + z$ then $x = y$.
8. $(\forall x, y, z)$ $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
9. $(\forall x, y)$ $x < y$ or $y < x$ or $x = y$ and no more than one of these holds.
10. $(\forall x, y, z)$ if $x < y$ then $z + x < z + y$
11. $(\forall x, y, z)$ if $x < y,\ z > 0$ then $z \cdot x < z \cdot y$

At this point the algebra is starting to look very much like it does for the rational numbers. We don't yet have additive inverses, but we will introduce them later (for now they just get in the way - pesky minus signs...)

Experience suggests that we should be able to have distances on the line corresponding to the non-negative rational numbers. Lets see how we might go about doing this.

Let m, n be natural numbers with $n \neq 0$. We want to define a length corresponding to $\frac{m}{n}$.

It is obvious how to define a length of 1, as we have a number in $1 \in \mathbb{F}$. So take a line of length 1 and split it up into n lines of equal length. Define the length of one of these lines to be the number $\frac{1}{n}$. We can then define $\frac{m}{n} = \frac{1}{n} + \cdots + \frac{1}{n}$ (with $\frac{1}{n}$ being summed m times).

It is easy to check that this is well defined (i.e. that $\frac{km}{kn}$ and $\frac{m}{n}$ correspond to the same number), and that the ordering, addition and multiplication of $\mathbb{F}$ are the same as those for the rationals. Also it is worth noting that, as one would expect, $nu = u + \cdots + u$ with u being summed n times, so that nu is the length of n lines of length u placed end on end and $\frac{1}{n} = n^{-1}$.

So, our numbers behave like the positive rationals and (effectively) contain them as a subset. Are the positive rationals enough then?

To start with, let us (finally) introduce negative numbers to our number system as well, in the following relatively obvious manner:

For each number u associate another number $-u$ that represents translation in the opposite direction to f_u. i.e. $f_{-u} = f_u^{-1}$. We then represent dilation by a factor -1 about a point x as 'sending y to the other side of x'. i.e. You draw the line from y to x, extend it a distance $d(x, y)$ past x and map y to that point. You then define dilation by a factor of u followed by dilation by a factor of -1. Ordering can merely be considered as saying $u > v$ if and only if given any point x on a line L, $g_u(x)$ lies to the right of $g_v(x)$.

Given all these definitions, you should now convince yourself that all the properties of proposition 1.2 hold, together $u + (-u) = 0$.

We now have a properly built up number system; we know that algebra must behave in a relatively familiar manner, and that our numbers contain the rational numbers.

So... given all this, what can we do with it exactly? Time for another section I guess.

1.2 Area and Pythagoras' Theorem

First of all, we want to be able to define some sense of area on shapes in the plane. A full treatment of this will need to wait until we have a proper treatment of integration, but for now we can at least define area for certain 'nice' shapes (where we regard a shape as an appropriate subset of the plane). Let the set of all subsets of $\mathbb{E}^2$ that have area be X. We will denote the area of a shape A by $\mu(A)$. As before we will start by defining certain properties that area, and shapes that have area, should have:

1. For every $A \in X$, $\mu(A) \geq 0$
2. If $A, B \in X$ with $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
3. Let A be a rectangle with sides of length a and b. Then $A \in X$ and $\mu(A) = ab$.
4. If $A, B \in X$ and only meet on their boundaries, then $A \cup B$ has area and $\mu(A \cup B) = \mu(A) + \mu(B)$.
5. If $A \in X$, and L is a line that intersects A, then let B and C be the parts of the shape on either side of the line, then B and C are both in X.
6. If $A \in X$ and B is obtained from A by rotations, translations and reflections then $B \in X$ and $\mu(B) = \mu(A)$

Of course, there are plenty of other shapes that we would like to assign area to - for example, the above collection of assumptions doesn't prove that the circle is in X. What we would like to be able to say was that we can 'fill up' the circle with polygons of increasing size which approximate it arbitrarily closely, so the circle must have area as well. Indeed, this is what we will eventually do (many chapters down the line). For now however this uses limiting ideas and properties of our number system which we do not have the tools to work with, so we will stick to this more restrictive set of assumptions.

Still, X is certainly a subset of all the shapes we would like to assign an area to, so it is sufficient to work with for now.

First of all, note that any right-angle triangle is in X.

Take a rectangle of sides a and b. Draw a line, L from two opposite corners. This cuts the rectangle up into two right-angled triangles each with sides a and b. Thus each of these triangles lies in X.

By a translation followed by two reflections (there will be a diagram here) we can transform one triangle into the other, so they must have a common area - call it k. Then, by our assumptions, $k + k = ab$, so $k = \frac{ab}{2}$.

Now, any right angled triangle with sides a, b can be transformed into any other by a series of translations, rotations, and reflections, so all right-angled triangles of sides a, b must have area $\frac{ab}{2}$.

Now take any triangle $\Delta = ABC$. Let L be the unique line perpendicular to BC that passes through A. There are three cases to consider:

1. If L passes through B or C then Δ is right-angled, so in X.
2. If L cuts BC but not at either end then let D be the point it intersects at. Then each of ABD and ACD are right-angled triangles, and only meet at AD, which is on their border. So $\Delta \in X$. Then if $h = d(A, D)$ these triangles have areas $\frac{1}{2}hd(B, D)$ and $\frac{1}{2}hd(C, D)$ respectively. Thus $\mu(\Delta) = \frac{1}{2}hd(B, D) + \frac{1}{2}hd(C, D) = \frac{1}{2}hd(B, C)$.
3. Else, extend the BC to the infinite line K. K is perpendicular to L, so they meet at a unique point D. By relabelling if necessary C lie between B and D. Then ABD is right-angled triangle, so lies in X and the line AC cuts it in two, into ABC and ADC. Thus, by property four, $\Delta \in X$. Now to work out it's area. Letting h be as above, we know the area of the big triangle ABD is $\frac{1}{2}hd(B, D)$ (as it is a right-angled triangle), and the area of the triangle ACD is $\frac{1}{2}hd(C, D)$, as it is also a right angled triangle. Thus, using the additive property for area,

$$\begin{aligned}
\mu(\Delta) + \mu(ACD) &= \mu(ACD) \\
\mu(\Delta) + \frac{1}{2}hd(C, D) &= \frac{1}{2}hd(B, D) \\
&= \frac{1}{2}(hd(B, C) + d(C, D) \\
&= \frac{1}{2}hd(B, C) + \frac{1}{2}hd(C, D) \\
\mu(\Delta) &= \frac{1}{2}hd(B, C)
\end{aligned}$$

Or, in other words, the area of a triangle is half height times base. Thrilling, isn't it...

So, any triangle is in X. From this we can go on to deduce that any polygon is in X. Thus we do have a reasonably extensive set of shapes we can work with.

So, assuming we have such defined, we find this gives us an additional constraint on our distance function d.

Theorem 1.1 *Pythagoras' Theorem*

Let the lines XY and YZ be perpendicular to eachother, then $d(X, Y)^2 + d(Y, Z)^2 = d(X, Z)^2$.

Proof: Let XYZ be a right-angled triangle as shown.

(Diagram)

QED.

So the existence of an area constrains our distance function significantly - as we will see shortly, it in fact uniquely specifies it.

Time for another exciting new section.

1.3 Cartesian Coordinates

You're no doubt familiar with - and bored to death of - Cartesian coordinates from school. They're not very exciting, but they are basically *the* tool one needs to put a handle on geometry and apply algebra to it in a useful manner.

Consider a line L. We can attach a co-ordinate - a way of using numbers to specify points on it - in an obvious manner.

Choose a direction on the line, and a base point e which we will call the origin. Given any x on the line, assign a number to it, which we will call $c(x)$ for now. Don't get too attached to that notation though - it's going to last all of a few paragraphs.

We define $c(e) = 0$. If x is to the right of e we define $c(x) = d(e, x)$, if x is to the left of e we define $c(x) = -d(e, x)$. It is then easy to check the following:

1. $c(x) > c(y)$ if and only if x lies to the right of y.
2. c is a bijection between L and $\mathbf{F}$.
3. If f_u is a translation, then for any x on L, $c(f_u(x)) = c(x) + u$
4. If g_u is a dilation about e then for any x on L, $c(g_u(x) = u \cdot c(x)$ (this includes u negative or zero).
5. $d(x, y) = |c(x) - c(y)|$, where $|t|$ is defined by $|t| = t$ if $t \geq 0$ else $|t| = -t$ and is called the absolute value of t.

Already the notion of a co-ordinate has given us a much more concrete handle on how our various abstract notions of translation, dilation, etc. work. Now lets see how we can generalise this to represent the entire plane.

To start with, consider our line, L, and some point not on the line, x. We will define the projection of x onto the line.

Consider a line T which is perpendicular to L and passes through x (there is a unique such line). The point where T intersects L is called the projection of x onto L, which we will denote by $\pi_L(x)$.

It is worth noting that $\pi_L(x)$ is the closest point to x on L, as by Pythagoras on L then $d(x,y)^2 = d(x,\pi_L(x))^2 + d(\pi_L(x),y)^2 \geq d(x,\pi_L(x))^2$, so $d(x,y) \geq d(x,\pi_L(x)$. This isn't particularly relevant, but it's a nice characterisation of the projection.

The projection onto a line suggests a natural way of assigning a co-ordinate to any point in the plane. If we take any line L and project x down onto L, we get the first co-ordinate of x. Then we have a unique line passing through x perpendicular to L, call it M. We can then assign co-ordinates on M to get the second co-ordinate.

The problem is, that doesn't quite work. As it stands our choice of the line M depends on x, so we could in principle have our co-ordinate co-ordinate depending on the first. It shouldn't take long to convince yourself that this isn't really a problem, but in order to make things nice and precise we'll try and pin it down a little more.

Any two lines perpendicular to L are parallel, so all we need to do is pick some specific M. There's an obvious choice to make - have it meet L where the L co-ordinate is 0. We can then define our co-ordinates on M with the origin at the intersection of M and L. We now call M, L our co-ordinate axes.

(Diagram)

From the diagram you can see that a point is uniquely specified by it's co-ordinates, and that any pair of numbers (x, y) specifies a point.

So, given our co-ordinate axes L, M we can use them to uniquely identify $\mathbb{E}^2$ with $\mathbb{F}^2$. This is a major step in our attempts to approach geometry from an algebraic point of view. When we have a co-ordinate system we will denote a point x by its co-ordinates, which we shall write as (x_1, x_2).

First lets note that we have now completely specified the distance between two points.

Consider two points x, z, and the point $y = (z_1, x_2)$. Then, as the line between x and y and the line between y and z are each parallel to the axes, the triangle formed by these three points is right-angled, with the right-angle at y. The length of the sides xy and yz are $|x_1 - z_1|$ and $|x_2 - z_2|$ respectively (by considering the projections of points onto the axes). Hence, by Pythagoras' theorem we have $d(x,z)^2 = |x_1 - z_1|^2 + |x_2 - z_2|^2 = (x_1 - z_1)^2 + (x_2 - z_2)^2$

So why does this specify the distance uniquely? Consider the equation $x^2 = u$. If there are any solutions to this then we can write $u = y^2$ for some y. So $x^2 = u$ if and only if $x^2 - y^2 = 0$. But $x^2 - y^2 = (x-y)(x+y)$. So $x = \pm y$. But we know that a distance is positive, so there is only one solution to $d(x,z)^2 = (x_1 - z_1)^2 + (x_2 - z_2)^2$.

Now lets look at our two main transformations: Translation and dilations.

Consider a line L with a direction, and a point x on L. Translate x along the line by a number v. If we then consider the projection of x onto the two co-ordinate axes, this gives a translation along each, by u_1 and u_2 respectively. It is easy to see that u_1, u_2 do not depend upon the choice of x, so the translation is the map $(x_1, x_2) \to (x_1 + u_1, x_2 + u_2)$. Further, any map of this form will give a translation of x along the line between x and $(x_1 + u_1, x_2 + u_2)$, so the translations are precisely the maps of this form. We will use the notation $x + u = (x_1 + u_1, x_2 + u_2)$. We now no longer have to specify a line for a translation, and can merely consider any map of the form $x \to x + u$ to be a translation.

Now consider dilations. First consider dilations about the origin by a factor v. Let x be a point. The projections of x along the two axes are dilated by a factor v, and are clearly dilated to the projections of the dilations of x. So we have x dilated to $(vx_1, vx_2) = vx$.

Now if we consider dilations about a point y, we can use what we have done previously. Translate y to the origin by the dilation $x \to x - y$. Then dilate by a factor v, and translate back by a factor of y. So we have $x \to v(x - y) + y$.

Finally, lets try and see how to characterise lines with our co-ordinate system.

Consider a line R and two points x, y on R. By dilating about y we can map x to any other point on the line, and any dilation about y will map x into the line. So the line is the set of points that we can dilate x to. i.e. the set $\{x + v(x - y) : v \in \mathbb{F}\}$.

We have now used our number system to completely characterise the plane geometry, so now we must determine the properties of our number system.

1.4 The Real Numbers

The first question we have to ask ourselves is if the rationals are enough. In order to resolve this question we shall first prove the following lemma:

Lemma 1.1

Let $k, n \in \mathbb{N}, x \in \mathbb{Q}$ such that $x^k = n$. Then $x \in Z$

Proof:

We may without loss of generality assume $x > 0$. So write $x = \frac{r}{s}$, with r and s coprime natural numbers.

Then $x^k = \frac{r^k}{s^k}$.

Suppose r^k and s^k had a common factor other than 1. Then there would be some prime p such that $p|r^k$ and $p|s^k$. Because p is prime we know that this would require that $p|r$ and $p|s$, whcih contradicts the fact that r and s are coprime. Hence r^k and s^k are coprime.

x^k is an integer, so we must have $s^k = 1$ (as x^k in integer implies that $s^k|r^k$, and they are coprime. So $s = 1$, and x is thus an integer.

QED.

Why is this useful? Because it immediately shows that there must be irrational numbers in our number system: Consider the points $(0, 1)$ and $(1, 0)$. Let d be the distance between them. Then $d^2 = 1^2 + 1^2 = 2$. But if d is rational then it is an integer, and there are no integers that square to 2. So d must be an irrational number.

But adding squares isn't enough. If you consider the smooth curve $\{(x, x^3) : x \in \mathbf{F}\}$ (where we define $x^3 = x \cdot x \cdot x$ as usual). It is geometrically obvious that this crosses the line $\{(x, 2) : x \in \mathbf{F}\}$ but, similarly to before, there is no rational x such that $x^3 = 2$.

So is it enough to add in roots of polynomials? That absence of such roots seems to be the major problem so far. At the moment it isn't obvious that there are any other problems, but further theory will reveal other, more subtle, difficulties. The major obstacle at the moment is that it is far from obvious how we would construct such a number system. We will thus take a different approach which will give us far more.

The next question we have to ask ourself is if we want to allow infinities and infinitesimals. This is in fact a genuine choice: There are number systems which are genuinely useful in analysis and calculus that have numbers which are 'infinitely big' or 'infinitely small'. While this is a valid approach, there are several problems with it:

Firstly, the construction of such numbers and the proofs of their various properties are genuinely hard. They involve some fairly advanced mathematics which you could quite plausibly complete a mathematics degree without ever encountering.

Secondly, on a practical level, the physical significance of infinitesimals isn't very plausible. Even the existence of irrational numbers is a little dubious - all the measurements we can make give rational numbers, and imprecisely at that. But with irrational numbers we believe that we could, in principle, construct them in real life if we had perfect measurement - e.g. You could get a length of $\sqrt(2)$ as the length of the hypotenuse of a right angled triangle of two sides exactly 1. Of course we couldn't measure the angle and sides perfectly, but in some idealised world we would expect to be able to obtain these numbers. Infinitesimals and infinities seem to be, even in principle, unmeasurable, so suggesting their existence seems little more than smoke and mirrors.

Thirdly, the methods of analysis given by infinitesimals don't seem to generalise nearly as naturally as the more traditional methods.

As you can probably tell, I don't like that way of doing things much. There are many strong arguments for the use of infinitesimals as well - they give a much more natural definition of various things, and once you have the basic theory sorted out a lot of the proofs are much simpler and more intuitive. If you're interested then it's certainly a subject worth looking into, but it's not the approach we're going to take.

So, we want to avoid infinities in our numbers. How best to do this?

If we think about two line segments L and M, of length a and b respectively, with $a < b$. By appropriate rotations, translations, etc. we may consider b to lie inside M. We want to have the notion that M is not infinitely larger than L. We can best express that by saying that if we keep adding copies of L end on end with eachother it will eventually be longer than M. i.e. for some natural number n we have $na > b$.

This suggests the following:

Definition 1.1 (The Axiom of Archimedes)

Let u, v be any two numbers greater than zero. There is a natural number n such that $nu > v$.

This then gives us:

Proposition 1.3

1. Let $u > 0$. There is a natural number n such that $\frac{1}{n} < u$
2. Let u be any number. There is a natural number n such that $n > u$
3. Let $u < v$. There is a rational number t such that $u < t < v$.

Proof:

1 and 2 are trivial consequences of the axiom of Archimedes, together with the fact that multiplication by rationals preserves ordering.

Pick a natural number n such that $\frac{1}{n} < (v - u)$. We know that there is some natural number k such that $k > v$ so $\frac{kn}{n} > v$. Thus the set $\{m \in \mathbb{N} : \frac{m}{n} < n\}$ is a bounded above set of natural numbers, so has a maximum element r.

Suppose $\frac{r}{n} \leq u$. Then $\frac{r+1}{n} \leq u + \frac{1}{n} < u + (v - u) = v$. This contradicts r being the maximal such element, so we must have $\frac{r}{n} > u$.

Hence we have $u < \frac{r}{n} < v$, as required.

QED

So there are rationals arbitrarily close to any other number. This will help us in building up our number system from the rationals, as will the following:

Proposition 1.4

Let $x \in \mathbb{F}$. Then x is uniquely determined by $\{t \in \mathbb{Q} : t < x\}$. i.e. if $\{t \in \mathbb{Q} : t < x\} = \{t \in \mathbb{Q} : t < y\}$ then $x = y$.

Proof:

Suppose $x \neq y$. Without loss of generality $x < y$. Then there is a rational z such that $x < z < y$, by the previous lemma. Then $z \in \{t \in \mathbb{Q} : t < y\}$ and $z \in \{t \notin \mathbb{Q} : t < x\}$. So $\{t \in \mathbb{Q} : t < x\} \neq \{t \in \mathbb{Q} : t < y\}$

QED

Note that this in fact shows slightly more: If $x < y$, then $\{t \in \mathbb{Q} : t < x\}$ is a proper subset of $\{t \in \mathbb{Q} : t < y\}$. It is easy to check that the converse also holds - if $\{t \in \mathbb{Q} : t < x\}$ is a proper subset of $\{t \in \mathbb{Q} : t < y\}$ then $x < y$. So sets of this form in some sense have the same ordering as $\mathbb{F}$, which suggests them as a possible candidate for the construction of $\mathbb{F}$.

The problem is that as of yet we don't know exactly which sets of rationals will correspond to members of $\mathbb{F}$. To start with we need the following:

Proposition 1.5

Let $x \in \mathbb{F}$, $X = \{t \in \mathbb{Q} : t < x\}$. Then

1. If $v \in X$ and $u < v$ then $u \in X$.
2. X has an upper bound in the rationals.
3. X has no greatest element.

The proof is left as an exercise.

Now suppose $X \subseteq \mathbb{Q}$ is a set with these three properties. Consider X^c. As X is bounded above, X^c is non-empty. If $y \in X^c$ and $t > y$ then $t \in X^c$ (else we would have $y \in X$). Suppose X^c had no least element; The numbers $\mathbb{F}$ correspond to a line, and we have effectively broken our line up into two disjoint pieces with a 'hole' in the middle. But our lines in geometry should consist of a single piece, so this doesn't correspond very well with our intuition of what a line is. So there should be a least element x of X^c. But then $X = \{t \in \mathbb{Q} : t < x\}$.

So we could consider sets of rationals satisfying these three properties as our model for $\mathbb{F}$. Unfortunately it turns out that this causes things to go slightly wrong when we try to define multiplication - basically the problem is that if you try to define multiplication in the natural way then the fact that two negative numbers multiply together to give a positive number causes things to go wrong. So we will go back to what we did earlier, and work only on the positive numbers to start with. We will then later reintroduce the negative numbers in a fairly simple way.

So, define $\mathbb{T}$ to be the set of non-empty subsets of $\{q \in \mathbb{Q} : q > 0\}$ which satisfy the above three properties. These will correspond to the strictly positive elements of $\mathbb{F}$.

Let $x, y \in \mathbb{T}$. We define:

$x < y$ if x is a proper subset of y.
$x + y = \{u + v : u \in x, \quad v \in y\}$
$x \cdot y = \{u \cdot v : u \in x, \quad v \in y\}$

Lemma 1.2

If we consider $+$, $\cdot$, $<$ as above, and using 1 as a short-hand for $\{t \in \mathbb{Q} : t < 1\}$, the following hold:

1. $(\forall x, y \in \mathbb{T})\ x + y \in \mathbb{T}$
2. $(\forall x, y \in \mathbb{T})\ x \cdot y \in \mathbb{T}$
3. $(\forall x, y, z \in \mathbb{T})\ \ x + (y + z) = (x + y) + z$
4. $(\forall x, y \in \mathbb{T})\ \ x + y = y + x$
5. $(\forall x \in \mathbb{T})\quad x \cdot 1 = x$
6. $(\forall x, y, z \in \mathbb{T})\ \ x \cdot (y \cdot z) = (x \cdot y) \cdot z$
7. $(\forall x, y \in \mathbb{T})\quad x \cdot y = y \cdot x$
8. $(\forall x, y, z \in \mathbb{T})$ if $x + z = y + z$ then $x = y$.
9. $(\forall x > y \in \mathbb{T})\exists z\ \ x = y + z$
10. $(\forall x, y, z \in \mathbb{T})\ \ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
11. $(\forall x, y \in \mathbb{T})$ $x < y$ or $y < x$ or $x = y$ and no more than one of these holds.
12. $(\forall x, y, z \in \mathbb{T})$ if $x < y$ then $z + x < z + y$
13. $(\forall x, y, z \in \mathbb{T})$ if $x < y,\ z > 0$ then $z \cdot x < z \cdot y$
14. Let $X \subseteq \mathbb{T}$ be non-empty and bounded above. It has a least upper bound.

Proof:

We will not bother to prove all of these, as most of them follow trivial from the definitions and corresponding properties for $\mathbb{Q}$.

(Proofs omitted for now)

So, now comes the question of defining negative numbers. We could consider representing numbers as $x - y$ where x, y are positive numbers. So we could consider our numbers as pairs of positive numbers with an appropriate equivalence relation. i.e. $(x, y) \backsim (u, v)$ if and only if $x - y = u - v$, or $x + v = y + u$ (as we don't have a subtraction defined. We can then define $(x, y) + (u, v) = (x+y, u+v)$, $(x, y)\cdot(u, v) = (x-y)(u-v) = (xu+yv)-(xv-yu) = (xu+yv, yv)$ and $(x, y) < (u, v)$ if $x + v < y + u$.

So, define $\mathbb{R}$ to be the set of equivalence classes of $\mathbb{T}^2$ with the relation $\backsim$, with the operations $+$ and $\cdot$, and the relation $<$ defined as above.

It is easy to check that these are well defined with respect to the equivalence relation, and that all the properties of lemma 1.2 still hold. In addition $\mathbb{R}$ has an additive identity, $0 = [(x, x)]_\backsim$, and additive inverses.

If we associate $x \in \mathbb{T}$ with $x_\mathbb{R} = [(2x, x)]_\backsim$ then it is easy to check then $(x + y)_\mathbb{R} = x_\mathbb{R} + y_\mathbb{R}$, etc. We can associate rationals with elements of $\mathbb{R}$ by first associating the positive rationals with elements of $\mathbb{T}$ by identifying a rational q with $\{t \in \mathbb{Q} : 0 < t \text{ and } t < q\} \in \mathbb{T}$. Negatives are then handled in the obvious way.

$\mathbb{R}$ will be the number system we shall use for all our geometry. We call the elements of $\mathbb{R}$ the *Real Numbers*. It is not immediately obvious that they have all the properties we want them to - indeed it is not at the moment entirely obvious that there are irrational real numbers. For now let the following example be at least suggestive that they are a step in the right direction:

Define $x \in \mathbb{T}$ by $x = \{t \in \mathbb{Q} : 0 < t \text{ and } t^2 < 2\}$.

$x^2 = x \cdot x \leq 2$ as if $u, v \in x$ then $uv < \max\{u, v\}^2 < 2$.

So $x^2 \leq 2$. Suppose $x^2 < 2$. Let $t \in \mathbb{T}$, $t < 2$. (We are being a bit loose with symbols here - $<$ in the context of $\mathbb{T}$ means $\subset$, but as we are thinking of elements of $\mathbb{T}$ as numbers we will use the subset notation). Consider $(x+t)^2 = x^2 + 2tx + t^2 \leq x^2 + t2(x + 1)$. Then if we choose t such that $t < \frac{2-x^2}{2(x+1)}$ we have $(x + t)^2 < x^2 + 2 - x^2 = 2$. As $x + t > x$ there is a rational number $q \in (x + t)$, $q \notin x$; but $q^2 \in (x + t)^2$, so $q^2 < 2$. Thus $q \in x$. This is a contradiction. Hence $x^2 = 2$. Note that this would work just as well if we replaced 2 with any element of $\mathbb{T}$. So every positive element of $\mathbb{R}$ has a square root.

With this in mind as an example, I assert that the real numbers are sufficient for doing geometry in. Thus we will identify our plane $\mathbb{E}^2$ with $\mathbb{R}^2$ by associating a point with it's co-ordinates (in some co-ordinate system - it doesn't matter which one, so we can assume a set of co-ordinates is given). We will study the geometry of $\mathbb{R}^2$ and show that it behaves entirely as we would expect euclidean geometry to do.

And here ends the first chapter. We have used our intuition about the geometry of the plane to develop the real number system, which will be the central object of the analysis in this book. The next few chapters will be spent exploring the properties of the real numbers, and using this to motivate more general ideas around which we will build our theory of real analysis.

Chapter 2

Fields and Orders

In this chapter we will introduce a much greater degree of abstraction than in the previous one. Now that we have constructed our main number system, we will take from it the key properties and use it to define a more general algebraic theory. We will introduce the notion of a field and other more general algebraic objects. We will finish by showing that the reals are uniquely determined by certain of their properties - that they are the unique complete ordered field - which will help us underline the relevant properties in developing our theory of analysis.

2.1 Fields

We first introduce the notion of a general system of numbers in which we can do algebra as we would expect.

If we are to do algebra there should be a multiplication, $+$ and an addition, $\cdot$ on our numbers. We want to define a sufficiently large class of objects on which these operations behave as we would expect them to.

Definition 2.1

Let $\mathbb{F}$ be a set with functions $+ : \mathbb{F}^2 \to \mathbb{F}$ and $\cdot : \mathbb{F}^2 \to \mathbb{F}$

We say $\mathbb{F}$ is a field (with operations $+$ and $\cdot$) if the following are satisfied:

1. $\mathbb{F}$ is an abelian group with operation $+$ and identity $0 \in \mathbb{F}$.
2. $\mathbb{F} \setminus \{0\}$ is an abelian group with operation $\cdot$ and identity $1 \in \mathbb{F}$
3. $\forall a, b, c \in \mathbb{F} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (Distributivity)

Given a field we denote the additive inverse of x by $-x$, and the multiplicative inverse of x by x^{-1} (if $x \neq 0$) (we know from group theory these inverses are unique). In particular, note that both $\mathbb{R}$ and $\mathbb{Q}$ are fields with the usual operations.

This section of the chapter will be concerned with algebra in fields and providing some - admittedly not very exciting - results and definitions which show that the algebraic manipulations we are used to - rearranging sums, etc. - will hold in the general context of a field. Some of the proofs will all be a bit sketchy, as they are mostly fairly trivial and something of a side issue from the main point of this book, but you should have no trouble filling in the details if you so desire.

While, officially, as good and conscientious mathematicians you should read through the proofs of these results and understand them, in actual practise you may find you don't gain a great deal from them. If you choose to merely skim this section of the chapter then you probably won't lose a great deal.

Proposition 2.1

1. $\forall x \in \mathbb{F} \quad 0 \cdot x = 0$
2. If $xy = 0$ then one of $x = 0$ or $y = 0$ must hold.

We will start by considering sums and products of more than two elements; at the moment we only have them defined for one.

For a finite sequence of numbers $x_m, \ldots, x_n$ with $m \leq n$ we define $x_m + \cdots x_n = \sum_{i=m}^{n} x_i$ inductively by:

$\sum_{i=m}^{m} x_i = x_m$
$\sum_{i=m}^{n+1} x_i = \sum_{i=m}^{n} x_i + x_{n+1}$

We define the product $\prod_{i=m}^{n} x_i$ similarly.

We would like to know that we can do the two main algebraic manipulations: Adding brackets in (or removing brackets) and rearranging terms.

e.g. is $x_0 + \cdots + x_n = (x_0 + \cdots + x_i) + (x_{i+1} + \cdots + x_n)$?

If you think about it, any way of adding in brackets is basically just a different way of breaking up the sum of $x_0m, ..., x_n$. You can't actually sum more than two things at once, so what you are doing is at each stage you're adding in brackets and summing the smaller part inductively. For our previously defined sum we make the break between $n-1$ and n - i.e. $x_m + \cdots x_n = (x_m + \cdots x_{n-1}) + x_n$. So what we're asking is whether it matters how we break up this sum.

So, define a sum of $x_m, \ldots, x_n$ inductively as follows:

A sum of x_i is x_i.

If a is a sum of $x_m, \dots x_i$ and b is a sum of $x_{i+1}, \dots x_m$ then $a + b$ is a sum of $x_m, \dots x_n$.

For example $\sum_{i=m}^n x_i$ is a sum of $x_m, \dots, x_n$.

Lemma 2.1

Any two sums of $x_m, \dots, x_n$ are equal.

Proof:

We will show that any sum of $x_m, \dots, x_n$ is equal to $\sum_{i=m}^n x_i$.

First we will need the following:

Let $m \leq k < n$. $\sum_{i=m}^n x_i = \sum_{i=m}^k x_i + \sum_{i=k+1}^n x_i$. This is a trivial proof by induction on n for fixed k.

We now prove the lemma by induction on the length of the sequence $x_m, \dots, x_n$ (where the length is defined to be $n - m$.

If $m = n$ (i.e. the length is zero) then we're done, as there is only one sum.

By the definition of sums, there is some k with $m \leq k < n$, a a sum for $x_m, \dots x_k$, b a sum for $x_{k+1}, \dots, x_n$ and $c = a + b$.

As $x_m, \dots x_k$ and $x_{k+1}, \dots, x_n$ are both sequences of length less than $n - m$ we have, by the inductive hypothesis that $a = \sum_{i=m}^k x_i$ and $b = \sum_{i=k+1}^n x_i$. Hence $c = \sum_{i=m}^k x_i + \sum_{i=k+1}^n x_i = \sum_{i=m}^n x_i$ (by the previous result).

So the result holds for all m, n, by induction.

QED

As we haven't used anything except the associativity of + a similar result holds for products. It will also hold in a general group.

The other thing we would like to be able to do is to rearrange terms at whim. i.e. we want the following result:

Lemma 2.2

Let ρ be a permutation of $\{m, \dots, n\}$. Then $\sum_{i=m}^n x_i = \sum_{i=m}^n x_{\rho(i)}$

Proof:

It is sufficient to consider the case where ρ is a transposition of two adjacent elements, as these generate the group of permutations. So let ρ permute k and $k + 1$.

By the previous result we have $\sum_{i=m}^n x_{\rho(i)} = \sum_{i=m}^{k-1} x_{\rho(i)} + x_{\rho(k)} + x_{\rho(k+1)} + \sum_{i=k+1}^n x_{\rho(i)}$

(Where we boldly introduce the notation that the sums on the left and right hand sides are zero if $m < k-1$ or $k+1 > n$ respectively)

So

$$\begin{aligned}
\sum_{i=m}^{n} x_{\rho(i)} &= \sum_{i=m}^{k-1} x_{\rho(i)} + x_{\rho(k)} + x_{\rho(k+1)} + \sum_{i=k+1}^{n} x_{\rho(i)} \\
&= \sum_{i=m}^{k-1} x_i + x_{k+1} + x_k + \sum_{i=k+1}^{n} x_i \\
&= \sum_{i=m}^{k-1} x_i + x_k + x_{k+1} + \sum_{i=k+1}^{n} x_i \\
&= \sum_{i=m}^{n} x_i
\end{aligned}$$

QED

Here we have only used commutativity and associativity, so the same result applies for products.

Note that this means that we can now define a sum more generally. If we have any finite set X and a function $f : X \to \mathbb{F}$ we can define $\sum_{x \in X} f(X)$ by listing X as $\{t_0, \ldots, t_n\}$ and then letting $x_i = f(t_i)$. We then define $\sum_{x \in X} f(X) = \sum_{i=0}^{n} x_i$. The previous lemma then gives that this is well defined, because any other listing of X would just result in a permutation of the x_i and so give the same value. In keeping with the notation we introduced in the middle of the last proof, we define the sum of an empty set to be zero.

From this we can now define things like $\sum_{i,j=0}^{n} x_{ij} = \sum_{(i,j) \in \{0,\ldots,n\}^2} x_i j$, and other similar notations.

The previous two results can be thought of as generalising the associativity and commutativity rules. Lets now try to generalise distributivity.

Lemma 2.3

Let $X_0, \ldots, X_n \subseteq \mathbb{F}$ be finite sets.

$\prod_{i=0}^{n} (\sum_{x_i \in X_i} x_i) = \sum_{x \in X_0 \times \ldots \times X_n} x_0 \cdots x_n$.

Proof:

This isn't really a very deep result. We prove it first for the $n = 1$ case, and the result will then follow by induction.

So consider X, Y finite sets (we have relabelled for convenience). We prove that $(\sum_{x \in X} x)(\sum_{y \in Y} y) = \sum_{x \in X,\ y \in Y} xy$.

We will do this by induction on $|Y|$. (Yes, you *are* going to be sick to death of induction by the time this chapter is over).

For $|Y| = 1$ the result is trivial - it is just saying that for a constant $y \sum_{x\in X} yx = y\sum_{x\in X} x$, which is an immediate consequence of distributivity and the definition.

If it is true when $|Y| = n$:

Let $|Y| = n+1$. Pick $w \in Y$ and let $Y' = Y - w$.

Then:

$$\begin{aligned}
(\sum_{x\in X} x)(\sum_{y\in Y} y) &= (\sum_{x\in X} x)((\sum_{y\in Y'} y) + w) \\
&= (\sum_{x\in X} x)(\sum_{y\in Y'} y) + w\sum_{x\in X} x \\
&= \sum_{x\in X,\ y\in Y'} xy + \sum_{x\in X} xw \\
&= \sum_{x\in X,\ y\in Y} xy
\end{aligned}$$

QED

For the next part it's important that we have the idea of an integer in the field. We define a map $\mathbb{Z} \to \mathbb{F}$ by sending 0 to $0_{\mathbb{F}}$, a positive integer $n \to n_{\mathbb{F}} = \overbrace{1 + \cdots + 1}^{n}$ and for a negative integer $-n \to (-n)_{\mathbb{F}} = -(n_{\mathbb{F}})$.

It is easy to check that $(m+n)_{\mathbb{F}} = m_{\mathbb{F}} + n_{\mathbb{F}}$ and $(mn)_{\mathbb{F}} = m_{\mathbb{F}} n_{\mathbb{F}}$. Using this as motivation we will usually abuse notation and write n for $n_{\mathbb{F}}$. However it should be noted that this representation is not unique - it is possible for a field to have $n_{\mathbb{F}} = 0$ for non-zero n. However fields in which this is the case aren't really very relevant to analysis, and we won't be looking at them more than in passing.

Definition 2.2

$\mathbb{F}$ is said to be of characteristic zero if $n \neq 0$ implies $n_{\mathbb{F}} \neq 0$. This implies that the map $n \to n_{\mathbb{F}}$ is injective.

We can also define integer powers as $x^n = \overbrace{x \cdots x}^{n}$ for $n > 0$, $x^0 = 1$ and $x^{-n} = (x^{-1})^n$. We can then check that $x^m x^n = x^{m+n}$ and $(x^m)^n = x^n$.

Theorem 2.1 *Binomial Theorem*

Let $x, y \in \mathbf{F}$. $(x+y)^n = \sum_{i=0}^{n} \ {}^nC_i x^r y^{n-i}$

Proof:

We start by relabelling $x = x_1,\ y = x_0$. Then

$(x+y)^n = \prod_{i=1}^{n}(x_0 + x_1)$ by the definition of t^n.

Using lemma 2.3 with $X_i = \{0, 1\}$ this gives us that $(x+y)^n = \sum_{(k_1,\ldots,k_n)\in\{0,1\}^n} x_{k_1} \cdots x_{k_n}$.

We know that a permutation of $\{1, \ldots, n\}$ preserves the product $x_{k_1} \cdots x_{k_n}$ so we may rearrange the product so all the terms with $k_i = 1$ are at the front. Thus if r_k is the number of terms with $k_i = 1$ then this product is equal to $x_0^{r_k}\ x_1^{n-r_k}$.

So $(x+y)^n = \sum_{k\in\{0,1\}^n} x_0^{r_k}\ x_1^{n-r_k}$.

If we now rearrange the sum to group together all the k with the same r_k, this becomes $(x+y)^n = \sum_{i=0}^{n} M_i x_0^i\ x_1^{n-i}$, where M_i is the number of k with $r_k = i$.

Consider the map $\phi\{0,1\}^n \to P(\{1, \ldots, n\})$ given by $\phi(k) = \{i : k_i = 1\}$. It is easy to check that this map is a bijection. Then $r_k = i$ iff $|\phi(k)| = i$, by definition. So the number of k which have $r_k = i$ is equal to the number of subsets of subsets of $\{1, \ldots, n\}$ of size i. This is ${}^nC_i$ by definition.

So, on replacing M_i by ${}^nC_i$, x_0 by x, and x_1 by y we get the desired result.

QED

2.2 Homomorphisms and Isomorphisms

Recall from group theory that a homomorphism is a function that preserves the algebraic structure of groups. We will define a homomorphism similarly for fields.

Definition 2.3

Let $\mathbb{F}$, $\mathbb{G}$ be fields. We say $f : \mathbb{F} \to \mathbb{G}$ is a homomorphism if:

1. $f(0) = 0$
2. $f(1) = 1$
3. $(\forall x, y \in \mathbb{F})\ \ f(x+y) = f(x) + f(y)$
4. $(\forall x, y \in \mathbb{F})\ \ f(x \cdot y) = f(x) \cdot f(y)$

If a homomorphism is bijective it is called an isomorphism. If there is an isomorphism $f : \mathbb{F} \to \mathbb{G}$ then $\mathbb{F}$ and $\mathbb{G}$ are said to be isomorphic. Isomorphic fields can be regarded as essentially the same field - the function f is just a relabelling of the elements.

Proposition 2.2

Let $f : \mathbb{F} \to \mathbb{G}$ be a homomorphism. Then for all $x \in \mathbb{F}$

1. $f(-x) = -f(x)$
2. $f(x^{-1}) = f(x)^{-1}$
3. $f(n_{\mathbb{F}}) = n_{\mathbb{G}}$
4. $f(x^n) = f(x)^n$

Lemma 2.4

Let $\mathbb{F}$, $\mathbb{G}$ be fields, $f : \mathbb{F} \to \mathbb{G}$ be a homomorphism. Then f is injective.

Proof:

Suppose $x \neq y$. Let $t = x - y \neq 0$. Then $f(t)f(t^{-1}) = f(tt^{-1}) = f(1) = 1 \neq 0$. Hence $f(t) \neq 0$.

Hence $f(x - y) \neq 0$ and so $f(x) = f(y) + f(x - y) \neq f(y)$.

QED

Definition 2.4

Let $\mathbb{F}$ be a field. $\mathbb{G} \subset \mathbb{F}$ is said to be a subfield of $\mathbb{F}$ if it is also a field (with the same operations as $\mathbb{F}$). It is easy to check that if $\mathbb{G}$ is non-empty then it is a subfield if and only if:

1. $x, y \in \mathbb{G} \implies x + y \in \mathbb{G}$
2. $x, y \in \mathbb{G} \implies x \cdot y \in \mathbb{G}$
3. $x \in \mathbb{G} \implies -x \in \mathbb{G}$
4. $x \in \mathbb{G} \implies x^{-1} \in \mathbb{G}$

It is an easy exercise that if $f : \mathbb{F} \to \mathbb{G}$ is a homomorphism, then $f(\mathbb{F})$ is a sub-field of $\mathbb{G}$, and f is an isomorphism between $\mathbb{F}$ and $f(\mathbb{F}$, as we know that f is injective and it is surjective onto $f(\mathbb{F})$ by definition.

Lemma 2.5

Let $\mathbb{F}$ be a field of characteristic zero. $\mathbb{F}$ has a subfield isomorphic to $\mathbb{Q}$.

Proof:

By the above observation it is sufficient to exhibit a homomorphism $f : \mathbb{Q} \to \mathbb{F}$.

Notice that there can only be one such homomorphism - the definition of a homomorphism forces it entirely.

If f were a homomorphism then we know that $f(1) = 1_\mathbb{F}$. Then for an integer m we must have $f(m) = m_\mathbb{F}$ by proposition 2.2. Also $f(\frac{1}{n}) = n_\mathbb{F}^{-1}$. Thus $f(\frac{m}{n}) = m_\mathbb{F} n_\mathbb{F}^{-1}$.

$$\begin{aligned}(km)_\mathbb{F}(kn)_\mathbb{F}^{-1} &= k_\mathbb{F} m_\mathbb{F} k_\mathbb{F}^{-1} n_\mathbb{F}^{-1} \\ &= k_\mathbb{F} k_\mathbb{F}^{-1} m_\mathbb{F} n_\mathbb{F}^{-1} \\ &= m_\mathbb{F} n_\mathbb{F}^{-1}\end{aligned}$$

So we know that if $\frac{m}{n} = \frac{i}{j}$ then $m_\mathbb{F} n_\mathbb{F}^{-1} = i_\mathbb{F} j_\mathbb{F}^{-1}$.

Because we know that for $n \neq 0$, $n_\mathbb{F} \neq 0$, we know that if $n \neq 0$ then $n_\mathbb{F}$ is invertible. Thus we may define a function f by: $f(\frac{m}{n}) = m_\mathbb{F} n_\mathbb{F}^{-1}$.

Trivially $f : 0,\ 1 \to 0_\mathbb{F},\ 1_\mathbb{F}$.

$$\begin{aligned}f(\frac{m}{n}\frac{r}{k}) &= f(\frac{mr}{nk}) \\ &= (mr)_\mathbb{F}(nk)_\mathbb{F}^{-1} \\ &= m_\mathbb{F} r_\mathbb{F} n_\mathbb{F}^{-1} k_\mathbb{F}^{-1} \\ &= (m_\mathbb{F} n_\mathbb{F}^{-1})(r_\mathbb{F} k_\mathbb{F}^{-1}) \\ &= f(\frac{m}{n}) f(\frac{r}{k})\end{aligned}$$

Similarly $f(x + y) = f(x) + f(y)$.

QED

Note that, from our observations in this proof, if $f : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism, then it is the identity.

This then makes more precise what we said in the previous chapter about the real numbers containing the rationals - the reals are a field of characteristic zero, so have a subfield isomorphic to the rationals. Like we did in the previous chapter, we will often abuse notation and say that any field of characteristic zero contains the rationals. In general when a field is isomorphic to a sub-field of another in a natural way, we will think of the second field containing the first.

2.3 Partial and Total Orders

We have previously seen a lot of examples of things which we might consider to be orderings - $<$ on the real numbers for example (or the rationals, integers, etc).

There are also things like subset inclusion which, while not the first example you would think of, bear a certain resemblance to an ordering as well. We will want to look at the notion of orderings in a slightly more general context.

Unfortunately when one gets *too* general there isn't a lot that you can say about things, so we need to look at some more specific cases as well. This unfortunately requires a large number of annoying definitions. When thinking about these definitions it will help to keep specific examples in mind of them, as otherwise you will likely get lost in trying to keep them straight.

Definition 2.5

Let X be a set with $\preceq$ a relation on it. $\preceq$ is said to be a partial order if:

1. $(\forall x \in X) \quad x \preceq x$. (Reflexive)
2. $(\forall x, y \in X) \quad$ if $x \preceq y$ and $y \preceq x$ then $x = y$. (Anti-symmetric)
3. $(\forall x, y, \in X)$ if $x \preceq y$ and $y \preceq z$ then $x \preceq z$. (Transitive)

If in addition $(\forall x, y \in X) \;\; x \preceq y$ or $y \preceq x$ then we say $\preceq$ is a total order.

For a partial order $\preceq$ we will write $x \prec y$ to mean $x \preceq y$ and $x \neq y$. (Similarly we will have $<$ corresponding to $\leq$, $\subset$ corresponding to $\subseteq$, etc. in this way).

We will write $\succeq$ for the reverse of $\preceq$. i.e. $x \succeq y$ means $y \preceq x$ (and similarily $\geq$, $\supseteq$ etc. It is easy to see that $\succeq$ is a partial order, and is a total order iff $\preceq$ is.

Examples

1. The relation $<$ is a total order on $\mathbb{N}$.
2. The relation $|$ (divides) is a partial order on $\mathbb{N} \setminus \{0\}$ but not a total order.
3. For any set T, the relation $\subseteq$ is a partial order on $\mathbb{P}(T)$, but unless T has only one member it is not a total order.

If $\preceq$ is a partial order on X then we say X is a partially ordered set (with partial order $\preceq$).

We're generally going to abuse notation and use the same symbol for different partial orders when considering different partially ordered sets, otherwise we would run out of symbols. Generally we will use $\preceq$ for any partial order, and $\leq$ for one which is also a total order. (Reserving the right to ignore this convention at whim of course, as with all good textbooks).

Definition 2.6

x is said to be a lower bound for Y if $\forall y \in Y \quad x \preceq y$. Similarly $x \in Y$ is said to be an upper bound for Y if $\forall y \in Y \quad y \preceq x$. If Y has an upper bounded it is said to be bounded above, and similarly if it has a lower bound it is bounded below.

x is said to be a least element of Y if $x \in Y$ and x is a lower bound for Y. Similarly a greatest element if $x \in Y$ and x is an upper bound for Y. It is easy to see that there can be at most one greatest or least element - If x, y are both least elements then $x \preceq y$ and $y \preceq x$, so $y = x$ (as $\preceq$ is a partial order) (similarly for greatest elements). If they exist then we write $min\ Y$ for the least element of Y and $max\ Y$ for the greatest.

If x is the least upper bound for a set Y we call it the supremum of Y, written $sup\ Y$. If x is the greatest lower bound for Y we call it the infimum, written $inf\ Y$.

A total order is said to be a well-order if every subset of X has a least element. It is said to be complete if every non-empty bounded above set has a least upper bound.

Example:

1. Every well-order is complete.
2. $\mathbb{N}$ is well-ordered by $\leq$
3. $\mathbb{Q}$ is not well-ordered by $\leq$, nor is the order is not complete.
4. $\mathbb{R}$ is not well-ordered by $\leq$, but the order is complete.

Proposition 2.3

Let X be a non-empty set with a complete total order $\leq$. Let $Y \subseteq X$ be non-empty and bounded below. Then Y has an infimum.

Proof:

Consider T, the set of lower bounds of Y. Y is non-empty, so T is bounded above. Y is bounded below, so T is non-empty. Let $t = sup\ T$. Suppose $y \in Y$. Then y is an upper bound for T, so as t is the least upper bound we have $t \leq y$. Thus $t \in T$. By definition t is an upper bound for T. Thus it is a greatest element for T, thus the greatest lower bound. Hence $t = inf\ Y$.

QED

Proposition 2.4

Let X be a finite set and $\leq$ a total order on it. Then $\leq$ is a well-order.

Proof:

As a subset of a finite set is itself finite and totally ordered by $\leq$, it is sufficient to show that every finite set with a total order $\leq$ has a least element.

We will prove this by induction on $n = |X|$.

For $n = 1$ it is trivial.

If it is true for n, let $|X|$ be a finite set of size $n+1$ totally ordered by $\leq$. Pick $x \in X$. If x is a least element of X then we're done. Else consider $X \setminus \{x\}$. This is a set of size n, so has a least element y. We must have $y \leq x$, else x would be a least element. Thus y is a least for X.

QED

Corollary: Every finite set totally ordered by $\leq$ has a greatest element, as $\geq$ is a total order on it as well.

Definition 2.7

Let X, Y be partially ordered sets. $f : X \to Y$ is said to be order preserving if $x \preceq y$ implies $f(x) \preceq f(y)$. If it is a bijection it is said to be an order isomorphism and X, Y are said to be isomorphic. As usual isomorphic sets are considered to be the same. As usual, isomorphism is an equivalence relation.

Definition 2.8

If $\leq$ is a total order on X, it is said to be dense if $\forall x, y$ with $x < y\ \exists z,\ x < z < y$.

Theorem 2.2

Let X be a countably infinite set totally ordered by $\preceq$, with least element u and greatest element v. Then X is order isomorphic to a subset of $Y = \{x \in \mathbb{Q} : 0 \leq x \leq 1\}$. If $\prec$ is a dense ordering then X is order isomorphic to Y.

Proof:

We will enumerate the two sets as $X = \{x_0, \ldots, x_n, \ldots\}$ and $\mathbb{Q} = \{a_0, \ldots, a_n, \}$. For convenience we let $x_0 = u,\ x_1 = v$ and $a_0 = 0, a_1 = 1$.

The idea is to move along the lists and define a function f inductively so that it slots x_n in into the right place on the list so that it is order preserving on each finite set $\{x_0, \ldots x_n\}$.

We will simultaneously define f inductively and show that it is injective and order preserving on $\{x_0, \ldots, x_n\}$ by:

$$f(x_0) = a_0$$
$$f(x_1) = a_1$$

f is thus clearly an order isomorphism on $\{x_0,\ x_1\}$.

If $f(x_m)$ is defined for $m < n$, and is an order isomorphism on $\{x_0, \dots, x_{n-1}\} then$ define $f(x_n)$ by:

Let $r = max\ \{x_i \quad : \ i < n\ x_i \prec x_n\}$, $s = min\ \{x_i \quad : \ i < n\ x_i \succ x_n\}$ - the two points on either side of x_n when considering $\{x_0, \dots, x_n\}$ being ordered by $\preceq$. (Note that the minimums and maximums are taken with respect to $\preceq$, not the index i).

We want to slot in $f(x_n)$ between $f(r)$ and $f(s)$ (both of which are already defined, because $r, s = x_i, x_j$ for some $i, j < n$).

Let $k = min\{m : a_m \notin \{f(x_i) : i < n\}\ f(r) < a_m < f(s)\}$

The set in the definition of k is non-empty as by hypothesis f is order-preserving and injective where it is defined so far, so $f(r) < f(s)$. Then $\frac{f(r)+f(s)}{2}$ is in the set.

Thus define $f(x_n) = a_k$.

By definition $a_k \notin \{f(x_i) : i < n\}$, so f is injective on $\{x_0, \dots, x_n\}$.

All that remains is to check that it is an order isomorphism. We know that it is true for x_i, x_j with $i, j < n$ so just need to check the case when one of i, j is n.

If $x_i < x_n$ then $x_i \leq r$. So $f(x_i) \leq f(r)$ (by hypothesis). But $f(r) < f(x_n)$. Thus $f(x_i) < f(x_n)$. Similarly if $x_i > x_n$.

Thus we have f injective and order-preserving on $\{x_0, \dots, x_n\}$ for each n, and hence on all of X. Thus X is isomorphic to $f(X) \subseteq \mathbb{Q}$.

Now suppose the ordering is dense. We will show that f is surjective.

Define $T_n = min\{m : a_m \notin \{f(x_0), \dots, f(x_n)\}\}$. We will show that for any m there is an n such that $T_n > m$. This then implies that $a_m \in f(X)$, and so that f is surjective.

Let $f(u) = max\ \{f(x_i) : f(x_i) < a_{T_n}\ i \leq n\}$, and $f(v) = min\ \{f(x_i) : f(x_i) < a_{T_n}\ i \leq n\}$.

We have $u < v$, thus because the ordering is dense there is a k with $u < x_k < v$. Define r_n to be the least such k. By definition of f we then have $f(x_{r_n}) = a_{T_n}$. So $a_{T_n} \in \{f(x_0), \dots, f(x_{r_n})\}$. Thus $T_{r_n} \geq T_n + 1$.

So $T_{\underbrace{r_{\dots r_0}}_{m}} > \overset{1+\dots+1}{\underbrace{m}} + T_0 = m + 1$

QED

Corollary: Let X be a countable densely ordered set with no greatest or least elements. Then X is isomorphic to $\mathbb{Q}$.

Proof:

Add greatest and least elements to it. Then it is order isomorphic to Y as above. Remove the endpoints and it it is order isomorphic to $\{x \in \mathbb{Q} : 0 < x < 1\}$. But $\mathbb{Q}$ also satisfies the hypothesis, so is isomorphic to this set. Hence X is isomorphic to $\mathbb{Q}$.

QED

The main reason why we care about this result (other than it being quite neat) is that it shows that $\mathbb{R}$ must be uncountable. Were it not then it would be order isomorphic to $\mathbb{Q}$, which it is not as $\mathbb{Q}$ is not complete. This shows that, although we only know of a few so far - the square roots - there must in fact be very many irrational numbers in $\mathbb{R}$.

Definition 2.9

When considering totally ordered sets it will be useful to consider a particular type of set - the intervals. Let X be a set totally ordered by $\leq$. $Y \subseteq X$ is said to be an interval if for $x, y \in Y$ with $x < y$ then for any z with $x < z < y$ we have $z \in y$. i.e. Y contains all points between x and y.

There are some obvious examples of intervals. For $a, b \in X$ we can define the sets $(a, b) = \{x : a < x < b\}$ and $[a, b] = \{x : a \leq x \leq b\}$. These are obviously intervals. We can also combine these to give $(a, b] = \{x : a < x \leq b\}$ and $[a, b)$ similarly.

We also define the intervals $(-\infty, b) = \{x : x < b\}$ and $(a, \infty, b) = \{x : a < x\}$, etc.

Proposition 2.5

Let X be a set totally ordered by $\leq$, where the ordering is complete. Let $Y \subseteq X$ be an interval. Then X has one of the above forms.

Proof:

We need to consider four cases, depending on whether Y is bounded above and below respectively. We shall only do the case where Y is bounded above and not below; the other cases will proceed in a virtually identical manner.

So, let Y be bounded above and not below. Let $b = sup\ Y$.

It is immediate from the definition that $Y \subseteq (-\infty, b]$. Suppose $x < b$; then as b is the least upper bound, x is not an upper bound for Y. So there must exist $d \in Y$ such that $x < d$. Further, Y is not bounded below, so x is not a lower bound for Y. Thus there exits $c \in Y$ such that $c < x$. Hence $\exists c, d \in Y \quad c < x < d$. Thus, since Y is an interval, $x \in Y$.

So we also have $(-\infty, b) \subseteq Y$. But $(-\infty, b] \setminus (-\infty, b) = \{b\}$, so there are only two remaining possibilities - $b \in Y$ or $b \notin Y$. This gives $Y = (-\infty, b)$ or $Y = (-\infty, b]$.

QED

This is not true in the general case where the ordering may not be complete - for example in $\mathbb{Q}$ the set $\{x : 0 < x,\ x^2 < 2\}$ is an interval, but isn't of the form $(0, a)$ for any a, because if it were then a would be a square root of 2.

Note: Some mathematicians use the term convex sets instead of intervals.

2.4 Ordered Fields

We now combine the two previous sections to give us a suitable context for the study of real numbers.

Definition 2.10

Let $\mathbb{F}$ be a field with a total order $\leq$ on it.

$\mathbb{F}$ is said to be an ordered field if:

1. $a \leq b$ implies $a + c \leq b + c$.
2. $a < b$ and $c > 0$ implies $ac < bc$.

Examples of ordered fields include $\mathbb{Q}$ and $\mathbb{R}$.

For two ordered fields a function between them is an isomorphism if it is a field isomorphism and order preserving.

We will now proceed to prove the big result of this chapter - any two complete ordered fields are isomorphic. First we will need some preliminary work.

Proposition 2.6

Let $\mathbb{F}$ be an ordered field. Then it is of characteristic zero.

Proof:

We first show that $1 > 0$ (no, really).

Suppose $1 < 0$. Then by adding -1 to both sides we get $0 < -1$. So we can multiply both sides by -1, preserving the ordering and get $0 < (-1)(-1) = 1$. This is a contradiction.

Now, adding $n_{\mathbb{F}}$ to both sides, we have $n + 1 > n$. A simple induction then gives that for $n > 0$ we have $n_{\mathbb{F}} > 0$ and in particular $n_{\mathbb{F}} \neq 0$. Thus $\mathbb{F}$ is of characteristic 0.

QED

In particular note that the isomorphism of the rationals into a complete ordered field is order preserving, so any ordered field has a subfield isomorphic to the rationals (as an ordered field).

Definition 2.11

An ordered field is said to be Archimedian if it satisfies the axiom of Archimedes. i.e. if for all $u, v > 0$ there exists an n such that $nu > v$.

Lemma 2.6

Let $\mathbb{F}$ be an Archimedian ordered field. Then

1. Let $u > 0$. There is a natural number n such that $\frac{1}{n} < u$
2. Let u be any number. There is a natural number n such that $n > u$
3. Let $u < v$. There is a rational number t such that $u < t < v$.

The proof is identical to that of **Proposition 1.3**.

Theorem 2.3 *Uniqueness of the Reals*

Let $\mathbb{F}$ and $\mathbb{G}$ be complete ordered fields. There exists an isomorphism $f : \mathbb{F} \to \mathbb{G}$.

Proof

Isomorphism is an equivalence relation, so it is sufficient to consider the case where $\mathbb{F} = \mathbb{R}$.

We start by defining a function on the positive reals (equivalently on $\mathbb{T}$) and extending it in the natural way.

Let $g : \mathbb{Q} \to \mathbb{G}$ be the usual homomorphism from $\mathbb{Q}$ into $\mathbb{G}$.

Let $x \in \mathbb{T}$. This is a bounded above non-empty set, so $\{g(t) : t \in x\}$ is a bounded non-empty subset of $\mathbb{G}$. Define $h(x) = sup\ f(x) = sup\ \{g(t) : t \in x\}$.

This is injective, as if $u, v \in \mathbb{T}$ with $u < v$ let $r \in v \setminus u$. Then $g(u) < f(r) < g(v)$ by construction. So $g(u) \neq g(v)$. This also gives that it is order preserving. The previous lemma gives that it is surjective onto $\mathbb{F}^+ = \{x \in \mathbb{F} : x > 0\}$ as $x = sup\ \{q \in \mathbb{Q} : q < x\}$. So g is an order isomorphism between $\mathbb{T}$ and $\mathbb{F}^+$.

Now to show that it preserves the algebraic structure. It is easy to check that for rational $q \in \mathbb{T}$ we have $g(q) = q$ (up to an appropriate abuse of notation), so $g(1) = 1$.

Let $X, Y \subseteq \mathbb{F}$ be bounded above and non-empty. We will show that if we define $X + Y = \{x + y : x \in X\ y \in Y\}$ and then $sup\ X + Y = sup\ X + sup\ Y$ and

First note that $sup\ X + sup\ Y$ is an upper bound for $X + Y$, as $x + y \leq x + sup\ Y \leq sup\ X + sup\ Y$.

Let d be an upper bound for $X + Y$. Then d is an upper bound for X, so $d \geq sup\ X$. We may thus write $d = c + sup\ X$. Suppose $c < supY$. We may find $x \in X$, $y \in Y$ such that $c < y$ and $sup\ X < x + y - c$ (as $sup\ X - (y - c)$ cannot be an upper bound for x, since $y - c > 0$). So $x + y > \ supX + c = d$. So d is not an upper bound for $X + Y$. This is a contradiction. Hence $c \geq sup\ Y$, so $d \geq sup\ X + \sup Y$. Thus $sup\ X + sup\ Y$ is the least upper bound for $X + Y$, so $sup\ X + Y = sup\ X + sup\ Y$.

Thus we have:

$$\begin{aligned} g(u+v) &= sup\ \{f(x) : x \in u+v\} \\ &= sup\ \{f(r+s) : r \in u\ s \in v\} \\ &= sup\ \{f(r) + f(s) : r \in u\ s \in v\} \\ &= sup\ \{x + y : x \in f(u)\ y \in f(v)\} \\ &= sup\ (f(u) + f(v)) \\ &= sup\ f(u) + sup\ f(v) \\ &= g(u) + g(v) \end{aligned}$$

We could prove that $g(u \cdot v) = g(u) \cdot g(v)$ in a similar fashion, but the details get a bit messier, so we shall take a different route.

Let $u \in \mathbb{T}$.

$$\begin{aligned} g(nu) &= g(\ sum_{i=1}^{n} u) \\ &= \sum_{i=1}^{n} g(u) \\ &= ng(u) \\ g(u) &= g(m\frac{1}{m}u) \\ &= mg(\frac{1}{m}u \\ g(\frac{1}{m}u) &= \frac{1}{m}u \\ g(\frac{n}{m}u) &= \frac{n}{m}u \end{aligned}$$

Hence for any rational $q \in \mathbb{T}$ and any $u \in \mathbb{T}$ we have $g(qu) = qg(u)$.

Fix $v \in \mathbb{T}$. The map $u \to uv$ is an order isomorphism of $\mathbb{T}$ to itself. Thus the map $h : u \to g(uv)$ is an order isomorphism from $\mathbb{T}$ to $\mathbb{F}^+$ (as it is a composition of isomorphisms). So, in particular, it preserves supremums.

Thus

$$\begin{aligned}
g(vu) &= h(u) \\
&= h(sup\ \{q \in \mathbb{Q} : q < u\}) \\
&= sup\ \{h(q) : q < u\} \\
&= sup\ \{g(qv) : q < u\} \\
&= sup\ \{g(q)g(v) : q < u\} \\
&= g(v)\ sup\ \{q : q < g(u)\} \\
&= g(v)g(u)
\end{aligned}$$

Where we could take the constant factor $g(v)$ outside the supremum as $g(v) > 0$, so the map $t \to t \cdot g(v)$ is an order isomorphism.

So we've shown that the ordering and the algebraic properties are preserved on $\mathbb{T}$. We now want to extend g to a map from $\mathbb{R}$ to $\mathbb{F}$.

Recall we defined $\mathbb{R}$ as equivalence classes of $\mathbb{T}^2$ where we think of (u, v) as $u-v$. So the only logical way to extend this is to define $f(u, v) = g(u) - g(v)$.

We first need to check that this is well defined. Suppose $(u, v) \sim (r, s)$. Then $u + s = r + v$. So $u = r + v - s$. So

$$\begin{aligned}
f(u, v) &= g(u) - g(v) \\
&= g(r + v - s) - g(v) \\
&= g(r) + g(v) - g(s) - g(v) \\
&= g(r) - g(s) \\
&= f(r, s)
\end{aligned}$$

So f inherits to the equivalence classes, and we may consider it as a function on $\mathbb{R}$. It is then a trivial exercise in the definitions to check that it is an isomorphism.

So $\mathbb{R}$ is isomorphic to $\mathbb{F}$.

QED

Exercise 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a field isomorphism (not neccesarily preserving order). Show that f is the identity.*

Hint: You may want to first show that f preserves order.

At first it may seem that this uniqueness result is rather boring and technical, but it really isn't - it lies at the heart of real analysis. It tells us two important things:

Firstly, that if we want to prove any result about the reals which is not true for ordered fields in general (and only depends on $+$, $\cdot$, and $\leq$), the only property we need is the completeness. This is very useful, because it immediately cuts out a lot of false avenues of exploration. We will of course often use previous results as well as the completeness property, but they too will depend on the completeness.

Secondly, it says that whenever we want a number system which has certain properties - specifically the idea of not having any 'gaps' in it (we will make this precise in the next chapter), the real numbers are the only choice. This makes them applicable to much wider areas than the geometric context we introduced them in. For example in classical physics we want to assign such numbers to measure time as well as distance, so the real numbers will represent time as well. From this we can then go on to represent things like velocity and acceleration with them as well.

Chapter 3

Continuity and the Real Numbers

In this chapter we shall actually begin to do some analysis. Now that we have developed a proper foundation for dealing with the real numbers, we want to explore real geometry. We shall introduce notions of continuity, topology, connectedness and limits, which will be central to our theory. A word of warning: Virtually all definitions in this chapter should be read as temporary definitions, in the rather restrictive case where we are only considering the real numbers. In the next chapter these ideas shall be generalised to a much broader context and new definitions - which will of course be equivalent in the special case we are considering this chapter - for the terms will be provided. However in most cases the new definitions (and the proofs of corresponding results) will be extremely close to the ones presented in this chapter, just in a slightly different setting.

3.1 Limits and Continuity

Recall the two problems we pointed out with rational numbers. These can be expressed in terms of the following graphs:

(Diagrams of $y = x^2$ and $y = x^3$)

They take values on either side of 2, but do not ever take the value 2.

This suggests that graphs and functions will be a good place to start our investigation of the geometry of $\mathbb{R}^2$. Certainly if we can find a problem like this for $\mathbb{R}$ it will show that the reals were in fact *not* a good starting place for geometry and we need to start again. This would mean that you have wasted a lot of time reading the last chapter, and that would be bad...

Certainly it's not true that for every function $f : \mathbb{R} \to \mathbb{R}$ it must pass through every point like that. For example if you consider the function $I_{(-\infty,0]}$ this takes the value 0 for $x \leq 0$, and 1 for $x > 0$, but never passes through $\frac{1}{2}$. This isn't a problem - the graph just jumps at 0. In mathematical terms, the graph is discontinuous at 0. So we need to consider a special type of function - continuous functions. The first thing we will need to do in this chapter is to try and define what we mean by a continuous function. After we have done this we will introduce some topological ideas which will explain why continuous functions on the reals are far more well behaved than they are for the rationals.

The first intuitive notion for a continuous function is that it is one which you can draw the graph of without taking your pencil off the page. This has two main problems - it is quite hard to pin down precisely what you mean by this, and it isn't very general. We would like, for example, to be able to define what it means for a function $f : \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ to be continuous. This is because later we will define more general curves to be continuous functions of this sort, and will develop some extremely valuable theories for it. Thus we want a notion of continuity which generalises easily.

So, as a starting point, we will go back to the idea of continuity as a function that does not jump. Again, it is rather difficult to say exactly what we mean by a function jumping.

For example consider the function $g = I_{\{x:x^2<2\}} : \mathbb{Q} \to \{0,1\} \subseteq \mathbb{Q}$. The graph is as follows:

(diagram)

So it certainly looks like the function jumps, but where does it jump? Intuitively it jumps at $\pm\sqrt(2)$, but $\sqrt(2) \notin \mathbb{Q}$. So there is nowhere in $\mathbb{Q}$ that the functions jump. More tellingly, the function $f(x) = x^2$ also jumps at $\pm\sqrt(2)$, but we *would* say this is continuous. Certainly everywhere that f is continuous, g is also continuous - they jump in the same places. So if we want it to be true that a function which is continuous at every point is continuous, we must conclude that g is also continuous. If we don't require this to be true, then our definition of continuity starts to rapidly depart from the intuitive one, and continuity starts to become quite hard to check.

Thus we will only consider the question of whether a function jumps at a point. If a function f doesn't jump at y, then we must have $f(x)$ approaching $f(y)$ (for some soon-to-be-precise definition of 'approaching').

In order to make sense of this idea we will need to recall that the distance between two real numbers (in the sense that we are considering $\mathbb{R}$ as a line in $(R)^2$) is given by $|x - y|$, where $x \to |x|$ is the absolute value function defined as $|x| = x$ if $x \geq 0$ or $|x| = -x$ if $x < 0$, so that $|x|$ is always positive.

So 'approaching' becomes the idea of a *limit*. We say $f(x) \to z$ as $x \to y$ (read '$f(x)$ tends to z as x tends to y) if we can make $f(x)$ arbitrarily close to z whenever x is sufficiently close to y, where close is defined with this distance. As usual we will write $f(x) \not\to z$ if it is not true that $f(x) \to z$.

Note however that the 'whenever' part of that is very important. For example if we have the function $g = I_{\mathbb{Q}}$ defined on $\mathbb{R}$, then this takes the value 1 arbitarily close to $x = 0$, but it also takes the value 0 arbitrarily close to $x = 0$. But we can't find a number $\delta > 0$ such that whenever $|x - 0| < \delta$. The point is that no matter how close we get to 0 (without actually reaching 0 itself), there will always be closer points which are 'far away' from 1, so $g(x) \not\to 1$ as $x \to 0$. Before we continue, we note two things:

Firstly, we do not want to count the value of f at y. For example if we were to define $f(x) = I_{\{1\}}$ we would want $f(x) \to 0$ as $x \to 0$, but $f(x) = 1$ would prevent this. If we didn't require this then the limit at y would be defined only if the function was continuous at y (as the limit would have to be arbitrarily close to $f(y)$, since for any requirement of being sufficiently close to y, y satisfies it).

Secondly, we do not need the function f to be defined everywhere, and in particular we do not need it to be defined at y. For example if we define a function on $[0, 1)$ it makes sense to consider the limit as $x \to 1$ (although this limit will of course not neccesarily exist).

We now define the idea of a limit precisely:

Definition 3.1

Let $X \subset \mathbb{R}$, $f(x) \to \mathbb{R}$ and $y \in \mathbb{R}$. We say $f(x) \to z$ as $x \to y$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Symbolically: $\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in X \;\; (0 < |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

The symbolic representation of statements like this is exceptionally useful, and you will need to learn to be able to read them easily. For now I will include both the written and symbolic forms of each statement, but eventually I will tend just to use symbolic statements.

If $y \in X$ and $f(x) \to f(y)$ as $x \to y$ then we say that f is continuous at y.

We now have a local definition of continuity. As a temporary definition, we will say that a function $f : X \to \mathbb{R}$ is continuous if it is continuous at each $x \in X$.

Note that the definition of a limit can be true vacuously - it won't always make sense to talk about limits. For example it make sense to consider the limit $x \to 1$ when $X = [0, 1)$, but it doesn't make sense to talk about the limit $x \to 3$, because there are no points in X which are near to 3.

In general it will only make sense if in the statement $\forall x \in X \;\; 0 < |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ there are actually x which satisfy that condition. This must hold for every δ, i.e. we must have $\forall \delta > 0 \; \exists x \in X \;\; 0 < |x - y| < \delta$. y is then called a *limit point* of X.

Suppose y is in fact a limit point of X. Is the limit unique? We will need the following:

Proposition 3.1

Let $x, y \in \mathbb{R}$. $|x+y| \leq |x| + |y|$

Proof:

Without loss of generality, by multiplying by -1 and relabelling, let $|x| \geq |y|$ and $x \geq 0$. Then if $y \geq 0$ we have $x + y \geq 0$ so $|x+y| = |x| + |y|$. If $y < 0$ then $|x+y| = x + y$ as we have $x \geq -y$ so $x + y \geq 0$. So $|x+y| = |x| - |y| \leq |x| \leq |x| + |y|$.

QED

Lemma 3.1

Suppose $f(x) \to u$ and $f(x) \to v$ as $x \to y$. Then $u = v$.

Proof:

Assume $u \neq v$. Then $|u - v| > 0$. Let $\epsilon = \frac{|u-v|}{3}$.

We can find δ_1 such that when $|x - y| < \delta_1$, $|f(x) - u| \leq \epsilon$ and δ_2 such that when $|x - y| < \delta_2$, $|f(x) - v| \leq \epsilon$. Let $|x - y| < min\ \{\delta_1, \delta_2\}$. Then $|u - v| = |(u - x) - (v - x)| \leq |u - x| + |v - x| < 2\epsilon = \frac{2}{3}|u - v| < |u - v|$. So $|u - v| < |u - v|$. This is a contradiction, so we must have $u = v$.

QED

When the limit exists, we thus write $\lim_{x \to y} f(x) = z$.

The above is what is typically known as an $\epsilon - \delta$ proof - proof of things by going back to the basic definition of limits in terms of ϵ and δ. They are a standard component of most treatments of analysis, and will rarely be used in this book. We will use them in establishing a couple basic results, and work from these results and more general theory to achieve our goals. This is basically just a stylistic difference, and it *is* important to be able to prove things in this fashion, so alternative proofs of results using the $\epsilon - \delta$ method will often be included as exercises.

3.2 Open and Closed Sets

In order to investigate the study of limits it will be useful to consider some particular types of set. As motivation consider the following:

Suppose $f : X \to Z \subseteq R$. If y is a limit point of X and $lim_{x \to y} = t$, it is clear that we needn't have $t \in Z$ - for example if $X = Z$ and f is the identity function, if $y \notin X$ then $t = y \notin Z$. However, if Z contains all it's limit points then we *do* know that $t \in Z$, which will be helpful in establishing certain things about the limits. Thus we define the following:

Definition 3.2

A set $X \subset \mathbb{R}$ is said to be closed if it contains all it's limit points. (i.e. X is closed if y a limit point of $X \implies y \in X$).

In working with closed sets (and later, in their own right), it will be very helpful to also have two other notions:

Let $x \in \mathbb{R}$ and $X \subset \mathbb{R}$. We say X is a neighbourhood of x if there exists $\delta > 0$ such that $|x - y| < \delta \implies y \in X$. i.e. X contains all points sufficiently close to x.

We say a set X is open if for every $x \in X$, X contains a neighbourhood of x (equivalently, X is a neighbourhood of x).

Note that x is a limit point of X iff every neighbourhood of x has a non-empty intersection with X $\{x\}$.

To get a better grip on how these behave, we prove the following lemma:

Lemma 3.2

1. Both $\emptyset$ and $\mathbb{R}$ are closed.
2. If $X_1, \ldots, X_n$ are closed then $X \bigcup_{i=1}^{n} X_i$ is closed.
3. If C is a non-empty collection of closed sets then $\bigcap C$ is closed. (Note that C can be finite, countable, uncountable - whatever).
4. X is closed iff X^c is open.
5. Both $\emptyset$ and $\mathbb{R}$ are open. (In particular note that a set can be both closed and open - this is important).
6. If $X_1, \ldots, X_n$ are open then $X \bigcap_{i=1}^{n} X_i$ is closed.
7. If C is a non-empty collection of open sets then $\bigcup C$ is closed.
8. If $X_1, \ldots, X_n$ are neighbourhoods of x then $\bigcap_{i=1}^{n} X_i$ is a neighbourhood of X.
9. X is a neighbourhood of x iff there exist a, b such that $a < x < b$ and $(a, b) \subseteq X$
10. If X is a neighbourhood of x and $X \subseteq Y$ then Y is a neighbourhood of x.
11. For any x, $\mathbb{R}$ is a neighbourhood of x.

Proof

We are actually going to prove these in reverse order.

11 and 10 are trivial from the definition.

To prove 9, we first show that (a, b) with $a < x < b$ is a neighbourhood of x. If we let $\delta = min\ \{x - a,\ b - x\} = -max\ \{a - x,\ x - b\}$ then if $|x - y| < \delta$

we have $a = x + (a - x) \leq x + max\ \{a - x, x - b\} = x - \delta < y < x + \delta = x + min\{x - a,\ b - x\} \leq x + b - x = b$. So $y \in (a, b)$. Hence (a, b) is a neighbourhood of x. Conversely, if X is a neighbourhood of x, let δ be such that $|x - y| < \delta \implies y \in X$. Then $a = x - \delta$, and $b = x + \delta$ are such that $a < x < b$ and $(a, b) \subseteq X$.

We will prove 8 for the case $n = 2$, and the general result follows by induction.

Let δ_1, δ_2 be such that for all y, $|x - y| < \delta_1 \implies y \in X_1$ and $|x - y| < \delta_2 \implies y \in X_2$. Define $\delta = min\ \{\delta_1, \delta_2\}$. If $|x - y| < \delta$ then $|x - y| < \delta_1$, so $y \in X_1$. Similarily $|x - y| < \delta_2$, so $y \in X_2$. Thus $y \in X_1 \cap X_2$. Hence $X_1 \cap X_2$ is a neighbourhood of x.

For 7, let C be such a collection and $x \in \bigcup C$. Then $\exists X \in C\ x \in X$. X is open, so is a neighbourhood of x. But $X \subseteq \bigcup C$. So $\bigcup C$ is a neighbourhood of X. Hence $\bigcup C$ is open.

6 follows from 8. Let $x \in \bigcap_{i=1}^{n} X_i$. Each X_i is a neighbourhood of x, so by 8 we have that $\bigcap_{i=1}^{n} X_i$ is a neighbourhood of x. Hence $\bigcap_{i=1}^{n} X_i$ is open.

5 is immediate from the definition.

For 4, recall our previous note that y is a limit point of X iff every neighbourhood of y has non-empty intersection with $X\ \{y\}$. Let $y \in X^c$. Then x is not a limit point of X, as X is closed, so must have a neighbourhood U disjoint from $X\ \{y\}$, and thus from X as $y \notin X$. Hence $U \cap X = \emptyset$, so $U \subseteq X^c$. Hence X^c contains a neighbourhood of y. Thus X^c is open. Conversely, if X^c is open let y be a limit point of X. Then X^c contains no neighbourhood of y (as every neighbourhood of y has non-empty intersection with X, so is not in X^c), so $y \notin X^c$. Hence $y \in X$. So X contains all of its limit points, and is thus closed.

1, 2 and 3 are now all simple applications of 4 and De Morgan's rules to the corresponding results for open sets.

QED

It is then easy to check from the above that, for example $[0, \infty)$ is closed (because it's compliment is $(-\infty, 0)$ and if x lies in $(-\infty, 0)$ then $x \in (x-1, 0) \subseteq (-\infty, 0)$), and indeed that an interval is closed iff it contains all it's endpoints and open if it doesn't contain either of it's endpoints. Also note that any finite set is closed, as every point is closed (because $\{x\}^c = (-\infty, x) \cup (x, \infty)$) and finite unions of closed sets are closed.

Also note that U is a neighbourhood of x iff it contains an open set V such that $V \ni x$.

Because the intersection of any arbitrary collection of closed sets is closed, we may define the *closure* of a set - the smallest closed set containing it. We define $\overline{X} = \bigcap\{Y \subseteq \mathbb{R}\ :\ Y \text{ is closed and } X \subseteq Y\}$. Then $\overline{X}$ is closed, $X \subseteq \overline{X}$ and for any closed set $Y \supset X$ we have $\overline{X} \subseteq Y$.

Lemma 3.3

1. Let $X \subseteq \mathbb{R}$. $x \in \overline{X}$ iff for every neighbourhood U of x, $U \cap X \neq \emptyset$.
2. Let $X \subset \mathbb{R}$ and let L be the set of limit points of X. Then $\overline{X} = X \cup L$.

Proof:

1. Suppose U was a neighbourhood of x such that $U \cap X = \emptyset$. Then U^c is a closed set containing X, so $\overline{X} \subseteq U^c$, and $x \notin \overline{X}$.

   Conversely, assume every neighbourhood U of x we have $U \cap X \neq \emptyset$. If $x \in X$ then we're done, as $X \subseteq \overline{X}$. Else we have for every neighbourhood of x, $U \cap (X \ \{x\}) \neq \emptyset$, so x is a limit point of X and thus of $\overline{X}$. As $\overline{X}$ is closed, this means we must have $x \in \overline{X}$.

2. Inclusion one way is trivial - $X \subset \overline{X}$, and any element of L is a limit point of $\overline{X}$, so is contained within it and thus $L \subseteq \overline{X}$. Hence $X \cup L \subseteq \overline{X}$. Conversely, suppose $x \in \overline{X} \setminus X$. Then by part 1 we have that for every neighbourhood U of x, $U \cap (X \ \{x\}) \neq \emptyset$, so x is a limit point of X and so $x \in L$. Hence $\overline{X} \subseteq X \cup L$.

Corollary: Let X be a non-empty set which is bounded above. $sup\ X \in \overline{X}$. Similarily if X is bounded below, $inf\ X \in \overline{X}$.

We can also define a similar object for open sets - the interior of X is the largest open set contained within X. We write it $\mathring{X}$ and it is defined by $\mathring{X} = \bigcup\{Y \subseteq X\ :\ Y \text{ is open }\}$. It is an easy check that X is a neighbourhood of x iff $x \in \mathring{X}$.

Finally, let us provide an actual use for these ideas.

Theorem 3.1 *Intermediate Value Theorem*

Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < y < f(b)$. There exists $c \in (a, b)$ with $f(c) = y$.

Proof:

Let $X = \{x \in [a, b] : f(x) \leq y\}$. Considering f restricted to X, we have $f(X) \subseteq (-\infty, y]$. Let $c = sup\ X$. $f(c) = \lim_{x \to c} f(x)$ with $x \in X$, so $f(c) \in (-\infty, y]$ as $(-\infty, y]$ is closed. Thus $c \in X$. Hence $c < b$, as $b \notin X$ and $f(c) \leq y$. Suppose $f(c) < y$.

But c is a limit point of $(c, b]$, and $f((c, b]) \subseteq (y, \infty)$ by the definiton of c (it is an upper bound for x). So $f(c) \subseteq [y, \infty)$. Hence $f(c) \subseteq (-\infty, y] \cap [y, \infty) = \{y\}$. Hence $f(c) = y$.

QED

Which is the result we were seeking.

It is my opinion that this is not a very good proof. It's not a difficult proof certainly, and it illustrates well why the idea of closed sets and limit points is a useful one, but it doesn't really tell us a lot about what's going on. In particular, it works in the specific case of the reals but will not generalise at all. The rest of this chapter will be spent in setting up a better framework for this which will more clearly show what is going on. It will also generalise in useful ways.

We will now link together the notions of continuity, open and closed sets.

Theorem 3.2

Let $f : \mathbb{R} \to \mathbb{R}$. The following are equivalent:

1. f is continuous.
2. For every open $U \subset \mathbb{R}$, $f^{-1}(U)$ is open.
3. For every closed $U \subset \mathbb{R}$, $f^{-1}(U)$ is closed.

textbfProof:

(1) $\iff$ (2):

Define $B(x, \epsilon) = \{y \in \mathbb{R} \ : \ |x - y| < \epsilon\}$ (this will be a standard notation).

We can rewrite the usual limit definition of continuity at y as $\forall \epsilon > 0 \ \exists \delta > 0 \ x \in B(y, \delta) \implies f(x) \in B(y, \epsilon)$. This can be further rewritten as saying for every neighbourhood Y of $f(y)$ there is a neighbourhood V of x such that $f(V) \subseteq Y$.

So, assume f is continuous. Let U be an open set, and pick $x \in f^{-1}(U)$. U is open and $f(x) \in U$, so U is a neighbourhood of $f(x)$. Thus, by continuity, there is a neighbourhood V of x such that $f(V) \subseteq U$. Hence $V \subseteq f^{-1}(U)$. So $f^{-1}(U)$ contains a neighbourhood of x. As x was arbitrary, this means that $f^{-1}(U)$ is open.

Conversely, suppose for every open U, $f^{-1}(U)$ is open. Let V be a neighbourhood of $f(x)$. Then there is an open U such that $f(x) \in U \subseteq V$. So $X = f^{-1}(U)$ is open and $x \in f^{-1}(U)$. So X is a neighbourhood of x and $f(X) \subseteq U \subseteq V$. Hence f is continuous at x. So f is continuous.

(2) $\iff$ (3) is trivial, as $f^{-1}(U^c) = (f^{-1}(U))^c$.

QED

From the above, we will generalise the notions of open and closed sets slightly. We want the above to hold for functions $f : X \to \mathbb{R}$, but will need to talk about something slightly different because as it stands it's not true - for example consider the identity function on $[0, 1]$. The inverse image of $\mathbb{R}$ is $[0, 1]$, which is not open. However it is open in $[0, 1]$. Fix $X \subset \mathbb{R}$. We say $Y \subseteq X$ is open in X if there is an open set U such that $Y = X \cap U$. Similarily Y is closed in X if there is a closed set U such that $Y = X \cap U$. It is easy to check that all the properties for open and closed sets in lemma 3.2 hold. Checking through the proof of the above we then have that it holds for $f : X \to Z$ where by open and closed we mean open and closed in X and Z respectively.

3.3 Connectedness

Recall when we were initially constructing the reals, we looked at sets of rationals, X, such that X is bounded above but has no greatest element, and if $x \in X$ and $y < x$ then $y \in X$. Such a set is an interval, and open (within $\mathbb{Q}$). We argued that the complement must have a least element when constructing the reals, else there would be a 'hole' in the number line. What if there wasn't a least element? Then X^c would also be open, so we would have written $\mathbb{Q}$ as a union of two non-empty disjoint open sets which in some sense disconnects $\mathbb{Q}$. These disconnections will turn out to be central to the problem we are considering.

Suppose we can write a set X as such a disjoint union of open sets. We can define a function $f : X \to \mathbb{R}$ by $f(x) = 0$ for $x \in U$ and $f(x) = 1$ for $x \in V$. Then this function is continuous, as the inverse image of open set is either $\emptyset$, U, V or X all of which are open. It also 'jumps' - it takes the values 0 and 1, with nothing in between. So the existence of disconnections of $\mathbb{Q}$ allows 'bad' continuous functions.

Conversely suppose we had a continuous function $f : X \to \mathbb{R}$ such that there exist $a, b \in X$ with $f(a) < y < f(b)$ and $y \notin f(X)$. Then $U = (-\infty, y) \cap f(X)$ and $V = (y, \infty) \cap f(X)$ are non-empty, open in $f(X)$, disjoint and their union is all of $f(X)$. Then by continuity $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. They are non-empty by hypothesis, are clearly disjoint and their union is the whole of $f(X)$.

So the question of continuous functions which jump is entirely equivalent to the question of write X as a disjoint union of two non-empty open sets. This motivates the following definition:

Definition 3.3

Let $X \subseteq \mathbb{R}$. If $U, V \subseteq X$ are non-empty disjoint open sets such that $U \cup V = X$ we say that U and V disconnect X. If there exists a disconnection of X then we say that X is disconnected; else we say that X is connected.

Theorem 3.3

If $X \subseteq \mathbb{R}$ is connected, and $f : X \to \mathbb{R}$ is continuous then $f(X)$ is connected.

Proof:

This is pretty much identical to what we discussed above. Suppose U, V disconnect $f(X)$. Then by the continuity of f, we must have $f^{-1}(U)$ and $f^{-1}(V)$ disconnecting X.

QED

In order to make this concept useful, we will need to show the following:

Theorem 3.4

Let $I \subseteq \mathbb{R}$ be an interval. Then I is connected.

Proof:

Suppose U, V disconnect I. At most one of U, V. Pick $y \in V$ that is not an end-point of I (which must exist, as a set consisting entirely of end-points isn't open). At least one of $U \cap (-\infty, y)$ and $U \cap (\infty, y)$ is non-empty. Without loss of generality suppose this was $U \cap (-\infty, y) = U'$ (the other case will follow in a virtually identical manner to this). Define $V' = V \cup (U \cap (\infty, y))$. Then U', V' disconnect I and U' is bounded above in I. Because I is an interval the ordering on it is complete, so let $x = sup\ U'$. We know that X is not an end-point of I, as $x \leq y$ and y is not an end-point.

U'^c is open, so U' is closed. Thus we must have $x \in U'$. But for $t > x$ we have $t \in V'$. So for any $(a, b) \ni x$ we must have $(a, b) \cap V' \neq \emptyset$. So x is a limit point of V'; but V' is similarily closed, so we must have $x \in V'$. So $x \in U' \cap V'$, which means that U' and V' aren't disjoint, which contradicts the hypothesis that U and V disconnected I.

QED

It is easy to see that if a set $X \subset \mathbb{R}$ is not an interval then it is disconnected, as if $a, b \in X$ with $a < c < b$ and $c \notin X$ then $(-\infty, c) \cap X$ and $(c, \infty) \cap X$ disconnect X. So we now know that a set is connected iff it is an interval.

Corollary: Let $f : [a, b] \to \mathbb{R}$ be continuous, and let y lie between $f(a)$ and $f(b)$ (i.e. $f(a) < y < f(b)$ or $f(b) < y < f(a)$). Then there exists $c \in (a, b)$ with $f(c) = y$.

Proof: $f([a, b])$ is connected, so an interval. Thus $y \in f([a, b])$. $f(a), f(b) \neq y$, so $y \in f((a, b))$. i.e. $\exists c \in (a, b)\ f(c) = y$.

QED

We have now established some of the basic tools we will use in studying analysis. In the next chapter we will take these ideas and generalise them, then apply the generalised ideas to the specific case of $\mathbb{R}$ to get various theorems about the continuous functions on $\mathbb{R}$.

Chapter 4

Real Valued Functions

In doing analysis most of our interest will be on real valued functions. However, at the moment we have a very short supply of such. We know about polynomials, and that's about it. We can of course also define some fairly obvious functions such as the following:

Definition 4.1 *1.*

$$\begin{aligned} |x| &= x \; x \geq 0 \\ &= -x \; x < 0 \end{aligned}$$

2.

$$\begin{aligned} x^+ &= x \; x \geq 0 \\ &= 0 \; x < 0 \end{aligned}$$

3.

$$\begin{aligned} x^- &= 0 \; x \geq 0 \\ &= -x \; x < 0 \end{aligned}$$

Note that $|x| = x^+ + x^-$, $x = x^+ - x^-$ and $x^- = (-x)^+$.

These are rather uninteresting functions, and use none of the theory we've developed so far. Note however that $x \to x^+$ is continuous (as it is monotone and surjective onto $[0, \infty)$). Thus $x \to x^-$ is continuous as it is a composition of continuous functions, and so $x \to |x|$, as it is a sum of continuous functions.

Recall that we showed that given any real number $y >$, there exists x such that $x^2 = y$. Further if we require x to be positive this number is unique.

We will quickly produce a new proof of a more general result using the ideas from the previous two chapters.

Proposition 4.1 *Let $n \in \mathbb{N}$ and $y \in [0, \infty)$. There exists a unique $x \in [0, \infty)$ with $x^n = y$*

Proof:

Consider the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^n$.

Note that f is monotone, as if $x > y$ then $f(x) - f(y) = (x - y) \sum_{i=0}^{n-1} x^r y^{n-1-r}$, which is a product of two positive terms, so positive. Thus $f(x) > f(y)$. This also shows that f is injective, giving the uniqueness part.

f is continuous. This can be proved by induction: It is true for $n = 1$. If it is true for $n - 1$ then $f(x) = xx^{n-1}$ is a product of continuous functions, and so continuous.

$f(x) \to \infty$ as $x \to \infty$, for let $M > 0$. Without loss of generality let $M > 1$. Then $\forall x > M,\ f(x) > f(M) \geq M$. So, in particular, for any $y \in [0, \infty)$ there exists t with $f(t) > y$.

So we have t with $f(0) < y < f(t)$. Thus, by the intermediate value theorem, there is some $x \in (0, t)$ with $f(x) = y$.

QED

Chapter 5

Abstract Topology

5.1 Topologies and Continuity

We want to consider a more general context in which to define continuity, limits closed sets, etc. From the last section we know that declaring what the open sets are is enough to define the others, so we will start by defining open sets more generally.

We want to retain the idea of a closure, so we need that the intersection of a collection of closed sets is closed. Equivalently that the union of a collection of open sets is open. In order to retain the idea of an open set being one which contains a neighbourhood of every point, we will need that $\emptyset$ is open, as it is true vacuously for the empty set, and that X is open, else some points would have no neighbourhood. Finally, we want that the intersection of two neighbourhoods of a point are also a neighbourhood of the point in order to retain the idea of a neighbourhood of x as a set containing all points sufficiently close to x. This is equivalent to requiring that the intersection of two open sets is open. So, we define the following:

Definition 5.1 *Let X be a set, and T a collection of subsets of X. We say T is a topology if:*

1. $\emptyset,\ X \in T$
2. If $C \subset T$ is a collection of elements of T then $\bigcup C \in T$.
3. If $U, V \in T$ then $U \cap V \in T$.

If T is a topology on X, then the ordered pair (X, T) is said to be a topological space. We will usually write this as X is a topological space with topology T, or often just X is a topological space if the topology is arbitrary or implicit.

The elements of a topology on X are called the open sets for the space.

A set $U \subseteq X$ is said to be closed if U^c is open.

Let $x \in X$. $U \subset X$ is said to be a neighbourhood of x if there is some open set V such that $x \in V \subseteq U$.

Let $Y \subseteq X$ and $x \in X$. x is a limit point of Y if every neighbourhood of x has non-empty intersection with Y.

This may seem like a rather large chunk of definitions all at once, but they really aren't anything more than a straightforward generalistion of the ideas we developed for the real numbers.

It is a trivial check that the following hold:

1. $\emptyset$, X are closed.
2. If C is a collection of closed sets then $\bigcap C$ is closed.
3. If X, Y are closed then $X \cup Y$ is closed.

Note that induction gives us that if $X_1, \ldots, X_n$ are open then $\bigcap_{i=1}^n X_i$ is open, and similarily if $X_1, \ldots, X_n$ are closed then $\bigcup_{i=1}^n X_i$ is closed.

Examples

1. For $\mathbb{R}$, the usual collection of open sets is a topology.
2. Indeed, for any set $X \subseteq \mathbb{R}$ the collection of sets which are open in X is a topology. In particular $\mathbb{Q}$ is a topological space with the usual open sets.
3. For any set X, the powerset of X is a topology on X, called the discrete topology.
4. For any set X, the set $\{\emptyset,\ X\}$ is a topology on X, called the indiscrete topology.
5. For any set X, the set $\{X \setminus Y\ :\ Y \subset X,\ \text{Y finite}\ \} \cup \{\emptyset\}$ is a topology. This is called the cofinite topology.

We must now make sure that these definitions behave in a way that we would expect.

Lemma 5.1

1. U is open iff for every $x \in U$, U contains a neighbourhood of X.
2. V is closed iff it contains all it's limit points.

Proof:

1. If U is open and $x \in U$ then U is a neighbourhood of x, so certainly contains a neighbourhood of x. Conversely, if for every $x \in U$ there is an open set V such that $x \in V \subseteq U$, then $U = \bigcup\{V \subseteq X \ : \ V \text{ is open }\}$, as this set is certainly contained within U and by hypothesis for each $x \in U$ there exists V such that $x \in V \subseteq \bigcup\{V \subseteq X \ : \ V \text{ is open }\}$. So U is a union of open sets, and thus open.

2. Suppose V is closed and $x \notin V$. Then V^c is an open set containing x, so a neighbourhood of x, which is disjoint from X. Thus x has a neighbourhood disjoint from V, so is not a limit point of V. Hence V contains all its limit points.

   Conversely, suppose V contains all it's limit points. Let $x \notin V$. Then x is not a limit point of V, so has a neighbourhood disjoint from X. Thus has a neighbourhood contained in X^c. Thus by the previous result, X^c is open. Hence X is closed.

QED

We define closure and interior in the same way as before. $\overline{Y} = \bigcap\{Z \ : \ Y \subseteq Z, \ Z \text{ is closed }\}$. $\mathring{Y} = \bigcup\{Z \ : \ Z \subseteq Y, \ Z \text{ is open }\}$.

As in the real case we have:

Lemma 5.2

1. Let $Y \subseteq X$. $x \in \overline{Y}$ iff for every neighbourhood U of x, $U \cap Y \neq \emptyset$.
2. If $Y \subset X$, let L be the set of limit points of Y. Then $\overline{Y} = Y \cup L$.

The proof of this is identical to that for the reals, so will not be repeated.

We now use the notion of a topology to generalise limits, continuity, etc. in the obvious way.

Definition 5.2

If X, Z are topological spaces, $Y \subseteq X$, $u \in X$, $v \in Z$ and $f : Y \to Z$, we say $f(x) \to v$ as $x \to u$ if for every neighbourhood V of v there is a neighbourhood U of u such that $f(U \ \{u\}) \subseteq V$.

The following lemma carries over from the real case:

Lemma 5.3

1. $f : X \to Y$ is continuous iff for each $x \in X$, f is continuous at x.
2. If $f : X \to Z$, $Y \subseteq X$ and $f(Y) \subseteq V \subseteq Z$, with V closed. If y is a limit point of Y and $f(x) \to t$ as $x \to y$ then $t \in V$.

However, unlike in the real case limits need *not* be unique. We will need to impose an additional condition on our topological spaces if we want them to be.

For example consider any set X with the indiscrete topology (for the example to be interesting we need $|X| \geq 2$, but it will still work without). Then given any function $f : X \to X$, for any $u, v \in X$ we have $f(x) \to v$ as $x \to u$ - the limits are anything but unique.

Looking back to our proof of the uniqueness of limits in the reals, we can rephrase it as the following:

Let $f : X \to \mathbb{R}$, with y a limit point of X. Suppos $f(x) \to u$ and $f(x) \to v$ as $x \to y$, with $u \neq v$. Then there are neighbourhoods U of u and V of v such that U and V are disjoint. Because $f(x) \to u$ and $f(x) \to v$ as $x \to y$ there are neighbourhoods X_1, X_2 of x such that $f(X_1 \ \{y\}) \subseteq U$ and $f(X_2 \ \{y\}) \subseteq V$. Then $X_1 \cap X_2$ is a neighbourhood of y, so there exists $x \in X_1 \cap X_2 \ \{y\}$ (as y is a limit point). Then $f(x) \in U$ and $f(x) \in V$. This contradicts the fact that U and V were disjoint.

This idea of disjoint neighbourhoods is thus sufficient to ensure distinct limits, and will often be useful in other contexts. We thus have the following definition:

Definition 5.3

Let X be a topological space. It is said to be Hausdorff (or to be a Hausdorff space) if for any $x, y \in X$ with $x \neq y$ there are neighbourhoods U of x and V of y.

Lemma 5.4

Let X, Z be topological spaces with $Y \subseteq Z$ and y, $u, v \in Z$, $f : Y \to Z$ and y a limit point of Y. If $f(x) \to u$ and $f(x) \to v$ as $x \to y$ then $u = v$.

The proof is as above.

Note: We do not require X to be Hausdorff in the above lemma.

Although we will not be assuming a space is Hausdorff unless specifically stated, most of the spaces we will consider are going to be hausdorff. It's a fairly easy requirement to check, and most interesting spaces satisfy it, but it gives a lot of extra strength to the theory.

Examples

1. Any subset of $\mathbb{R}$ with the usual topology is Hausdorff.
2. Any set with the discrete topology is Hausdorff.
3. Any set with more than one element and the indiscrete topology is not Hausdorff.

4. An infinite set with the cofinite topology is not Hausdorff.

The notions of continuity will inherit in a similar fashion:

Definition 5.4

Let X, Y be topological spaces. We say $f : X \to Y$ is continuous if for every open $U \subseteq Y$, $f^{-1}(U)$ is open (in X).

Let X, Y be topological spaces, $x \in X$ and $f : X \to Y$. We say f is continuous at x if for every neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subseteq V$. Note that this may not be quite the same as $f(y) \to f(x)$ as $y \to x$, as we could have x not being a limit point of $X\ \{x\}$. This happens, for example, in the discrete topology.

Lemma 5.5

Let X, Y be topological spaces, $f : X \to Y$. The following are equivalent:

1. f is continuous
2. For each $x \in X$ it is continuous at x.
3. For any closed $U \subseteq Y$, $f^{-1}(U)$ is closed.

The proof is the same as for the reals, so is again omitted.

In the reals we had the notion of subsets of a set $W \subseteq \mathbb{R}$ being open in W but not neccesarily in $\mathbb{R}$. We will want to generalise this to topological spaces.

Definition 5.5

Let X be a topological space and $W \subseteq X$. We define the subset topology on W by saying that $Y \subseteq W$ is open in W if there is a set $U \subseteq X$ which is open in X and $Y = W \cap U$.

Proposition 5.1

Let X be a topological space, and $W \subseteq X$.

1. U is closed in W iff there is some set V which is closed in X such that $U = V \cap W$.
2. If W is open in X and U is open in W then U is open in X.
3. If W is closed in X and U is closed in W then U is closed in X.

4. Let Y be another topological space and $f : X \to Y$ continuous. Then $f|_W$ is continuous (with respect to the subspace topology)
5. Let Y be another topological space and $f : Y \to X$ continuous. If $f(Y) \subseteq W$ then $f : Y \to W$ is continuous.

Proof:

The proofs of these are all rather trivial.

1. Suppose U is closed in W. Then $W \setminus U$ is open in W, so there is some open set V in X such that $W \setminus U = W \cap U$. Then V^c is closed and $W \cap V^c = W \setminus (W \cap V) = W \setminus (W \setminus U) = U$.

   The converse is similar; If $U = W \cap V$ for V closed in X, then $U^c = W \setminus U = W \setminus (W \cap V) = W \cap (V^c)$, so U^c is open in W, and hence U is closed.
2. Suppose W was open in X and U open in W. Then there is a V open in X such that $W = V \cap W$. So W is an intersection of two open sets, and thus open.
3. This is pretty much immediate from the definition of

A demonstration of the power of characterising continuous functions in terms of open and closed sets comes in the following two lemmas:

Lemma 5.6

Let X, Y, Z be topological spaces with $f : X \to Y$ and $g : Y \to Z$ continuous functions. The composition $g \cdot f$ is continuous.

Proof:

$g \cdot f : X \to Z$.

Let U be open in Z. Then $(g \cdot f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Because g is continuous, $g^{-1}(U)$ is open. f is also continuous, so f^{-1} of an open set is open. Hence $(g \cdot f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open.

So the inverse image of any open set is open. i.e. $g \cdot f$ is continuous.

QED

The proof wouldn't be *difficult* with the $\epsilon - \delta$ definition of continuity, but it would be messy. This way however is immediate from the definition, and fairly transparent.

The following is even more useful:

Lemma 5.7 *Pasting Lemma*

Let X be a topological space, with $U, V \subseteq X$ closed sets such that $X = U \cup V$. Let Y be another topological space with $f : U \to Y$ and $g : V \to Y$ continuous (with respect to the subset topology) and $\forall x \in U \cap V \ f(x) = g(x)$. Then the function $h : X \to Y$ given by $h(x) = f(x)$ for $x \in U$ and $h(x) = g(x)$ for $x \in V$ is continuous.

Proof:

We will show that the inverse images of closed sets are closed.

Let $W \subseteq Y$ be closed. Then $h^{-1}(W) = f^{-1}(W) \cup g^{-1}(W)$. $f^{-1}(W)$ is closed in U by the continuity of f, and as U is closed it must also be closed in X. Similarily $g^{-1}(V)$ is closed in X. Thus $h^{-1}(W)$ is the union of two closed sets, so closed.

Hence h is continuous.

QED

We can easily extend this to an arbitrary finite collection of closed sets.

A special case of this is the following:

Let $a, b, c \in \mathbb{R}$ with $a < b < c$. Let $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ be continuous with $f(b) = g(b)$. Then h defined as above is continuous.

We could prove this version with the $\epsilon - \delta$ method, but the proof would probably even take slightly longer than our proof of the pasting lemma, and is a much less general result (even if we only consider topological spaces which are subsets of $\mathbb{R}$). This characterisation of continuity has given us a very nice and general way of 'gluing together' continuous functions on different parts of the space.

5.2 Constructing Topologies

So far we only know of a rather limited class of topologies - the subsets of $\mathbb{R}$. Sure there are rather a lot of them, but so far topology appears to only be a different way of looking at things rather than a generalisation. In this section we will look at other examples of topological spaces, with some examples as to why they are interesting things to study.

We shall need the following (rather trivial) result:

Proposition 5.2

Let X be a set and C be a non-empty collection of topologies on X, then $T = \bigcap C$ is also a topology on X.

Proof:

The proof of this is almost trivial.

$\emptyset$ and X are members of each element of C, so are members of T.

Suppose $U, V \in T$. Then for each $S \in C$ we have $U, V \in S$. S is a topology, so $U \cap V \in S$. Thus $U \cap V \in T$. Similarily for unions. Thus T is a topology.

QED

Why is this interesting? Because we will often want to consider the smallest topology that satisfies a given property. This won't always exist of course, but when it does the above proposition will usually give us one.

Corollary 1 *Let K be a collection of subsets of X. There is a topology T such that $K \subseteq T$ and for any topology S such that $K \subseteq S$ we have $T \subseteq S$. T is called the topology generated by K.*

Proof: Consider the collection of topologies containing K (this collection is non-empty as the power set of X is one such topology). Their intersection works for T.

QED

This is often how we will define topologies - we consider which sets we want to be open, and then take the smallest topology in which they are open.

Exercise 5.1 *Let $K = \{(a, b) : a, b \in \mathbb{R},\ a < b\}$. Show that the topology generated by K is the usual topology on $\mathbb{R}$.*

Knowing that a set generates a topology can tell us various useful things. The following lemma is a good example:

Lemma 5.8 *Let X, Y be topological spaces, K be a set that generates the topology of Y, call it T, and $f : X \to Y$. f is continuous iff $\forall U \in K \quad f^{-1}(U)$ is open.*

Proof:

One way is clear. If f is continuous then $\forall U \in K \quad f^{-1}(U)$ as any element of K is open in Y.

For the other, let $S = \{U \in T \ : \ f^{-1}(U) \text{ open in } X\ \}$.

We will show that S is a topology containing K. From this it will follow that $T \subseteq S$, but as we already know that $S \subseteq T$ this will show $S = T$. Hence for all open U, $f^{-1}(U)$ is open, and thus f is continuous.

Clearly $\emptyset \in S$, and $f^{-1}(Y) = X$ which is open.

Suppose $U, V \in S$. Then $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ which is open as $f^{-1}(U)$, $f^{-1}(V)$ are.

Similarily if $\{U_\alpha \ : \ \alpha \in A\} \subseteq S$ then $f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$ which is open as each of the $f^{-1}(U_\alpha)$ is.

This proves our claim, and thus by the above comment proves the lemma.

QED

For example if we have a function $f : X \to \mathbb{R}$ we need only show that for all $a < b$, $f^{-1}(a, b)$ is open in X.

Exercise 5.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be monotone and surjective. Show that it is continuous.*

Note that not all collections of sets will neccesarily generate topologies in an 'interesting' way. For example if they miss points - if $x \notin \bigcup K$ then the topology on X won't tell us a lot about x. We would like K to, in some sense, generate the neighbourhoods of a point. So we will want to consider K such that $\bigcup K = X$.

Also, looking at the previous exercise, let us recall something from the last chapter. A set U is open iff $\forall x \in U \ \ \exists (a, b)$ such that $x \in (a, b) \subseteq U$. So letting K be the set of such intervals (which generates the topology) this is equivalent to saying that U is open iff $\exists V \in K$ with $x \in V \subseteq U$. This is quite a nice characterisation of the topology generated by K, but won't be true for all such K. Let S be the collection of all sets $Y \subseteq X$ such that $\forall y \in Y \ \ \exists V \in K$ with $y \in V \subseteq Y$. This is clearly contained in the topology generated by K, as every element of S is a union of elements of K, so must be open in any topology containing K. So in order to make the desired result hold we need that if $U, V \in S$ then $U \cap V \in S$.

This motivates the following definitions:

Definition 5.6

Let X be a set, and K a collection of subsets of X.

K is said to be a sub-basis if $\bigcup K = X$.

K is said to be a basis if $\bigcup K = X$ and $\forall U, V \in K$ we have $\forall x \in U \cap V \ \ \exists W \in K$ such that $x \in W \subseteq U \cap V$.

Note that if K is a sub-basis, then $\{U_1 \cap \cdots \cap U_n \ : \ n \in \mathbb{N} \ \ U_1, \ldots, U_n \in K\}$ is a basis.

The above discussion (and some easy checks) gives the following:

Proposition 5.3 *Let K be a basis of subsets of X and T be the topology generated by K. Let $x \in X$. U is a neighbourhood of x iff $\exists V \in K \ \ x \in V \subseteq U$. Thus U is open iff $\forall x \in U \ \ \exists V \in K$ with $x \in V \subseteq U$.*

Not all sets we use to generate a topology will be bases, or even sub-bases, but it will often be convenient to consider ones which are.

As an example of a useful generating set which may not be a basis, let X be a set, Y a topological space and F a collection of functions $f : X \to Y$. Then the topology generated by $K = \{f^{-1}(U) \;:\; f \in F,\ U \subseteq Y \text{ open }\}$. Then K is the smallest topology such that every member of F is continuous.

5.3 The Subspace Topology

Given a set of $Y \subseteq \mathbb{R}$ we were able to turn it into a topological space by considering sets of the form $U \cap Y$ with U open in $\mathbb{R}$. We can repeat exactly the same construction given any other topological space.

Proposition 5.4 *Let X be a topological space, with topology T and $Y \subseteq X$.*

1. *The set $T_Y = \{U \cap Y \;:\; U \in T\}$ is a topology on Y. This is called the* Subspace Topology*.*
2. *Let B be a basis generating T. $\{U \cap Y \;:\; U \in B\}$ is a basis for Y, generating T_Y.*

The proof is left as an exercise.

Lemma 5.9 *Let X, Z be a topological space, $Y \subseteq X$, $f : X \to Z$ continuous. Then $f|_Y : Y \to Z$ is continuous (with respect to the subspace topology).*

Proof:

$f|_Y^{-1}(U) = Y \cap f^{-1}(U)$

QED

Exercise 5.3 *Give an example of topological spaces X, Z, a set $Y \subseteq X$ and a continuous function $g : Y \to Z$ such that there is no continuous $f : X \to Z$ with $f|_Y = g$.*

5.4 The Order Topology

Looking at how we constructed the topology on $\mathbb{R}$, the only property of it that we used was its order. Our construction amounts to taking the topology generated by the basis of sets (a, b). In the more general case of a total order this won't neccesarily be a basis - e.g. given $[0, 1]$ with the usual ordering, no such interval contains 0 or 1. If we consider sets of the form $(-\infty, b)$ and (a, ∞) then these *will* always form a sub-basis for a topology (and even a basis if we thrown in sets of the form (a, b)). This is called the *Order Topology*.

We have two important lemmas about these:

Lemma 5.10 *Let X be a totally ordered set and $I \subset X$ an interval. The order topology on I is the same as the subset topology*

Proof:

QED

5.5 The Product Topology

In the last chapter we proved in exercises from the $\epsilon - \delta$ definition of continuity that if f and g were continuous functions then $f+g$ was a continuous function. It wasn't very difficult, but was a little messy and uninformative. Using the work from this section we'd like to provide a better proof.

Suppose we could put a topology on $\mathbb{R}^2$ such that when f and g were continuous $x \to (f(x), g(x))$ was also continuous, and such that $+ : \mathbb{R}^2 \to \mathbb{R}$ was continuous. Then $f+g$ would be a composition of continuous functions, and so continuous, which would be a significantly nicer proof.

We will discuss in general how to put a topology on $X \times Y$ when X and Y are topological spaces.

We want the property that $f : Z \to X$, $g : Z \to Y$ are continuous if and only if $f \times g : Z \to X \times Y$, $t \to (f(t), g(t))$ is continuous.

We will start with the easier direction. If we want $f \times g$ continuous to imply $f,\ g$ are continuous, and obvious way to do it is to require continuity of the projections maps

$$\pi_1 : X \times Y \to X$$
$$\pi_1 : (x, y) \to x$$
$$\pi_2 : X \times Y \to Y$$
$$\pi_2 : (x, y) \to x$$

Then $f = \pi_1 \cdot (f \times g)$, $g = \pi_2 \cdot (f \times g)$ are compositions of continuous functions, and so are continuous.

Given a collection of functions we want to be continuous, we know how to generate a topology from this - the smallest topology containing the preimages of open sets in the image spaces. Surprisingly this will in fact work, as the following lemma shows:

Lemma 5.11 *Let T be the smallest topology on $X \times Y$ such that π_1, π_2 are continuous.*

1. *T is generated by the basis $P = \{U \times V \ : \ U \subseteq X, V \subseteq Y,\ U, V \text{ open } \}$*

2. *Let $f : Z \to X$, $g : Z \to Y$ be continuous. $f \times g$ is continuous.*

Proof:

1 is a trivial consequence of the definitions. The preimages of open sets under π_1, π_2 are of the form $U \times Y$ and $X \times V$ respectively. Then $U \times V = U \times Y \cap X \times V$, so is open in the resulting topology. P is closed under intersection and contains each of the original generating sets $U \times Y$ and $X \times V$, so is a basis for the topology.

We will prove 2 using 1 and Lemma 4.8. Let $U \times V \in P$.

$$\begin{aligned}(f \times g)^{-1}(U \times V) &= \{z : (f(z), g(z)) \in U \times V\} \\ &= \{z : f(z) \in U \text{ and } g(z) \in V\} \\ &= \{z : f(z) \in U\} \cap \{z : g(z) \in V\} \\ &= f^{-1}(U) \cap g^{-1}(V)\end{aligned}$$

By the continuity of f and g, each of the two sets $f^{-1}(U)$, $g^{-1}(V)$ are open, and hence their intersection is open. Thus for all $U \times V \in P$, $(f \times g)^{-1}(U \times V)$ is open. But P generates the topology, and hence for any open set $A \in T$, $(f \times g)^{-1}(A)$ is open. Thus $f \times g$ is continuous.

QED

We call the topology T the *Product Topology*, and whenever we talk about $X \times Y$ as a topological space, we will always mean the product topology unless specifically stated otherwise.

We can extend this in an obvious way to finite products. The product topology is on $X_1 \times \cdots \times X_n$ is the smallest topology such that each of the $\pi_i : (x_1, \ldots, x_n) \to x_i$ is continuous. It is generated by sets of the form $U_1 \times \cdots \times U_n$ and $f_1, \ldots, f_n$ are continuous if and only if $f_1 \times \cdots \times f_n$ is. The definition of the product topology (smallest topology such that the projections are continuous) still works for an arbitrary product, but consider the following exercise:

Exercise 5.4 1. *Show that for an infinite product $\prod_{a \in A} X_a$ a set of the form $\prod_{a \in A} U_a$ with $U_a \subseteq X_a$ open need not itself be open in the product topology.*

2. *Find a suitable basis for the product topology similar to that in the previous lemma.*

3. *Show that $\{f_a : a \in A\}$ are continuous iff $\prod_{a \in A} f_a$ is continuous.*

Proposition 5.5 $+ : \mathbb{R}^2 \to \mathbb{R}$ *is continuous.*

Proof:

The sets of the form $(-\infty, b)$ and (a, ∞) generate the topology of $\mathbb{R}$, so we need only show that the preimage of one of these is open.

Consider $+^{-1}(\infty, b) = \{(x, y) : x + y < b\}$. Let $u < \frac{b-(x+y)}{2}$. Suppose $a < x + u$, $b < y + u$. Then $a + b < (x + u) + (y + u) = x + y + 2u < x + y + b - (x + y) = b$. Further, $b > 0$. Thus we have $(x, y) \in (-\infty, x + u) \times (-\infty, y + u) \subseteq +^{-1}(\infty, b)$. Hence $+^{-1}(\infty, b)$ is a neighbourhood of each of its elements, and so is open.

Similarily $+^{-1}(a, \infty)$ is open.

Hence $+$ is continuous. Thus in particular if f, g are continuous $f + g = + \cdot (f \times g)$ is continuous.

QED

Before we show a similar result for the product, we will take a bit of a digression.

Chapter 6

Compactness

A common problem we are interested in is to find maximum and minimum values of a real valued function. Clearly, it is not always possible to do this: We can easily construct continuous functions on $(a, b), (a, b]$ and (a, ∞) which are unbounded.

Some thought will show that it is much harder to think of such an example for $[a, b]$. Certainly none of the standard examples of continuous functions we've seen so far seem to work - polynomials, powers, exponentials, inverses, and various compositions thereof. This leads us to ask if such a function can in fact exist.

Clearly from our definition of continuity a function $f : [a, b] \to \mathbf{R}$ must, for each $x \in [a, b]$, be bounded on some neighbourhood of x, as there must be a neighbourhood, U, of x such that $f(U) \subseteq (f(x) - 1, f(x) + 1)$. Can we find a function so that, despite it being bounded on a neighbourhood of each point, is unbounded on $[a, b]$ as a whole? In other words, can we cover $[a, b]$ by open sets such that it is bounded on each open set, but not bounded on the whole interval. Certainly this is possible for the rationals. Consider:

$$
\begin{aligned}
f &: [0, 2]_{\mathbb{Q}} \to \mathbb{Q} \\
f(x) &= \frac{1}{2 - x^2}
\end{aligned}
$$

,

This is continuous as there is no rational x with $x^2 = 2$, but is clearly unbounded. As discussed the problem is that, while there is no one one point around which the function is unbounded, the function can still become globally unbounded. However this isn't the same as the issue with $\mathbb{Q}$ being disconnected - it's not purely a matter of 'holes', as $(0, 1)$ is connected and has unbounded functions on it.

We can thus rephrase the problem as follows: Can we write $[a, b] = \bigcup_{\alpha \in A} U_\alpha$, with each U_α open with f bounded on, but $\sup\{|f(U_\alpha)| : \alpha \in A\}$ unbounded?

If A is finite we cannot do this, as any finite set has a maximum element. In fact if *any* finite collection of the U_α covered $[a,b]$ then f would be bounded, as we could then write $[a,b] = \bigcup_{\alpha \in B} U_\alpha$ with f bounded on each U_α and B finite.

It thus seems reasonable to ask whether we can cover $[a,b]$ with open sets in such a way that no finite collection of the open sets covers it. First, lets see if a negative answer would help us.

Let $f : [a,b] \to \mathbf{R}$ be continuous. Define $U_(r,s) = f^{-1}(r,s)$ for $r,s \in \mathbf{R}, r < s$. These cover $[a,b]$, so a finite collection of them cover [a, b]. The union of this collection is an open interval (q,t), so $f([a,b]) \subseteq (q,t)$, thus f is bounded on $[a,b]$. So all that remains to prove (or disprove) is that it is impossible to have such a collection of open sets. This motivates the following theorem:

Theorem 6.1 *Let $\{U_\alpha \ : \ \alpha \in A\}$ be a collection of open sets (in $[a,b]$) such that $\bigcup_{\alpha \in A} U_\alpha = [a,b]$. There is a finite $B \subseteq A$ such that $\bigcup_{\alpha \in B} U_\alpha = [a,b]$.*

Proof So, let $[a,b] = \bigcup_{\alpha \in A} U_\alpha$, with each U_α open. Consider $[a,c] \subseteq [a,b]$ such that $[a,c]$ is covered by a finite number of the U_α.

Let $F = \{c : [a,c] \subseteq [a,b]$ such that $[a,c]$ is covered by a finite number of the $U_\alpha\}, y = Sup(F)$.

By assumption there is some $\beta \in A$ such that $a \in U_\beta$. U_β is open, so $\exists d > a$ $[a,d) \subseteq U_\beta$. Let $a < c < d.[a,c]$ is covered by U_β, so by a finite number of the U_α, so $c \in F$. Thus $y \geq c > a$.

If $x < y$ then $[a,x]$ is covered by a finite number of the U_α, as by the definition of the supremum $\exists t \in F, x < t \leq y$, and any collection of U_α that covers $[a,t]$ covers $[a,x]$

Again, by assumption there is some $\beta \in A, d < y$ such that $y \in U_\beta \supseteq (d,y]$. Then $[a,d]$ is covered by a finite collection of U_α, say $\{U_\alpha : \alpha \in C\}.C \cup \{\beta\}$ then covers $[a,y]$.

Now suppose $y < b$. There is a $\beta \in A, c < y < d$ such that $y \in U_\beta \supseteq (c,d]$. Let $\{U_\alpha : \alpha \in D\}$ cover $[a,y]$, and $x \in (y,d)$. Then x is covered by $\{U_\alpha : \alpha \in C\} \cup \beta$, which is a finite set, so $x \in F$. This contradicts y being the supremum. Thus $y = b$, and $[a,b]$ is covered by a finite number of U_α

QED

Note that we haven't really used many of the properties of **R** in this proof - it will in fact hold for any order topology such that the ordering is complete, dense, and has a greatest and least element. This won't really be important, but it is nice to prove things with the minimal logical assumptions.

We will use this to motivate some definitions.

Definition 6.1 *Let X be a topological space with topology T, $Y \subseteq X$. An open cover of Y is a set $S \subseteq T$ such that $Y \subseteq \bigcup S$. If S is an open cover of Y a* *subcover of S is a set $R \subseteq S$ such that R is also an open cover of Y. Y is said to be compact if every open cover of it has a finite subcover.*

Note that $Y \subseteq X$ is compact in X iff it is compact with the subspace topology.

We can thus restate the previous result as:

Theorem 6.2 *Heine-Borel Theorem. Let $a, b \in \boldsymbol{R}, a < b.[a, b]$ is compact.*

The following theorem is probably one of the most useful features of compact spaces:

Theorem 6.3 *Let X, Y be topological spaces $f : X \to Y$ continuous, $Z \subseteq X$ compact. $f(Z)$ is compact.*

Proof:

Let $\{U_\alpha : \alpha \in A\}$ be an open cover of $f(X)$. $\{f^{-1}(U_\alpha) : \alpha \in A\}$ is then an open cover of X (as f is continuous, so each is open), so has a finite subcover $\{U_\alpha : \alpha \in B\}$. $\{U_\alpha : \alpha \in B\}$ is then a finite subcover of the original cover. Hence f(Y) is compact.

Corollary 2 *Let X and Y be homeomorphic topological spaces. X is compact iff Y is compact.*

Now we would like a simple categorisation of what the compact subsets of **R** are.

Theorem: Let X be a compact topological space, $Y \subseteq X$ closed. Y is compact.

Proof:

Let $\{U_\alpha : \alpha \in A\}$ be an open cover of Y. $\{U_\alpha : \alpha \in A\} \cup \{Y^c\}$ is an open cover of X, so has a finite subcover $\{U_\alpha : \alpha \in B\} \cup \{Y^c\}$. $\{U_\alpha : \alpha \in B\}$ is then a finite subcover of Y.

Corollary: Any closed bounded subset of **R** is compact.

Proof: Let $Y \subseteq \mathbf{R}$ be closed and bounded. $\exists a, b \in \mathbf{R}, Y \subseteq [a, b]$. $[a, b]$ is closed in **R**, so Y is closed in $[a, b]$ iff it is closed in **R**. Thus Y is a closed subset of a compact space (as $[a, b]$ is compact), and so compact.

Theorem: Let X) be a hausdorff topological space, $Y \subseteq X$ compact. Y is closed.

Proof:

Let $x \notin Y$. For each $y \in Y$ there are open sets U, V such that $x \in U, y \in V, U \cap V = \emptyset$ (by the definition of a hausdorff topological space). Thus $\{U : U$

open, $\exists V \ni x, U \cap V = \emptyset$, V open } is an open cover of Y. Let $\{U_i : 1 \leq i \leq n\}$ be a subcover, with corresponding open sets $V_1, \ldots, V_n$ such that $V_i \ni x$ and $V_i \cap U_i = \emptyset$. Then $\bigcap_{i=1}^n V_i$ is a neighbourhood of x disjoint from Y. Hence Y is closed.

QED.

Exercise 6.1 *Find an example of a topological space X with $Y \subseteq X$ compact and not closed.*

Exercise 6.2 *Let X, Y be topological spaces with Y hausdorff and X compact. Let $f : X \to Y$ be a continuous bijection. Show that f^{-1} is continuous, and thus f is a homeomorphism.*

Exercise 6.3 *Let X be a compact hausdorff space, with topology T. Show that any topology finer than T is not compact, and any topology coarser than T is not hausdorff. (* Hint: Use the previous exercise.

In a metric space we have some extra information about compact sets.

Proposition 6.1 *Let X be a metric space, and $Y \subseteq X$ compact. Y is bounded.*

Proof:

Fix $x \in X$. Then $\{B(x, r) : r > 0\}$ has a finite subcover. This is a $\subseteq$-chain, so a finite subcover contains a maximal element. Thus for some r, $X \subseteq B(x, r)$. i.e. X is bounded.

QED

Theorem 6.4 $X \subseteq \mathbb{R}$ *is compact iff it is closed and bounded.*

Proof:

Our previous results immediately give that X compact implies X closed and bounded. Suppose X is closed and bounded. Then it is contained in some interval $[a, b]$, which is compact. Thus it is a closed subset of a compact space, and so compact.

QED

Corollary 3 *: Let X be a compact topological space, $f : X \to \boldsymbol{R}$ continuous. f is bounded and attains it's bounds (i.e. $\exists x, y \in X f(X) \subseteq [f(x), f(y)]$).*

Proof: f(X) is closed, so contains is supremum and infimum.

Corollary: Let $a, b \in \mathbf{R}, f : [a, b] \to \mathbf{R}$ continuous. $\exists c, d \in \mathbf{R}, f([a, b]) = [c, d]$).

Proof: $f([a, b])$ is connected, so an interval. $f([a, b])$ is compact, so closed and bounded, so of the form $[c, d]$.

6.1 Compactness of Products

We know how compactness behaves when taking subspaces. Now how does it behave when we consider products?

We first need the following lemma:

Lemma 6.1 *The Tube Lemma*

Let X, Y be topological spaces with Y compact, $x_0 \in X$ an W an open set in $X \times Y$ such that $\{x_0\} \times Y \subseteq U$.

There exists $N \subseteq X$ open such that $x_0 \in N$, and $N \times Y \subseteq U$.

Proof

$\{x_0\} \times Y$ is homeomorphic to Y by the map $(x_0, y) \to y$, and so is compact.

Consider the collection of sets $\{U \times V \ : \ U \times V \subseteq W \ , \ U \times V \cap \{x_0\} \times Y \neq \emptyset\}$. This covers $\{x_0\} \times Y$, so by compactness has a finite subcover, say $U_1 \times V_1, \ldots U_1 \times V_n$.

Let $N = \bigcap_{i=1}^{n} U_i$. I claim this works.

Certainly N is open, and $x_0 \in U_i$ by hypothesis, so $x_0 \in N$.

Let $(x, y) \in W \times Y$. Consider (x_0, y). This is in some $U_i \times V_i$, so $Y_i \in V_i$. But by construction we have for all k, $x \in U_k$. Hence, in particular, $x \in U_i$. Thus $x \times y \in U_i \times V_i \subseteq W$. Hence $\{x_0\} \times Y \subseteq N \times Y \subseteq W$.

QED

This will now give us an important theorem:

Theorem 6.5 *Tychonoff's Theorem for $X \times Y$.*

Let X, Y be compact topological spaces. $X \times Y$ is compact.

Proof:

Let C be an open cover.

Fix $x \in X$. $\{x\} \times Y$. This is compact, so a finite subcover of C, say $A_1, \ldots, A_n$ covers it. Let $W = \bigcup A_n$. Then by the tube lemma there is some $N \subseteq X$ such that $x \times y \subseteq N \times Y \subseteq W$. Hence x has a neighbourhood N such that $N \times Y$ is covered by a finite subcover of C.

Let $D = \{U \subseteq X \ : \ U$ open and $U \times Y$ is covered by a finite subcover of $C\}$. By the above, D is an open cover of X. Thus it has a finite subcover $B_1, \ldots, B_m$.

So we can write $X \times Y = \bigcup_{k=1}^{m} B_k \times Y$. Each of these can be covered by finitely many elements of C, and so the whole of $X \times Y$ can be covered by finitely many elements of C. Hence C is compact.

QED

Corollary 4 *Let $X_1, \ldots, X_n$ be compact. $X_1 \times \cdots \times X_n$ is compact.*

Proof is by induction on n, using the previous result.

This gives us the following, which is often extremely useful.

Theorem 6.6 *$X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof:

The proof is more or less the same as for $\mathbb{R}$. If X is bounded then it is contained in some $[a_1, b_1] \times \cdots \times [a_n, b_n]$, which is compact by Tychonoff's theorem. Thus it is a closed subset of a compact space, and so compact.

QED

Exercise 6.4 *Let X be a metric space and $A, B \subseteq X$ disjoint closed subsets with A compact. Show that $d(A, B) > 0$. Find an example to show this does not hold if we drop the requirement that A is compact.*

Chapter 7

Problems of Optimisation, Differentiation

7.1 Functions on the Reals

We have shown that for $[a, b] \subseteq \mathbf{R}$ if we have a continuous function $f : [a, b] \to \mathbf{R}$ then f attains both it's maximum and minimum values. The question is, how do we find these values? Consider the following model problem:

Let

$$f : [-3, 3] \to \mathbf{R}$$

$$f(x) = 3x^4 + 4x^3 + 6x^2 + 1$$

What is the smallest value that f takes?

At the moment, although we know that f does indeed take a minimum value, it is far from clear how we would go about calculating it. Plotting a graph of the function suggests that it takes it miminum value at zero, but this is hardly a proof.

Notice that we can write $f(x) = 1 + x^2(3x^2 + 4x + 6).x \to 3x^2 + 4x + 6$ is a continuous function, which takes the value 6 at zero, so there is a neighbourhood U of zero such that $\forall x \in U 3x^2 + 4x + 6 > 3$. Thus $\forall x \in U f(x) > 1 + 3x^2 \geq 1$.

So, 1 is the minimum value that f takes on some neighbourhood of 0. We just don't know whether or not it is a minimum for f on the whole of $[-3, 3]$. This leads us to consider the following definitions:

Definition: Let $\chi = (X, T)$ be a topological space, $f : X \to \mathbf{R}, x \in X$. We say x is a local minimum if there is a neighbourhood U of x such that $\forall y \in U f(y) \geq f(x)$. x is a global minimum if $\forall y \in X, f(y) \geq f(x)$. Clearly a global minimum is a local minimum. Global and local maximums are defined similarily.

Let $f : [a, n] \to \mathbf{R}$ be a polynomial, and let $y \in \mathbf{R}$ be a local minimum or maximum. for f. By the remainder theorem we can write $f(x) = g(x)(x - y) + f(y)$, for some polynomial g.

Let y be an interior point of $[a, b]$ (i.e. $\exists (c, d), y \in (c, d) \subseteq [a, b]$). Suppose $g(y) \neq 0$. Without loss of generality let $g(y) > 0$ (else we can consider -f). Let V be a neighbourhood of y. We can choose some neighbourhood, U of y such that $\forall x \in U \;\; \frac{1}{2}g(y) < g(x) < \frac{3}{2}g(y)$ and $U \subseteq V$. Now if $U \ni x > y,\; f(x) = g(x)(x - y) + f(y) > \frac{1}{2}g(y)(x - y) + f(y) > f(y)$. So y is not a local maximum (as V is arbuitray). Similarily considering $x < y$, y is not a local minimum.

Thus for any extrema we have $g(y) = 0$. This will prove to be a useful way of finding maxima and minima. The question becomes how do we calculate $g(y)$ easily.

Rewrite the equation as $(x - y)g_y(x) = f(x) - f(y)$.

Let $f(x) = \sum_{i=0}^{n} a_n x^n$

$$\begin{aligned} f(x) - f(y) &= \sum_{i=0}^{n} a_n (x^n - y^n) \\ &= \sum_{i=0}^{n} a_n (x - y) \sum_{j=0}^{n-1} x^j y^{n-1-j} \\ &= (x - y) \sum_{i=0}^{n} a_n \sum_{j=0}^{n-1} x^j y^{n-1-j} \end{aligned}$$

So

$$g_y(x) = \sum_{i=0}^{n} a_n \sum_{j=0}^{n-1} x^j y^{n-1-j}$$

$$\begin{aligned} g_y(y) &= \sum_{i=0}^{n} a_n \sum_{j=0}^{n-1} y^j y^{n-1-j} \\ &= \sum_{i=0}^{n} a_n \sum_{j=0}^{n-1} y^{n-1} \\ &= \sum_{i=0}^{n} a_n y^{n-1} \sum_{j=0}^{n-1} 1 \\ &= \sum_{i=0}^{n} n a_n y^{n-1} \end{aligned}$$

Define $f'(x) = g_x(x)$. From our previous discussion we know that the extrema for f are either at a, b or a zero of f'.

Thus, finding the extrema for a polynomial of degree n is reduced to the problem of finding the roots of a polynomial of degree $n-1$ which, while not trivial, is substantially easier.

Thus, returning to our original model problem, we now have a way to solve it.

$$f(x) = 3x^4 + 4x^3 + 6x^2 + 1$$

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 + 12x \\ &= 12x(x^2 + x + 1) \\ &= 12x((x + \frac{1}{2})^2 + \frac{3}{4}) \end{aligned}$$

So the only zero of $f'(x)$ is at $x = 0$. Thus the minimum value comes is either at -3, 0 or 3. A little thought will show that it in fact has to be a minimum in this case, but it's also easy just to check the three values:

$$f(0) = 1$$

$$f(3) = 406$$

$$f(-3) = 190$$

So the minimum value f takes is 1, and it only takes it at zero.

Can we generalise this method to deal with functions other than polynomials? A fairly obvious place to start would be the following:

Definition: Let $X \subseteq \mathbf{R}, y \in X, f : X \to \mathbf{R}$. We say f is differentiable at y if we can write $f(x) = f(y) + (x-y)k + (x-y)g(x-y)$ where $k \in \mathbf{R}$, and $g(t) \to 0$ as $t \to 0$. We say f is differentiable if for each $x \in X$ it is differentiable at x.

(X will usually be an interval in the cases we will deal with).

We will often use the notation $o(h(x))$ to mean a function $g(x)$ such that $\frac{h(x)}{g(x)} \to 0$ as $x \to 0$. So the above could be written as $f(x) = f(y) + (x-y)k + o(|x-y|)$.

Note that if f is differentiable at y then it is continuous at y, as $lim_{x \to y} f(x) = lim_{x \to y}(f(y) + (x-y)k + (x-y)g(x-y)) = f(y) + 0 + 0 = f(y)$.

Suppose we have $f(x) = f(y) + (x-y)k + (x-y)g(x-y), k \in \mathbf{R},\ g(t) \to 0$ as $t \to 0$ and $f(x) = f(y) + (x-y)s + (x-y)h(x-y), s \in \mathbf{R},\ h(t) \to 0$ as $t \to 0$.

$$f(y) + (x-y)k + (x-y)g(x-y) = f(y) + (x-y)s + (x-y)h(x-y)$$

$$(x-y)(k-s) = (x-y)(h(x-y) - g(x-y))$$

$$k - s = h(x-y) - g(x-y) \to 0 \text{ as } x \to y$$

$$k - s = 0$$

$$k = s$$

This gives us the following definition:

Definition: Let f be differentiable at y. The derivative of f at y, denoted $f'(y)$ is the unique number such that $f(x) = f(y) + (x-y)f'(y) + (x-y)g(x-y)$ such that $g(t) \to 0$ as $t \to 0$. If f is differentiable on all of X, then the derivative defines a new function $f' : X \to \mathbf{R}$.

We can thus restate our previous discussion of maxima and minima in the more general setting:

Theorem 7.1 *Let $f : X \to \mathbb{R}$ be differentiable and have a local maxima or minima at an interior point $x \in X$. Then $f'(x) = 0$.*

The following results give us easy ways of calculating derivatives, as well as showing that functions are differentiable:

Lemma: $f(x)$ is differentiable at y with $f'(y) = k$ iff $lim_{x \to y} \frac{f(x)-f(y)}{x-y} = k$.

Theorem: Let $f, g : X \to \mathbf{R}$ be differentiable. Then:

1. $f + g$ is differentiable and $(f+g)' = f' + g'$
2. fg is differentiable and $(fg)' = fg' + f'g$

Theorem 7.2 *The Chain Rule Let $f : X \to \boldsymbol{R}, g : f(X) \to \boldsymbol{R}$ be differentiable. $g \cdot f$ is differentiable and $(g \cdot f)' = f'g' \cdot f$.*

At the moment none of these actually give us any new differentiable functions, simply convenient ways to calculate the derivatives of existing ones, as the only examples of differentiable functions we know of right now are polynomials. There are also other easy examples, e.g.

$$\begin{aligned} f(x) &= x^3 \quad (x \geq 0) \\ &= x^2 \quad (x \leq 0) \end{aligned}$$

f is clearly differentiable everywhere except as zero, as it is a polynomial on a neighbourhood of each such point. It is then easy to check that f is differentiable at zero.

The following theorem will give us some fundamentally different examples:

Theorem: Let $I \subseteq \mathbf{R}$ be an interval, $f : I \to \mathbf{R}$ differentiable and injective, with $f'(x) \neq 0$. $f^{-1} : f(I) \to \mathbf{R}$ is differentiable and $f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$.

This then gives us that each of the functions $f : x \to x^{\frac{1}{n}}$ $(n \in \mathbf{N})$ is differentiable on $(0, \infty)$, and $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$, and an application of the chain rule then gives us that if $g : x \to x^{\frac{m}{n}}$ $(m, n \in \mathbf{N})$.

Also let

$$f(x) = x^{-1}, \quad y \in (0, \infty)$$

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &= \frac{1}{x - y}(x^{-1} - y^{-1}) \\ &= \frac{1}{x - y}\frac{y - x}{xy} \\ &= \frac{-1}{xy} \\ &\to \tfrac{-1}{x^2} \text{ as } x \to y \end{aligned}$$

Thus using the chain rule we get that $\frac{d}{dx}x^a = ax^{a-1}$ for any rational a. Note that this does *not* show the result for irrational a - although we would expect it to hold, we shall have to wait a while before we can prove that.

Does the derivative of a function tell us anything interesting about it other than where extrema might occur? It seems like it should - it contains a whole lot of information other than where it's zeroes are. Intuitively the derivative of a function at the point x corresponds to the slope of a tangent to the graph at x. This suggests a lot of intuitive ideas about the derivative, but at the moment we don't know if they hold. For example, suppose a function is defined on an interval and it's derivative is zero everywhere. Does this imply the function is constant?

For this we will need two theorems:

Theorem 7.3 *(Rolle's Theorem): Let $f : [a, b] \to \boldsymbol{R}$ be differentiable. If $f(a) = f(b)$ then $\exists c \in (a, b), f'(c) = 0$.*

Proof:

f is continuous, so it attains it's maximum and minimum values. If these are both attained at $f(a)$ then the function is constant, so certainly has zero derivative somewhere in (a, b).

Theorem (Mean Value Theorem): Let $f : [a, b] \to \mathbf{R}$ be differentiable. Then $\exists c \in (a, b), f(b) - f(a) = (b - a)f'(c) = 0$

Corollaries: Let $I \subseteq \mathbf{R}$ be an interval, $f : I \to \mathbf{R}$ differentiable.

1. If $\forall x \in I, f'(x) = 0$ then f is constant.
2. If $\forall x \in I, f'(x) > 0$ then f is strictly increasing.

3. If $\forall x \in I, f'(x) > 0$ then f is strictly increasing.

At the moment you may be thinking that this theory is all very well, but we *still* don't know of any differentiable functions except for polynomials. This is true, but the theory is still remarkably useful for determining things about polynomials that we wouldn't otherwise be able to do. Here is an example:

Consider the polynomial $f(x) = 3x^4 + 7x^3 - 3x^2 + 4$. How many zeroes does it have? We know that it could, in principle, have up to four, but may have fewer - or indeed none at all.

First we calculate the derivative.

$$\begin{aligned} f'(x) &= 12x^3 + 21x^2 - 6x \\ &= 3x(4x^2 + 7x - 2) \\ &= 3x(4x - 1)(x + 2) \end{aligned}$$

This is an easy polynomial to understand. In fact all we need to know is where it takes values less than, greater to or equal to zero. This is easy to find by considering the signs of the individual terms:

$$\begin{array}{ll} f'(x) < 0 & x < -2 \\ f'(x) > 0 & -2 < x < 0 \\ f'(x) < 0 & 0 < x < \dfrac{1}{4} \\ f'(x) > 0 & \dfrac{1}{4} < x \end{array}$$

With the only zeroes being at $\frac{1}{4}, -2$ and 0.

Lets take a look at some values of f.

$$\begin{gathered} f(-3) = 31 \\ f(-2) = -16 < 0 \\ f(0) = 4 \\ f(\frac{1}{4}) = 3 + \frac{239}{256} > 0 \end{gathered}$$

The intermediate value theorem gives us at least two zeroes between -3 and 0. Let u and v be two distinct zeroes, and suppose there was a third, t. Without loss of generality, by relabelling, let $u < t < v$. Rolle's theorem would give a zero of $f'(x)$ between each of u and t and v and t. But $f'(x)$ has only one zero in $(-3, 0)$. Hence there are exactly two zeroes in this region.

Our corollary to the mean value theorem shows that f is decreasing on $(0, \frac{1}{4})$, so for all x in this region $f(x) > f(\frac{1}{4}) > 0$. Similarily f is increasing on $(\frac{1}{4}, \infty)$, so in this region $f(x) > f(\frac{1}{4}) > 0$. Thus f has exactly two roots, and these lie in $(-3, -2)$ and $(-2, 0)$ respectively.

This of course all seems like a lot of work for something you can just prove by drawing a graph, but the point is that this is evidence that the graph drawing *works*, which wasn't neccesarily the case - e.g. It doesn't work for the rationals, so is good evidence that the real numbers are indeed the right setting in which to do geometry.

Chapter 8

Linear Algebra and Analysis

We often want to consider optimisation problems for functions with domains other than subsets of $\mathbb{R}$, e.g. what is the maximum value the function $f(x, y) = x^2 - 4xy + y^2$ takes on $[0, 1]^2$? We know it must have one, by the Heine-Borel theorem for $\mathbb{R}^n$, but how do we find it?

For $\mathbb{R}$ we had the notion of differentiation. Can we generalise this to work for functions $f : \mathbb{R}^n \to \mathbb{R}$?

The current definition won't work. At the moment we have f is differentiable at y if $f(x) = f(y) + k(x - y) + o(|x - y|)$

This doesn't make a lot of sense. We don't have an obvious notion of $|z|$ in $\mathbb{R}^n$, and $k(x - y)$ is an element of $\mathbb{R}^n$, not a scalar. We do have a topology on $\mathbb{R}^n$, so the idea of a limit at least makes sense.

In order to make the definition of a deriative generalise to $\mathbb{R}^n$ we will need to find something to replace k and decide what the function $||$ is supposed to be.

The notions of addition and multiplication by a scalar do however make sense in an obvious way. If $k \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots x_n), y = (y_1, \ldots, y_n)$, we define $x + y = (x_1 + y_1, \ldots, x_n + y_n)$ and $kx = (kx_1, \ldots, kx_n)$.

Proposition 8.1 *Let* $x, y, z \in \mathbb{R}^n, \lambda, \mu \in \mathbb{R}$. *Denote* $(0, \ldots 0) \in \mathbb{R}^n$ *by* 0 *(it will usually be clear from context where we intend our* 0 *to lie), and* $-x = (-x_1, \ldots, x_n)$

1. $(x + y) + z = x + (y + z)$
2. $x + y = y + x$
3. $x + 0 = x$

4. $x + (-x) = 0$

5. $\lambda(x + y) = \lambda x + \lambda y$

6. $(\lambda + \mu)x = \lambda x + \mu x$

7. $1x = x$

8. $\lambda(\mu x) = (\lambda\mu)x$

Proof: An exercise for the reader, if you are very bored.

Exercise 8.1 *Show that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous.*

As usual, we would like to generalise the above.

Definition 8.1 *Given a field $\mathbb{F}$, we define a vector space over $\mathbb{F}$ to be a set V with an addition $+$ and a scalar multiplication by elements of $\mathbb{F}$ satisfying the conditions of the previous proposition. i.e. V is an abelian group under $+$ and*

1. $\lambda(x + y) = \lambda x + \lambda y$

2. $(\lambda + \mu)x = \lambda x + \mu x$

3. $1x = x$

4. $\lambda(\mu x) = (\lambda\mu)x$

If $W \subseteq V$ and is also a vector space with the same operations, it is called a subspace of V.

So Proposition 7.1 says that $\mathbb{R}^n$ is a vector space with the operations we have defined.

Examples

1. Any field is a vector space over itself with the usual operations.
2. If $\mathbb{F}$ is a subfield of $\mathbb{G}$ then $\mathbb{G}$ is a vector space over $\mathbb{F}$
3. Let X be a set. The set of functions $g : X \to \mathbb{F}$ is a vector space over $\mathbb{F}$ with the operations $(g + h)(x) = f(x) + g(x)$ and $(\lambda g)(x) = \lambda g(x)$
4. If $\mathbb{F} = \mathbb{R}$ and X has a topology on it, the set of continuous functions $g : X \to \mathbb{R}$ is a subspace of the space of functions $g : X \to \mathbb{R}$. We denote this space by $C(X)$

We now want to figure out what to replace the scalar k with. As it takes a vector $x - y$ and outputs a scalar, it must be some sort of function of $x - y$. Further, in order to preserve the sense of differentiation from $\mathbb{R}$ we want our function to be 'linear'. We shall use the following definition:

Definition 8.2 *Let V, W be vector spaces over a field $\mathbb{F}$. A map $T : V \to W$ is said to be linear if $T(x+y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$*

So the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point y is as linear map $D : \mathbb{R}^n \to \mathbb{R}$.

8.1 Norms and Distance

Now we just have to make sense of the notion of $||$ in $\mathbb{R}^n$. As we will want to talk about both real numbers and elements of $\mathbb{R}^n$ at once, to avoid confusion we shall introduce the notation $||||$ to replace $||$ in $\mathbb{R}^n$. In keeping with the notion from $\mathbb{R}$, $|x|$ should be the distance from x to the origin. Recall (from pythagoras theorem) that in $\mathbb{R}^2$ the distance from x to the origin is $(x_1^2 + x_2^2)^{\frac{1}{2}}$. This suggests the natural generalisation of letting $||x|| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$

Before we go further with it, we will need to prove various things about this function. In doing so it will be useful to consider one further generalisation: Let $x, y \in \mathbb{R}^n$. Define $< x, y >= \sum_{i=1}^n x_i y_i$. Then $||x||^2 =< x, x >$

Proposition 8.2

1. *$< x, y >=< y, x >$*
2. *$< x, y + z >=< x, y > + < x, z >$*
3. *$< x, \lambda y >= \lambda < x, y >$*
4. *$< x, x >\geq 0$ with equality iff $x = 0$*

The proof is left as an exercise.

Given a vector space over $\mathbb{R}$, something satisfying the above is called an inner product. A vector space with an inner product is called an inner product space.

Theorem 8.1 *Cauchy-Schwartz Inequality*

Let $< , >$ be an inner product on a vector space over $\mathbb{R}$, V. Let $||x|| =< x, x >^{\frac{1}{2}}$.

Then $| < x, y > | \leq ||x||||y||$

Equality occurs iff one of them is a scalar multiple of the other.

Proof:

Without loss of generality, we may consider both $x, y \neq 0$.

Consider $< x - ty, x - ty > \geq 0$ with equality iff $x = ty$.

Simple algebra gives that $< x - ty, x - ty >= t^2||y||^2 - 2t < x, y > + ||x||^2$. This is a quadratic equation, so the number of roots depends on the sign of the discriminant, which is $4(< x, y >^2 - ||x||^2||y||^2)$. There is at most one root, so we know this must be ≤ 0, with equality iff there is a root.

Thus $< x, y >^2 \leq ||x||^2||y||^2$ with equality iff there is a root. i.e. $| < x, y > | \leq ||x||||y||$ with equality iff $\exists t \ x = ty$.

QED

Corollary 5 *Let $< , >$ be an inner product, and define $||x|| =< x, x >^{\frac{1}{2}}$. Then $\forall x, y \ ||x + y|| \leq ||x|| + ||y||$.*

Proof:

$$\begin{aligned} ||x+y||^2 &= ||x||^2 + 2 < x, y > + ||y||^2 \\ &\leq ||x||^2 + 2||x||||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2 \\ ||x+y|| &\leq ||x|| + ||y|| \end{aligned}$$

The result of this corollary is called the triangle inequality. It can be viewed as saying that the length of any side of a triangle is shorter than the sum of the other two lengths.

(Picture)

Generalising yet more, we define the following:

Definition 8.3 *Given a vector space V over $\mathbb{R}$, a norm is a function $|| \cdot || : V \to \mathbb{R}$ such that:*

1. $||x|| \geq 0$ *with equality iff* $x = 0$
2. $||\lambda x|| = |\lambda|||x||$
3. $||x + y|| \leq ||x|| + ||y||$

A vector space with a norm is called a normed space. Our previous theorem shows that every inner product space is a normed space in a natural way.

Examples:

1. Given any inner product, $||x|| =< x, x >^{\frac{1}{2}}$ is a norm.

2. $||x||_1 = \sum_{i=1}^n |x_i|$ is also a norm on $\mathbb{R}^n$
3. As is $||x||_\infty = \max_i |x_i|$
4. $||f|| = \sup_x |f(x)|$ is a norm on $C([0,1])$.

Proposition 8.3 *Let $||||$ be a norm.*

1. $||x-y|| \leq ||x-z|| + ||y-z||$

2. $\forall x, y \; |||x|| - ||y||| \leq ||x-y||$

Proof:

1.

$$\begin{aligned} ||x-y|| &= ||(x-z)-(y-z)|| \\ &\leq ||x-z|| + ||y-z|| \end{aligned}$$

2.

$$\begin{aligned} ||x|| &= ||y+(x-y)|| \\ &\leq ||y|| + ||x-y|| \\ ||x|| - ||y|| &\leq ||x-y|| \end{aligned}$$

Similarily $||y|| - ||x|| \leq ||x-y||$

Hence $-||x-y|| \leq ||x|| - ||y|| \leq ||x-y||$

So $|||x|| - ||y||| \leq ||x-y||$

QED

Proposition 8.4 *Let $||\cdot||$ be defined as above on $\mathbb{R}^n$. It is continuous with respect to the product topology.*

Proof:

It is an easy check (by induction) that the map $y \to \sum_{i=1}^n y_i$ is continuous. Each of the maps $x \to x_i^2$ is continuous, so the map $x \to (x_1^2, \ldots, x_n^2)$ is continuous. Hence so is the map $x \to \sum_{i=1}^n x_i^2$, as it is a composition of continuous maps. Thus $x \to (\sum_{i=1}^n x_i^2)^{\frac{1}{2}} = ||x||$ is a composition of continuous functions, and so continuous.

QED

Corollary 6 *The sets* $B(x, \epsilon) = \{y \ : \ |x - y| < \epsilon\}$ *are open in* $\mathbb{R}^n$

Before we continue discussing differentiability, we will take a quick look at the links between norms and topology.

Theorem 8.2 $U \subseteq \mathbb{R}^n$ *is open if and only if* $\forall x \in U \ \exists \epsilon > 0$ *such that* $||x-y|| < \epsilon \implies y \in U$.

Proof:

Let T be the usual topology and $S = \{U \subseteq \mathbb{R}^n \ : \ \forall x \in U \ \exists \epsilon > 0$ such that $||x - y|| < \epsilon \implies y \in U\}$. Clearly S is a topology. We will show that $T = S$.

Claim: The sets $B(x, \epsilon)$ generate S.

It is obvious that every open set is a union of sets of such form. It thus suffices to check that these sets are open.

Let $y \in B(x, \epsilon)$. Let $0 < \delta < \epsilon - ||x - y||$. Then if $||z - y|| < \delta$ we have $||x - z|| \leq ||x - y|| + ||y - z|| < ||x - y|| + \epsilon - ||x - y|| = \epsilon$.

Further, given that each of these sets is the preimage of $(-\infty, \epsilon)$ under the map $y \to ||y - x||$ and that this map is continuous with respect to T, we know that these sets are open in T. Thus we must have $S \subseteq T$.

Now note that all the projection functions are continuous with respect to S, as $|\pi_i(x) - \pi_i(y)| \leq ||x - y||$. Let $V \subseteq \mathbb{R}$ be open and $\pi_i(x) \in V$. There exists some $\epsilon > 0$ such that $|\pi_i(x) - y| < \epsilon$ implies $y \in V$. Hence $||x - y|| < \epsilon$ implies $\pi_i(y) \in V$, and so $y \in \pi_i^{-1}(V)$. Thus $\pi_i^{-1}(V)$ is open. Hence π_i is continuous. T was the smallest topology such that the π_i were continuous, so we have $T \subseteq S$. Hence $T = S$ and the theorem is proved.

QED

Given any normed space over $\mathbb{R}$, the above defines a topology on it. Unless we explicitly say otherwise this will always be the topology meant.

Before we get back to differentiation, we will just need one more

Chapter 9

Fixed Point Problems

You may be familiar with the newton-raphson method for finding roots of equations. You take an initial value x_0, find the tangent at that point, and calculate where this tangent crosses the x axis. You then take this as your new value of x_1 and repeat the process. In principle the x_n should provide increasingly good approximations to a root of the function.

Lets state this a bit more precisely:

Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable, $x_0 \in \mathbf{R}$.

Define $F(x) = x - \frac{f(x)}{f'(x)}$ (this is the equation we get for finding the root of the tangent). Let $x_n = F^n(x_0)$.

The newton raphson method works when x_n converges to x, such that $f(x) = 0$.

First of all, what do we mean by x_n converging to x? Simply that x_n provide an arbitrarily good approximation to x as n gets large. In particular x_n is close to x for *all* sufficiently large x. As usual we will replace 'close' with 'inside an arbitrary neighbourhood' to get a definition for a general topological space.

Definition 9.1 *Let x_n be a sequence in a topological space x. We say $x_n \to x$ if for every neighbourhood U of x, x_n is eventually in U. i.e. There exists N such that for all $n \geq N$, $x_n \in U$.*

The following will be important:

Theorem 9.1 *Let $x_n \to x$ and f be continuous at x. $f(x_n) \to f(x)$*

Proof:

Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of x, so x_n is eventually in it. Thus $f(x_n)$ is eventually in V. Hence $f(x_n) \to x$.

QED

Suppose we do have x_n converging to x for some x (not neccesarily a root of f). If we have f' continuous and non-zero on some neighbourhood, U, of x, then we have F continuous on a neighbourhood of x. Discard initial terms of x_n so all remaining terms lie in such a neighbourhood (we can do that, as we know that there is some N such that for all $n > N,\ x_n \in U$). Thus we have

$$\begin{aligned} F(x) &= F(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} F(x_n) \\ &= \lim_{n\to\infty} F^{n+1}(x_0) \\ &= \lim_{n\to\infty} F^{n+1}(x_0) \\ &= \lim_{n\to\infty} F^{n}(x_0) \\ &= \lim_{n\to\infty} x_n \\ &= x \end{aligned}$$

But if $F(x) = x$ then $x = x - \frac{f(x)}{f'(x)}$ so $f(x) = 0$.

So, if the sequence converges it reaches a fixed point of F, and any fixed point of F is a root of f. Also note that any root of f is a fixed point of F - thus the problem of finding fixed points of F is completely equivalent to finding roots of f.

Now suppose that there is indeed a root of f, call it t.

$$\begin{aligned} |x_n - t| &= |x_n - F(t)| \\ &= |F(x_{n-1}) - F(t)| \\ &= |x_{n-1} - t||F'(c)| \quad \text{some } c \in \mathbf{R} \end{aligned}$$

(Assuming that is that F is differentiable, and using the mean value theorem).

$$\begin{aligned} F'(x) &= 1 - \frac{f'(x)}{f'(x)} + \frac{f''(x)f(x)}{(f'(x))^2} \\ &= \frac{f''(x)f(x)}{(f'(x))^2} \end{aligned}$$

Lets assume that $|F'(x)|$ is bounded by some value K. This is fairly reasonable as it holds for polynomials that have non-zero derivatives.

We now have:

$$\begin{aligned}
|x_n - t| &= |x_{n-1} - t||F'(c)| \quad \text{some } c \in \mathbf{R} \\
&\leq |x_{n-1} - t|K \\
&\leq |x_0 - t|K^n
\end{aligned}$$

So if we now make yet another assumption, that K ¡ 1, we have $x_n \to t$. (Note that this will also work if we only require the condition $K < 1$ on some neighbourhood of t and choose x_0 sufficiently close to x).

We call maps such as F, such that F is lipschitz with a lipschitz constant of less than 1 contractions, and it will prove to be very useful in the study of fixed point problems. Fixed point problems will turn out to be equivalent to a lot of other problems in analysis, and thus rather useful things to be able to handle.

Let us do something similar to what we did for the newton Raphson method. Pick $x_0 \in \mathbf{R}$ and define $x_n = F^n(x_0)$. Now let $m, n \in \mathbf{N},\ m > n$. Let $m = n+k$.

$$\begin{aligned}
|x_n - x_m| &= |F^m(x_0) - F^n(x_0)| \\
&= |F^{n+k}(x_0) - F^n(x_0)| \\
&= |F^n(F^k(x_0)) - F^n(x_0)| \\
&\leq K^n|F^k(x_0) - x_0| \\
&= K^n|x_k - x_0| \\
&\leq K^n \sum_{i=0}^{k-1} |x_{i+1} - x_i| \\
&\leq K^n \sum_{i=0}^{k-1} K^i|x_1 - x_0| \\
&\leq K^n \sum_{i=0}^{\infty} K^i|x_1 - x_0| \\
&= \frac{K^n|x_1 - x_0|}{1 - k} \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

So, we can make the elements of x_n lie in an increasingly small area. Does this mean it converges? In fact it does:

First note that if there is a limit point of $\{x_n : n \in \mathbf{N}\}$ then it converges, for if x is such a limit point then we can choose N such that for every $m, n > N_1$ we have $|x_m - x_n| < \frac{\varepsilon}{2}$. We can also find $k > N$ such that $|x_k - x| < \frac{\varepsilon}{2}$, as infinitely many of the x_n lie in $B(x, \frac{\varepsilon}{2})$. Thus for all $n > N$, $|x_n - x| \leq |x_n - x_k| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So, the case where this set has a limit point is covered. Now assume that it doesn't have one:

Note that the set $Y = \{x_n : n \in \mathbf{N}\}$ must be bounded, so lies in some compact set X. It has no limit points, and thus vacuously contains all its limit points. Thus Y is closed, and so compact. But each point in it must be open, as if it had no neighbourhood disjoint from the rest of the set it would be a limit point of the set. Thus $\{\{x_n\} : n \in \mathbf{N}\}$ is an open cover of Y, so has a finite subcover. So Y must be a finite set. Now if we choose N so that for all $m, n > N$, $|x_m - x_n| < \min_{x,y \in Y \quad x \neq y} |x - y|$ we must have x_n taking only one value after N, so it is certainly convergent.

So, we now know that the sequence x_n converges to a limit x, and as previously noted $F(x) = x$ (i.e. x is a fixed point of F). In other words, every contraction mapping has a fixed point.

Further, this point is unique, for if we have another fixed point $y \neq x$ then $|x - y| = |F(x) - F(y)| \leq K|x - y| < |x - y|$, which is a contradiction.

Given the above, we can generalise this without any difficulty to more general metric spaces:

Let (X, d) be a metric space.

Definition 9.2 *Let $(x_n)_{n \geq 0}$ be a sequence taking values in X. It is said to be cauchy if $\forall \varepsilon > 0\ \exists N \in \boldsymbol{N}\ \forall m, n > N\ d(x_m, x_n) < \varepsilon$.*

A metric space X is said to be complete if every cauchy sequence converges. (We will provide a revised definition of this in the next chapter, but it will prove to be equivalent).

Definition 9.3 *$F : X \to X$ is said to be a contraction if $\exists K < 1 \quad \forall x, y \in X \quad d(F(x), F(y)) \leq K d(x, y)$.*

We can then restate our previous results as:

Theorem 9.2 *$\boldsymbol{R}$ is complete (as a metric space).*

Theorem 9.3 *(Contraction Mapping Theorem): Let X be a complete metric space, and $F : X \to X$ a contraction. F has a unique fixed point.*

The following version will also sometimes be useful:

Corollary 7 *Let X be a complete metric space and $F : X \to X$ with F^n a contraction for some n. F has a unique fixed point.*

Proof:

F^n is a contraction, so has a unique fixed point x.

$$\begin{aligned} F^n(F(x)) &= F^{n+1}(x) \\ &= F(F^n(x)) \\ &= F(x) \end{aligned}$$

So $F(x)$ is a fixed point of F^n, and thus by uniqueness $F(x) = x$. Further if y is a fixed point of F, then $F^n(y) = y$, so $y = x$.

QED

Note that, as well as giving us a proof of the existence of a solution, the contraction mapping theorem also gives us a way to find arbitrarily good approximations to the solution, which is all we really need for numerical work.

Chapter 10

Notions of Convergence

We have so far encountered two notions of convergence. Firstly the notion of a limit of a function as it approaches a point, and secondly the notion of the limit of a sequence.

From the first we shall extract a more general notion of convergence - that of a filter. We shall then relate this to the convergence of sequences, and go on to another notion of convergence from that: That of a net. There will be a sort of duality between these two concepts which will make them essentially equivalent, but they will prove to be more useful in different situations.

10.1 Filters

First we shall try to generalise the notion of the convergence of functions.

Let X be a topological space, with $x \in X$. Let $N_x = \{A \subseteq X \ : \ \exists U$ open, with $U \subseteq A \ \}$, the set of neighbourhoods of x. Recall the following:

1. $X \in N_x$
2. If $A \in N_x$ and $A \subseteq B$ then $B \in N_x$.
3. If $A, B \in N_x$ then $A \cap B \in N_x$.

Further recall that U is open iff it is a neighbourhood of each of its points. i.e. iff $\forall x \in U, \ \exists A \in N_x, A \subseteq U$.

Proposition 10.1 *Let X be a set. Suppose for each $x \in X$ we have a collection of sets K_x satisfying the above three conditions, and further that $\forall A \in K_x \ x \in A$. There is a unique topology T on X such that for every x, $K_x = N_x$ in this topology.*

Proof:

Uniqueness is obvious from the above observation - the neighbourhoods of points uniquely determine the topology.

Suppose we have such K_x. Define $T = \{U \subseteq X \ : \ \forall x \in U \ \exists A \in K_x \ K_x \subseteq U\}$.

It is easy to verify that T is a topology, and that $K_x = N_x$ in this topology.

QED

This suggests that such objects are useful to study.

Definition 10.1 *Let X be a set, and F a collection of subsets of X. We say F is a filter if:*

1. $X \in F$
2. $\emptyset \notin F$
3. *If $A \in F$ and $A \subseteq B$ then $B \in F$*
4. *If $A, B \in F$ then $A \cap B \in F$*

N_x is then called the neighbourhood filter of x.

Like with topologies we will often want to consider sets generating a filter.

Proposition 10.2 *Let K be a collection of subsets of X with $\forall n, \ \forall A_1, \ldots, A_n \in K \ \bigcap_{i=1}^n A_i \neq \emptyset$. There is a unique filter F such that $K \subseteq F$ and for any filter G with $K \subseteq G$ we have $F \subseteq G$. Further, if $\forall n, \ \forall A_1, \ldots, A_n \in K \ \bigcap_{i=1}^n A_i \in K$ then $F = \{U \subseteq X \ : \ \exists A \in K \ : A \subseteq U\}$*

Proof:

$F = \{U \subseteq X \ : \ \exists A_1, \ldots, A_n \in K \ : \bigcap_{i=1}^n A_i \subseteq U\}$ can easily be verified to be such a filter.

QED

Definition 10.2 *In the above lemma, K is said to be a sub-base for F, or a base if $\forall n, \ \forall A_1, \ldots, A_n \in K \ \bigcap_{i=1}^n A_i \in K$. A base for N_x is said to be a neighbourhood base for x.*

Further recall that if $f : X \to Y$ we say $f(t) \to y$ as $t \to x$ iff for every $V \in N_y$ there is a $U \in N_x$ with $f(U) \subseteq f(V)$. $f(U) \cap f(V) = f(U \cap V)$, so $\{f(U) \ : \ U \in N_x\}$ forms a filter base for some filter, which we will denote $f(N_x)$. $f(N_x) = \{V \subseteq Y \ : \exists U \in N_x \ f(U) \subseteq V\}$. So the condition for $f(t) \to y$ as $t \to x$ becomes $N_y \subseteq f(N_x)$.

This motivates the following definition:

Definition 10.3 *Let F be a filter on X and $f : X \to Y$. The filter generated by the base $\{f(U) : U \in F\}$ is denoted $f(F)$.*

Definition 10.4 *Let F be a filter on a topological space X and $x \in X$. We say F converges to x, written $F \to x$, if $N_x \subseteq F$.*

Definition 10.5 *If F, G are filters with $F \subseteq G$ we say that G extends F.*

Thus $f(t) \to y$ as $t \to x$ is the same as saying $f(N_x) \to y$.

In particular we have:

Proposition 10.3 *Let X, Y be topological spaces. $f : X \to Y$ is continuous iff whenever $F \to x$ then $f(F) \to f(x)$.*

Proof:

It is trivial that if $G \to y$ and $G \subseteq H$ then $H\ toy$. We thus have from the above that f is continuous at x iff whenever $F \to x$ $f(F) \to f(x)$. f is continuous iff it is continuous at every x, and thus the result follows.

QED

Various concepts we've met so far have natural representations in terms of a filter:

Theorem 10.1 *Let X be a topological space. It is hausdorff iff filters have unique limits. i.e. iff whenever $F \to x$ and $F \to y$ then $x = y$.*

Proof:

Suppose X is hausdorff. Let $x \neq y$. There are elements $U \in N_x$, $V \in N_y$ such that $U \cap V = \emptyset$. Thus we cannot have both U and V in a filter F, as then $\emptyset \in F$. So we cannot have $F \to x$ and $F \to y$.

Conversely, suppose X is not hausdorff. Then there exist x, y with $x \neq y$ such that for any $U \in N_x$, $V \in N_y$ we have $U \cap V \neq \emptyset$. This implies that $N_x \cup N_y$ is a sub-base for some filter F. Then $F \to x$ and $F \to y$.

QED

So a space being hausdorff is precisely the requirement that limits are unique.

Compactness too takes a nice form:

Theorem 10.2 *Let X be a topological space. X is compact iff every filter can be extended to a convergent filter.*

Proof:

Suppose X is compact. Let F be a filter. $\{\overline{U} : U \in F\}$ is a collection of closed sets with the finite intersection property. Thus it has non-empty intersection. Let x be in the intersection.

For each neighbourhood V of x and every $U \in F$, we must have $U \cap V \neq \emptyset$, as $x \in \overline{U}$. Thus $N_x \cup F$ forms a sub-base for some filter G, with $F \subseteq G$ and $G \to x$.

Conversely suppose every filter F can be extended to a convergent filter. Let K be a collection of sets with the finite intersection property. Then K forms a sub-base for some filter F. F can be extended to a convergent filter $G \supseteq N_x$ for some x. Thus we must have for every $A \in K$ and every $U \in N_x$, $U \cap A \neq \emptyset$. Thus for every $A \in K$ we must have $x \in A$, as A was closed, and so $x \in \bigcap K$, and in particular $\bigcap K \neq \emptyset$. Thus X is compact.

QED

With a function defined on a set A and x a limit point of A, we can talk about the limit as $t \to x$ of $f(t)$. We would like to be able to do the same with filters. We first prove the following proposition:

Proposition 10.4 *Define $N_{x,A} = \{U \cap A : U \text{ is a neighbourhood of } x\}$. If $x \in \overline{A}$ then this is a filter.*

Let $f : A \to Y$. $f(t) \to y$ as $t \to x$ iff $f(N_{x,A}) \to y$.

The proof is left as an exercise.

Given a filter F on A, it seems reasonable to say $A \to x$ with $x \in X$ iff $i(F) \to x$, where $i : A \to X$ is the inclusion mapping. This is equivalent to requiring that $N_{x,A} \subseteq F$.

We have the following easy theorem:

Theorem 10.3 *Let $A \subseteq X$, $x \in X$. $x \in \bar{(}A)$ iff x is the limit of a filter in A.*

Proof:

If $x \notin \overline{A}$ then $\emptyset \in N_{x,A}$, and so no filter in A can converge to x.

Conversely, if $x \in \overline{A}$ then $N_{x,A}$ is a filter in A converging to x.

QED

10.2 Nets

We wish to generalise the idea of a sequence. Lets look at what we have currently. We have the natural numbers $\mathbb{N}$ with an order $\geq$ and a topological space

X. A sequence is then a function $x : \mathbb{N} \to X$ (where we denote $x(n)$ by x_n. We say $x_n \to x$ if for every neighbourhood U of x, x_α is eventually in U. i.e. there exists N such that $\forall n \geq N$ $x_n \in U$.

The most obvious first step to generalising this would be to replace $\geq$ and $\mathbb{N}$ with an arbitrary partially ordered set. Unfortunately this doesn't quite work. The following example shows why:

Consider the set $X = \mathbb{N} \times \{0, 1\}$ with the relation $(n, x) \succ (m, y)$ iff $n \geq m$ and $x = y$. Define a function $t : X \to \mathbb{R}$ by $t(n, x) = x$. Then according to the above definition we have $t \to 0$ and $t \to 1$. This should strike us as bad, seeing as $\mathbb{R}$ is hausdorff.

One obvious way to fix the above problem would be to have every two elements comparable - only considering total orders. This is overly restrictive, and proves not to be strong enough in an arbitrary topological space (not that this really concerns us - it's perfectly good in a metric space: The main objection is that it's too restrictive to be substantially more useful than a sequence). It proves to be enough that given any two elements x, y there is a third element z such that $z \succ x, y$. This ensures that the above situation cannot occur.

We also drop the requirement that if $x \succ y$ and $y \succ x$ then $x = y$. It's essentially unnecesary, and will occasionally be convenient not to have it.

Definition 10.6 *A directed set is a set A with a relation $\succ$ such that:*

1. $x \succ x$
2. *If $x \succ y$ and $y \succ z$ then $x \succ z$*
3. *For and x, y there is a z such that $z \succ x$ and $z \succ y$.*

A function who's domain is a directed set is called a net. We will usually use sequence notation for it; i.e. if $x : A \to Y$ is a net, then we shall write $x(\alpha) = x_\alpha$.

Let $x : A \to Y$ with Y a topological space and $y \in Y$. We say $x_\alpha \to y$ if for every neighbourhood U of y there is some $\beta \in A$ such that $\forall \alpha \succ \beta$ $x_\alpha \in U$.

We could go through with this definition and prove a lot of results about nets similar to those about filters, but the following will be more useful:

Definition 10.7 *Let A be a directed set and $x : A \to Y$ a net. The* Frechet Filter *is the set $F_x = \{U \subseteq Y \ : \ x$ is eventually in $U\}$.*

Theorem 10.4 *Let $x : A \to Y$ be a net. The Frechet Filter, F_x, defined above is a filter and given any topology on Y, $F_x \to y$ iff $x_\alpha \to y$. Further let $f : Y \to Z$. $F_{f \cdot x} = f(F_x)$.*

Proof:

The only part of verifying that F_x is a filter that is at all non-obvious is showing it is closed under pairwise intersection.

Let $U, V \in F_x$. There exists x, y such that for $\alpha \succ x$ we have $x_\alpha \in U$ and for $\alpha \succ y$, $x_\alpha \in V$. As A is a directed set there is some z such that $z \succ y$ and $z \succ x$. Thus for $\alpha \succ z$, $x_\alpha \in U$ and $x_\alpha \in V$, so in $U \cap V$. Thus $U \cap V \in F_x$.

Now let Y have a topology. Suppose $F_x \to y$. Then $N_y \subseteq F_x$, so for every neighbourhood U of y, x_α is eventually in U, as $U \in F_x$. Conversely if $x_\alpha \to y$ then let U be a neighbourhood of y. x_α is eventually in it, so $U \in F_x$. Thus $N_y \subseteq F_x$ and $F_x \to y$.

Let $f : Y \to Z$.

$$\begin{aligned}
F_{f \cdot x} &= \{U \ : \ f(x_\alpha) \text{ is eventually in } U\} \\
&= \{U \ : \ x_\alpha \text{ is eventually in } f^{-1}(U)\} \\
&= \{U \ : \ f^{-1}(U) \in F_x\} \\
&= f(F_x)
\end{aligned}$$

QED

It is clear that distinct nets can give the same filter - e.g. just by relabelling A. However every filter does arise as the Frechet Filter of some net.

Proposition 10.5 *Let G be a filter on Y. There is a net $x : D \to Y$ such that $G = F_x$.*

Proof:

Let $D_x = \{(U, x) \ : \ U \in F, \ x \in U\}$, ordered by $(U, x) \succ (V, y)$ iff $U \subseteq V$. (Note that this is an example of where we don't want to require antisymmetry). This is clearly a directed set, as F was a filter.

Define $t_x : D \to Y$ by $t_x(U, s) = s$.

Trivially $G \subseteq F_{t_x}$, as if $U \in G$ then pick $x \in U$. For any $\alpha = (V, y) \succ (U, x)$ we have $t_\alpha = y \in V \subseteq U$. So t_α is eventually in U. Suppose t is eventually in U. There exists $V \in G$ such that for any $W \subseteq V$, $W \subseteq U$. i.e. $V \subseteq U$, and so $U \in G$.

Thus $G = F_{t_x}$.

QED

The above two theorems give that essentially any theorem that holds for filters holds for nets, and vice versa. So we have the following theorems essentially for free.

So we have the following:

Theorem 10.5 *1. $f(t) \to y$ as $t \to x$ iff for any net with $x_\alpha \to x$ we have $f(x_\alpha) \to f(x)$*

2. f is continuous iff whenever $x_\alpha \to x$ then $f_\alpha \to f(x)$.

3. X is hausdorff iff every net has at most one limit.

These are immediate consequences of the corresponding theorems for filters.

There is another way to set up this equivalence.

Definition 10.8 *Let A be a directed set. A tail of A is a set of the form $T_\beta = \{\alpha : \alpha \succ \beta\}$. A tail of a net $x : A \to Y$ is the image of a tail of A, and is denoted by T^x_β*

Proposition 10.6 *Let $x : A \to X$ be a net. $\{T^x_\beta : \beta \in A\}$ forms a base for F_x.*

We also had theorems involving the extension of filters. i.e. when we have two filters F, G with $F \subseteq G$. What is the corresponding concept for net?

Lets look at t_F and t_G as defined above.

(Fix this later)

10.3 Completeness

For sequences we had the notion of a cauchy sequence, and defined a metric space to be complete if every cauchy sequence converged. We can do the same with both of the above notions of convergence. (We will define it so that the concept of being cauchy is the same in all the different methods, with the above correspondence). For nets we define it exactly as for sequences.

Definition 10.9 *Let x_α be a net in a metric space X. x_α is said to be cauchy if $\forall \epsilon > 0\ \exists \gamma\ \forall \alpha, \beta > \gamma\ d(x_\alpha, x_\beta) < \epsilon$.*

Proposition 10.7 *x_α is cauchy if and only if:*

1. $\forall \epsilon > 0\ \exists \alpha \quad diam(T^x_\alpha) < \epsilon$
2. $\forall \epsilon > 0\ \exists U \in F_x \quad diam(U) < \epsilon$

This motivates the definition:

Definition 10.10 *A filter F is said to be cauchy if for every $\epsilon > 0$ there exists $U \in F$ such that $diam(U) < \epsilon$.*

Proposition 10.8
1. *Let K be a filter base for a filter F. F is cauchy iff $\forall \epsilon > 0,\ \exists U \in K,\ diam(U) < \epsilon$.*
2. *Let x_α be a net. x_α is cauchy iff F_x is.*

Proof:

1. *If K contains arbitrarily small sets then so does F, as $K \subseteq F$. Thus the implication from left to right is immediate.*

   Conversely, suppose F is cauchy. Then $\forall \epsilon > 0\ \exists U \in F,\ diam(U) < \epsilon$. K is a filter base, so $\forall U \in F\ \ \exists V \in K\ \ V \subseteq U$. Hence $\exists V \in K\ \ diam(V) < diam(U)$. Thus $\forall \epsilon > 0\ \exists\ V \in K\ \ diam(V) < \epsilon$.

2. *We know that x_α is cauchy iff $\forall \epsilon > 0\ \exists \alpha\quad diam(T^x_\alpha) < \epsilon$. The set $K = \{\ T^x_\alpha\ \}$ is a base for F_x. Hence by the previous part the result is proved.*

The filter definition proves to often be the easiest to work with, so we shall often use it for proving our theorems.

Proposition 10.9 *Every convergent filter or net is cauchy.*

Proof:

Let F be a convergent filter, say $F \to x$. Fix $\epsilon > 0$. Then $B(x, \frac{\epsilon}{2}) \in F$ and $\mathrm{diam}(B(x, \frac{\epsilon}{2})) < \epsilon)$. So F is cauchy.

The result

Theorem 10.6 *Let X be a metric space. The following are equivalent:*

1. *Every cauchy sequence converges.*
2. *Every cauchy filter converges.*
3. *Every cauchy net converges.*

Proof:

The discussion so far shows that 2 and 3 are equivalent. A sequence is a particular type of net, so if every cauchy net converges then every cauchy sequence converges. Thus (3) $\implies$ (1).

Suppose every cauchy sequence converges. Let F be a cauchy filter. For every $\epsilon > 0$ there is a set in F of diameter less than 2^{-n}. Pick a sequence of such sets

U_n. We may assume the U_n are closed, as diam(X) =diam$(\overline{X})$, and that for $m \geq n$ we have $U_m \supseteq U_n$. Pick a sequence $x_n \in U_n$. Then for $m > n$, $x_m \in U_n$, so $d(x_n, x_m) < 2^{-n}$. Thus given $\epsilon > 0$, by making n large enough we can pick N such that for all $m, n > N$ we have $d(x_n, x_m) < 2^{-N} < \epsilon$. So x_n is cauchy, and thus convergent, say to x. Then the U_n were closed, so $\forall n \; x \in U_n$. In particular $U_n \subseteq B(x, 2^{-n})$. Let U be a neighbourhood of x. There exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Pick n such that $2^{-}n < \epsilon$, and then we have $U_n \subseteq U$, so $U \in F$. Hence $N_x \subseteq F$, so $F \to x$. Thus we have (1) $\implies$ (2) and the theorem is proved.

QED

Proposition 10.10 *Let F be a cauchy filter and $F \subseteq G$, with $G \to x$. Then $F \to x$.*

Proof:

Pick $U \in F$ with diam$(U) < \epsilon$. G is a filter and $U \in F$, so for every neighbourhood V of x, $V \in G$ and so $V \cap U \neq \emptyset$. Thus x is a limit point of U, and so $\forall y \in U \; d(x,y) \leq diam(U) < \epsilon$. Thus $U \subseteq B(x, \epsilon)$, and so $B(x, \epsilon \in F$. These sets generate N_x, so $N_x \subseteq F$.

QED

Corollary 8 *Let X be a compact metric space. X is complete.*

Theorem 10.7 *Let X be a metric space and $Y \subseteq X$. If Y is complete then it is closed. If X is complete and Y is closed then Y is complete.*

Proof:

If Y is not closed then let x be a limit point of Y such that $x \notin Y$. Then $N_{x,\ A}$ is a cauchy filter in Y which does not converge (by uniqueness of limits in $Y \cup \{x\}$)

Conversely, suppose Y is closed and X is complete. Let F be a cauchy filter on Y. This generates a cauchy filter G on X. G converges, say to x, as X is complete. As G was the filter on X generated by F, this means that $F \to x$. But Y is closed, so $x \in Y$. Thus $F \to x \in y$.

QED

Chapter 11

Integration and Differential Equations

In physics and various areas of maths, we often want to describe a problem in terms of its local behaviour - e.g. in terms of a differential equation - and get a general solution from this.

For example, there is the simple harmonic oscillator. Solutions to a differential equation of the form

$$\frac{d^2 f}{dx^2} + \omega^2 f = 0$$

. This crops up a lot, in all sorts of areas.

At the moment it is far from obvious that these equations have any solutions at all, or when they do to what extent the solution is unique. In this chapter we shall attempt to answer the question of existence and uniqueness for a certain class of differential equation.

11.1 The Gauge Integral

We will first look at the most basic type of differential equation:

Let $g : (a, b) \to \mathbb{R}$. We want $f : [a, b] \to \mathbb{R}$ with f continuous on $[a, b]$ and differentiable on (a, b) with $\frac{df}{dx} = g(x)$. We call f a *primitive*, or antiderivative, for g.[1]

Note that this is not possible for every g. e.g. consider $g : (-1, 1) \to \mathbb{R}$ with $g(0) = 1$, $g(x) = 0$ for $x \neq 0$. This cannot be the derivative of any function.

Suppose f satisfies the above. It is clear that for any real number c, $f + c$ also

[1] It is more normal to consider $g : [a, b] \to \mathbb{R}$, and we will do that later. For now it is more convenient to look at it this way.

satisfies it. Further, if h satisfies it, then $\frac{d}{dx}(f-h) = \frac{df}{dx} - \frac{dh}{dx} = g(x) - g(x) = 0$, so $f - h$ is constant. Thus if there is a solution f, the set of solutions is $\{f + c \ : \ c \in \mathbb{R}\}$.

Suppose f was a solution. Fix a and b with $a < b$ and without loss of generality let $f(a) = 0$ (we may do this by adding $-f(a)$, as this still gives a solution).

We know from the mean value theorem that there is some c in (a, b) with $f(b) = (b-a)g(c)$. Thus without knowing what c is, for any $x \in (a, b)$ we may use $(b-a)g(x)$ as an approximation for $f(b)$ (albeit not a very good one).

If we add in a point y with $a < x_0 < b$, we have $f(b) = (b-y)f(c) + (y-a)f(d)$, with $c \in (y, b)$ and $d \in (a, y)$. Again, letting c and d range freely over their intervals this should provide an approximation to $f(b)$, as long as g does not vary too much over the interval. We can continue this process.

Definition 11.1 *A tagged partition of $[a, b]$ is a pair of finite sequences $x_0, \ldots, x_n, v_1, \ldots, v_n$ with $a = x_0 < x_1 < \ldots < x_n = b$ and $\forall i \ x_{i-1} \leq v_i \leq x_i$.*

For a function $f : [a, b] \to \mathbb{R}$ and a tagged partition π of $[a, b]$, we define the riemann sum to be $\sum_\pi f = \sum_{i=1}^n (x_i - x_{i-1}) f(v_i)$

(Note that this doesn't *exactly* correspond to what we've just discussed - it is often convenient to allow v_i to lie on the endpoints of the interval, so we have extended the definition slightly)

For 'nice' g, the riemann sums should provide an arbitrarily good approximation to $f(b)$ [2]. i.e. we want the riemann sums to converge to $f(b)$.

Definition 11.2 *Let (x_i, v_i) and (y_i, u_i) be tagged partitions of $[a, b]$. Say $(x_i, v_i) \succ (y_i, u_i)$ if $\{y_i\} \subseteq \{x_i\}$. Trivially, this makes the set of all tagged partitions of $[a, b]$ into a directed set.*

Given a function $f : [a, b] \to \mathbb{R}$ and a tagged partition π of $[a, b]$, then $\pi \to \sigma_\pi f$ is a net. If this net converges we call the limit the Riemann integral of f, and denote it by $\int_a^b f$, or $R \int_a^b f$ if we want to be clear what type of integral we're talking about (as we'll meet two other formulations later).

This proves not to be good enough. For example if g is unbounded. Consider $g : (0, 1) \to \mathbb{R}$, $g(x) = x^{-\frac{1}{2}}$. g has a primitive - e.g. $f(x) == x^{\frac{1}{2}}$. However it is not riemann integrable, because we can bring v_1 arbitrarily close to 0, no matter how small δ is, making the riemann sums arbitrarily large. We need to find some way of restricting the intervals so that when $g(v_i)$ is large, $x_i - x_{i-1}$ is small. We shall do this as follows:

Definition 11.3 *Let $\delta : [a, b] \to \mathbb{R}$ be a strictly positive function (we will often call such a Gauge). We say a tagged partition x_i, v_i is* δ-fine *if $x_i - x_{i-1} < \delta(v_i)$.*

[2] note that by restricting the problem to $[a, x]$ this will allow us to work out $f(x)$, so the problem loses no generality by only considering $f(b)$

Before we proceed any further we will need to prove the following:

Lemma 11.1 ***Cousin's Lemma***

Let $\delta : [a, b] \to (0, \infty)$*. There exists a* δ*-fine partition of* $[a, b]$*.*

Proof:

Let $x = \sup\{y \ : \ \exists \text{ a } \delta\text{-fine partition of } [a, y] \ \}$.

(This set is non-empty, as there is of course a δ-fine partition of $[a, a]$).

Claim: There exists a δ-fine partition of $[a, x]$.

Chapter 12

Infinite Sums and Power Series

Let $f : \mathbb{R} \to \mathbb{R}$. Suppose we know that f is differentiable at x. This tells us about the local behaviour of f near x. If we know that f is differentiable in some neighbourhood of x, we can ask about the local behaviour of f' near x to get more information about f. Lets for now assume that f is in fact infinitely differentiable, and we get the sequence of numbers $f(x),\ f'(x),\ f''(x),\ \dots\ f^{(r)},\ \dots$. This should in principle tell us a great deal about the local behaviour of f. From this, how much can we deduce about the behaviour of f globally?

In general, the answer turns out to be 'not much'. Consider the following function:

$$f(x) = e^{-\frac{1}{x^2}} \quad x \neq 0$$
$$f(0) = 0$$

An easy induction shows that f is infinitely differentiable, with derivative

$$f^{(r)}(x) = p_r(x^{-1})f(x) \quad x \neq 0$$
$$f^{(r)}(0) = 0$$

For some polynomials p_r.

So the function f has all its derivatives equal to 0 at $x = 0$, but is not constant.

However, it turns out that for a surprising number of functions we can extract a good deal of information about f from its derivatives - indeed for a certain class of functions, the global behaviour is determined uniquely by the derivatives at one point.

Let's start with looking at polynomials around $x = 0$.

First note that

$$\begin{aligned} \frac{d^n}{dx^n} x^m_{|x=0} &= 0 \quad m \neq n \\ &= m! \quad m = n \end{aligned}$$

Let $f(x) = \sum_{i=0}^n a_i x^i$. Then for $r \leq n$, we have from the above that $f^{(r)}(x) = r! a_r$.

Hence $f(x) = \sum_{r=0}^n f^{(r)}(0) \frac{x^r}{r!}$

When f is infinitely differentiable but not a polynomial, we can still form these polynomials. Define $S_n(x) = \sum_{r=0}^n f^{(r)}(a) \frac{x^r}{r!}$. What do these polynomials tell us about f?

Lemma 12.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be $n+1$ times differentiable.*

$$f(x) = \sum_{r=0}^n f^{(r)}(a) \frac{(x-a)^r}{r!} + \frac{1}{n!} \int_a^x (x-t)^n$$