# Some Lemmas

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The following lemma comes up when defining topological entropy.

**Lemma 1.**

*Let $a_n$ be a sequence of non-negative real numbers such that*

$$\forall m, n \; a_{m+n} \leq a_m + a_n$$

*Then*

$$\frac{a_n}{n} \to \inf_{n \geq 1} \frac{a_n}{n}$$

*Proof.* Let $I = \inf\limits_{n \geq 1} \frac{a_n}{n}$.

Note that $\frac{a_{kn}}{kn} \leq \frac{a_n}{n}$. So, suppose $\frac{a_n}{n} \leq I + \epsilon$. Then whenever $n|m$ we have $\frac{a_m}{m} \leq I + \epsilon$.

So, suppose $\epsilon > 0$. We can find $N$ such that $\frac{a_N}{N} \leq I + \frac{1}{2}\epsilon$, and thus for all $n$ such that $N|n$ we have $\frac{a_n}{n} < I + \frac{1}{2}\epsilon$.

Let $n \geq N$. We may write $n = kN + r$, where $0 \leq r < N - 1$. So

$$\begin{aligned}
\frac{a_n}{n} &\leq \frac{a_{kN}}{n} + \frac{a_r}{n} \\
&\leq \frac{a_{kN}}{kN} + \frac{Na_1}{n} \\
&< \frac{1}{2}\epsilon + \frac{Na_1}{n}
\end{aligned}$$

Thus picking $N'$ big enough such that when $n \geq N'$ we have $\frac{Na_1}{n} < \frac{1}{2}\epsilon$, whenever $n \geq \max\{N, N'\}$, we have $I \leq \frac{a_n}{n} \leq I + \epsilon$ and so $|I - \frac{a_n}{n}| < \epsilon$.

Hence $\frac{a_n}{n} \to I$.

□

The following is a slight generalisation of the well known Cesaro summation method. The normal version is simply the case $b_n = n$.

**Lemma 2 (Cesaro's Lemma).** *Let $b_n$ be a monotonic increasing sequence of positive real numbers such that $b_n \to \infty$. Let $x_n \to x$. Then,*

$$\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \to x \quad \textit{as } n \to \infty$$

*Proof.* Let $\epsilon > 0$. Pick $N$ such that whenever $n \geq N$ we have $x_n \geq x - \epsilon$.

So for $n \geq N$ we have

$$\begin{aligned}
\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k &= \frac{1}{b_n}\sum_{k=1}^{N}(b_k - b_{k-1})x_k + \frac{1}{b_n}\sum_{k=N+1}^{n}(b_k - b_{k-1})x_k \\
&\geq \frac{1}{b_n}\sum_{k=1}^{N}(b_k - b_{k-1})x_k + \frac{1}{b_n}\sum_{k=N+1}^{n}(b_k - b_{k-1})(x - \epsilon) \\
&= \frac{1}{b_n}\sum_{k=1}^{N}(b_k - b_{k-1})x_k + \frac{1}{b_n}(b_n - b_N)(x - \epsilon)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\liminf_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k &\geq \frac{1}{b_n}\sum_{k=1}^{N}(b_k - b_{k-1})x_k + \frac{1}{b_n}(b_n - b_N)(x - \epsilon) \\
&= 0 + x - \epsilon \\
&= x - \epsilon
\end{aligned}$$

So

$$\liminf_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \ \geq \ x$$

But similarly we have

$$\limsup_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \ \leq \ x$$

So

$$x \leq \liminf_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \ \leq \limsup_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \ \leq \ x$$

Thus

$$\liminf_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \ = \limsup_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k = x$$

So

$$\frac{1}{b_n}\sum_{k=1}^{n}(b_k - b_{k-1})x_k \to x$$

□

**Lemma 3 (The Dini Lemma).** *Let $X$ be a compact topological space and $f_n : X \to \mathbb{R}$ be a monotonic decreasing sequence of non-negative continuous functions such that $f_n \to f$ pointwise. Then*

$$\sup_{x\in X} f_n(x) \to \sup_{x\in X} f(x)$$

*(Note that continuity of $f$ is not assumed).*

*Proof.*

Note that

$$\sup_{x\in X} f_n(x)$$

is monotone decreasing and bounded below, so it converges. Say

$$\sup_{x\in X} f_n(x) \to M$$

Suppose $t < M$.

Let $L_n = \{x \in X \ : \ f_n(x) \geq t\}$.

Then $L_n$ is closed. It is non-empty because $\sup_{x\in X} f_n(x) \geq M > t$. Because the sequence $f_n$ is monotone decreasing we have $L_n$ is as well (with respect to $\subseteq$). Thus by compactness $\bigcap L_n \neq \emptyset$. Let $x \in \bigcap L_n$. Then $\forall n \ f_n(x) \geq t$. Thus $f_n(x) \geq t$.

Hence

$$\forall t < M, \ \sup_{x \in X} f(x) \geq t$$

Thus

$$\sup_{x \in X} f(x) \geq M$$

But

$$\forall x \ f(x) \leq f_n(x)$$

Hence

$$\sup_{x \in X} f(x) \leq \sup_{x \in X} f_n(x) \to M$$

Thus

$$\sup_{x \in X} f(x) = M$$

and so

$$\sup_{x \in X} f_n(x) \to \sup_{x \in X} f(x)$$

$\square$

**Corollary 3.1 (Dini's Theorem).** *Let $X$ be a compact topological space and $f_n : X \to \mathbb{R}$ be a monotonic decreasing sequence of non-negative continuous functions such that $f_n \to f$ pointwise with $f$ continuous. Then $f_n \to f$ uniformly.*

*Proof.*

Apply the lemma to $f_n - f$.

$\square$

**Lemma 4 (Abel's Theorem).** *If $\sum a_n = l$ then $\sum a_n t^n \to l$ as $t \to 1^-$.*

*Proof.* Define $x_n = \sum\limits_{k=0}^{n}$. Then, taking $x_{-1} = 0$, we have:

$$\begin{aligned}
f(t) &= \sum a_n t^n \\
&= \sum (x_n - x_{n-1}) t^n \\
&= \sum x_n t^n - \sum x_{n-1} t^n \\
&= (1-t) \sum x_n
\end{aligned}$$

Define $\alpha = \sum a_n$ and $r_n = \sum\limits_{k \geq n} a_n$.

Note that $(1-t)\sum t^n = 1$.

We have

$$\begin{aligned}
\alpha - f(t) &= (1-t) \sum (\alpha - x_n) t^n \\
&= (1-t) \sum r_n t^n \\
|\alpha - f(t)| &\leq (1-t) \sum_{n=0}^{N} |r_n| t^n + (1-t) \sum_{n=N+1} |r_n| t^n
\end{aligned}$$

Fix $\epsilon > 0$. Now choose $N$ such that $r_{N+1} < \frac{1}{2}\epsilon$. We thus have that

$$\begin{aligned}
|\alpha - f(t) &\leq (1-t) \sum_{n=0}^{N} |r_n| t^n + \frac{1}{2}\epsilon \\
&\leq (1-t)M + \frac{1}{2}\epsilon
\end{aligned}$$

Thus for $t > \frac{1}{2M}\epsilon$ we have $|\alpha - f(t)| < \epsilon$.

Hence $f(t) \to \alpha$ as desired.

$\square$

**Corollary 4.1.** *Let $f$ be an analytic function with a taylor series of about zero of $\sum a_n z^n$, with radius of convergence $R$. Suppose $|z| = R$ and $\sum a_n z^n$ converges. Then $\sum a_n z^n = f(z)$.*

*Proof.* Apply Abel's theorem and note that $\sum a_n z^n t^n = f(tz)$. By continuity $f(tz) \to f(z)$ as $t \to 1^-$. $\square$