Gaussian conditioning and LDL decomposition

September 11, 2022

1 Gaussian decomposition

2 Gaussian elimination

3 LDL decomposition

Suppose one of the diagonal block matrix is invertible. Here we assume the lower part of the block is diagonable. Then we can eliminate upper block by applying upper triangular matrix.

4 Conditioning Gaussian

In general, the joint distribution is splitted into two probabilities. One is conditional probability and the other is marginal distribution.

$$p(x_1, x_2) = p(x_1|x_2)p(x_2). \tag{1}$$

The joint distribution is written by Gaussian. We know the conditional and marginals are Gaussian as well. Then our goal is to find their means and covariances. Let those means and covariances $mu_{1|2}, \Sigma_{1|2}, mu_2, \Sigma_2$. The equation 1 reads

$$\exp\left[-\frac{1}{2}x^T\Sigma^{-1}x\right] = \exp\left[-\frac{1}{2}{x_{1|2}}^T{\Sigma_{1|2}}^{-1}x_{1|2} + \frac{1}{2}{x_2}^T{\Sigma_2}^{-1}x_2\right] \tag{2}$$

where we applied the exponential rules: $(\exp a)(\exp b) = \exp(a+b)$.

$$-\frac{1}{2}x^T\Sigma^{-1}x = -\frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3}$$

The goal is to decompose in a way that one term is pure in x_2, and the other is the linear mixture of x_1 and x_2.

$$\frac{1}{2}{x_\star}^T{\Sigma_\star}^{-1}x_\star + \frac{1}{2}{x_2}^T\Sigma^{-1}x_2 \tag{4}$$

Once this decomposition is achieved we have decomposition in probability: