Finding conditional distribution = completing squares

When we want to find conditional and mariginal distribution, we focus on the exponent and decompose it into two quadratics.

$$\frac{1}{2}{x_\star}^T {\Sigma_\star}^{-1} x_\star + \frac{1}{2}{x_2}^T \Sigma^{-1} x_2. \tag{1}$$

This decomposition is usually done by completeing squares. This method is 1. tiresome and moreover, requires finding inverse separately.

Finding conditional = Block diagonalization

$$x_\star{}^T \Sigma_\star{}^{-1} x_\star + \frac{1}{2} x_2{}^T \Sigma_2{}^{-1} x_2. \tag{2}$$

In matrix notation, we can write this with a block diagonal matrix:

$$\begin{bmatrix} x_\star{}^T \\ x_2{}^T \end{bmatrix} \begin{bmatrix} \Sigma_\star{}^{-1} & 0 \\ 0 & \Sigma_2{}^{-1} \end{bmatrix} \begin{bmatrix} x_\star \\ x_2 \end{bmatrix} \tag{3}$$

Since inverting the block diagonal matrix is reduced to the inverting the component matrices at the diagonal,

$$\begin{bmatrix} x_\star{}^T \\ x_2{}^T \end{bmatrix} \begin{bmatrix} \Sigma_\star & 0 \\ 0 & \Sigma_2 \end{bmatrix}^{-1} \begin{bmatrix} x_\star \\ x_2 \end{bmatrix}. \tag{4}$$

Thus our problem is reduced to finding the block diagonal matrix *that keeps second dimension identical*.

Keeping lower part identical = Gaussian elimination with upper triangular matrix

Suppose Σ_{22} is invertible. Then we can eliminate the upper off-diagonal blcok as

$$\begin{bmatrix} \mathbf{1}_n & -\Sigma_{12}{\Sigma_{22}}^{-1} \\ 0 & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}{\Sigma_{22}}^{-1}\Sigma_{21} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{5}$$