Ron Kenett, Shelemyahu Zacks, Peter Gedeck

Modern Statistics: A Computer Based Approach with Python

Solutions

April 6, 2022

Springer Nature

Contents

Chapter 1
Analyzing Variability: Descriptive Statistics

Import required modules and define required functions

```
import math
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import mistat
from scipy import stats

def trim_std(data, alpha):
  """ Calculate trimmed standard deviation """
  data = np.array(data)
  data.sort()
  n = len(data)
  low = int(n * alpha) + 1
  high = int(n * (1 - alpha))
  return data[low:(high + 1)].std()
```

Exercise 1.1 In the present problem we are required to generate at random 50 integers from the set $\{1, 2, 3, 4, 5, 6\}$. To do this we can use the `random.choices` method from the `random` package.

Use this method of simulation and count the number of times the different integers have been repeated. This counting can be done by using the Counter class from the `collections` package.

How many times you expect each integer to appear if the process generates the numbers at random?

Solution 1.1 `random.choices` selects k values from the list using sampling with replacement.

```
import random
random.seed(1)
values = random.choices([1, 2, 3, 4, 5, 6], k=50)
```

`Counter` counts the number of occurrences of a given value in a list.

```
from collections import Counter
Counter(values)
```

```
Counter({1: 9, 6: 9, 5: 8, 2: 10, 3: 10, 4: 4})
```

The expected frequency in each cell, under randomness is $50/6 = 8.3$. You will get different numerical results, due to randomness.

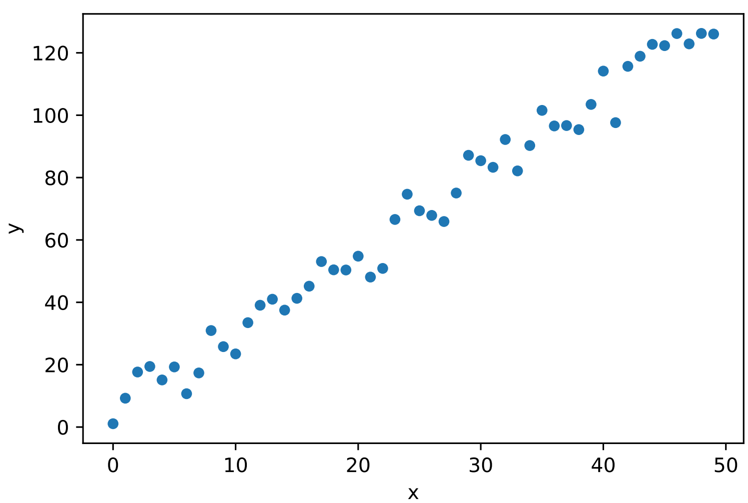
Exercise 1.2 Construct a sequence of 50 numbers having a linear trend for deterministic components with random deviations around it. This can be done by using these Python commands. We use Python's list comprehension to modify the elements of the list

```
random.seed(1)
x = list(range(50))
y = [5 + 2.5 * xi for xi in x]
y = [yi + random.uniform(-10, 10) for yi in y]
```

By plotting y versus x one sees the random variability around the linear trend.

Solution 1.2 The Python function `range` is an iterator. As we need a list of values, we need to explicitly convert it.

```
x = list(range(50))
y = [5 + 2.5 * xi for xi in x]
y = [yi + random.uniform(-10, 10) for yi in y]
pd.DataFrame({'x': x, 'y': y}).plot.scatter(x='x', y='y')
plt.show()
```



Exercise 1.3 Generate a sequence of 50 random binary numbers $(0, 1)$, when the likelihood of 1 is p using the command `binom.rvs(1, p, size=50)`.

Do this for the values $p = 0.1, 0.3, 0.7, 0.9$. Count the number of 1's in these random sequences, by summing the result sequence.

Solution 1.3 In Python

```
from scipy.stats import binom
np.random.seed(1)

for p in (0.1, 0.3, 0.7, 0.9):
  X = binom.rvs(1, p, size=50)
  print(p, sum(X))
```

```
0.1 4
0.3 12
0.7 33
0.9 43
```

Notice that the expected values of the sums are 5, 15, 35 and 45.

Exercise 1.4 The following are two sets of measurements of the weight of an object, which correspond to two different weighing instruments. The object has a true weight of 10 kg.
Instrument 1:

9.490950	10.436813	9.681357	10.996083	10.226101	10.253741
10.458926	9.247097	8.287045	10.145414	11.373981	10.144389
11.265351	7.956107	10.166610	10.800805	9.372905	10.199018
9.742579	10.428091				

Instrument 2:

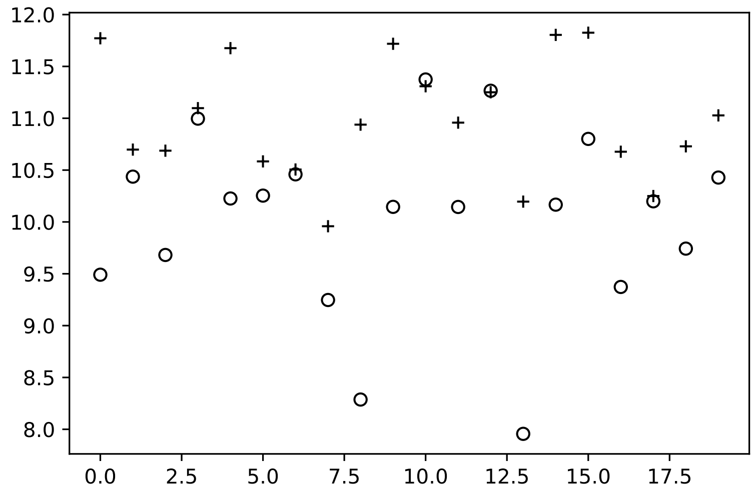
11.771486	10.697693	10.687212	11.097567	11.676099	10.583907
10.505690	9.958557	10.938350	11.718334	11.308556	10.957640
11.250546	10.195894	11.804038	11.825099	10.677206	10.249831
10.729174	11.027622				

Which instrument seems to be more accurate? Which instrument seems to be more precise?

Solution 1.4 We can plot the data and calculate mean and standard deviation.

```
inst1 = [9.490950, 10.436813, 9.681357, 10.996083, 10.226101, 10.253741,
         10.458926, 9.247097, 8.287045, 10.145414, 11.373981, 10.144389,
         11.265351, 7.956107, 10.166610, 10.800805, 9.372905, 10.199018,
         9.742579, 10.428091]
inst2 = [11.771486, 10.697693, 10.687212, 11.097567, 11.676099,
         10.583907, 10.505690, 9.958557, 10.938350, 11.718334,
         11.308556, 10.957640, 11.250546, 10.195894, 11.804038,
         11.825099, 10.677206, 10.249831, 10.729174, 11.027622]
ax = pd.Series(inst1).plot(marker='o', linestyle='none',
                           fillstyle='none', color='black')
pd.Series(inst2).plot(marker='+', linestyle='none', ax=ax,
                      fillstyle='none', color='black')
plt.show()

print('mean inst1', np.mean(inst1))
print('stdev inst1', np.std(inst1, ddof=1))
print('mean inst2', np.mean(inst2))
print('stdev inst2', np.std(inst2, ddof=1))
```

```
mean inst1 10.03366815
stdev inst1 0.8708144577963102
mean inst2 10.98302505
stdev inst2 0.5685555119253366
```



As shown in the following Figure, the measurements on Instrument 1, ○, seem to be accurate but less precise than those on Instrument 2, +. Instrument 2 seems to have an upward bias (inaccurate). Quantitatively, the mean of the measurements on Instrument 1 is $\bar{X}_1 = 10.034$ and its standard deviation is $S_1 = 0.871$. For Instrument 2 we have $\bar{X}_2 = 10.983$ and $S_2 = 0.569$.

Exercise 1.5 The quality control department of a candy factory uses a scale to verify compliance of the weight of packages. What could be the consequences of problems with the scale accuracy, precision and stability.

Solution 1.5 If the scale is inaccurate it will show on the average a deterministic component different than the nominal weight. If the scale is imprecise, different weight measurements will show a high degree of variability around the correct nominal weight. Problems with stability arise when the accuracy of the scale changes with time, and the scale should be recalibrated.

Exercise 1.6 Draw a random sample with replacement (RSWR) of size $n = 20$ from the set of integers $\{1, 2, \cdots, 100\}$.

Solution 1.6 The method `random.choices` creates a random selection with replacement. Note that in `range(start, end)` the `end` argument is excluded. We therefore need to set it to 101.

```
import random
random.choices(range(1, 101), k=20)
```

```
[6, 88, 57, 20, 51, 49, 36, 35, 54, 63, 62, 46, 3, 23, 18, 59, 87, 80,
80, 82]
```

Exercise 1.7 Draw a random sample without replacement (RSWOR) of size $n = 10$ from the set of integers $\{11, 12, \cdots, 30\}$.

Solution 1.7 The method `random.choices` creates a random selection without replacement.

```
import random
random.sample(range(11, 31), 10)
```

```
[19, 12, 13, 28, 11, 18, 26, 23, 15, 14]
```

Exercise 1.8 (i) How many words of 5 letters can be composed ($N = 26$, $n = 5$)?
(ii) How many words of 5 letters can be composed, if all letters are different?
(iii) How many words of 5 letters can be written if the first and the last letters are x?
(iv) An electronic signal is a binary sequence of 10 zeros or ones. How many different signals are available?
(v) How many electronic signals in a binary sequence of size 10 are there in which the number 1 appears exactly 5 times?

Solution 1.8 **(i)** $26^5 = 11,881,376$; **(ii)** $7,893,600$; **(iii)** $26^3 = 17,576$; **(iv)** $2^{10} = 1,024$; **(v)** $\binom{10}{5} = 252$.

Exercise 1.9 For each of the following variables state whether it is discrete or continuous:

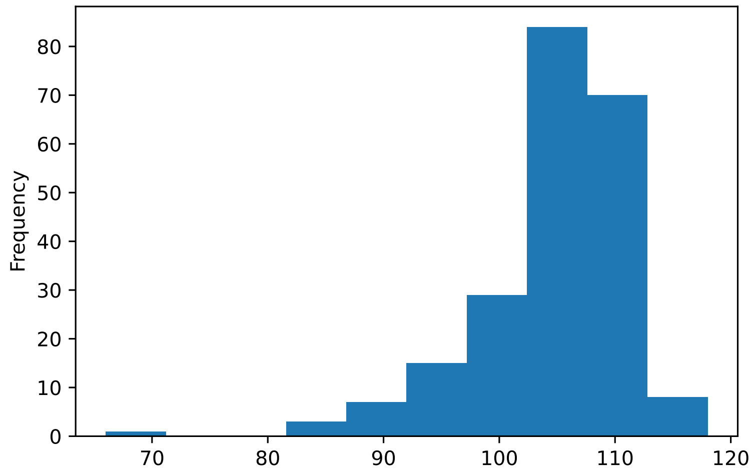
(i) The number of "heads" among the results of 10 flippings of a coin;
(ii) The number of blemishes on a ceramic plate;
(iii) The thickness of ceramic plates;
(iv) The weight of an object.

Solution 1.9 (i) discrete;
(ii) discrete;
(iii) continuous;
(iv) continuous.

Exercise 1.10 Data file **FILMSP.csv** contains data gathered from 217 rolls of film. The data consists of the film speed as measured in a special lab. Prepare a histogram of the data.

Solution 1.10

```
filmsp = mistat.load_data('FILMSP')
filmsp.plot.hist()
plt.show()
```



Exercise 1.11 Data file **COAL.csv** contains data on the number of yearly disasters in coal mines in England. Prepare a table of frequency distributions of the number of coalmine disasters.

Solution 1.11

```
coal = mistat.load_data('COAL')
pd.DataFrame(coal.value_counts(sort=False))
```

```
   COAL
3    17
6     3
4     6
0    33
5     5
2    18
7     1
1    28
```

Exercise 1.12 Data file **CAR.csv** contains information on 109 different car models. For each car there are values of five variables

1. Number of cylinders (4, 6, 8)
2. Origin (1, 2, 3)
3. Turn Diameter [m]
4. Horsepower [HP]
5. Number of miles/gallon in city driving [mpg].

Prepare frequency distributions of variables 1, 2, 3, 4, 5.

Solution 1.12 For (1) and (2), we can use the pandas `value_counts` method. e.g.:

```
car = mistat.load_data('CAR')
car['cyl'].value_counts(sort=False)
```

```
4    66
6    30
8    13
Name: cyl, dtype: int64
```

(i) Frequency distribution of number of cylinders:

Cyl	4	6	8	Total
Frequency	66	30	13	109

(ii) Frequency distribution of car's origin:

Origin	U.S.	Europe	Asia	Total
Frequency	58	14	37	109

For (3) to (5), we need to bin the data first. We can use the pandas `cut` method for this.

(iii) Frequency distribution of Turn diameter: We determine the frequency distribution on 8 intervals of length 2, from 28 to 44.

```
pd.cut(car['turn'], bins=range(28, 46, 2)).value_counts(sort=False)
```

```
(28, 30]     3
(30, 32]    16
(32, 34]    16
(34, 36]    26
(36, 38]    20
(38, 40]    18
(40, 42]     8
(42, 44]     2
Name: turn, dtype: int64
```

Note that the bin intervals are open on the left and closed on the right.

(iv) Frequency distribution of Horsepower:

```
pd.cut(car['hp'], bins=range(50, 275, 25)).value_counts(sort=False)
```

```
(50, 75]       7
(75, 100]     34
(100, 125]    22
(125, 150]    18
(150, 175]    17
(175, 200]     6
(200, 225]     4
(225, 250]     1
Name: hp, dtype: int64
```

(v) Frequency Distribution of MPG:

```
pd.cut(car['mpg'], bins=range(9, 38, 5)).value_counts(sort=False)
```

```
(9, 14]      1
(14, 19]    42
(19, 24]    41
(24, 29]    22
(29, 34]     3
Name: mpg, dtype: int64
```

Exercise 1.13 Compute the following five quantities for the data in file **FILMSP.csv**

(i) Sample minimum, $X_{(1)}$;
(ii) Sample first quartile, Q_1;
(iii) Sample median, M_e;
(iv) Sample third quartile, Q_3;
(v) Sample maximum, $X_{(217)}$.
(vi) The .8-quantile.
(vii) The .9-quantile.
(viii) The .99-quantile.

Show how you get these statistics by using the formulae. The order statistics of the sample can be obtained by first ordering the values of the sample.

Solution 1.13

```
filmsp = mistat.load_data('FILMSP')
filmsp = filmsp.sort_values(ignore_index=True)  # sort and reset index

print(filmsp.quantile(q=[0, 0.25, 0.5, 0.75, 1.0]))
print(filmsp.quantile(q=[0.8, 0.9, 0.99]))
```

```
0.00     66.0
0.25    102.0
0.50    105.0
0.75    109.0
1.00    118.0
Name: FILMSP, dtype: float64
0.80    109.8
0.90    111.0
0.99    114.0
Name: FILMSP, dtype: float64
```

Here is a solution that uses pure Python. Note that the `pandas` quantile implements different interpolation methods which will lead to differences for smaller datasets. We therefore recommend using the library method and select the method that is most suitable for your use case.

```
def calculate_quantile(x, q):
  idx = (len(x) - 1) * q
  left = math.floor(idx)
  right = math.ceil(idx)
  return 0.5 * (x[left] + x[right])

for q in (0, 0.25, 0.5, 0.75, 0.8, 0.9, 0.99, 1.0):
  print(q, calculate_quantile(filmsp, q))
```

```
0 66.0
0.25 102.0
0.5 105.0
0.75 109.0
0.8 109.5
0.9 111.0
0.99 114.0
1.0 118.0
```

Exercise 1.14 Compute with Python the indices of skewness and kurtosis of the **FILMSP.csv**, using the given formulas.

Interpret the skewness and kurtosis of this sample in terms of the shape of the distribution of film speed.

Solution 1.14

```
filmsp = mistat.load_data('FILMSP')
n = len(filmsp)
mean = filmsp.mean()
deviations = [film - mean for film in filmsp]
S = math.sqrt(sum(deviation**2 for deviation in deviations) / n)

skewness = sum(deviation**3 for deviation in deviations) / n / (S**3)
kurtosis = sum(deviation**4 for deviation in deviations) / n / (S**4)
print('Python:\n',
      f'Skewness: {skewness}, Kurtosis: {kurtosis}')

print('Pandas:\n',
      f'Skewness: {filmsp.skew()}, Kurtosis: {filmsp.kurtosis()}')
```

```
Python:
 Skewness: -1.8098727695275856, Kurtosis: 9.014427238360716
Pandas:
 Skewness: -1.8224949285588137, Kurtosis: 6.183511188870432
```

The distribution of film speed is negatively skewed and much steeper than the normal distribution. Note that the calculated values differ between the methods.

Exercise 1.15 Compare the means and standard deviations of the number of miles per gallon/city of cars by origin (1 = US; 2 = Europe; 3 = Asia) according to the data of file **CAR.csv**.

Solution 1.15 The pandas `groupby` method groups the data based on the value. We can then calculate individual statistics for each group.

```
car = mistat.load_data('CAR')
car['mpg'].groupby(by=car['origin']).mean()
car['mpg'].groupby(by=car['origin']).std()
# calculate both at the same time
print(car['mpg'].groupby(by=car['origin']).agg(['mean', 'std']))
```

```
             mean       std
origin
1       20.931034  3.597573
2       19.500000  2.623855
3       23.108108  4.280341
```

Exercise 1.16 Compute the coefficient of variation of the Turn Diameter of US made cars (Origin = 1) in file **CAR.csv**.

Solution 1.16 We first create a subset of the data frame that contains only US made cars and then calculate the statistics for this subset only.

```
car = mistat.load_data('CAR')
car_US = car[car['origin'] == 1]
gamma = car_US['turn'].std() / car_US['turn'].mean()
```

Coefficient of variation gamma = 0.084.

Exercise 1.17 Compare the mean $\bar{X}$ and the geometric mean G of the Turn Diameter of US made and Japanese cars in **CAR.csv**.

Solution 1.17

```
car = mistat.load_data('CAR')

car_US = car[car['origin'] == 1]
car_Asia = car[car['origin'] == 3]
print('US')
print('mean', car_US['turn'].mean())
print('geometric mean', stats.gmean(car_US['turn']))
print('Japanese')
print('mean', car_Asia['turn'].mean())
print('geometric mean', stats.gmean(car_Asia['turn']))
```

```
US
mean 37.203448275862065
geometric mean 37.06877691910792
Japanese
mean 33.04594594594595
geometric mean 32.97599107553825
```

We see that $\bar{X}$ is greater than G. The cars from Asia have smaller mean turn diameter.

Exercise 1.18 Compare the prediction proportions to the actual frequencies of the intervals

$$\bar{X} \pm kS, \quad k = 1, 2, 3$$

for the film speed data, given in **FILMSP.csv** file.

Solution 1.18

```
filmsp = mistat.load_data('FILMSP')

Xbar = filmsp.mean()
S = filmsp.std()
print(f'mean: {Xbar}, stddev: {S}')
expected = {1: 0.68, 2: 0.95, 3: 0.997}
for k in (1, 2, 3):
  left = Xbar - k * S
  right = Xbar + k * S
  proportion = sum(left < film < right for film in filmsp)
  print(f'X +/- {k}S: ',
        f'actual freq. {proportion}, ',
        f'pred. freq. {expected[k] * len(filmsp):.2f}')
```

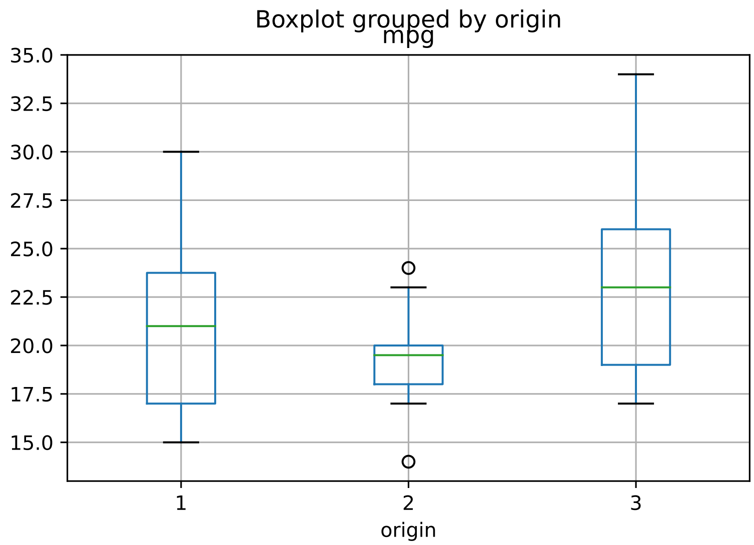
```
mean: 104.59447004608295, stddev: 6.547657682704987
X +/- 1S:  actual freq. 173,  pred. freq. 147.56
X +/- 2S:  actual freq. 205,  pred. freq. 206.15
X +/- 3S:  actual freq. 213,  pred. freq. 216.35
```

The discrepancies between the actual frequencies to the predicted frequencies are due to the fact that the distribution of film speed is neither symmetric nor bell-shaped.

Exercise 1.19 Present side by side the box plots of Miles per Gallon/City for cars by origin. Use data file **CAR.csv**.

Solution 1.19

```
car = mistat.load_data('CAR')
car.boxplot(column='mpg', by='origin')
plt.show()
```



Exercise 1.20 Prepare a stem-leaf diagram of the piston cycle time in file **OTURB.csv**. Compute the five summary statistics $(X_{(1)}, Q_1, M_e, Q_3, X_{(n)})$ from the stem-leaf.

Solution 1.20

```
oturb = mistat.load_data('OTURB')
mistat.stemLeafDiagram(oturb, 2, leafUnit=0.01)
```

```
   4 2 3444
  18 2 55555666677789
  40 3 0000001111112222333345
(15) 3 566677788899999
  45 4 00022334444
  34 4 566888999
  25 5 0112333
  18 5 6789
  14 6 01122233444
   3 6 788
```

- $X_{(1)} = 0.23$,
- $Q_1 = X_{(25.25)} = X_{(25)} + 0.25(X_{(26)} - X_{(25)}) = 0.31$,
- $M_3 = X_{(50.5)} = 0.385$,
- $Q_3 = X_{(75.75)} = 0.49 + 0.75(0.50 - 0.49) = 0.4975$,
- $X_{(n)} = 0.68$.

Exercise 1.21 Compute the trimmed mean $\bar{T}_{.10}$ and trimmed standard-deviation, $S_{.10}$ of the piston cycle time of file **OTURB.csv**.

Solution 1.21

```
from scipy.stats import trim_mean
```

```
oturb = mistat.load_data('OTURB')
print(f'T(0.1) = {trim_mean(oturb, 0.1)}')
print(f'S(0.1) = {trim_std(oturb, 0.1)}')
```

```
T(0.1) = 0.40558750000000005
S(0.1) = 0.09982289003530202
```

$\bar{T}_\alpha = 0.4056$ and $S_\alpha = 0.0998$, where $\alpha = 0.10$.

Exercise 1.22 The following data is the time (in sec.) to get from 0 to 60 mph for a sample of 15 German made cars and 20 Japanese cars

German made cars			Japanese made cars			
10.0	10.9	4.8	9.4	9.5	7.1	8.0
6.4	7.9	8.9	8.9	7.7	10.5	6.5
8.5	6.9	7.1	6.7	9.3	5.7	12.5
5.5	6.4	8.7	7.2	9.1	8.3	8.2
5.1	6.0	7.5	8.5	6.8	9.5	9.7

Compare and contrast the acceleration times of German and Japanese made cars, in terms of their five summary statistics.

Solution 1.22

```
germanCars = [10, 10.9, 4.8, 6.4, 7.9, 8.9, 8.5, 6.9, 7.1,
              5.5, 6.4, 8.7, 5.1, 6.0, 7.5]
japaneseCars = [9.4, 9.5, 7.1, 8.0, 8.9, 7.7, 10.5, 6.5, 6.7,
                9.3, 5.7, 12.5, 7.2, 9.1, 8.3, 8.2, 8.5, 6.8, 9.5, 9.7]
# convert to pandas Series
germanCars = pd.Series(germanCars)
japaneseCars = pd.Series(japaneseCars)
# use describe to calculate statistics
comparison = pd.DataFrame({
  'German': germanCars.describe(),
  'Japanese': japaneseCars.describe(),
})
print(comparison)
```

```
          German   Japanese
count  15.000000  20.000000
mean    7.373333   8.455000
std     1.780235   1.589596
min     4.800000   5.700000
25%     6.200000   7.175000
50%     7.100000   8.400000
75%     8.600000   9.425000
max    10.900000  12.500000
```

Exercise 1.23 Summarize variables Res 3 and Res 7 in data set HADPAS.csv by computing sample statistics, histograms and stem and leaf diagrams.

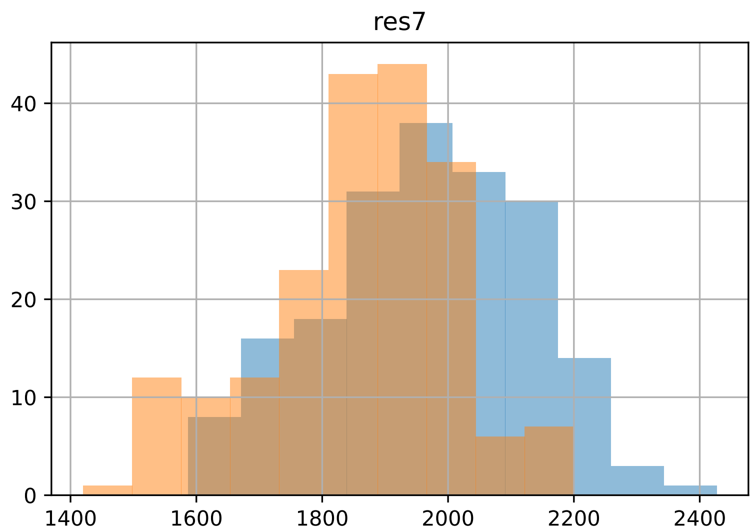
Solution 1.23 Sample statistics:

```
hadpas = mistat.load_data('HADPAS')
sampleStatistics = pd.DataFrame({
  'res3': hadpas['res3'].describe(),
  'res7': hadpas['res7'].describe(),
})
print(sampleStatistics)
```

```
              res3         res7
count   192.000000   192.000000
mean   1965.239583  1857.776042
std     163.528165   151.535930
min    1587.000000  1420.000000
25%    1860.000000  1772.250000
50%    1967.000000  1880.000000
75%    2088.750000  1960.000000
max    2427.000000  2200.000000
```

Histogram:

```
ax = hadpas.hist(column='res3', alpha=0.5)
hadpas.hist(column='res7', alpha=0.5, ax=ax)
plt.show()
```



We overlay both histograms in one plot and make them transparent (`alpha`).
Stem and leaf diagrams:

```
print('res3')
mistat.stemLeafDiagram(hadpas['res3'], 2, leafUnit=10)
print('res7')
mistat.stemLeafDiagram(hadpas['res7'], 2, leafUnit=10)
```

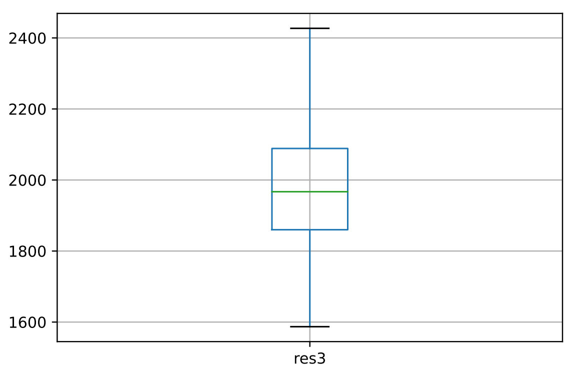
```
res3
        1     15    8
        6     16    01124
       14     16    56788889
       22     17    00000234
       32     17    5566667899
       45     18    0011112233444
       60     18    556666677888899
       87     19    000000111111223333344444444
     (18)     19    566666666667888889
       87     20    0000000012222233333334444
       64     20    5555666666667778999
       44     21    0000011222233344444
```

```
      25     21   566667788888
      13     22   000111234
       4     22   668
       0     23
       1     24   2
res7
       1     14   2
       9     15   11222244
      15     15   667789
      23     16   00012334
      30     16   5566799
      40     17   0022233334
      54     17   66666666777999
      79     18   0000222222222233344444444
    (28)     18   5555556666666778888888999999
      85     19   0000000011111122222222233333444444
      52     19   566666667777888888889999
      28     20   0000111222333444
      12     20   678
       9     21   1123344
       2     21   8
       1     22   0
```

Exercise 1.24 Are there outliers in the Res 3 data of **HADPAS.csv**? Show your calculations.

Solution 1.24

```
hadpas.boxplot(column='res3')
plt.show()
```



Lower whisker starts at $\max(1587, 1511.7) = 1587 = X_{(1)}$; upper whisker ends at $\min(2427, 2440.5) = 2427 = X_{(n)}$. There are no outliers.

Chapter 2
Probability Models and Distribution Functions

Import required modules and define required functions

```
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
```

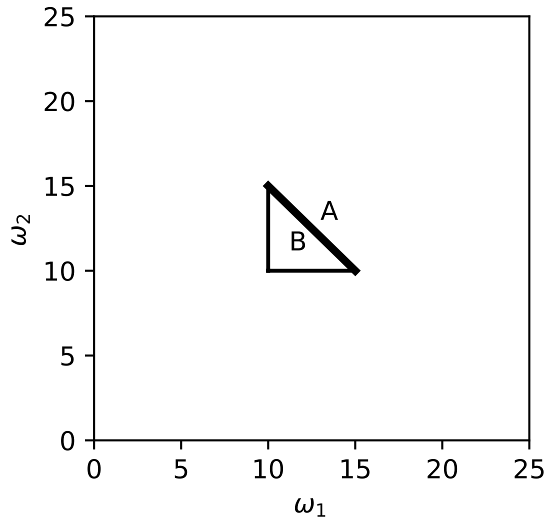
Exercise 2.1 An experiment consists of making 20 observations on the quality of chips. Each observation is recorded as G or D.

(i) What is the sample space, S, corresponding to this experiment?
(ii) How many elementary events in S?
(iii) Let A_n, $n = 0, \cdots, 20$, be the event that exactly n G observations are made. Write the events A_n formally. How many elementary events belong to A_n?

Solution 2.1 **(i)** $\mathcal{S} = \{(w_1, \ldots, w_{20}); w_j = G, D, j = 1, \ldots, 20\}$.
(ii) $2^{20} = 1,048,576$.
(iii) $A_n = \left\{(w_1, \ldots, w_{20}) : \sum_{j=1}^{20} I\{w_j = G\} = n\right\}, n = 0, \ldots, 20,$
where $I\{A\} = 1$ if A is true and $I\{A\} = 0$ otherwise. The number of elementary events in A_n is $\binom{20}{n} = \frac{20!}{n!(20-n)!}$.

Exercise 2.2 An experiment consists of 10 measurements $w_1, \cdots, w_{10}$ of the weights of packages. All packages under consideration have weights between 10 and 20 pounds. What is the sample space $\mathcal{S}$? Let $A = \{(w_1, w_2, \cdots, w_{10}) : w_1 + w_2 = 25\}$. Let $B = \{(w_1, \cdots, w_{10}) : w_1 + w_2 \leq 25\}$. Describe the events A and B graphically. Show that $A \subset B$.

Solution 2.2 $\mathcal{S} = \{(\omega_1, \ldots, \omega_{10}) : 10 \leq \omega_i \leq 20, i = 1, \ldots, 10\}$. Looking at the (ω_1, ω_2) components of A and B we have the following graphical representation:



If $(\omega_1, \ldots, \omega_{10}) \in A$ then $(\omega_1, \ldots, \omega_{10}) \in B$. Thus $A \subset B$. $A \cap B = A$.

Exercise 2.3 Strings of 30 binary $(0, 1)$ signals are transmitted.

(i) Describe the sample space, S.
(ii) Let A_{10} be the event that the first 10 signals transmitted are all 1's. How many elementary events belong to A_{10}.
(iii) Let B_{10} be the event that exactly 10 signals, out of 30 transmitted, are 1's. How many elementary events belong to B_{10}? Does $A_{10} \subset B_{10}$?

Solution 2.3 **(i)** $\mathcal{S} = \{(i_1, \ldots, i_{30}) : i_j = 0, 1,\ j = 1, \ldots, 30\}$.
(ii) $A_{10} = \{(1, 1, \ldots, 1, i_{11}, i_{12}, \ldots, i_{30}) : i_j = 0, 1,\ j = 11, \ldots, 30\}$. $|A_{10}| = 2^{20} = 1,048,576$. ($|A_{10}|$ denotes the number of elements in A_{10}.)
(iii) $B_{10} = \{(i_1, \ldots, i_{30}) : i_j = 0, 1$ and $\sum_{j=1}^{30} i_j = 10\}$, $|B_{10}| = \binom{30}{10} = 30,045,015$. $A_{10} \not\subset B_{10}$, in fact, A_{10} has only one element belonging to B_{10}.

Exercise 2.4 Prove DeMorgan laws

(i) $(A \cup B)^c = A^c \cap B^c$.
(ii) $(A \cap B)^c = A^c \cup B^c$.

Solution 2.4 $\mathcal{S} = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$, a union of mutually disjoint sets.
(a) $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$. Hence, $(A \cup B)^c = A^c \cap B^c$.
(b)

$$\begin{aligned}(A \cap B)^c &= (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)\\ &= (A \cap B^c) \cup A^c\\ &= A^c \cup B^c.\end{aligned}$$

Exercise 2.5 Consider Exercise [3.1] Show that the events $A_0, A_1, \cdots, A_{20}$ are a partition of the sample space S.

Solution 2.5 As in Exercise **3.1**, $A_n = \left\{(\omega_1, \ldots, \omega_{20}) : \sum_{i=1}^{20} I\{\omega_i = G\} = n\right\}$, $n = 0, \ldots, 20$. Thus, for any $n \neq n'$, $A_n \cap A_{n'} = \emptyset$, moreover $\bigcup_{n=0}^{20} A_n = \mathcal{S}$. Hence $\{A_0, \ldots, A_{20}\}$ is a partition.

{exc:partition-union}

Exercise 2.6 Let $A_1, \cdots, A_n$ be a partition of S. Let B be an event. Show that $B = \bigcup_{i=1}^{n} A_i B$, where $A_i B = A_i \cap B$, is a union of disjoint events.

Solution 2.6 $\bigcup_{i=1}^{n} A_i = \mathcal{S}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

$$\begin{aligned} B = B \cap \mathcal{S} = B \cap \left(\bigcup_{i=1}^{n} A_i \right) \\ = \bigcup_{i=1}^{n} A_i B. \end{aligned}$$

Exercise 2.7 Develop a formula for the probability $\Pr\{A \cup B \cup C\}$, where A, B, C are arbitrary events.

Solution 2.7

$$\begin{aligned} \Pr\{A \cup B \cup C\} &= \Pr\{(A \cup B) \cup C\} \\ &= \Pr\{(A \cup B)\} + \Pr\{C\} - \Pr\{(A \cup B) \cap C\} \\ &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} + \Pr\{C\} \\ &\quad - \Pr\{A \cap C\} - \Pr\{B \cap C\} + \Pr\{A \cap B \cap C\} \\ &= \Pr\{A\} + \Pr\{B\} + \Pr\{C\} - \Pr\{A \cap B\} \\ &\qquad - \Pr\{A \cap C\} - \Pr\{B \cap C\} + \Pr\{A \cap B \cap C\}. \end{aligned}$$

Exercise 2.8 Show that if $A_1, \cdots, A_n$ is a partition, then for any event B, $P\{B\} = \sum_{i=1}^{n} P\{A_i B\}$. [Use the result of 2.6.] {exc:partition-union}

Solution 2.8 We have shown in Exercise 2.6 that $B = \bigcup_{i=1}^{n} A_i B$. Moreover, since $\{A_1, \ldots, A_n\}$ is a partition, $A_i B \cap A_j B = (A_i \cap A_j) \cap B = \emptyset \cap B = \emptyset$ for all $i \neq j$. Hence, from Axiom 3 {exc:partition-union}

$$\Pr\{B\} = \Pr\left\{ \bigcup_{i=1}^{n} A_i B \right\} = \sum_{i=1}^{n} \Pr\{A_i B\}.$$

Exercise 2.9 An unbiased die has the numbers $1, 2, \cdots, 6$ written on its faces. The die is thrown twice. What is the probability that the two numbers shown on its upper face sum up to 10?

Solution 2.9

$$\mathcal{S} = \{(i_1, i_2) : i_j = 1, \ldots, 6, \;\; j = 1, 2\}$$
$$A = \{(i_1, i_2) : i_1 + i_2 = 10\} = \{(4,6), (5,5), (6,4)\}$$
$$\Pr\{A\} = \frac{3}{36} = \frac{1}{12}.$$

{exc:ttf-exponential}

Exercise 2.10 The time till failure, T, of electronic equipment is a random quantity. The event $A_t = \{T > t\}$ is assigned the probability $\Pr\{A_t\} = \exp\{-t/200\}$, $t \geq 0$. What is the probability of the event $B = \{150 < T < 280\}$?

Solution 2.10

$$\Pr\{B\} = \Pr\{A_{150}\} - \Pr\{A_{280}\} = \exp\left(-\frac{150}{200}\right) - \exp\left(-\frac{280}{200}\right) = 0.2258.$$

Exercise 2.11 A box contains 40 parts, 10 of type A, 10 of type B, 15 of type C and 5 of type D. A random sample of 8 parts is drawn without replacement. What is the probability of finding two parts of each type in the sample?

Solution 2.11

$$\frac{\binom{10}{2}\binom{10}{2}\binom{15}{2}\binom{5}{2}}{\binom{40}{8}} = 0.02765$$

Exercise 2.12 How many samples of size $n = 5$ can be drawn from a population of size $N = 100$,

(i) with replacement?
(ii) without replacement?

Solution 2.12 (i) $100^5 = 10^{10}$; **(ii)** $\binom{100}{5} = 75{,}287{,}520$.

Exercise 2.13 A lot of 1,000 items contain $M = 900$ "good" ones, and 100 "defective" ones. A random sample of size $n = 10$ is drawn from the lot. What is the probability of observing in the sample at least 8 good items,

(i) when sampling is with replacement?
(ii) when sampling is without replacement?

Solution 2.13 $N = 1{,}000$, $M = 900$, $n = 10$.

$$\textbf{(i)}\ \Pr\{X \geq 8\} = \sum_{j=8}^{10} \binom{10}{j} (0.9)^j (0.1)^{10-j} = 0.9298.$$
$$\textbf{(ii)}\ \Pr\{X \geq 8\} = \sum_{j=8}^{10} \frac{\binom{900}{j}\binom{100}{10-j}}{\binom{1000}{10}} = 0.9308.$$

Exercise 2.14 In continuation of the previous exercise, what is the probability of observing in an RSWR at least one defective item.

Solution 2.14 $1 - (0.9)^{10} = 0.6513$.

{exc:ttf-exponential} **Exercise 2.15** Consider the problem of Exercise 2.10. What is the conditional probability $\Pr\{T > 300 \mid T > 200\}$.

Solution 2.15 $\Pr\{T > 300 \mid T > 200\} = 0.6065$

Exercise 2.16 A point (X, Y) is chosen at random within the unit square, i.e.

$$S = \{(x, y) : 0 \le x, y \le 1\}.$$

Any set A contained in S having area given by

$$\text{Area}\{A\} = \iint_A \mathrm{d}x\,\mathrm{d}y$$

is an event, whose probability is the area of A. Define the events

$$\begin{aligned} B &= \left\{(x, y) : x > \frac{1}{2}\right\} \\ C &= \{(x, y) : x^2 + y^2 \le 1\} \\ D &= \{(x, y) : (x + y) \le 1\}. \end{aligned}$$

(i) Compute the conditional probability $\Pr\{D \mid B\}$.
(ii) Compute the conditional probability $\Pr\{C \mid D\}$.

Solution 2.16 **(i)** $\Pr\{D \mid B\} = \frac{1}{4}$; **(ii)** $\Pr\{C \mid D\} = 1$.

Exercise 2.17 Show that if A and B are independent events then A^c and B^c are also independent events.

Solution 2.17 Since A and B are independent, $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$. Using this fact and DeMorgan's Law,

$$\begin{aligned} \Pr\{A^c \cap B^c\} &= \Pr\{(A \cup B)^c\} \\ &= 1 - \Pr\{A \cup B\} \\ &= 1 - (\Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}) \\ &= 1 - \Pr\{A\} - \Pr\{B\} + \Pr\{A\}\Pr\{B\} \\ &= \Pr\{A^c\} - \Pr\{B\}(1 - \Pr\{A\}) \\ &= \Pr\{A^c\}(1 - \Pr\{B\}) \\ &= \Pr\{A^c\}\Pr\{B^c\}. \end{aligned}$$

Since $\Pr\{A^c \cap B^c\} = \Pr\{A^c\}\Pr\{B^c\}$, A^c and B^c are independent.

Exercise 2.18 Show that if A and B are disjoint events then A and B are dependent events.

Solution 2.18 We assume that $\Pr\{A\} > 0$ and $\Pr\{B\} > 0$. Thus, $\Pr\{A\}\Pr\{B\} > 0$. On the other hand, since $A \cap B = \emptyset$, $\Pr\{A \cap B\} = 0$.

Exercise 2.19 Show that if A and B are independent events then

$$\begin{aligned}\Pr\{A \cup B\} &= \Pr\{A\}(1 - \Pr\{B\}) + \Pr\{B\} \\ &= \Pr\{A\} + \Pr\{B\}(1 - \Pr\{A\}).\end{aligned}$$

Solution 2.19

$$\begin{aligned}\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \\ &= \Pr\{A\} + \Pr\{B\} - \Pr\{A\}\Pr\{B\} \\ &= \Pr\{A\}(1 - \Pr\{B\}) + \Pr\{B\} \\ &= \Pr\{B\}(1 - \Pr\{A\}) + \Pr\{A\}.\end{aligned}$$

Exercise 2.20 A machine which tests whether a part is defective, D, or good, G, may err. The probabilities of errors are given by

$$\begin{aligned}\Pr\{A \mid G\} &= .95, \\ \Pr\{A \mid D\} &= .10,\end{aligned}$$

where A is the event "the part is considered G after testing." If $\Pr\{G\} = .99$, what is the probability of D given A?

Solution 2.20 By Bayes' theorem,

$$\Pr\{D \mid A\} = \frac{\Pr\{A \mid D\}\Pr\{D\}}{\Pr\{A \mid D\}\Pr\{D\} + \Pr\{A \mid G\}\Pr\{G\}} = \frac{0.10 \times 0.01}{0.10 \times 0.01 + 0.95 \times 0.99} = 0.0011.$$

Additional problems in combinatorial and geometric probabilities

Exercise 2.21 Assuming 365 days in a year, if there are 10 people in a party, what is the probability that their birthdays fall on different days? Show that if there are more than 22 people in the party, the probability is greater than 1/2 that at least 2 will have birthdays on the same day.

Solution 2.21 Let n be the number of people in the party. The probability that all their birthdays fall on different days is $\Pr\{D_n\} = \prod_{j=1}^{n}\left(\frac{365 - j + 1}{365}\right)$.

(i) If $n = 10$, $\Pr\{D_{10}\} = 0.8831$.

(ii) If $n = 23$, $\Pr\{D_{23}\} = 0.4927$. Thus, the probability of at least 2 persons with the same birthday, when $n = 23$, is $1 - \Pr\{D_{23}\} = 0.5073 > \frac{1}{2}$.

Exercise 2.22 A number is constructed at random by choosing 10 digits from $\{0, \ldots, 9\}$ with replacement. We allow the digit 0 at any position. What is the probability that the number does not contain 3 specific digits?

Solution 2.22 $\left(\frac{7}{10}\right)^{10} = 0.02825.$

{exc:seven-digit-phone}

Exercise 2.23 A caller remembers all the 7 digits of a telephone number, but is uncertain about the order of the last four. He keeps dialing the last four digits at random, without repeating the same number, until he reaches the right number. What is the probability that he will dial at least ten wrong numbers?

Solution 2.23 $\prod_{j=1}^{10}\left(1-\dfrac{1}{24-j+1}\right)=0.5833.$

Exercise 2.24 One hundred lottery tickets are sold. There are four prizes and ten consolation prizes. If you buy 5 tickets, what is the probability that you win:

(i) one prize?
(ii) a prize and a consolation prize?
(iii) Something?

Solution 2.24 **(i)** $\dfrac{\binom{4}{1}\binom{86}{4}}{\binom{100}{5}}=0.1128$; **(ii)** $\dfrac{\binom{4}{1}\binom{10}{1}\binom{86}{3}}{\binom{100}{5}}=0.0544$; **(iii)** $1-\dfrac{\binom{86}{5}}{\binom{100}{5}}=0.5374$.

Exercise 2.25 Ten PCB's are in a bin, two of these are defectives. The boards are chosen at random, one by one, without replacement. What is the probability that exactly five good boards will be found between the drawing of the first and second defective PCB?

Solution 2.25 $\dfrac{4}{\binom{10}{2}}=0.0889.$

Exercise 2.26 A random sample of 11 integers is drawn without replacement from the set $\{1,2,\ldots,20\}$. What is the probability that the sample median, Me, is equal to the integer k? $6\le k\le 15$.

Solution 2.26 The sample median is $X_{(6)}$, where $X_{(1)}<\cdots<X_{(11)}$ are the ordered sample values. $\Pr\{X_{(6)}=k\}=\dfrac{\binom{k-1}{5}\binom{20-k}{5}}{\binom{20}{11}}$, $k=6,\ldots,15$. This is the probability distribution of the sample median. The probabilities are

k	6	7	8	9	10	11	12	13	14	15
Pr	.01192	.04598	.09902	.15404	.18905	.18905	.15404	.09902	.04598	.01192

Exercise 2.27 A stick is broken at random into three pieces. What is the probability that these pieces can be the sides of a triangle? {exc:stick-pieces}

Solution 2.27 Without loss of generality, assume that the stick is of length 1. Let x, y and $(1-x-y)$, $0<x,y<1$, be the length of the 3 pieces. Obviously, $0<x+y<1$. All points in $\mathcal{S}=\{(x,y):x,y>0,x+y<1\}$ are uniformly distributed. In order that the three pieces can form a triangle, the following three conditions should be satisfied:

(i) $x+y>(1-x-y)$

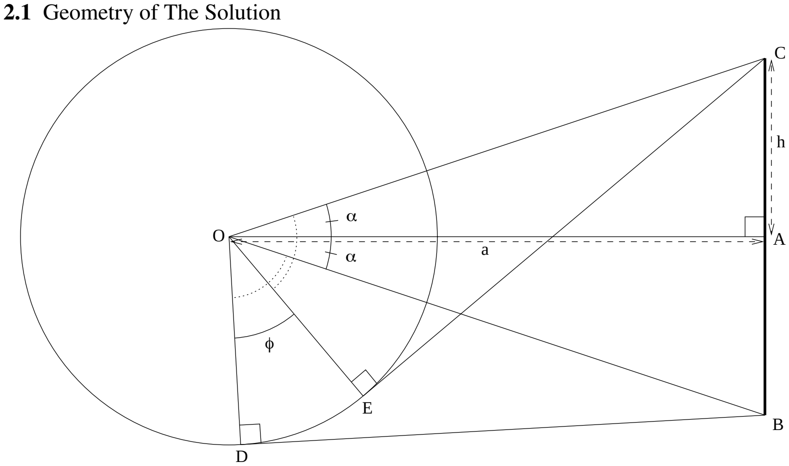
(ii) $x + (1 - x - y) > y$
(iii) $y + (1 - x - y) > x$.

The set of points (x, y) satisfying (i), (ii) and (iii) is bounded by a triangle of area 1/8. $\mathcal{S}$ is bounded by a triangle of area 1/2. Hence, the required probability is 1/4.

Exercise 2.28 A particle is moving at a uniform speed on a circle of unit radius and is released at a random point on the circumference. Draw a line segment of length $2h$ ($h < 1$) centered at a point A of distance $a > 1$ from the center of the circle, O. Moreover the line segment is perpendicular to the line connecting O with A. What is the probability that the particle will hit the line segment? [The particle flies along a straight line tangential to the cirlce.]

{exc:geometrySolution} **Solution 2.28** Consider Figure 2.1. Suppose that the particle is moving along the circumference of the circle in a counterclockwise direction. Then, using the notation in the diagram, $\Pr\{\text{hit}\} = \phi/2\pi$. Since $OD = 1 = OE$, $OB = \sqrt{a^2 + h^2} = OC$ and the lines $\overline{DB}$ and $\overline{EC}$ are tangential to the circle, it follows that the triangles ΔODB and ΔOEC are congruent. Thus $m(\angle\, DOB) = m(\angle\, EOC)$, and it is easily seen that $\phi = 2\alpha$. Now $\alpha = \tan^{-1}\left(\dfrac{h}{a}\right)$, and hence, $\Pr\{\text{hit}\} = \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{h}{a}\right)$.

{exc:geometrySolution} **Fig. 2.1** Geometry of The Solution



Exercise 2.29 A block of 100 bits is transmitted over a binary channel, with probability $p = 10^{-3}$ of bit error. Errors occur independently. Find the probability that the block contains at least three errors.

Solution 2.29 $1-(0.999)^{100}-100\times(0.001)\times(0.999)^{99}-\binom{100}{2}\times(0.001)^2(0.999)^{98} = 0.0001504$.

Exercise 2.30 A coin is tossed repeatedly until 2 "heads" occur. What is the probability that 4 tosses are required.

Solution 2.30 The probability that n tosses are required is $p(n) = \binom{n-1}{1}\left(\frac{1}{2}\right)^n$, $n \geq 2$. Thus, $p(4) = 3 \cdot \frac{1}{2^4} = \frac{3}{16}$.

Exercise 2.31 Consider the sample space S of all sequences of 10 binary numbers (0-1 signals). Define on this sample space two random variables and derive their probability distribution function, assuming the model that all sequences are equally probable.

Solution 2.31 $\mathcal{S} = \{(i_1, \ldots, i_{10}) : i_j = 0, 1, j = 1, \ldots, 10\}$. One random variable is the number of 1's in an element, i.e., for $\omega = (i_1, \ldots, i_{10})$ $X_1(\omega) = \sum_{j=1}^{10} i_j$. Another random variable is the number of zeros to the left of the 1st one, i.e., $X_2(\omega) = \sum_{j=1}^{10} \prod_{k=1}^{j}(1 - i_k)$. Notice that $X_2(\omega) = 0$ if $i_1 = 1$ and $X_2(\omega) = 10$ if $i_1 = i_2 = \ldots = i_{10} = 0$. The probability distribution of X_1 is $\Pr\{X_1 = k\} = \binom{10}{k}/2^{10}$, $k = 0, 1, \ldots, 10$. The probability distribution of X_2 is

$$\Pr\{X_2 = k\} = \begin{cases} \left(\frac{1}{2}\right)^{k+1}, & k = 0, \ldots, 9 \\ \left(\frac{1}{2}\right)^{10}, & k = 10. \end{cases}$$

Exercise 2.32 The number of blemishes on a ceramic plate is a discrete random variable. Assume the probability model, with p.d.f.

$$p(x) = e^{-5}\frac{5^x}{x!}, \quad x = 0, 1, \cdots$$

(i) Show that $\sum_{x=0}^{\infty} p(x) = 1$
(ii) What is the probability of at most 1 blemish on a plate?
(iii) What is the probability of no more than 7 blemishes on a plate?

Solution 2.32 **(i)** Since $\sum_{x=0}^{\infty} \frac{5^x}{x!} = e^5$, we have $\sum_{x=0}^{\infty} p(x) = 1.$; **(ii)** $\Pr\{X \leq 1\} = e^{-5}(1 + 5) = 0.0404$; **(iii)** $\Pr\{X \leq 7\} = 0.8666$.

{exc:mixed-cdf-example}

Exercise 2.33 Consider a distribution function of a mixed type with c.d.f.

$$F_x(x) = \begin{cases} 0, & \text{if } x < -1 \\ .3 + .2(x + 1), & \text{if } -1 \leq x < 0 \\ .7 + .3x, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x. \end{cases}$$

(i) What is $\Pr\{X = -1\}$?

(ii) What is $\Pr\{-.5 < X < 0\}$?
(iii) What is $\Pr\{0 \le X < .75\}$?
(iv) What is $\Pr\{X = 1\}$?
(v) Compute the expected value, $E\{X\}$ and variance, $V\{X\}$.

Solution 2.33 **(i)** $\Pr\{X = -1\} = 0.3$; **(ii)** $\Pr\{-0.5 < X < 0\} = 0.1$; **(iii)** $\Pr\{0 \le X < 0.75\} = 0.425$; **(iv)** $\Pr\{X = 1\} = 0$; **(v)** $E\{X\} = -0.25$, $V\{X\} = 0.4042$.

Exercise 2.34 A random variable has the Rayleigh distribution, with c.d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ \\ 1 - e^{-x^2/2\sigma^2}, & x \ge 0 \end{cases}$$

where σ^2 is a positive parameter. Find the expected value $E\{X\}$.

Solution 2.34

$$E\{X\} = \int_0^\infty (1 - F(x))\,\mathrm{d}x = \int_0^\infty e^{-x^2/2\sigma^2}\,\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}.$$

Exercise 2.35 A random variable X has a discrete distribution over the integers $\{1, 2, \dots, N\}$ with equal probabilities. Find $E\{X\}$ and $V\{X\}$.

Solution 2.35

$$E\{X\} = \frac{1}{N}\sum_{i=1}^{N} i = \frac{N+1}{2}; \quad E\{X^2\} = \frac{1}{N}\sum_{i=1}^{N} i^2 = \frac{(N+1)(2N+1)}{6}$$

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{2(N+1)(2N+1) - 3(N+1)^2}{12} = \frac{N^2-1}{12}.$$

Exercise 2.36 A random variable has expectation $\mu = 10$ and standard deviation $\sigma = 0.5$. Use Chebychev's inequality to find a lower bound to the probability

$$\Pr\{8 < X < 12\}.$$

Solution 2.36 $\Pr\{8 < X < 12\} = \Pr\{|X - 10| < 2\} \ge 1 - \frac{V\{X\}}{4} = 1 - \frac{0.25}{4} = 0.9375$.

Exercise 2.37 Consider the random variable X with c.d.f.

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), \quad -\infty < x < \infty.$$

Find the .25-th, .50-th and .75-th quantiles of this distribution.

Solution 2.37 Notice that $F(x)$ is the standard Cauchy distribution. The p-th quantile, x_p, satisfies the equation $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x_p) = p$, hence $x_p = \tan\left(\pi\left(p - \frac{1}{2}\right)\right)$. For $p = 0.25, 0.50, 0.75$ we get $x_{.25} = -1,\ x_{.50} = 0,\ x_{.75} = 1$, respectively.

Exercise 2.38 Show that the central moments μ_l^* relate to the moments μ_l around the origin, by the formula

$$\mu_l^* = \sum_{j=0}^{l-2}(-1)^j\binom{l}{j}\mu_{l-j}\mu_1^j + (-1)^{l-1}(l-1)\mu_1^l.$$

Solution 2.38 $\mu_l^* = E\{(X-\mu_1)^l\} = \sum_{j=0}^{l}(-1)^j\binom{l}{j}\mu_1^j\mu_{l-j}$.

When $j = l$ the term is $(-1)^l\mu_1^l$. When $j = l-1$ the term is $(-1)^{l-1}l\mu_1^{l-1}\mu_1 = (-1)^{l-1}l\mu_1^l$. Thus, the sum of the last 2 terms is $(-1)^{l-1}(l-1)\mu_1^l$ and we have
$\mu_l^* = \sum_{j=0}^{l-2}(-1)^j\binom{l}{j}\mu_1^j\mu_{l-j} + (-1)^{l-1}(l-1)\mu_1^l$.

Exercise 2.39 Find the expected value μ_1 and the second moment μ_2 of the random variable whose c.d.f. is given in Exercise 2.33. {exc:mixed-cdf-example}

Solution 2.39 We saw in the solution of Exercise 2.33 that $\mu_1 = -0.25$. Moreover, {exc:mixed-cdf-example}
$\mu_2 = V\{X\} + \mu_1^2 = 0.4667$.

Exercise 2.40 A random variable X has a continuous uniform distribution over the interval (a, b), i.e.,

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X. Find the mean and variance by differentiating the m.g.f.

Solution 2.40 $M_X(t) = \dfrac{1}{t(b-a)}(e^{tb} - e^{ta}), \quad -\infty < t < \infty, \quad a < b.$

$E\{X\} = \dfrac{a+b}{2};\ V\{X\} = \dfrac{(b-a)^2}{12}.$

Exercise 2.41 Consider the moment generating function, m.g.f. of the exponential distribution, i.e.,

$$M(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

(i) Find the first four moments of the distribution, by differentiating $M(t)$.
(ii) Convert the moments to central moments.
(iii) What is the index of kurtosis β_4?

Solution 2.41 **(i)** For $t < \lambda$ we have $M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$,

$$\begin{aligned}
M'(t) &= \frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}, & \mu_1 &= M'(0) = \frac{1}{\lambda}\\
M''(t) &= \frac{2}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-3}, & \mu_2 &= M''(0) = \frac{2}{\lambda^2}\\
M^{(3)}(t) &= \frac{6}{\lambda^3}\left(1-\frac{t}{\lambda}\right)^{-4}, & \mu_3 &= M^{(3)}(0) = \frac{6}{\lambda^3}\\
M^{(4)}(t) &= \frac{24}{\lambda^4}\left(1-\frac{t}{\lambda}\right)^{-5}, & \mu_4 &= M^{(4)}(0) = \frac{24}{\lambda^4}.
\end{aligned}$$

(ii) The central moments are

$$\begin{aligned}
\mu_1^* &= 0,\\
\mu_2^* &= \frac{1}{\lambda^2},\\
\mu_3^* &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 = \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3},\\
\mu_4^* &= \mu_4 - 4\mu_3\mu_1 + 6\mu_2(\mu_1)^2 - 3\mu_1^4 = \frac{1}{\lambda^4}(24 - 4\cdot 6 + 6\cdot 2 - 3) = \frac{9}{\lambda^4}.
\end{aligned}$$

(iii) The index of kurtosis is $\beta_4 = \dfrac{\mu_4^*}{(\mu_2^*)^2} = 9$.

Exercise 2.42 Using Python, prepare a table of the p.d.f. and c.d.f. of the binomial distribution $B(20, .17)$.

Solution 2.42 `scipy.stats.binom` provides the distribution information.

```
x = list(range(15))
table = pd.DataFrame({
  'x': x,
  'p.d.f.': [stats.binom(20, 0.17).pmf(x) for x in x],
  'c.d.f.': [stats.binom(20, 0.17).cdf(x) for x in x],
})
print(table)
```

```
     x        p.d.f.    c.d.f.
0    0  2.407475e-02  0.024075
1    1  9.861947e-02  0.122694
2    2  1.918921e-01  0.314586
3    3  2.358192e-01  0.550406
4    4  2.052764e-01  0.755682
5    5  1.345426e-01  0.890224
6    6  6.889229e-02  0.959117
7    7  2.822094e-02  0.987338
8    8  9.392812e-03  0.996731
9    9  2.565105e-03  0.999296
10  10  5.779213e-04  0.999874
11  11  1.076086e-04  0.999981
12  12  1.653023e-05  0.999998
13  13  2.083514e-06  1.000000
14  14  2.133719e-07  1.000000
```

Exercise 2.43 What are the 1st quantile, Q_1, median, Me, and 3rd quantile, Q_3, of $B(20, .17)$?

Solution 2.43 $Q_1 = 2$, Med $= 3$, $Q_3 = 4$.

Exercise 2.44 Compute the mean $E\{X\}$ and standard deviation, σ, of $B(45, .35)$.

Solution 2.44 $E\{X\} = 15.75$, $\sigma = 3.1996$.

Exercise 2.45 A PCB is populated by 50 chips which are randomly chosen from a lot. The probability that an individual chip is non-defective is p. What should be the value of p so that no defective chip is installed on the board is $\gamma = .99$? [The answer to this question shows why the industry standards are so stringent.]

Solution 2.45 Pr{no defective chip on the board} $= p^{50}$. Solving $p^{50} = 0.99$ yields $p = (0.99)^{1/50} = 0.999799$.

{exc:binomial-convergence-poisson}

Exercise 2.46 Let $b(j; n, p)$ be the p.d.f. of the binomial distribution. Show that as $n \to \infty$, $p \to 0$ so that $np \to \lambda$, $0 < \lambda < \infty$, then

$$\lim_{\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}} b(j; n, p) = e^{-\lambda}\frac{\lambda^j}{j!}, \quad j = 0, 1, \ldots .$$

Solution 2.46 Notice first that $\lim_{\substack{n\to\infty \\ np\to\lambda}} b(0; n, p) = \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$. Moreover, for all $j = 0, 1, \cdots, n-1$, $\frac{b(j+1; n, p)}{b(j; n, p)} = \frac{n-j}{j+1}\cdot\frac{p}{1-p}$. Thus, by induction on j, for $j > 0$

$$\begin{aligned}
\lim_{\substack{n\to\infty \\ np\to\lambda}} b(j; n, p) &= \lim_{\substack{n\to\infty \\ np\to\lambda}} b(j-1; n, p)\frac{n-j+1}{j}\cdot\frac{p}{1-p} \\
&= e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}\lim_{\substack{n\to\infty \\ np\to\lambda}}\frac{(n-j+1)p}{j\left(1-\frac{\lambda}{n}\right)} \\
&= e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}\cdot\frac{\lambda}{j} = e^{-\lambda}\frac{\lambda^j}{j!}.
\end{aligned}$$

Exercise 2.47 Use the result of the previous exercise to find the probability that a block of 1,000 bits, in a binary communication channel, will have less than 4 errors, when the probability of a bit error is $p = 10^{-3}$.

Solution 2.47 Using the Poisson approximation, $\lambda = n\cdot p = 1000\cdot 10^{-3} = 1$. $\Pr\{X < 4\} = e^{-1}\sum_{j=0}^{3}\frac{1}{j!} = 0.9810$.

Exercise 2.48 Compute $E\{X\}$ and $V\{X\}$ of the hypergeometric distribution $H(500, 350, 20)$.

Solution 2.48 $E\{X\} = 20\cdot\frac{350}{500} = 14$; $\quad V\{X\} = 20\cdot\frac{350}{500}\cdot\frac{150}{500}\left(1 - \frac{19}{499}\right) = 4.0401$.

Exercise 2.49 A lot of size $N = 500$ items contains $M = 5$ defective ones. A random sample of size $n = 50$ is drawn from the lot without replacement (RSWOR). What is the probability of observing more than 1 defective item in the sample?

Solution 2.49 Let X be the number of defective items observed.
$\Pr\{X > 1\} = 1 - \Pr\{X \leq 1\} = 1 - H(1; 500, 5, 50) = 0.0806.$

{ex:pcb-rectification} **Exercise 2.50** Consider Example 2.23. What is the probability that the lot will be rectified if $M = 10$ and $n = 20$?

Solution 2.50

$$\begin{aligned}\Pr\{R\} &= 1 - H(3; 100, 10, 20) + \sum_{i=1}^{3} h(i; 100, 10, 20)\,[1 - H(3-i; 80, 10-i, 40)]\\ &= 0.87395.\end{aligned}$$

Exercise 2.51 Use the m.g.f. to compute the third and fourth central moments of the Poisson distribution $P(10)$. What is the index of skewness and kurtosis of this distribution?

Solution 2.51 The m.g.f. of the Poisson distribution with parameter λ, $P(\lambda)$, is

$$\begin{aligned}M(t) &= \exp\{-\lambda(1-e^t)\},\ -\infty < t < \infty.\ \text{Accordingly,}\\ M'(t) &= \lambda M(t) e^t\\ M''(t) &= (\lambda^2 e^{2t} + \lambda e^t) M(t)\\ M^{(3)}(t) &= (\lambda^3 e^{3t} + 3\lambda^2 e^{2t} + \lambda e^t) M(t)\\ M^{(4)}(t) &= (\lambda^4 e^{4t} + 6\lambda^3 e^{3t} + 7\lambda^2 e^{2t} + \lambda e^t) M(t).\end{aligned}$$

The moments and central moments are

$$\begin{aligned}\mu_1 &= \lambda & \mu_1^* &= 0\\ \mu_2 &= \lambda^2 + \lambda & \mu_2^* &= \lambda\\ \mu_3 &= \lambda^3 + 3\lambda^2 + \lambda & \mu_3^* &= \lambda\\ \mu_4 &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda & \mu_4^* &= 3\lambda^2 + \lambda.\end{aligned}$$

Thus, the indexes of skewness and kurtosis are $\beta_3 = \lambda^{-1/2}$ and $\beta_4 = 3 + \dfrac{1}{\lambda}$.
For $\lambda = 10$ we have $\beta_3 = 0.3162$ and $\beta_4 = 3.1$.

Exercise 2.52 The number of blemishes on ceramic plates has a Poisson distribution with mean $\lambda = 1.5$. What is the probability of observing more than 2 blemishes on a plate?

Solution 2.52 Let X be the number of blemishes observed. $\Pr\{X > 2\} = 0.1912$.

Exercise 2.53 The error rate of an insertion machine is 380 PPM (per 10^6 parts inserted). What is the probability of observing more than 6 insertion errors in 2 hours of operation, when the insertion rate is 4,000 parts per hour?

Solution 2.53 Using the Poisson approximation with $N = 8000$ and $p = 380 \times 10^{-6}$, we have $\lambda = 3.04$ and $\Pr\{X > 6\} = 0.0356$, where X is the number of insertion errors in 2 hours of operation.

Exercise 2.54 In continuation of the previous Exercise, let N be the number of parts inserted until an error occurs. What is the distribution of N? Compute the expected value and the standard deviation of N.

Solution 2.54 The distribution of N is geometric with $p = 0.00038$. $E\{N\} = 2631.6$, $\sigma_N = 2631.08$.

Exercise 2.55 What are Q_1, Me and Q_3 of the negative binomial N.B. (p, k) with $p = 0.01$ and $k = 3$?

Solution 2.55 Using Python we obtain that for the $NB(p, k)$ with $p = 0.01$ and $k = 3$, $Q_1 = 170$, $Me = 265$, and $Q_3 = 389$.

```
stats.nbinom.ppf([0.25, 0.5, 0.75], 3, 0.01)
```

```
array([170., 265., 389.])
```

{exc:mgf-nb}

Exercise 2.56 Derive the m.g.f. of $NB(p, k)$.

Solution 2.56 By definition, the m.g.f. of $NB(p, k)$ is

$$M(t) = \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} p^k ((1-p)e^t)^i,$$

for $t < -\log(1-p)$.

Thus $M(t) = \dfrac{p^k}{(1-(1-p)e^t)^k} \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} (1-(1-p)e^t)^k ((1-p)e^t)^i$.

Since the last infinite series sums to one, $M(t) = \left[\dfrac{p}{1-(1-p)e^t}\right]^k$, $t < -\log(1-p)$.

Exercise 2.57 Differentiate the m.g.f. of the geometric distribution, i.e.,

$$M(t) = \frac{pe^t}{(1-e^t(1-p))}, \quad t < -\log(1-p),$$

to obtain its first four moments, and derive then the indices of skewness and kurtosis.

Solution 2.57 $M(t) = \dfrac{pe^t}{1-(1-p)e^t}$, for $t < -\log(1-p)$. The derivatives of $M(t)$ are

$$
\begin{aligned}
M'(t) &= M(t)(1-(1-p)e^t)^{-1} \\
M''(t) &= M(t)(1-(1-p)e^t)^{-2}(1+(1-p)e^t) \\
M^{(3)}(t) &= M(t)(1-(1-p)e^t)^{-3} \cdot [(1+(1-p)e^t)^2 + 2(1-p)e^t] \\
M^{(4)}(t) &= M(t)(1-(1-p)e^t)^{-4}[1+(1-p)^3e^{3t} + 11(1-p)e^t + 11(1-p)^2e^{2t}].
\end{aligned}
$$

The moments are

$$
\begin{aligned}
\mu_1 &= \frac{1}{p} \\
\mu_2 &= \frac{2-p}{p^2} \\
\mu_3 &= \frac{(2-p)^2 + 2(1-p)}{p^3} = \frac{6-6p+p^2}{p^3} \\
\mu_4 &= \frac{11(1-p)(2-p) + (1-p)^3 + 1}{p^4} = \frac{24-36p+14p^2-p^3}{p^4}.
\end{aligned}
$$

The central moments are

$$
\begin{aligned}
\mu_1^* &= 0 \\
\mu_2^* &= \frac{1-p}{p^2}, \\
\mu_3^* &= \frac{1}{p^3}(1-p)(2-p), \\
\mu_4^* &= \frac{1}{p^4}(9-18p+10p^2-p^3).
\end{aligned}
$$

Thus the indices of skewness and kurtosis are $\beta_3^* = \dfrac{2-p}{\sqrt{1-p}}$ and $\beta_4^* = \dfrac{9-9p+p^2}{1-p}$.

Exercise 2.58 The proportion of defective RAM chips is $p = 0.002$. You have to install 50 chips on a board. Each chip is tested before its installation. How many chips should you order so that, with probability greater than $\gamma = .95$ you will have at least fifty good chips to install?

Solution 2.58 If there are n chips, $n > 50$, the probability of at least 50 good ones is $1 - B(49; n, 0.998)$. Thus, n is the smallest integer > 50 for which $B(49; n, 0.998) < 0.05$. It is sufficient to order 51 chips.

Exercise 2.59 The random variable X assumes the values $\{1, 2, \ldots\}$ with probabilities of a geometric distribution, with parameter p, $0 < p < 1$. Prove the "memoryless" property of the geometric distribution, namely:

$$P[X > n + m \mid X > m] = P[X > n],$$

for all $n, m = 1, 2, \ldots$.

Solution 2.59 If X has a geometric distribution then, for every j, $j = 1, 2, \ldots$ $\Pr\{X > j\} = (1 - p)^j$. Thus,

$$\begin{aligned} \Pr\{X > n + m \mid X > m\} &= \frac{\Pr\{X > n + m\}}{\Pr\{X > m\}} \\ &= \frac{(1 - p)^{n+m}}{(1 - p)^m} \\ &= (1 - p)^n \\ &= \Pr\{X > n\}. \end{aligned}$$

Exercise 2.60 Let X be a random variable having a continuous c.d.f. $F(x)$. Let $Y = F(X)$. Show that Y has a uniform distribution on $(0, 1)$. Conversely, if U has a uniform distribution on $(0, 1)$ then $X = F^{-1}(U)$ has the c.d.f. $F(x)$.

Solution 2.60 For $0 < y < 1$, $\Pr\{F(X) \le y\} = \Pr\{X \le F^{-1}(y)\} = F(F^{-1}(y)) = y$. Hence, the distribution of $F(X)$ is uniform on $(0, 1)$. Conversely, if U has a uniform distribution on $(0, 1)$, then

$$\Pr\{F^{-1}(U) \le x\} = \Pr\{U \le F(x)\} = F(x).$$

Exercise 2.61 Compute the expected value and the standard deviation of a uniform distribution $U(10, 50)$.

Solution 2.61 $E\{U(10, 50)\} = 30$; $V\{U(10, 50)\} = \dfrac{1600}{12} = 133.33$; $\sigma\{U(10, 50)\} = \dfrac{40}{2\sqrt{3}} = 11.547$.

Exercise 2.62 Show that if U is uniform on $(0, 1)$ then $X = -\log(U)$ has an exponential distribution $E(1)$.

Solution 2.62 Let $X = -\log(U)$ where U has a uniform distribution on (0,1).

$$\begin{aligned} \Pr\{X \le x\} &= \Pr\{-\log(U) \le x\} \\ &= \Pr\{U \ge e^{-x}\} \\ &= 1 - e^{-x}. \end{aligned}$$

Therefore X has an exponential distribution $E(1)$.

Exercise 2.63 Use Python to compute the probabilities, for $N(100, 15)$, of

(i) $92 < X < 108$;

(ii) $X > 105$;
(iii) $2X + 5 < 200$.

Solution 2.63 **(i)** $\Pr\{92 < X < 108\} = 0.4062$; **(ii)** $\Pr\{X > 105\} = 0.3694$; **(iii)** $\Pr\{2X + 5 < 200\} = \Pr\{X < 97.5\} = 0.4338$.

```
rv = stats.norm(100, 15)
print('(i)', rv.cdf(108) - rv.cdf(92))
print('(ii)', 1 - rv.cdf(105))
print('(iii)', rv.cdf((200 - 5)/2))
```

```
(i) 0.4061971427922976
(ii) 0.36944134018176367
(iii) 0.43381616738909634
```

Exercise 2.64 The .9-quantile of $N(\mu, \sigma)$ is 15 and its .99-quantile is 20. Find the mean μ and standard deviation σ.

Solution 2.64 Let z_α denote the α quantile of a $N(0, 1)$ distribution. Then the two equations $\mu + z_{.9}\sigma = 15$ and $\mu + z_{.99}\sigma = 20$ yield the solution $\mu = 8.8670$ and $\sigma = 4.7856$.

Exercise 2.65 A communication channel accepts an arbitrary voltage input v and outputs a voltage $v + E$, where $E \sim N(0, 1)$. The channel is used to transmit binary information as follows:

(i) to transmit 0, input $-v$
(ii) to transmit 1, input v
(iii) The receiver decides a 0 if the voltage Y is negative, and 1 otherwise.

What should be the value of v so that the receiver's probability of bit error is $\alpha = .01$?

Solution 2.65 Due to symmetry, $\Pr\{Y > 0\} = \Pr\{Y < 0\} = \Pr\{E < v\}$, where $E \sim N(0, 1)$. If the probability of a bit error is $\alpha = 0.01$, then $\Pr\{E < v\} = \Phi(v) = 1 - \alpha = 0.99$.

Thus $v = z_{.99} = 2.3263$.

Exercise 2.66 Aluminum pins manufactured for an aviation industry have a random diameter, whose distribution is (approximately) normal with mean of $\mu = 10$ [mm] and standard deviation $\sigma = 0.02$ [mm]. Holes are automatically drilled on aluminum plates, with diameters having a normal distribution with mean μ_d [mm] and $\sigma = 0.02$ [mm]. What should be the value of μ_d so that the probability that a pin will not enter a hole (too wide) is $\alpha = 0.01$?

Solution 2.66 Let X_p denote the diameter of an aluminum pin and X_h denote the size of a hole drilled in an aluminum plate. If $X_p \sim N(10, 0.02)$ and $X_h \sim N(\mu_d, 0.02)$ then the probability that the pin will not enter the hole is $\Pr\{X_h - X_p < 0\}$. Now $X_h - X_p \sim N(\mu_d - 10, \sqrt{0.02^2 + 0.02^2})$ and for $\Pr\{X_h - X_p < 0\} = 0.01$, we obtain $\mu_d = 10.0658$ mm. (The fact that the sum of two independent normal random variables is normally distributed should be given to the student since it has not yet been covered in the text.)

Exercise 2.67 Let $X_1, \ldots, X_n$ be a random sample (i.i.d.) from a normal distribution $N(\mu, \sigma^2)$. Find the expected value and variance of $Y = \sum_{i=1}^n iX_i$.

Solution 2.67 For $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$, $Y = \sum_{i=1}^n iX_i \sim N(\mu_Y, \sigma_Y^2)$ where $\mu_Y = \mu \sum_{i=1}^n i = \mu \dfrac{n(n+1)}{2}$ and $\sigma_Y^2 = \sigma^2 \sum_{i=1}^n i^2 = \sigma^2 \dfrac{n(n+1)(2n+1)}{6}$.

Exercise 2.68 Concrete cubes have compressive strength with log-normal distribution LN$(5, 1)$. Find the probability that the compressive strength X of a random concrete cube will be greater than 300 [kg/cm^2].

Solution 2.68 $\Pr\{X > 300\} = \Pr\{\log X > 5.7038\} = 1 - \Phi(0.7038) = 0.24078$.

Exercise 2.69 Using the m.g.f. of $N(\mu, \sigma)$, derive the expected value and variance of $LN(\mu, \sigma)$. [Recall that $X \sim e^{N(\mu,\sigma)}$.]

Solution 2.69 For $X \sim e^{N(\mu,\sigma)}$, $X \sim e^Y$ where $Y \sim N(\mu, \sigma)$, $M_Y(t) = e^{\mu t + \sigma^2 t^2/2}$.

$$\xi = E\{X\} = E\{e^Y\} = M_Y(1) = e^{\mu + \sigma^2/2}.$$

Since $E\{X^2\} = E\{e^{2Y}\} = M_Y(2) = e^{2\mu + 2\sigma^2}$ we have

$$\begin{aligned} V\{X\} &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\ &= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \\ &= \xi^2(e^{\sigma^2} - 1). \end{aligned}$$

Exercise 2.70 What are Q_1, Me and Q_3 of $E(\beta)$?

Solution 2.70 The quantiles of $E(\beta)$ are $x_p = -\beta \log(1-p)$. Hence, $Q_1 = 0.2877\beta$, $Me = 0.6931\beta$, $Q_3 = 1.3863\beta$.

Exercise 2.71 Show that if the life length of a chip is exponential $E(\beta)$ then only 36.7% of the chips will function longer than the mean time till failure β.

Solution 2.71 If $X \sim E(\beta)$, $\Pr\{X > \beta\} = e^{-\beta/\beta} = e^{-1} = 0.3679$.

Exercise 2.72 Show that the m.g.f. of $E(\beta)$ is $M(t) = (1 - \beta t)^{-1}$, for $t < \dfrac{1}{\beta}$.

Solution 2.72 The m.g.f. of $E(\beta)$ is

$$\begin{aligned} M(t) &= \frac{1}{\beta} \int_0^\infty e^{tx - x/\beta}\, \mathrm{d}x \\ &= \frac{1}{\beta} \int_0^\infty e^{-\frac{(1-t\beta)}{\beta} x}\, \mathrm{d}x \\ &= (1 - t\beta)^{-1}, \quad \text{for } t < \frac{1}{\beta}. \end{aligned}$$

Exercise 2.73 Let X_1, X_2, X_3 be independent random variables having an identical exponential distribution $E(\beta)$. Compute $\Pr\{X_1 + X_2 + X_3 \geq 3\beta\}$. {exc:prob-ind-exp-rv}

Solution 2.73 By independence,

$$\begin{aligned} M_{(X_1+X_2+X_3)}(t) &= E\{e^{t(X_1+X_2+X_3)}\} \\ &= \prod_{i=1}^{3} E\{e^{tX_i}\} \\ &= (1-\beta t)^{-3}, \quad t < \frac{1}{\beta}. \end{aligned}$$

{exc:mgf-ind-exp-rv} Thus $X_1 + X_2 + X_3 \sim G(3, \beta)$, (see Exercise 2.76).
Using the formula of the next exercise,

$$\begin{aligned} \Pr\{X_1 + X_2 + X_3 \geq 3\beta\} &= \Pr\{\beta G(3,1) \geq 3\beta\} \\ &= \Pr\{G(3,1) \geq 3\} \\ &= e^{-3} \sum_{j=0}^{2} \frac{3^j}{j!} \\ &= 0.4232. \end{aligned}$$

Exercise 2.74 Establish the formula

$$G(t; k, \frac{1}{\lambda}) = 1 - e^{-\lambda t} \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!},$$

by integrating in parts the p.d.f. of

$$G\left(k; \frac{1}{\lambda}\right).$$

Solution 2.74

$$\begin{aligned}
G(t;k,\lambda) &= \frac{\lambda^k}{(k+1)!}\int_0^t x^{k-1}e^{-\lambda x}\,\mathrm{d}x \\
&= \frac{\lambda^k}{k!}t^k e^{-\lambda t} + \frac{\lambda^{k+1}}{k!}\int_0^t x^k e^{-\lambda x}\,\mathrm{d}x \\
&= \frac{\lambda^k}{k!}t^k e^{-\lambda t} + \frac{\lambda^{k+1}}{(k+1)!}t^{k+1}e^{-\lambda t} + \frac{\lambda^{k+2}}{(k+1)!}\int_0^t x^{k+1}e^{-\lambda x}\,\mathrm{d}x \\
&= \cdots \\
&= e^{-\lambda t}\sum_{j=k}^{\infty}\frac{(\lambda t)^j}{j!} \\
&= 1 - e^{-\lambda t}\sum_{j=0}^{k-1}\frac{(\lambda t)^j}{j!}.
\end{aligned}$$

Exercise 2.75 Use Python to compute $\Gamma(1.17)$, $\Gamma\left(\frac{1}{2}\right)$, $\Gamma\left(\frac{3}{2}\right)$.

Solution 2.75 $\Gamma(1.17) = 0.9267$, $\Gamma\left(\frac{1}{2}\right) = 1.77245$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = 0.88623$.

```
from scipy.special import gamma
print(gamma(1.17), gamma(1 / 2), gamma(3 / 2))
```

```
0.9266996106177159 1.7724538509055159 0.8862269254527579
```

{exc:mgf-ind-exp-rv}

Exercise 2.76 Using m.g.f., show that the sum of k independent exponential random variables, $E(\beta)$, has the gamma distribution $G(k,\beta)$.

Solution 2.76 The moment generating function of the sum of independent random variables is the product of their respective m.g.f.'s. Thus, if $X_1,\cdots,X_k$ are i.i.d. $E(\beta)$, using the result of Exercise 2.73, $M_S(t) = \prod_{i=1}^k (1-\beta t)^{-1} = (1-\beta t)^{-k}$, $t < \frac{1}{\beta}$, where $S = \sum_{i=1}^k X_i$. On the other hand, $(1-\beta t)^{-k}$ is the m.g.f. of $G(k,\beta)$. {exc:prob-ind-exp-rv}

Exercise 2.77 What is the expected value and variance of the Weibull distribution $W(2, 3.5)$?

Solution 2.77 The expected value and variance of $W(2, 3.5)$ are

$$\begin{aligned}
E\{W(2,3.5)\} &= 3.5\times\Gamma\left(1+\frac{1}{2}\right) = 3.1018, \\
V\{W(2,3.5)\} &= (3.5)^2\left[\Gamma\left(1+\frac{2}{2}\right) - \Gamma^2\left(1+\frac{1}{2}\right)\right] = 2.6289.
\end{aligned}$$

Exercise 2.78 The time till failure (days) of an electronic equipment has the Weibull distribution $W(1.5, 500)$. What is the probability that the failure time will not be before 600 days?

Solution 2.78 Let T be the number of days until failure. $T \sim W(1.5, 500) \sim 500W(1.5, 1)$.

$$\Pr\{T \geq 600\} = \Pr\left\{W(1.5, 1) \geq \frac{6}{5}\right\} = e^{-(6/5)^{1.5}} = 0.2686.$$

Exercise 2.79 Compute the expected value and standard deviation of a random variable having the Beta distribution Beta$\left(\frac{1}{2}, \frac{3}{2}\right)$.

Solution 2.79 Let $X \sim$ Beta$\left(\frac{1}{2}, \frac{3}{2}\right)$.

$$E\{X\} = \frac{1/2}{\frac{1}{2} + \frac{3}{2}} = \frac{1}{4}, \quad V\{X\} = \frac{\frac{1}{2} \cdot \frac{3}{2}}{2^2 \cdot 3} = \frac{1}{16} \text{ and } \sigma\{X\} = \frac{1}{4}.$$

Exercise 2.80 Show that the index of kurtosis of Beta(ν, ν) is $\beta_2 = \dfrac{3(1 + 2\nu)}{3 + 2\nu}$.

Solution 2.80 Let $X \sim$ Beta(ν, ν). The first four moments are

$$\begin{aligned}
\mu_1 &= \nu/2\nu = \frac{1}{2} \\
\mu_2 &= \frac{B(\nu + 2, \nu)}{B(\nu, \nu)} = \frac{\nu + 1}{2(2\nu + 1)} \\
\mu_3 &= \frac{B(\nu + 3, \nu)}{B(\nu, \nu)} = \frac{(\nu + 1)(\nu + 2)}{2(2\nu + 1)(2\nu + 2)} \\
\mu_4 &= \frac{B(\nu + 4, \nu)}{B(\nu, \nu)} = \frac{(\nu + 1)(\nu + 2)(\nu + 3)}{2(2\nu + 1)(2\nu + 2)(2\nu + 3)}.
\end{aligned}$$

The variance is $\sigma^2 = \dfrac{1}{4(2\nu + 1)}$ and the fourth central moment is
$\mu_4^* = \mu_4 - 4\mu_3 \cdot \mu_1 + 6\mu_2 \cdot \mu_1^2 - 3\mu_1^4 = \dfrac{3}{16(3 + 8\nu + 4\nu^2)}$.

Finally, the index of kurtosis is $\beta_2 = \dfrac{\mu_4^*}{\sigma^4} = \dfrac{3(1 + 2\nu)}{3 + 2\nu}$.

Exercise 2.81 The joint p.d.f. of two random variables (X, Y) is

$$f(x, y) = \begin{cases} \frac{1}{2}, & \text{if } (x, y) \in S \\ 0, & \text{otherwise} \end{cases}$$

where S is a square of area 2, whose vertices are $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

(i) Find the marginal p.d.f. of X and of Y.
(ii) Find $E\{X\}$, $E\{Y\}$, $V\{X\}$, $V\{Y\}$.

Solution 2.81 Let (X, Y) have a joint p.d.f.

$$f(x, y) = \begin{cases} \frac{1}{2}, & (x, y) \in S \\ 0, & \text{otherwise.} \end{cases}$$

(i) The marginal distributions of X and Y have p.d.f.'s
$f_X(x) = \frac{1}{2} \int_{-1+|x|}^{1-|x|} \mathrm{d}y = 1 - |x|, -1 < x < 1,$ and by symmetry,
$f_Y(y) = 1 - |y|, -1 < y < 1.$
(ii) $E\{X\} = E\{Y\} = 0, \quad V\{X\} = V\{Y\} = 2\int_0^1 y^2(1-y)\,\mathrm{d}y = 2B(3, 2) = \frac{1}{6}.$

Exercise 2.82 Let (X, Y) have a joint p.d.f.

$$f(x, y) = \begin{cases} \frac{1}{y} \exp\left\{-y - \frac{x}{y}\right\}, & \text{if } 0 < x, y < \infty \\ \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{COV}(X, Y)$ and the coefficient of correlation ρ_{XY}.

Solution 2.82 The marginal p.d.f. of Y is $f(y) = e^{-y}$, $y > 0$, that is, $Y \sim E(1)$. The conditional p.d.f. of X, given $Y = y$, is $f(x \mid y) = \frac{1}{y} e^{-x/y}$ which is the p.d.f. of an exponential with parameter y. Thus, $E\{X \mid Y = y\} = y$, and $E\{X\} = E\{E\{X \mid Y\}\} = E\{Y\} = 1$. Also,

$$\begin{aligned} E\{XY\} &= E\{YE\{X \mid Y\}\} \\ &= E\{Y^2\} \\ &= 2. \end{aligned}$$

Hence, $\mathrm{cov}(X, Y) = E\{XY\} - E\{X\}E\{Y\} = 1$. The variance of Y is $\sigma_Y^2 = 1$. The variance of X is

$$\begin{aligned} \sigma_X^2 &= E\{V\{X \mid Y\}\} + V\{E\{X \mid Y\}\} \\ &= E\{Y^2\} + V\{Y\} \\ &= 2 + 1 = 3. \end{aligned}$$

The correlation between X and Y is $\rho_{XY} = \frac{1}{\sqrt{3}}$.

Exercise 2.83 Show that the random variables (X, Y) whose joint distribution is defined in Example 2.27 are dependent. Find $\mathrm{COV}(X, Y)$. {ex:bivariate-dist-marginal}

Solution 2.83 Let (X, Y) have joint p.d.f. $f(x, y) = \begin{cases} 2, & \text{if } (x, y) \in T \\ 0, & \text{otherwise.} \end{cases}$

The marginal densities of X and Y are

$f_X(x) = 2(1 - x), \quad 0 \le x \le 1$

$f_Y(y) = 2(1 - y), \quad 0 \le y \le 1.$

Notice that $f(x, y) \ne f_X(x) f_Y(y)$ for $x = \frac{1}{2}, y = \frac{1}{4}$. Thus, X and Y are dependent.

$\text{cov}(X, Y) = E\{XY\} - E\{X\}E\{Y\} = E\{XY\} - \frac{1}{9}.$

$$\begin{aligned} E\{XY\} &= 2 \int_0^1 x \int_0^{1-x} y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 x(1 - x)^2 \, \mathrm{d}x \\ &= B(2, 3) = \frac{1}{12}. \end{aligned}$$

Hence, $\text{cov}(X, Y) = \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}.$

{ex:pair-rv-JN} **Exercise 2.84** Find the correlation coefficient of N and J of Example 2.31.

Solution 2.84 $J \mid N \sim B(N, p)$; $N \sim P(\lambda)$. $E\{N\} = \lambda$, $V\{N\} = \lambda$, $E\{J\} = \lambda p$.

$$\begin{aligned} V\{J\} &= E\{V\{J \mid N\}\} + V\{E\{J \mid N\}\} \\ &= E\{Np(1 - p)\} + V\{Np\} \\ &= \lambda p(1 - p) + p^2\lambda = \lambda p. \end{aligned}$$

$$\begin{aligned} E\{JN\} &= E\{NE\{J \mid N\}\} \\ &= pE\{N^2\} \\ &= p(\lambda + \lambda^2) \end{aligned}$$

Hence, $\text{cov}(J, N) = p\lambda(1 + \lambda) - p\lambda^2 = p\lambda$ and $\rho_{JN} = \dfrac{p\lambda}{\lambda\sqrt{p}} = \sqrt{p}$.

Exercise 2.85 Let X and Y be independent random variables, $X \sim G(2, 100)$ and $W(1.5, 500)$. Find the variance of XY.

Solution 2.85 Let $X \sim G(2, 100) \sim 100G(2, 1)$ and $Y \sim W(1.5, 500) \sim 500W(1.5, 1)$. Then $XY \sim 5 \times 10^4 G(2, 1) \cdot W(1.5, 1)$ and $V\{XY\} = 25 \times 10^8 \cdot V\{GW\}$, where $G \sim G(2, 1)$ and $W \sim W\left(\frac{3}{2}, 1\right)$.

$$\begin{aligned} V\{GW\} &= E\{G^2\}V\{W\} + E^2\{W\}V\{G\} \\ &= 6\left(\Gamma\left(1 + \frac{4}{3}\right) - \Gamma^2\left(1 + \frac{2}{3}\right)\right) + 2 \cdot \Gamma^2\left(1 + \frac{2}{3}\right) \\ &= 3.88404. \end{aligned}$$

Thus $V\{XY\} = 9.7101 \times 10^9$.

{ex:insert-machine-trinomial} **Exercise 2.86** Consider the trinomial distribution of Example 2.33.

(i) What is the probability that during one hour of operation there will be no more than 20 errors.
(ii) What is the conditional distribution of wrong components, given that there are 15 misinsertions in a given hour of operation?
(iii) Approximating the conditional distribution of (ii) by a Poisson distribution, compute the conditional probability of no more than 15 wrong components?

Solution 2.86 Using the notation of Example 2.33, {ex:insert-machine-trinomial}
(i) $\Pr\{J_2 + J_3 \le 20\} = B(20; 3500, 0.005) = 0.7699$.
(ii) $J_3 \mid J_2 = 15 \sim$ Binomial $B\left(3485, \dfrac{0.004}{0.999}\right)$.
(iii) $\lambda = 3485 \times \dfrac{0.004}{0.999} = 13.954$, $\Pr\{J_2 \le 15 \mid J_3 = 15\} \approx P(15; 13.954) = 0.6739$.

Exercise 2.87 In Continuation of Example 2.34, compute the correlation between J_1 and J_2. {ex:pdf-hypergeom-joint}

Solution 2.87 Using the notation of Example 2.34, the joint p.d.f. of J_1 and J_2 is {ex:pdf-hypergeom-joint}

$$p(j_1, j_2) = \frac{\binom{20}{j_1}\binom{50}{j_2}\binom{30}{20-j_1-j_2}}{\binom{100}{20}}, \quad 0 \le j_1, j_2;\ j_1 + j_2 \le 20.$$

The marginal distribution of J_1 is $H(100, 20, 20)$. The marginal distribution of J_2 is $H(100, 50, 20)$. Accordingly,

$$V\{J_1\} = 20 \times 0.2 \times 0.8 \times \left(1 - \frac{19}{99}\right) = 2.585859,$$

$$V\{J_2\} = 20 \times 0.5 \times 0.5 \times \left(1 - \frac{19}{99}\right) = 4.040404.$$

The conditional distribution of J_1, given J_2, is $H(50, 20, 20 - J_2)$. Hence,

$$\begin{aligned} E\{J_1 J_2\} &= E\{E\{J_1 J_2 \mid J_2\}\} \\ &= E\left\{J_2(20 - J_2) \times \frac{2}{5}\right\} \\ &= 8E\{J_2\} - 0.4E\{J_2^2\} \\ &= 80 - 0.4 \times 104.040404 = 38.38381 \end{aligned}$$

and $\mathrm{cov}(J_1, J_2) = -1.61616$.

Finally, the correlation between J_1 and J_2 is $\rho = \dfrac{-1.61616}{\sqrt{2.585859 \times 4.040404}} = -0.50$.

Exercise 2.88 In a bivariate normal distribution, the conditional variance of Y given X is 150 and the variance of Y is 200. What is the correlation ρ_{XY}?

Solution 2.88 $V\{Y \mid X\} = 150, \quad V\{Y\} = 200, \quad V\{Y \mid X\} = V\{Y\}(1-\rho^2)$. Hence $|\rho| = 0.5$. The sign of ρ cannot be determined.

Exercise 2.89 $n = 10$ electronic devices start to operate at the same time. The times till failure of these devices are independent random variables having an identical $E(100)$ distribution.

(i) What is the expected value of the first failure?
(ii) What is the expected value of the last failure?

Solution 2.89 **(i)** $X_{(1)} \sim E\left(\dfrac{100}{10}\right)$, $E\{X_{(1)}\} = 10.$; **(ii)** $E\{X_{(10)}\} = 100\sum_{i=1}^{10}\dfrac{1}{i} = 292.8968$.

Exercise 2.90 A factory has $n = 10$ machines of a certain type. At each given day, the probability is $p = .95$ that a machine will be working. Let J denote the number of machines that work on a given day. The time it takes to produce an item on a given machine is $E(10)$, i.e., exponentially distributed with mean $\mu = 10$ [min]. The machines operate independently of each other. Let $X_{(1)}$ denote the minimal time for the first item to be produced. Determine

(i) $P[J = k, X_{(1)} \le x]$, $k = 1, 2, \ldots$
(ii) $P[X_{(1)} \le x \mid J \ge 1]$.

Notice that when $J = 0$ no machine is working. The probability of this event is $(0.05)^{10}$.

Solution 2.90 $J \sim B(10, 0.95)$. If $\{J = j\}$, $j > 1$, $X_{(1)}$ is the minimum of a sample of j i.i.d. $E(10)$ random variables. Thus $X_{(1)} \mid J = j \sim E\left(\dfrac{10}{j}\right)$.

(i) $\Pr\{J = k, X_{(1)} \le x\} = b(k; 10, 0.95)(1 - e^{-\frac{kx}{10}})$, $k = 1, 2, \cdots, 10$.
(ii) First note that $\Pr\{J \ge 1\} = 1 - (0.05)^{10} \approx 1$.

$$\begin{aligned}\Pr\{X_{(1)} \le x \mid J \ge 1\} &= \sum_{k=1}^{10} b(k; 10, 0.95)(1 - e^{-\frac{kx}{10}})\\ &= 1 - \sum_{k=1}^{10}\binom{10}{k}(0.95e^{-\frac{x}{10}})^k(0.05)^{(10-k)}\\ &= 1 - [0.05 + 0.95e^{-x/10}]^{10} + (0.05)^{10}\\ &= 1 - (0.05 + 0.95e^{-x/10})^{10}.\end{aligned}$$

Exercise 2.91 Let $X_1, X_2, \ldots, X_{11}$ be a random sample of exponentially distributed random variables with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$.

(i) What is the p.d.f. of the median $Me = X_{(6)}$?
(ii) What is the expected value of Me?

Solution 2.91 The median is $Me = X_{(6)}$.

(a) The p.d.f. of Me is $f_{(6)}(x) = \dfrac{11!}{5!5!}\lambda(1 - e^{-\lambda x})^5 e^{-6\lambda x}$, $x \ge 0$.
(b) The expected value of Me is

$$\begin{aligned}
E\{X_{(6)}\} &= 2772\lambda \int_0^\infty x(1-e^{-\lambda x})^5 e^{-6\lambda x}\,\mathrm{d}x \\
&= 2772\lambda \sum_{j=0}^{5} (-1)^j \binom{5}{j} \int_0^\infty x e^{-\lambda x(6+j)}\,\mathrm{d}x \\
&= 2772 \sum_{j=0}^{5} (-1)^j \binom{5}{j} \frac{1}{\lambda(6+j)^2} \\
&= 0.73654/\lambda.
\end{aligned}$$

Exercise 2.92 Let X and Y be independent random variables having an $E(\beta)$ distribution. Let $T = X + Y$ and $W = X - Y$. Compute the variance of $T + \frac{1}{2}W$.

Solution 2.92 Let X and Y be i.i.d. $E(\beta)$, $T = X + Y$ and $W + X - Y$.

$$\begin{aligned}
V\left\{T + \frac{1}{2}W\right\} &= V\{\frac{3}{2}X + \frac{1}{2}Y\} \\
&= \beta^2 \left(\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) \\
&= 2.5\beta^2.
\end{aligned}$$

Exercise 2.93 Let X and Y be independent random variables having a common variance σ^2. What is the covariance $\mathrm{cov}(X, X+Y)$?

Solution 2.93 $\mathrm{cov}(X, X+Y) = \mathrm{cov}(X, X) + \mathrm{cov}(X, Y) = V\{X\} = \sigma^2$.

Exercise 2.94 Let (X, Y) have a bivariate normal distribution. What is the variance of $\alpha X + \beta Y$?

Solution 2.94 $V\{\alpha X + \beta Y\} = \alpha^2\sigma_X^2 + \beta^2\sigma_Y^2 + 2\alpha\beta\mathrm{cov}(X, Y) = \alpha^2\sigma_X^2 + \beta^2\sigma_Y^2 + 2\alpha\beta\rho_{XY}\sigma_X\sigma_Y$.

Exercise 2.95 Let X have a normal distribution $N(\mu, \sigma)$. Let $\Phi(z)$ be the standard normal c.d.f. Verify that $E\{\Phi(X)\} = P\{U < X\}$, where U is independent of X and $U \sim N(0, 1)$. Show that

$$E\{\Phi(X)\} = \Phi\left(\frac{\eta}{\sqrt{1+\sigma^2}}\right).$$

Solution 2.95 Let $U \sim N(0, 1)$ and $X \sim N(\mu, \sigma)$. We assume that U and X are independent. Then $\Phi(X) = \Pr\{U < X \mid X\}$ and therefore

$$\begin{aligned}
E\{\Phi(X)\} &= E\{\Pr\{U < X \mid X\}\} \\
&= \Pr\{U < X\} \\
&= \Pr\{U - X < 0\} \\
&= \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right).
\end{aligned}$$

The last equality follows from the fact that $U - X \sim N(-\mu, \sqrt{1+\sigma^2})$.

Exercise 2.96 Let X have a normal distribution $N(\mu, \sigma)$. Show that

$$E\{\Phi^2(X)\} = \Phi_2\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\mu}{\sqrt{1+\sigma^2}}; \frac{\sigma^2}{1+\sigma^2}\right).$$

Solution 2.96 Let U_1, U_2, X be independent random variables; U_1, U_2 i.i.d. $N(0,1)$. Then

$\Phi^2(X) = \Pr\{U_1 \le X, U_2 \le X \mid X\}$. Hence

$E\{\Phi^2(X)\} = \Pr\{U_1 \le X, U_2 \le X\} = \Pr\{U_1 - X \le 0, U_2 - X \le 0\}$.

Since $(U_1 - X, U_2 - X)$ have a bivariate normal distribution with means $(-\mu, -\mu)$ and variance-covariance matrix $V = \begin{bmatrix} 1+\sigma^2 & \sigma^2 \\ \sigma^2 & 1+\sigma^2 \end{bmatrix}$, it follows that

$$E\{\Phi^2(X)\} = \Phi_2\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\mu}{\sqrt{1+\sigma^2}}; \frac{\sigma^2}{1+\sigma^2}\right).$$

Exercise 2.97 X and Y are independent random variables having Poisson distributions, with means $\lambda_1 = 5$ and $\lambda_2 = 7$, respectively. Compute the probability that $X + Y$ is greater than 15.

Solution 2.97 Since X and Y are independent, $T = X + Y \sim P(12)$, $\Pr\{T > 15\} = 0.1556$.

Exercise 2.98 Let X_1 and X_2 be independent random variables having continuous distributions with p.d.f. $f_1(x)$ and $f_2(x)$, respectively. Let $Y = X_1 + X_2$. Show that the p.d.f. of Y is

$$g(y) = \int_{-\infty}^{\infty} f_1(x) f_2(y - x)\,\mathrm{d}x.$$

[This integral transform is called the convolution of $f_1(x)$ with $f_2(x)$. The convolution operation is denoted by $f_1 * f_2$.]

Solution 2.98 Let $F_2(x) = \int_{-\infty}^{x} f_2(z)\,\mathrm{d}z$ be the c.d.f. of X_2. Since $X_1 + X_2$ are independent

$$\begin{aligned}\Pr\{Y \le y\} &= \int_{-\infty}^{\infty} f_1(x) \Pr\{X_2 \le y - x\}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_1(x) F_2(y - x)\,\mathrm{d}x.\end{aligned}$$

Therefore, the p.d.f. of Y is

$$\begin{aligned}g(y) &= \frac{d}{dy}\Pr\{Y \le y\} \\ &= \int_{-\infty}^{\infty} f_1(x) f_2(y - x)\,\mathrm{d}x.\end{aligned}$$

Exercise 2.99 Let X_1 and X_2 be independent random variables having the uniform distributions on $(0, 1)$. Apply the convolution operation to find the p.d.f. of $Y = X_1 + X_2$.

Solution 2.99 Let $Y = X_1 + X_2$ where X_1, X_2 are i.i.d. uniform on (0,1). Then the p.d.f.'s are

$$f_1(x) = f_2(x) = I\{0 < x < 1\}$$

$$g(y) = \begin{cases} \int_0^y \mathrm{d}x = y, & \text{if } 0 \le y < 1 \\ \\ \int_{y-1}^1 \mathrm{d}x = 2 - y, & \text{if } 1 \le y \le 2. \end{cases}$$

Exercise 2.100 Let X_1 and X_2 be independent random variables having a common exponential distribution $E(1)$. Determine the p.d.f. of $U = X_1 - X_2$. [The distribution of U is called bi-exponential or Laplace and its p.d.f. is $f(u) = \dfrac{1}{2}e^{-|u|}$.]

Solution 2.100 X_1, X_2 are i.i.d. $E(1)$. $U = X_1 - X_2$. $\Pr\{U \le u\} = \int_0^\infty e^{-x} \Pr\{X_1 \le u + x\}\,\mathrm{d}x$. Notice that $-\infty < u < \infty$ and $\Pr\{X_1 \le u + x\} = 0$ if $x + u < 0$. Let $a^+ = \max(a, 0)$. Then

$$\begin{aligned} \Pr\{U \le u\} &= \int_0^\infty e^{-x}(1 - e^{-(u+x)^+})\,\mathrm{d}x \\ &= 1 - \int_0^\infty e^{-x-(u+x)^+}\,\mathrm{d}x \\ &= \begin{cases} 1 - \frac{1}{2}e^{-u}, & \text{if } u \ge 0 \\ \\ \frac{1}{2}e^{-|u|}, & \text{if } u < 0 \end{cases} \end{aligned}$$

Thus, the p.d.f. of U is $g(u) = \dfrac{1}{2}e^{-|u|}$, $-\infty < u < \infty$.

Exercise 2.101 Apply the central limit theorem to approximate $P\{X_1 + \cdots + X_{20} \le 50\}$, where $X_1, \cdots, X_{20}$ are independent random variables having a common mean $\mu = 2$ and a common standard deviation $\sigma = 10$.

Solution 2.101 $T = X_1 + \cdots + X_{20} \mathrel{\dot\sim} N(20\mu, \sqrt{20\sigma^2})$. $\Pr\{T \le 50\} \approx \Phi\left(\dfrac{50 - 40}{44.7214}\right) = 0.5885$.

Exercise 2.102 Let X have a binomial distribution $B(200, .15)$. Find the normal approximation to $\Pr\{25 < X < 35\}$.

Solution 2.102 $X \sim B(200, 0.15)$. $\mu = np = 30$, $\sigma = \sqrt{np(1 - p)} = 5.0497$.

$$\Pr\{25 < X < 35\} \approx \Phi\left(\frac{34.5 - 30}{5.0497}\right) - \Phi\left(\frac{25.5 - 30}{5.0497}\right) = 0.6271.$$

Exercise 2.103 Let X have a Poisson distribution with mean $\lambda = 200$. Find, approximately, $\Pr\{190 < X < 210\}$.

Solution 2.103 $X \sim P(200)$. $\quad \Pr\{190 < X < 210\} \approx 2\Phi\left(\dfrac{9.5}{\sqrt{200}}\right) - 1 = 0.4983$.

Exercise 2.104 $X_1, X_2, \cdots, X_{200}$ are 200 independent random variables have a common Beta distribution $B(3, 5)$. Approximate the probability $\Pr\{|\bar{X}_{200} - .375| < 0.2282\}$, where

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad n = 200.$$

Solution 2.104 $X \sim \text{Beta}(3,5)$. $\mu = E\{X\} = \dfrac{3}{8} = 0.375$. $\sigma = \sqrt{V\{X\}} = \left(\dfrac{3\cdot 5}{8^2\cdot 9}\right)^{1/2} = 0.161374$.

$$\Pr\{|\bar{X}_{200} - 0.375| < 0.2282\} \approx 2\Phi\left(\frac{\sqrt{200}\cdot 0.2282}{0.161374}\right) - 1 = 1.$$

Exercise 2.105 Use Python to compute the .95-quantiles of $t[10]$, $t[15]$, $t[20]$.

Solution 2.105 $t_{.95}[10] = 1.8125, \quad t_{.95}[15] = 1.7531, \quad t_{.95}[20] = 1.7247.$

```
print(stats.t.ppf(0.95, 10))
print(stats.t.ppf(0.95, 15))
print(stats.t.ppf(0.95, 20))
```

```
1.8124611228107335
1.7530503556925547
1.7247182429207857
```

Exercise 2.106 Use Python to compute the .95-quantiles of $F[10,30]$, $F[15,30]$, $F[20,30]$.

Solution 2.106 $F_{.95}[10,30] = 2.1646, \quad F_{.95}[15,30] = 2.0148, \quad F_{.95}[20,30] = 1.9317.$

```
print(stats.f.ppf(0.95, 10, 30))
print(stats.f.ppf(0.95, 15, 30))
print(stats.f.ppf(0.95, 20, 30))
```

```
2.164579917125473
2.0148036912954885
1.931653475236928
```

Exercise 2.107 Show that, for each $0 < \alpha < 1$, $t^2_{1-\alpha/2}[n] = F_{1-\alpha}[1,n]$.

Solution 2.107 The solution to this problem is based on the fact, which is not discussed in the text, that $t[\nu]$ is distributed like the ratio of two independent random variables, $N(0,1)$ and $\sqrt{\chi^2[\nu]/\nu}$. Accordingly, $t[n] \sim \dfrac{N(0,1)}{\sqrt{\frac{\chi^2[n]}{n}}}$, where $N(0,1)$ and $\chi^2[n]$ are independent. $t^2[n] \sim \dfrac{(N(0,1))^2}{\frac{\chi^2[n]}{n}} \sim F[1,n]$. Thus, since $\Pr\{F[1,n] \le F_{1-\alpha}[1,n]\} = 1-\alpha$.

$$\Pr\{-\sqrt{F_{1-\alpha}[1,n]} \le t[n] \le \sqrt{F_{1-\alpha}[1,n]}\} = 1-\alpha.$$

It follows that $\sqrt{F_{1-\alpha}[1,n]} = t_{1-\alpha/2}[n]$,

or $F_{1-\alpha}[1,n] = t^2_{1-\alpha/2}[n]$. (If you assign this problem, please inform the students of the above fact.)

Exercise 2.108 Verify the relationship

$$F_{1-\alpha}[\nu_1,\nu_2] = \frac{1}{F_\alpha[\nu_2,\nu_1]}, \quad 0 < \alpha < 1,$$

$\nu_1, \nu_2 = 1, 2, \cdots$.

Solution 2.108 A random variable $F[\nu_1,\nu_2]$ is distributed like the ratio of two independent random variables $\chi^2[\nu_1]/\nu_1$ and $\chi^2[\nu_2]/\nu_2$. Accordingly, $F[\nu_1,\nu_2] \sim \dfrac{\chi^2[\nu_1]/\nu_1}{\chi^2[\nu_2]/\nu_2}$ and

$$\begin{aligned}
1-\alpha &= \Pr\{F[\nu_1,\nu_2] \le F_{1-\alpha}[\nu_1,\nu_2]\} \\
&= \Pr\left\{\frac{\chi_1^2[\nu_1]/\nu_1}{\chi_2^2[\nu_2]/\nu_2} \le F_{1-\alpha}[\nu_1,\nu_2]\}\right\} \\
&= \Pr\left\{\frac{\chi_2^2[\nu_2]/\nu_2}{\chi_1^2[\nu_1]/\nu_1} \ge \frac{1}{F_{1-\alpha}[\nu_1,\nu_2]}\right\} \\
&= \Pr\left\{F[\nu_2,\nu_1] \ge \frac{1}{F_{1-\alpha}[\nu_1,\nu_2]}\right\} \\
&= \Pr\left\{F[\nu_2,\nu_1] \ge F_\alpha[\nu_2,\nu_1]\right\}.
\end{aligned}$$

Hence $F_{1-\alpha}[\nu_1,\nu_2] = \dfrac{1}{F_\alpha[\nu_2,\nu_1]}$.

Exercise 2.109 Verify the formula

$$V\{t[\nu]\} = \frac{\nu}{\nu-2}, \quad \nu > 2.$$

Solution 2.109 Using the fact that $t[\nu] \sim \dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}$, where $N(0,1)$ and $\chi^2[\nu]$ are independent,

$$\begin{aligned}
V\{t[\nu]\} &= V\left\{\frac{N(0,1)}{\sqrt{\chi^2[\nu]/\nu}}\right\} \\
&= E\left\{V\left\{\frac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\}\right\} + V\left\{E\left\{\frac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\}\right\}.
\end{aligned}$$

By independence, $V\left\{\dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\} = \dfrac{\nu}{\chi^2[\nu]}$, and $E\left\{\dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\} = 0$.

Thus, $V\{t[\nu]\} = \nu E\left\{\frac{1}{\chi^2[\nu]}\right\}$. Since $\chi^2[\nu] \sim G\left(\frac{\nu}{2}, 2\right) \sim 2G\left(\frac{\nu}{2}, 1\right)$,

$$E\left\{\frac{1}{\chi^2[\nu]}\right\} = \frac{1}{2} \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty x^{\nu-2} e^{-x}\,\mathrm{d}x$$
$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{\nu}{2} - 1\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \frac{1}{2} \cdot \frac{1}{\frac{\nu}{2} - 1} = \frac{1}{\nu - 2}.$$

Finally, $V\{t[\nu]\} = \frac{\nu}{\nu - 2}$, $\nu > 2$.

Exercise 2.110 Find the expected value and variance of $F[3, 10]$.

Solution 2.110 $E\{F[3, 10]\} = \frac{10}{8} = 1.25, \quad V\{F[3, 10]\} = \frac{2 \cdot 10^2 \cdot 11}{3 \cdot 8^2 \cdot 6} = 1.9097.$

Chapter 3
Statistical Inference and Bootstrapping

Import required modules and define required functions

```
import random
import math
import numpy as np
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
import pingouin as pg
import mistat
```

Exercise 3.1 The consistency of the sample mean, $\bar{X}_n$, in RSWR, is guaranteed by the WLLN, whenever the mean exists. Let $M_l = \frac{1}{n}\sum_{i=1}^{n} X_i^l$ be the sample estimate of the l-th moment, which is assumed to exist ($l = 1, 2, \cdots$). Show that M_r is a consistent estimator of μ_r.

Solution 3.1 By the WLLN, for any $\epsilon > 0$, $\lim_{n\to\infty} \Pr\{|M_l - \mu_l| < \epsilon\} = 1$. Hence, M_l is a consistent estimator of the l-th moment.

Exercise 3.2 Consider a population with mean μ and standard deviation $\sigma = 10.5$. Use the CLT to find, approximately how large should the sample size, n, be so that $\Pr\{|\bar{X}_n - \mu| < 1\} = .95$.

Solution 3.2 Using the CLT, $\Pr\{|\bar{X}_n - \mu| < 1\} \approx 2\Phi\left(\frac{\sqrt{n}}{\sigma}\right) - 1$. To determine the sample size n so that this probability is 0.95 we set $2\Phi\left(\frac{\sqrt{n}}{\sigma}\right) - 1 = 0.95$ and solve for n. This gives $\frac{\sqrt{n}}{\sigma} = z_{.975} = 1.96$. Thus $n \geq \sigma^2(1.96)^2 = 424$ for $\sigma = 10.5$.

Exercise 3.3 Let $X_1, \cdots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma)$. What is the moments equation estimator of the p-th quantile $\xi_p = \mu + z_p\sigma$?

Solution 3.3 $\hat{\xi}_p = \bar{X}_n + z_p\hat{\sigma}_n$, where $\hat{\sigma}_n^2 = \frac{1}{n}\sum(X_i - \bar{X}_n)^2$.

Exercise 3.4 Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution. What is the moments equations estimator of the correlation ρ?

Solution 3.4 Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with density $f(x, y; \mu, \eta, \sigma_X, \sigma_Y, \rho)$ as given in Eq. (4.6.6). Let $Z_i = X_i Y_i$ for $i = 1, \ldots, n$. Then the first moment of Z is given by

$$\begin{aligned}
\mu_1(F_Z) &= E\{Z\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y; \mu, \eta, \sigma_X, \sigma_Y, \rho)\,\mathrm{d}x\,\mathrm{d}y \\
&= \mu\eta + \rho\sigma_X\sigma_Y.
\end{aligned}$$

Using this fact, as well as the first 2 moments of X and Y, we get the following moment equations:

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} X_i = \mu && \frac{1}{n}\sum_{i=1}^{n} Y_i = \eta \\
&\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \sigma_X^2 + \mu^2 && \frac{1}{n}\sum_{i=1}^{n} Y_i^2 = \sigma_Y^2 + \eta^2 \\
&\frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \mu\eta + \rho\sigma_X\sigma_Y.
\end{aligned}$$

Solving these equations for the 5 parameters gives

$$\hat{\rho}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \bar{X}\bar{Y}}{\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right) \cdot \left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2\right)\right]^{1/2}}, \quad \text{or equivalently,}$$

$$\hat{\rho}_n = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \cdot \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)^{1/2}}.$$

Exercise 3.5 Let $X_1, X_2, \ldots, X_n$ be a sample from a beta distribution Beta(ν_1, ν_2); $0 < \nu_1, \nu_2 < \infty$. Find the moment-equation estimators of ν_1 and ν_2.

Solution 3.5 The two first moments are

$$\mu_1 = \frac{\nu_1}{\nu_1 + \nu_2}, \qquad \mu_2 = \frac{\nu_1(\nu_1 + 1)}{(\nu_1 + \nu_2)(\nu_1 + \nu_2 + 1)}.$$

Equating the theoretical moments to the sample moments $M_1 = \frac{1}{n}\sum_{i=1}^n X_i$ and $M_2 = \frac{1}{n}\sum_{i=1}^n X_i^2$, we otain with $\hat{\sigma}_n^2 = M_2 - M_1^2$ the moment equation estimators.

$$\hat{\nu}_1 = M_1(M_1 - M_2)/\hat{\sigma}_n^2 \quad \text{and} \quad \hat{\nu}_2 = (M_1 - M_2)(1 - M_1)/\hat{\sigma}_n^2.$$

Exercise 3.6 Let $\bar{Y}_1, \cdots, \bar{Y}_k$ be the means of k independent RSWR from normal distributions, $N(\mu, \sigma_i)$, $i = 1, \cdots, k$, with common means and variances σ_i^2 **known**. Let $n_1, \cdots, n_k$ be the sizes of these samples. Consider a weighted average $\bar{Y}_w = \dfrac{\sum_{i=1}^k w_i \bar{Y}_i}{\sum_{i=1}^k w_i}$, with $w_i > 0$. Show that for the estimator $\bar{Y}_w$ having smallest variance, the required weights are $w_i = \dfrac{n_i}{\sigma_i^2}$.

Solution 3.6 $V\{\bar{Y}_w\} = \left(\sum_{i=1}^k w_i^2 \dfrac{\sigma_i^2}{n_i}\right) \Big/ \left(\sum_{i=1}^k w_i\right)^2$. Let $\lambda_i = \dfrac{w_i}{\sum_{j=1}^k w_j}$, $\sum_{i=1}^k \lambda_i = 1$. We find weights λ_i which minimize $V\{\bar{Y}_w\}$, under the constraint $\sum_{i=1}^k \lambda_i = 1$. The Lagrangian is $L(\lambda_1, \ldots, \lambda_k, \rho) = \sum_{i=1}^k \lambda_i^2 \frac{\sigma_i^2}{n_i} + \rho\left(\sum_{i=1}^k \lambda_i - 1\right)$. Differentiating with respect to λ_i, we get

$$\frac{\partial}{\partial \lambda_i} L(\lambda_1, \ldots, \lambda_k, \rho) = 2\lambda_i \frac{\sigma_i^2}{n_i} + \rho, i = 1, \ldots, k \quad \text{and} \quad \frac{\partial}{\partial \rho} L(\lambda_1, \ldots, \lambda_k, \rho) = \sum_{i=1}^k \lambda_i - 1.$$

Equating the partial derivatives to zero, we get $\lambda_i^0 = -\frac{\rho}{2}\frac{n_i}{\sigma_i^2}$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i^0 = -\frac{\rho}{2}\sum_{i=1}^k \frac{n_i}{\sigma_i^2} = 1$.

Thus, $-\dfrac{\rho}{2} = \dfrac{1}{\sum_{i=1}^k n_i/\sigma_i^2}$, $\lambda_i^0 = \dfrac{n_i/\sigma_i^2}{\sum_{j=1}^k n_j/\sigma_j^2}$, and therefore $w_i = n_i/\sigma_i^2$.

Exercise 3.7 Using the formula

$$\hat{\beta}_1 = \sum_{i=1}^n w_i Y_i,$$

with $w_i = \dfrac{x_i - \bar{x}_n}{SS_x}$, $i = 1, \ldots, n$, for the LSE of the slope β in a simple linear regression, derive the formula for $V\{\hat{\beta}_1\}$. We assume that $V\{Y_i\} = \sigma^2$ for all $i = 1, \cdots, n$. You can refer to Chapter 4 for a detailed exposition of linear regression. {ch:regression}

Solution 3.7 Since the Y_i are uncorrelated,
$V\{\hat{\beta}_1\} = \sum_{i=1}^n w_i^2 V\{Y_i\} = \sigma^2 \sum_{i=1}^n \dfrac{(x_i - \bar{x}_n)^2}{SS_x^2} = \dfrac{\sigma^2}{SS_x}$, where $SS_x = \sum_{i=1}^n (x_i - \bar{x}_n)^2$.

{exc:cov-beta0-beta1}

Exercise 3.8 In continuation of the previous Exercise, derive the formula for the variance of the LSE of the intercept β_0 and $\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1)$.

Solution 3.8 Let $w_i = \dfrac{x_i - \bar{x}_n}{SS_x}$ for $i = 1, \cdots, n$ where $SS_x = \sum_{i=1}^{n}(x_i - \bar{x}_n)^2$. Then we have

$$\sum_{i=1}^{n} w_i = 0, \quad \text{and} \quad \sum_{i=1}^{n} w_i^2 = \frac{1}{SS_x}.$$

Hence,

$$\begin{aligned}
V\{\hat{\beta}_0\} &= V\{\bar{Y}_n - \hat{\beta}_1 \bar{x}_n\} \\
&= V\left\{\bar{Y}_n - \left(\sum_{i=1}^{n} w_i Y_i\right)\bar{x}_n\right\} \\
&= V\left\{\sum_{i=1}^{n}\left(\frac{1}{n} - w_i \bar{x}_n\right) Y_i\right\} \\
&= \sum_{i=1}^{n}\left(\frac{1}{n} - w_i \bar{x}_n\right)^2 \sigma^2 \\
&= \sigma^2 \sum_{i=1}^{n}\left(\frac{1}{n^2} - \frac{2 w_i \bar{x}_n}{n} + w_i^2 \bar{x}_n^2\right) \\
&= \sigma^2\left(\frac{1}{n} - \frac{2}{n}\bar{x}_n \sum_{i=1}^{n} w_i + \bar{x}_n^2 \sum_{i=1}^{n} w_i^2\right) \\
&= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}_n^2}{SS_x}\right).
\end{aligned}$$

Also

$$\begin{aligned}
\operatorname{cov}(\hat{\beta}_0, \hat{\beta}_1) &= \operatorname{cov}\left(\sum_{i=1}^{n}\left(\frac{1}{n} - w_i \bar{x}_n\right) Y_i, \sum_{i=1}^{n} w_i Y_i\right) \\
&= \sigma^2 \sum_{i=1}^{n}\left(\frac{1}{n} - w_i \bar{x}_n\right) w_i \\
&= -\sigma^2 \frac{\bar{x}_n}{SS_x}.
\end{aligned}$$

Exercise 3.9 Show that the correlation between the LSE's, $\hat{\beta}_0$ and $\hat{\beta}_1$ in the simple linear regression is

$$\rho = -\frac{\bar{x}_n}{\left(\frac{1}{n}\sum x_i^2\right)^{1/2}}.$$

Solution 3.9 The correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$ is

$$\rho_{\hat{\beta}_0,\hat{\beta}_1} = -\frac{\sigma^2 \bar{x}_n}{\sigma^2 SS_x \left[\left(\frac{1}{n} + \frac{\bar{x}_n^2}{SS_x}\right)\frac{1}{SS_x}\right]^{1/2}}$$
$$= -\frac{\bar{x}_n}{\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)^{1/2}}.$$

Exercise 3.10 Let $X_1, \cdots, X_n$ be i.i.d. random variables having a Poisson distribution $P(\lambda)$, $0 < \lambda < \infty$. Show that the MLE of λ is the sample mean $\bar{X}_n$.

Solution 3.10 $X_1, X_2, \ldots, X_n$ i.i.d., $X_1 \sim P(\lambda)$. The likelihood function is

$$L(\lambda; X_1, \ldots, X_n) = e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n X_i!}$$

Thus, $\frac{\partial}{\partial\lambda}\log L(\lambda; X_1, \ldots, X_n) = -n + \frac{\sum_{i=1}^n X_i}{\lambda}$. Equating this to zero and solving for λ, we get $\hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}_n$.

Exercise 3.11 Let $X_1, \cdots, X_n$ be i.i.d. random variables from a gamma distribution, $G(\nu, \beta)$, with **known** ν. Show that the MLE of β is $\hat{\beta}_n = \frac{1}{\nu}\bar{X}_n$, where $\bar{X}_n$ is the sample mean. What is the variance of $\hat{\beta}_n$?

Solution 3.11 Since ν is known, the likelihood of β is $L(\beta) = C_n \frac{1}{\beta^{n\nu}} e^{-\sum_{i=1}^n X_i/\beta}$, $0 < \beta < \infty$ where C_n does not depend on β. The log-likelihood function is

$$l(\beta) = \log C_n - n\nu\log\beta - \frac{1}{\beta}\sum_{i=1}^n X_i.$$

The score function is $l'(\beta) = -\frac{n\nu}{\beta} + \frac{\sum_{i=1}^n X_i}{\beta^2}$. Equating the score to 0 and solving for β, we obtain the MLE $\hat{\beta} = \frac{1}{n\nu}\sum_{i=1}^n X_i = \frac{1}{\nu}\bar{X}_n$. The variance of the MLE is $V\{\hat{\beta}\} = \frac{\beta^2}{n\nu}$.

Exercise 3.12 Consider Example 3.4. Let $X_1, \cdots, X_n$ be a random sample from a negative-binomial distribution, N.B.$(2, p)$. Show that the MLE of p is {ex:neg-binom-dist}

$$\hat{p}_n = \frac{2}{\bar{X}_n + 2},$$

where $\bar{X}_n$ is the sample mean.

(i) On the basis of the WLLN show that $\hat{p}_n$ is a consistent estimator of p [Hint: $\bar{X}_n \to E\{X\} = (2-p)/p$ in probability as $n \to \infty$].

(ii) Using the fact that if $X_1, \cdots, X_n$ are i.i.d. like N.B.(k, p) then $T_n = \sum_{i=1}^n X_i$ is distributed like N.B.(nk, p), and the results of Example 3.4, show that for large values of n, {ex:neg-binom-dist}

$$\text{Bias}(\hat{p}_n) \cong \frac{3p(1-p)}{4n} \quad \text{and}$$
$$V\{\hat{p}_n\} \cong \frac{p^2(1-p)}{2n}.$$

{exc:mgf-nb} **Solution 3.12** We proved in Exercise 2.56 that the m.g.f. of $NB(2, p)$ is

$$M_X(t) = \frac{p^2}{(1-(1-p)e^t)^2}, \quad t < -\log(1-p).$$

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $NB(2, p)$, then the m.g.f. of $T_n = \sum_{i=1}^n X_i$ is

$$M_{T_n}(t) = \frac{p^{2n}}{(1-(1-p)e^t)^{2n}}, \quad t < -\log(1-p).$$

{ex:neg-binom-dist} Thus, $T_n \sim NB(2n, p)$. According to Example 3.4, the MLE of p, based on T_n (which is a sufficient statistic) is $\hat{p}_n = \dfrac{2n}{T_n + 2n} = \dfrac{2}{\bar{X}_n + 2}$, where $\bar{X}_n = T_n/n$ is the sample mean.

(i) According to the WLLN, $\bar{X}_n$ converges in probability to $E\{X_1\} = \dfrac{2(1-p)}{p}$. Substituting $2\dfrac{1-p}{p}$ for $\bar{X}_n$ in the formula of $\hat{p}_n$ we obtain $p^* = \dfrac{2}{2 + 2\frac{1-p}{p}} = p$. This shows that the limit in probability as $n \to \infty$, of $\hat{p}_n$ is p.

{ex:neg-binom-dist} **(ii)** Substituting $k = 2n$ in the formulas of Example 3.4 we obtain

$$\text{Bias}(\hat{p}_n) \approx \frac{3p(1-p)}{4n} \quad \text{and} \quad V\{\hat{p}_n\} \approx \frac{p^2(1-p)}{2n}.$$

Exercise 3.13 Let $X_1, \ldots, X_n$ be a random sample from a shifted exponential distribution

$$f(x; \mu, \beta) = \frac{1}{\beta} \exp\left\{-\frac{x-\mu}{\beta}\right\}, \quad x \geq \mu,$$

where $0 < \mu, \beta < \infty$.

(i) Show that the sample minimum $X_{(1)}$ is an MLE of μ.

(ii) Find the MLE of β.

(iii) What are the variances of these MLE's?

Solution 3.13 The likelihood function of μ and β is

$$L(\mu, \beta) = I\{X_{(1)} \geq \mu\} \frac{1}{\beta^n} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^n (X_{(i)} - X_{(1)}) - \frac{n}{\beta}(X_{(1)} - \mu)\right\},$$

for $-\infty < \mu \leq X_{(1)}, \quad 0 < \beta < \infty$.

(i) $L(\mu, \beta)$ is maximized by $\hat{\mu} = X_{(1)}$, that is,

$$L^*(\beta) = \sup_{\mu \leq X_{(1)}} L(\mu, \beta) = \frac{1}{\beta^n} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^{n} (X_{(i)} - X_{(1)}) \right\}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the ordered statistics.

(ii) Furthermore $L^*(\beta)$ is maximized by $\hat{\beta}_n = \frac{1}{n} \sum_{i=2}^{n} (X_{(i)} - X_{(1)})$. The MLEs are $\hat{\mu} = X_{(1)}$, and $\hat{\beta}_n = \frac{1}{n} \sum_{i=2}^{n} (X_{(i)} - X_{(1)})$.

(iii) $X_{(1)}$ is distributed like $\mu + E\left(\frac{\beta}{n}\right)$, with p.d.f.

$$f_{(1)}(x; \mu, \beta) = I\{x \geq \mu\} \frac{n}{\beta} e^{-\frac{n}{\beta}(x-\mu)}.$$

Thus, the joint p.d.f. of $(X_1, \ldots, X_n)$ is factored to a product of the p.d.f. of $X_{(1)}$ and a function of $\hat{\beta}_n$, which does not depend on $X_{(1)}$ (nor on μ). This implies that $X_{(1)}$ and $\hat{\beta}_n$ are independent. $V\{\hat{\mu}\} = V\{X_{(1)}\} = \frac{\beta^2}{n^2}$. It can be shown that $\hat{\beta}_n \sim \frac{1}{n} G(n-1, \beta)$. Accordingly, $V\{\hat{\beta}_n\} = \frac{n-1}{n^2}\beta^2 = \frac{1}{n}\left(1 - \frac{1}{n}\right)\beta^2$.

Exercise 3.14 We wish to test that the proportion of defective items in a given lot is smaller than $P_0 = 0.03$. The alternative is that $P > P_0$. A random sample of $n = 20$ is drawn from the lot **with** replacement (RSWR). The number of observed defective items in the sample is $X = 2$. Is there sufficient evidence to reject the null hypothesis that $P \leq P_0$?

Solution 3.14 In sampling with replacement, the number of defective items in the sample, X, has the binomial distribution $B(n, p)$. We test the hypotheses $H_0 : p \leq 0.03$ against $H_1 : p > 0.03$. H_0 is rejected if $X > B^{-1}(1-\alpha, 20, 0.03)$. For $\alpha = 0.05$ the rejection criterion is $k_\alpha = B^{-1}(0.95, 20, 0.03) = 2$. Since the number of defective items in the sample is $X = 2$, H_0 is not rejected at the $\alpha = 0.05$ significance level.

Exercise 3.15 Compute and plot the operating characteristic curve OC(p), for binomial testing of $H_0 : P \leq P_0$ versus $H_1 : P > P_0$, when the hypothesis is accepted if 2 or less defective items are found in a RSWR of size $n = 30$.

Solution 3.15 The OC function is $OC(p) = B(2; 30, p)$, $0 < p < 1$. A plot of this OC function is given in Figure 3.1. {fig:PlotOCcurveBinomial_2_30}

```
/usr/local/lib/python3.9/site-packages/outdated/utils.py:14:
OutdatedPackageWarning: The package pingouin is out of date. Your
version is 0.5.0, the latest is 0.5.1.
Set the environment variable OUTDATED_IGNORE=1 to disable these
warnings.
  return warn(
```

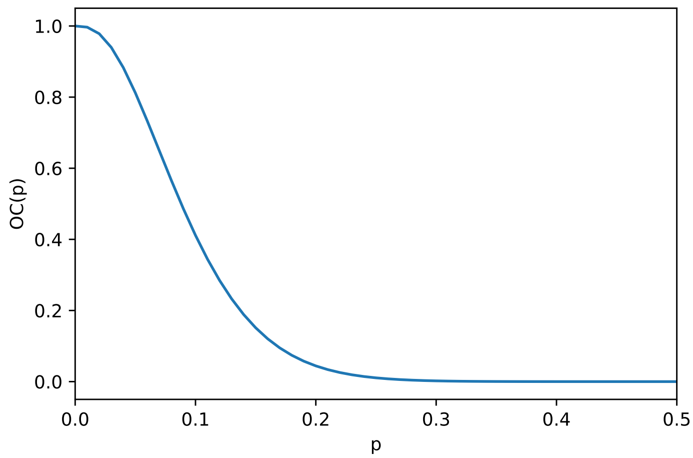


Fig. 3.1 The OC Function $B(2; 30, p)$.

{fig:PlotOCcurveBinomial_2_30}

Exercise 3.16 For testing the hypothesis $H_0 : P = 0.01$ versus $H_1 : P = 0.03$, concerning the parameter P of a binomial distribution, how large should the sample be, n, and what should be the critical value, k, if we wish that error probabilities will be $\alpha = 0.05$ and $\beta = 0.05$? [Use the normal approximation to the binomial.]

Solution 3.16 Let $p_0 = 0.01$, $p_1 = 0.03$, $\alpha = 0.05$, $\beta = 0.05$. According to Eq. (3.3.12), the sample size n should satisfy

$$1 - \Phi\left(\frac{p_1 - p_0}{\sqrt{p_1 q_1}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{p_1 q_1}}\right) = \beta$$

or, equivalently,

$$\frac{p_1 - p_0}{\sqrt{p_1 q_1}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{p_1 q_1}} = z_{1-\beta}.$$

This gives

$$\begin{aligned} n &\approx \frac{(z_{1-\alpha}\sqrt{p_0 q_0} + z_{1-\beta}\sqrt{p_1 q_1}\,)^2}{(p_1 - p_0)^2} \\ &= \frac{(1.645)^2(\sqrt{0.01 \times 0.99} + \sqrt{0.03 \times 0.97})^2}{(0.02)^2} = 494. \end{aligned}$$

For this sample size, the critical value is $k_\alpha = np_0 + z_{1-\alpha}\sqrt{np_0 q_0} = 8.58$. Thus, H_0 is rejected if there are more than 8 “successes” in a sample of size 494.

Exercise 3.17 As will be discussed in Chapter 10 in the Industrial Statistics book, the Shewhart 3-σ control charts, for statistical process control provide repeated tests of the hypothesis that the process mean is equal to the nominal one, μ_0. If a sample mean $\bar{X}_n$ falls outside the limits $\mu_0 \pm 3\dfrac{\sigma}{\sqrt{n}}$, the hypothesis is rejected.

(i) What is the probability that $\bar{X}_n$ will fall outside the control limits when $\mu = \mu_0$?
(ii) What is the probability that when the process is in control, $\mu = \mu_0$, all sample means of 20 consecutive independent samples, will be within the control limits?
(iii) What is the probability that a sample mean will fall outside the control limits when μ changes from μ_0 to $\mu_1 = \mu_0 + 2\dfrac{\sigma}{\sqrt{n}}$?
(iv) What is the probability that, a change from μ_0 to $\mu_1 = \mu_0 + 2\dfrac{\sigma}{\sqrt{n}}$, will not be detected by the next 10 sample means?

Solution 3.17 $\bar{X}_n \sim N(\mu, \dfrac{\sigma}{\sqrt{n}})$.

(i) If $\mu = \mu_0$ the probability that $\bar{X}_n$ will be outside the control limits is

$$\Pr\left\{\bar{X}_n < \mu_0 - \frac{3\sigma}{\sqrt{n}}\right\} + \Pr\left\{\bar{X}_n > \mu_0 + \frac{3\sigma}{\sqrt{n}}\right\} = \Phi(-3) + 1 - \Phi(3) = 0.0027.$$

(ii) $(1 - 0.0027)^{20} = 0.9474$.

(iii) If $\mu = \mu_0 + 2(\sigma/\sqrt{n})$, the probability that $\bar{X}_n$ will be outside the control limits is

$$\Phi(-5) + 1 - \Phi(1) = 0.1587.$$

(iv) $(1 - 0.1587)^{10} = 0.1777$.

Exercise 3.18 Consider the data in file **SOCELL.csv**. Use Python to test whether the mean ISC at time t_1 is significantly smaller than 4 (Amp). [Use 1-sample t-test.]

Solution 3.18 We can run the 1-sample t-test in Python as follows:

```
socell = mistat.load_data('SOCELL')
t1 = socell['t1']

statistic, pvalue = stats.ttest_1samp(t1, 4.0)
# divide pvalue by two for one-sided test
pvalue = pvalue / 2
print(f'pvalue {pvalue:.2f}')
```

```
pvalue 0.35
```

The hypothesis $H_0 : \mu \geq 4.0$ amps is not rejected.

Exercise 3.19 Is the mean of ISC for time t_2 significantly larger than 4 (Amp)?

Solution 3.19 We can run the 1-sample t-test in Python as follows:

```
socell = mistat.load_data('SOCELL')
t2 = socell['t2']

statistic, pvalue = stats.ttest_1samp(t2, 4.0)
```

```
# divide pvalue by two for one-sided test
pvalue = pvalue / 2
print(f'pvalue {pvalue:.2f}')
```

```
pvalue 0.03
```

The hypothesis $H_0 : \mu \geq 4.0$ amps is rejected at a 0.05 level of significance.

Exercise 3.20 Consider a one sided t-test based on a sample of size $n = 30$, with $\alpha = 0.01$. Compute the OC(δ) as a function of $\delta = (\mu - \mu_0)/\sigma$, $\mu > \mu_0$.

Solution 3.20 Let $n = 30$, $\alpha = 0.01$. The $OC(\delta)$ function for a one-sided t-test is

$$\begin{aligned} OC(\delta) &= 1 - \Phi\left(\frac{\delta\sqrt{30} - 2.462 \times (1 - \frac{1}{232})}{(1 + \frac{6.0614}{58})^{1/2}}\right) \\ &= 1 - \Phi(5.2117\delta - 2.3325). \end{aligned}$$

```
delta = np.linspace(0, 1.0, 11)

a = np.sqrt(30)
b = 2.462 * ( 1 - 1/232)
f = np.sqrt(1 + 6.0614 / 58)
OC_delta = 1 - stats.norm.cdf((a * delta - b) / f)
```

Values of $OC(\delta)$ for $\delta = 0, 1(0.1)$ are given in the following table.

δ	OC(δ)
0.0	0.990164
0.1	0.964958
0.2	0.901509
0.3	0.779063
0.4	0.597882
0.5	0.392312
0.6	0.213462
0.7	0.094149
0.8	0.033120
0.9	0.009188
1.0	0.001994

Exercise 3.21 Compute the OC function for testing the hypothesis $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$, when $n = 31$ and $\alpha = 0.10$.

Solution 3.21 Let $n = 31$, $\alpha = 0.10$. The OC function for testing $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$ is

$$
\begin{aligned}
OC(\sigma^2) &= \Pr\left\{S^2 \le \frac{\sigma_0^2}{n-1}\chi^2_{0.9}[n-1]\right\} \\
&= \Pr\left\{\chi^2[30] \le \frac{\sigma_0^2}{\sigma^2}\chi^2_{0.9}[30]\right\} \\
&= 1 - P\left(\frac{30}{2} - 1; \frac{\sigma_0^2}{\sigma^2} \cdot \frac{\chi^2_{0.9}[30]}{2}\right).
\end{aligned}
$$

```
sigma2 = np.linspace(1, 2, 11)
OC_sigma2 = 1 - stats.poisson.cdf(30 / 2 - 1,
                    stats.chi2(30).ppf(0.90) / (2 * sigma2))
```

The values of $OC(\sigma^2)$ for $\sigma^2 = 1, 2(0.1)$ are given in the following table: (Here $\sigma_0^2 = 1$.)

σ^2	OC(σ^2)
1.0	0.900000
1.1	0.810804
1.2	0.700684
1.3	0.582928
1.4	0.469471
1.5	0.368201
1.6	0.282781
1.7	0.213695
1.8	0.159540
1.9	0.118063
2.0	0.086834

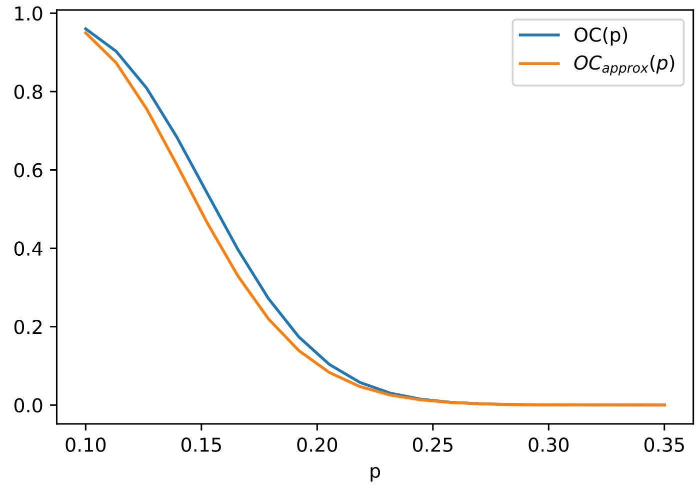
Exercise 3.22 Compute the OC function in testing $H_0 : p \le p_0$ versus $H_1 : p > p_0$ in the binomial case, when $n = 100$ and $\alpha = 0.05$.

Solution 3.22 The OC function, for testing $H_0 : p \le p_0$ against $H_1 : p > p_0$ is approximated by

$$
OC(p) = 1 - \Phi\left(\frac{p - p_0}{\sqrt{pq}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{pq}}\right),
$$

for $p \ge p_0$. In Figure 3.2 we present the graph of $OC(p)$, for $p_0 = 0.1$, $n = 100$, $\alpha = 0.05$ both using the exact solution and the normal approximation. {fig:OC_OCapprox}

```
n = 100
p0 = 0.1
alpha = 0.05

c_alpha = stats.binom(n, p0).ppf(1 - alpha)
p = np.linspace(0.1, 0.35, 20)
OC_exact = stats.binom(n, p).cdf(c_alpha)
```



{fig:OC_OCapprox} **Fig. 3.2** Comparison exact to normal approximation of $OC(p)$

```
z_ma = stats.norm.ppf(1 - alpha)
q0 = 1- p0
pq = p * (1 - p)
OC_approx = 1 - stats.norm.cdf((p - p0) * np.sqrt((n / pq)) -
                               z_ma * np.sqrt(p0 * q0 / pq))
df = pd.DataFrame({
  'p': p,
  'OC(p)': OC_exact,
  '$OC_{approx}(p)$': OC_approx,
})
df.plot.line(x='p', y=['OC(p)', '$OC_{approx}(p)$'])
plt.show()
```

Exercise 3.23 Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma)$. For testing $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$ we use the test which reject H_0 if $S_n^2 \geq \dfrac{\sigma_0^2}{n-1}\chi^2_{1-\alpha}[n-1]$, where S_n^2 is the sample variance. What is the power function of this test?

Solution 3.23 The power function is

$$\begin{aligned}\psi(\sigma^2) &= \Pr\left\{S^2 \geq \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha}[n-1]\right\}\\ &= \Pr\left\{\chi^2[n-1] \geq \frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}[n-1]\right\}.\end{aligned}$$

Exercise 3.24 Let $S_{n_1}^2$ and $S_{n_2}^2$ be the variances of two independent samples from normal distributions $N(\mu_i, \sigma_i)$, $i = 1, 2$. For testing $H_0 : \dfrac{\sigma_1^2}{\sigma_2^2} \leq 1$ against $H_1 : \dfrac{\sigma_1^2}{\sigma_2^2} >$

1, we use the F-test, which rejects H_0 when $F = \dfrac{S_{n_1}^2}{S_{n_2}^2} > F_{1-\alpha}[n_1 - 1, n_2 - 1]$. What is the power of this test, as a function of $\rho = \sigma_1^2/\sigma_2^2$?

Solution 3.24 The power function is $\psi(\rho) = \Pr\left\{F[n_1 - 1, n_2 - 1] \geq \dfrac{1}{\rho} F_{1-\alpha}[n_1 - 1, n_2 - 1]\right\}$, for $\rho \geq 1$, where $\rho = \dfrac{\sigma_1^2}{\sigma_2^2}$.

Exercise 3.25 A random sample of size $n = 20$ from a normal distribution gave the following values: 20.74, 20.85, 20.54, 20.05, 20.08, 22.55, 19.61, 19.72, 20.34, 20.37, 22.69, 20.79, 21.76, 21.94, 20.31, 21.38, 20.42, 20.86, 18.80, 21.41. Compute

(i) Confidence interval for the mean μ, at level of confidence $1 - \alpha = .99$.
(ii) Confidence interval for the variance σ^2, at confidence level $1 - \alpha = .99$.
(iii) A confidence interval for σ, at level of confidence $1 - \alpha = .99$.

Solution 3.25 **(i)** Using the following Python commands we get a 99% C.I. for μ:

```
data = [20.74, 20.85, 20.54, 20.05, 20.08, 22.55, 19.61, 19.72,
        20.34, 20.37, 22.69, 20.79, 21.76, 21.94, 20.31, 21.38,
        20.42, 20.86, 18.80, 21.41]
alpha = 1 - 0.99

df = len(data) - 1
mean = np.mean(data)
sem = stats.sem(data)

print(stats.t.interval(1 - alpha, df, loc=mean, scale=sem))
```

```
(20.136889216656858, 21.38411078334315)
```

Confidence Intervals

Variable	N	Mean	StDev	SE Mean	99.0% C.I.
Sample	20	20.760	0.975	0.218	(20.137, 21.384)

(ii) A 99% C.I. for σ^2 is $(0.468, 2.638)$.

```
var = np.var(data, ddof=1)
print(df * var / stats.chi2(df).ppf(1 - alpha/2))
print(df * var / stats.chi2(df).ppf(alpha/2))
```

```
0.46795850248657883
2.6380728125212016
```

(iii) A 99% C.I. for σ is $(0.684, 1.624)$.

Exercise 3.26 Let C_1 be the event that a confidence interval for the mean, μ, covers it. Let C_2 be the event that a confidence interval for the standard deviation σ covers it. The probability that both μ **and** σ are simultaneously covered is

$$\Pr\{C_1 \cap C_2\} = 1 - \Pr\{\overline{C_1 \cap C_2}\}$$
$$= 1 - \Pr\{\bar{C}_1 \cup \bar{C}_2\} \geq 1 - \Pr\{\bar{C}_1\} - \Pr\{\bar{C}_2\}.$$

This inequality is called the **Bonferroni inequality**. Apply this inequality and the results of the previous exercise to determine the confidence interval for $\mu + 2\sigma$, at level of confidence not smaller than 0.98.

Solution 3.26 Let $(\underline{\mu}_{.99}, \bar{\mu}_{.99})$ be a confidence interval for μ, at level 0.99. Let $(\underline{\sigma}_{.99}, \bar{\sigma}_{.99})$ be a confidence interval for σ at level 0.99. Let $\underline{\xi} = \underline{\mu}_{.99} + 2\underline{\sigma}_{.99}$ and $\bar{\xi} = \bar{\mu}_{.99} + 2\bar{\sigma}_{.99}$. Then

$$\Pr\{\underline{\xi} \leq \mu + 2\sigma \leq \bar{\xi}\} \geq \Pr\{\underline{\mu}_{.99} \leq \mu \leq \bar{\mu}_{.99},\ \underline{\sigma}_{.99} \leq \sigma \leq \bar{\sigma}_{.99}\} \geq 0.98.$$

Thus, $(\underline{\xi}, \bar{\xi})$ is a confidence interval for $\mu + 2\sigma$, with confidence level greater or equal to 0.98. Using the data of the previous problem, a 98% C.I. for $\mu + 2\sigma$ is $(21.505, 24.632)$.

Exercise 3.27 20 independent trials yielded $X = 17$ successes. Assuming that the probability for success in each trial is the same, θ, determine the confidence interval for θ at level of confidence 0.95.

Solution 3.27 Let $X \sim B(n, \theta)$. For $X = 17$ and $n = 20$, a confidence interval for θ, at level 0.95, is (0.6211, 0.9679).

```
alpha = 1 - 0.95
X = 17
n = 20
F1 = stats.f(2*(n-X+1), 2*X).ppf(1 - alpha/2)
F2 = stats.f(2*(X+1), 2*(n-X)).ppf(1 - alpha/2)
pL = X / (X + (n-X+1) * F1)
pU = (X+1) * F2 / (n-X + (X+1) * F2)
print(pL, pU)
```

```
0.6210731734546859 0.9679290628145363
```

Exercise 3.28 Let $X_1, \cdots, X_n$ be a random sample from a Poisson distribution with mean λ. Let $T_n = \sum_{i=1}^n X_i$. Using the relationship between the Poisson and the gamma c.d.f. we can show that a confidence interval for the mean λ, at level $1 - \alpha$, has lower and upper limits, λ_L and λ_U, where

$$\lambda_L = \frac{1}{2n}\chi^2_{\alpha/2}[2T_n + 2], \quad \text{and}$$
$$\lambda_U = \frac{1}{2n}\chi^2_{1-\alpha/2}[2T_n + 2].$$

The following is a random sample of size $n = 10$ from a Poisson distribution 14, 16, 11, 19, 11, 9, 12, 15, 14, 13. Determine a confidence interval for λ at level of confidence 0.95. You can calculate the confidence intervals using either the exact value $\chi^2_p[\nu]$ or use an approximation. For large number of degrees of freedom $\chi^2_p[\nu] \approx \nu + z_p\sqrt{2\nu}$, where z_p is the p-th quantile of the standard normal distribution.

Solution 3.28 From the data we have $n = 10$ and $T_{10} = 134$. For $\alpha = 0.05$, $\lambda_L = 11.319$ and $\lambda_U = 15.871$.

```
X = [14, 16, 11, 19, 11, 9, 12, 15, 14, 13]
alpha = 1 - 0.95
T_n = np.sum(X)

# exact solution
print(stats.chi2(2 * T_n + 2).ppf(alpha/2) / (2 * len(X)))
print(stats.chi2(2 * T_n + 2).ppf(1 - alpha/2) / (2 * len(X)))

# approximate solution
nu = 2 * T_n + 2
print((nu + stats.norm.ppf(alpha/2) * np.sqrt(2 * nu)) / (2 * len(X)))
print((nu + stats.norm.ppf(1-alpha/2) * np.sqrt(2 * nu)) / (2 * len(X)))
```

```
11.318870163746238
15.870459268116013
11.222727638613012
15.777272361386988
```

Exercise 3.29 The mean of a random sample of size $n = 20$, from a normal distribution with $\sigma = 5$, is $\bar{Y}_{20} = 13.75$. Determine a $1 - \beta = .90$ content tolerance interval with confidence level $1 - \alpha = .95$.

Solution 3.29 For $n = 20$, $\sigma = 5$, $\bar{Y}_{20} = 13.75$, $\alpha = 0.05$ and $\beta = 0.1$, the tolerance interval is (3.33, 24.17).

Exercise 3.30 Use the **YARNSTRG.csv** data file to determine a (.95,.95) tolerance interval for log-yarn strength. [Hint: Notice that the interval is $\bar{Y}_{100} \pm kS_{100}$, where $k = t(.025, .025, 100)$.]

Solution 3.30 Use the following Python code:

```
yarnstrg = mistat.load_data('YARNSTRG')
n = len(yarnstrg)
Ybar = yarnstrg.mean()
S = yarnstrg.std()

alpha, beta = 0.025, 0.025
z_1a = stats.norm.ppf(1-alpha)
z_1b = stats.norm.ppf(1-beta)
z_a = stats.norm.ppf(alpha)
z_b = stats.norm.ppf(beta)

t_abn = (z_1b/(1 - z_1a**2/(2*n)) +
         z_1a*np.sqrt(1 + z_b**2/2 - z_1a**2/(2*n)) /
             (np.sqrt(n)*(1 - z_1a**2/(2*n))))

print(Ybar - t_abn*S, Ybar + t_abn*S)
```

```
0.7306652047594424 5.117020795240556
```

From the data we have $\bar{X}_{100} = 2.9238$ and $S_{100} = 0.9378$.

$$t(0.025, 0.025, 100) = \frac{1.96}{1 - 1.96^2/200} + \frac{1.96(1 + \frac{1.96^2}{2} - \frac{1.96^2}{200})^{1/2}}{10(1 - \frac{1.96^2}{200})}$$
$$= 2.3388.$$

The tolerance interval is $(0.7306, 5.1171)$.

Exercise 3.31 Use the minimum and maximum of the log-yarn strength (see previous problem) to determine a distribution free tolerance interval. What are the values of α and β for your interval. How does it compare with the interval of the previous problem?

Solution 3.31 From the data, $Y_{(1)} = 1.151$ and $Y_{(100)} = 5.790$. For $n = 100$ and $\beta = 0.10$ we have $1 - \alpha = 0.988$. For $\beta = 0.05$, $1 - \alpha = 0.847$, the tolerance interval is $(1.151, 5.790)$. The nonparametric tolerance interval is shorter and is shifted to the right with a lower confidence level.

Exercise 3.32 Make a normal Q-Q plot to test, graphically, whether the ISC-t_1 of data file **SOCELL.csv**, is normally distributed.

Solution 3.32 The following is a normal probability plot of ISC-t_1:

{fig:qqplotISCT1Socell} According to Figure 3.3, the hypothesis of normality is not rejected.

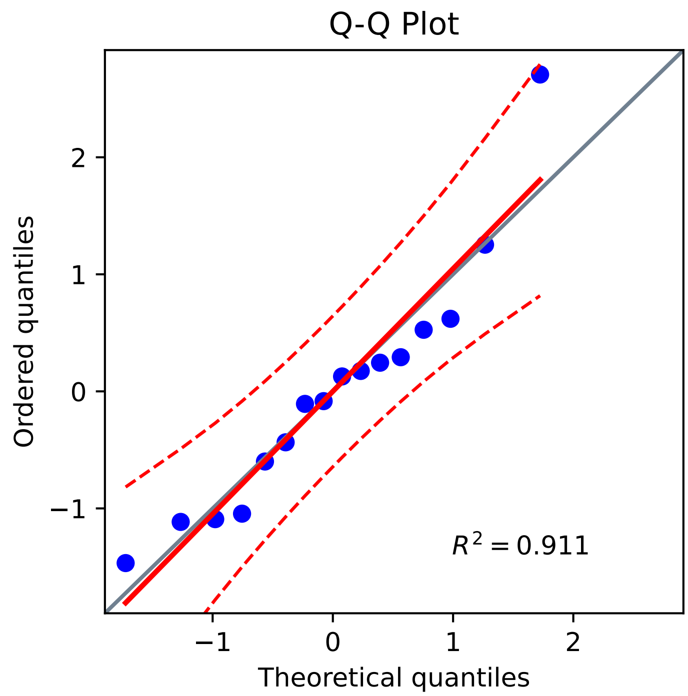
Exercise 3.33 Using Python and data file **CAR.csv**.

(i) Test graphically whether the turn diameter is normally distributed.
(ii) Test graphically whether the log (horse-power) is normally distributed.

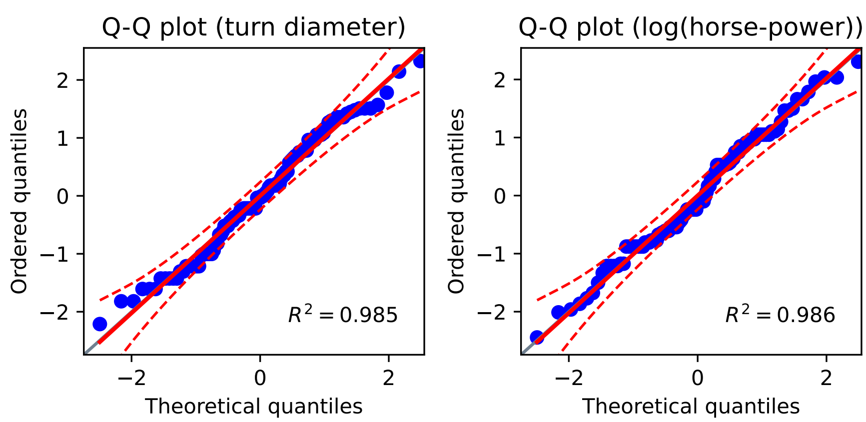
{fig:qqplotCar} **Solution 3.33** As is shown in the normal probability plots in Figure 3.4, the hypothesis of normality is not rejected in either case.

Exercise 3.34 Use the **CAR.csv** file. Make a frequency distribution of turn-diameter, with $k = 11$ intervals. Fit a normal distribution to the data and make a chi-squared test of the goodness of fit.

Solution 3.34 Frequency distribution for turn diameter:



{fig:qqplotISCT1Socell} **Fig. 3.3** Q-Q plot of ISC-t_1 (**SOCELL.csv**)



{fig:qqplotCar} **Fig. 3.4** Q-Q plot of **CAR.csv** data

Interval	Observed	Expected	$(0-E)^2/E$
- 31	11	8.1972	0.9583
31 - 32	8	6.3185	0.4475
32 - 33	9	8.6687	0.0127
33 - 34	6	10.8695	2.1815
34 - 35	18	12.4559	2.4677
35 - 36	8	13.0454	1.9513
36 - 37	13	12.4868	0.0211
37 - 38	6	10.9234	2.2191
38 - 39	9	8.7333	0.0081
39 - 40	8	6.3814	0.4106
40 -	13	9.0529	1.7213
Total	109	—	12.399

The expected frequencies were computed for $N(35.5138, 3.3208)$. Here $\chi^2 = 12.4$, d.f. = 8 and the P value is 0.134. The differences from normal are not significant.

```
            observed   expected  (O-E)^2/E
[28, 31)          11   8.197333   0.958231
[31, 32)           8   6.318478   0.447500
[32, 33)           9   8.668695   0.012662
[33, 34)           6  10.869453   2.181487
[34, 35)          18  12.455897   2.467673
[35, 36)           8  13.045357   1.951317
[36, 37)          13  12.486789   0.021093
[37, 38)           6  10.923435   2.219102
[38, 39)           9   8.733354   0.008141
[39, 40)           8   6.381395   0.410550
[40, 44)          13   9.052637   1.721231
chi2-statistic of fit 12.398987638400024
chi2[8] for 95% 15.50731305586545
p-value of observed statistic 0.1342700576126994
```

Exercise 3.35 Using Python and the **CAR.csv** data file. Compute the K.S. test statistic D_n^* for the turn-diameter variable, testing for normality. Compute k_α^* for $\alpha = .05$. Is D_n^* significant?

Solution 3.35 In Python:

```
KstestResult(statistic=0.07019153486614366, pvalue=0.6303356787948367)
k_alpha 0.08514304524687971
```

$D_{109}^* = 0.0702$. For $\alpha = 0.05$ $k_\alpha^* = 0.895/\left(\sqrt{109} - 0.01 + \frac{0.85}{\sqrt{109}}\right) = 0.0851$. The deviations from the normal distribution are not significant.

Exercise 3.36 The daily demand (loaves) for whole wheat bread at a certain bakery has a Poisson distribution with mean $\lambda = 100$. The loss to the bakery for undemanded unit at the end of the day is $C_1 = \$0.10$. On the other hand the penalty for a shortage of a unit is $C_2 = \$0.20$. How many loaves of whole wheat bread should be baked every day?

Solution 3.36 For $X \sim P(100)$, $n^0 = P^{-1}\left(\frac{0.2}{0.3}, 100\right) = 100 + z_{.67} \times 10 = 105$.

Exercise 3.37 A random variable X has the binomial distribution $B(10, p)$. The parameter p has a beta prior distribution $\text{Beta}(3, 7)$. What is the posterior distribution of p, given $X = 6$?

Solution 3.37 Given $X = 6$, the posterior distribution of p is Beta(9,11).

Exercise 3.38 In continuation to the previous exercise, find the posterior expectation and posterior standard deviation of p.

Solution 3.38 $E\{p \mid X = 6\} = \dfrac{9}{20} = 0.45$ and $V\{p \mid X = 6\} = \dfrac{99}{20^2 \times 21} = 0.0118$ so $\sigma_{p|X=6} = 0.1086$.

Exercise 3.39 A random variable X has a Poisson distribution with mean λ. The parameter λ has a gamma, $G(2, 50)$, prior distribution.

(i) Find the posterior distribution of λ given $X = 82$.
(ii) Find the .025-th and .975-th quantiles of this posterior distribution.

Solution 3.39 Let $X \mid \lambda \sim P(\lambda)$ where $\lambda \sim G(2, 50)$.

(i) The posterior distribution of $\lambda \mid X = 82$ is $G\left(84, \dfrac{50}{51}\right)$.

(ii) $G_{.025}\left(84, \dfrac{50}{51}\right) = 65.6879$ and $G_{.975}\left(84, \dfrac{50}{51}\right) = 100.873$.

Exercise 3.40 A random variable X has a Poisson distribution with mean which is either $\lambda_0 = 70$ or $\lambda_1 = 90$. The prior probability of λ_0 is $1/3$. The losses due to wrong actions are $r_1 = \$100$ and $r_2 = \$150$. Observing $X = 72$, which decision would you take?

Solution 3.40 The posterior probability for $\lambda_0 = 70$ is

$$\pi(72) = \frac{\frac{1}{3} p(72; 70)}{\frac{1}{3} p(72; 70) + \frac{2}{3} p(72; 90)} = 0.771.$$

$H_0 : \lambda = \lambda_0$ is accepted if $\pi(X) > \dfrac{r_0}{r_0 + r_1} = 0.4$. Thus, H_0 is accepted.

Exercise 3.41 A random variable X is normally distributed, with mean μ and standard deviation $\sigma = 10$. The mean μ is assigned a prior normal distribution with mean $\mu_0 = 50$ and standard deviation $\tau = 5$. Determine a credibility interval for μ, at level 0.95. Is this credibility interval also a HPD interval?

Solution 3.41 The credibility interval for μ is (43.235,60.765). Since the posterior distribution of μ is symmetric, this credibility interval is also a HPD interval.

Exercise 3.42 Read file **CAR.csv** in Python using `mistat.load_data`. There are five variables stored in columns. Write a function which samples 64 values from column `mpg` (MPG/City), with replacement and store in a variable. Let $k1$ be the mean of the sample. Execute this function $M = 200$ times to obtain a sampling distribution of the sample means. Check graphically whether this sampling distribution is approximately normal. Also check whether the standard deviation of the sampling distribution is approximately $S/8$, where S is the standard deviation of `mpg`.

Solution 3.42 In Python:

```
random.seed(1)
car = mistat.load_data('CAR')

def mean_of_sample_with_replacement(x, n):
  sample = random.choices(x, k=n)
  return np.mean(sample)

means = [mean_of_sample_with_replacement(car['mpg'], 64)
         for _ in range(200)]

fig, ax = plt.subplots(figsize=[4, 4])
pg.qqplot(means, ax=ax)
plt.show()
```

{fig:qqplotSampleMeansMPG} As Figure 3.5 shows, the resampling distribution is approximately normal.

```
S = np.std(car['mpg'], ddof=1)
print('standard deviation of mpg', S)
print('S/8', S/8)
S_resample = np.std(means, ddof=1)
print('S.E.\{X\}', S_resample)
```

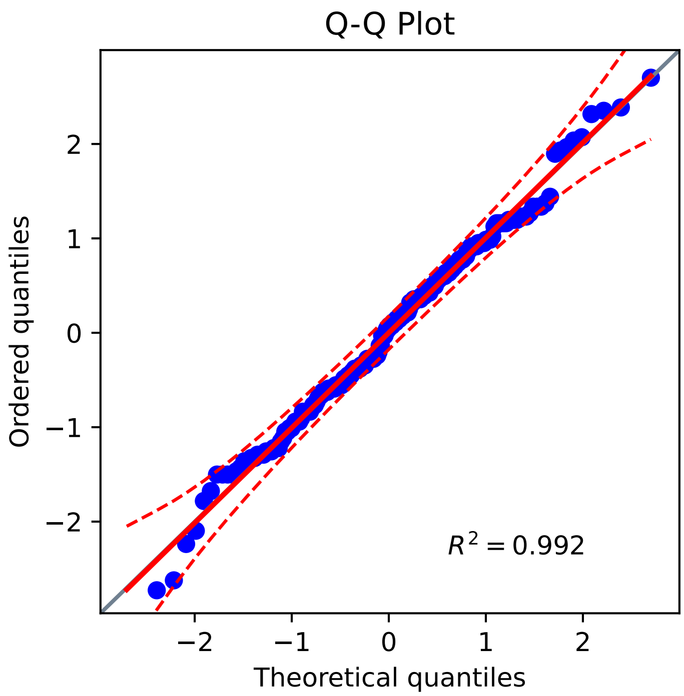
```
standard deviation of mpg 3.9172332424696052
S/8 0.48965415530870066
S.E.\{X\} 0.44718791755315535
```

Executing the macro 200 times, we obtained $\text{S.E.}\{\bar{X}\} = \frac{S}{8} = 0.48965$. The standard deviation of the resampled distribution is 0.4472. This is a resampling estimate of $\text{S.E.}\{\bar{X}\}$.

Exercise 3.43 Read file **YARNSTRG.csv** using Python. Use bootstrap sampling $M = 500$ times, to obtain confidence intervals for the mean. Use samples of size $n = 30$. Check in what proportion of samples the confidence intervals cover the mean.

Solution 3.43 In our particular execution with $M = 500$, we have a proportion $\hat{\alpha} = 0.07$ of cases in which the bootstrap confidence intervals do not cover the mean of `yarnstrg`, $\mu = 2.9238$. This is not significantly different from the nominal $\alpha = 0.05$. The determination of the proportion $\hat{\alpha}$ can be done by using the following commands:

```
random.seed(1)
yarnstrg = mistat.load_data('YARNSTRG')
```



{fig:qqplotSampleMeansMPG} **Fig. 3.5** *Q*-*Q* Plot of mean of samples from `mpg` data

```
def confidence_interval(x, nsigma=2):
  sample_mean = np.mean(x)
  sigma = np.std(x, ddof=1) / np.sqrt(len(x))
  return (sample_mean - 2 * sigma, sample_mean + 2 * sigma)

mean = np.mean(yarnstrg)
outside = 0
for _ in range(500):
  sample = random.choices(yarnstrg, k=30)
  ci = confidence_interval(sample)
  if mean < ci[0] or ci[1] < mean:
    outside += 1

hat_alpha = outside / 500

ci = confidence_interval(yarnstrg)
print(f' Mean: {mean}')
print(f' 2-sigma-CI: {ci[0]:.1f} - {ci[1]:.1f}')
print(f' proportion outside: {hat_alpha:.3f}')
```

```
Mean: 2.9238429999999993
2-sigma-CI: 2.7 - 3.1
proportion outside: 0.068
```

Exercise 3.44 The average turn diameter of 58 US made cars, in data file **CAR.csv**, is $\bar{X} = 37.203$ [m]. Is this mean significantly larger than 37 [m]? In order to check this, use Python. After loading the data, you will need to filter the dataset to extract the data for the 58 US made cars (origin = 1).

Write a function which samples with replacement from the turn column 58 values, and store them in a list. Repeat this 100 times. An estimate of the P-value is the proportion of means smaller than 36, greater than $2 \times 37.203 - 37 = 37.406$. What is your estimate of the P-value?

Solution 3.44 In Python:

```
random.seed(1)
car = mistat.load_data('CAR')
us_cars = car[car['origin'] == 1]
us_turn = list(us_cars['turn'])

sample_means = []
for _ in range(100):
  x = random.choices(us_turn, k=58)
  sample_means.append(np.mean(x))

is_larger = sum(m > 37.406 for m in sample_means)
ratio = is_larger / len(sample_means)
print(ratio)
```

```
0.23
```

We obtained $\tilde{P} =$ 0.23. The mean $\bar{X} = 37.203$ is **not** significantly larger than 37.

Exercise 3.45 You have to test whether the proportion of non-conforming units in a sample of size $n = 50$ from a production process is significantly greater than $p = 0.03$. Use Python to determine when should we reject the hypothesis that $p \leq 0.03$ with $\alpha = 0.05$.

Solution 3.45 Let X_{50} be the number of non-conforming units in a sample of $n = 50$ items. We reject H_0, at level of $\alpha = 0.05$, if $X_{50} > B^{-1}(0.95, 50, 0.03) = 4$. The criterion k_α is obtained by using the Python commands:

```
stats.binom(50, 0.03).ppf(0.95)
```

```
4.0
```

{exc:bootstrap_mean_sd_cyclt}

Exercise 3.46 Generate 1000 bootstrap samples of the sample mean and sample standard deviation of the data in **CYCLT.csv** on 50 piston cycle times.

(i) Compute 95% confidence intervals for the sample mean and sample standard deviation.
(ii) Draw histograms of the EBD of the sample mean and sample standard deviation.

Solution 3.46 i Calculation of 95% confidence intervals:

```
random.seed(1)
cyclt = mistat.load_data('CYCLT')

B = pg.compute_bootci(cyclt, func=np.mean, n_boot=1000,
```

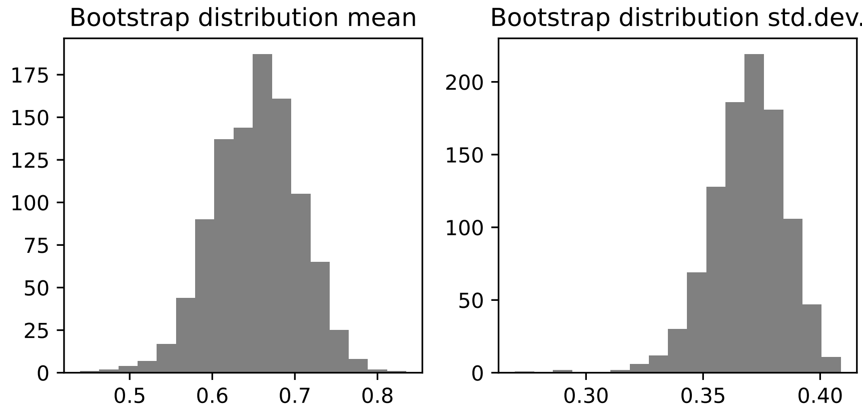


Fig. 3.6 Histograms of EBD for **CYCLT.csv** data

{fig:histEBD_CYCLT}

```
                      confidence=0.95, return_dist=True, seed=1)

ci_mean, dist_mean = B
print(f' Mean: {np.mean(dist_mean):.3f}')
print(f' 95%-CI: {ci_mean[0]:.3f} - {ci_mean[1]:.3f}')

B = pg.compute_bootci(cyclt, func=lambda x: np.std(x, ddof=1), n_boot=1000,
                      confidence=0.95, return_dist=True, seed=1)

ci_std, dist_std = B
print(f' Mean: {np.mean(dist_std):.3f}')
print(f' 95%-CI: {ci_std[0]:.3f} - {ci_std[1]:.3f}')
```

```
Mean: 0.653
95%-CI: 0.540 - 0.740
Mean: 0.369
95%-CI: 0.340 - 0.400
```

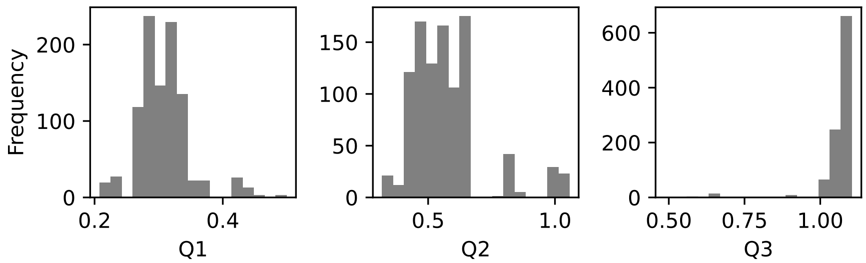
ii Histograms of the EBD, see Figure 3.6.

{fig:histEBD_CYCLT}

```
fig, axes = plt.subplots(figsize=[6, 3], ncols=2)
axes[0].hist(dist_mean, color='grey', bins=17)
axes[1].hist(dist_std, color='grey', bins=17)
axes[0].set_title('Bootstrap distribution mean')
axes[1].set_title('Bootstrap distribution std.dev.')
plt.tight_layout()
plt.show()
```

Exercise 3.47 Use Python to generate 1000 bootstrapped quartiles of the data in **CYCLT.csv**.

(i) Compute 95% confidence intervals for the 1st quartile, the median and the 3rd quartile.
(ii) Draw histograms of the bootstrap quartiles.



{fig:histEBD_CYCLT_quartiles} **Fig. 3.7** Histograms of EBD for quartiles for **CYCLT.csv** data

Solution 3.47 In Python:

```
cyclt = mistat.load_data('CYCLT')
ebd = {}
for quantile in (1, 2, 3):
  B = pg.compute_bootci(cyclt, func=lambda x: np.quantile(x, 0.25 * quantile),
                        n_boot=1000, confidence=0.95, return_dist=True, seed=1)
  ci, dist = B
  ebd[quantile] = dist
  print(f'Quantile {quantile}: {np.mean(dist):.3f} 95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
```

```
Quantile 1: 0.308 95%-CI: 0.240 - 0.420
Quantile 2: 0.575 95%-CI: 0.350 - 1.010
Quantile 3: 1.060 95%-CI: 0.920 - 1.090
```

{fig:histEBD_CYCLT_quartiles} **ii** Histograms of the EBD, see Figure 3.7.

```
fig, axes = plt.subplots(figsize=[6, 2], ncols=3)
axes[0].hist(ebd[1], color='grey', bins=17)
axes[1].hist(ebd[2], color='grey', bins=17)
axes[2].hist(ebd[3], color='grey', bins=17)
axes[0].set_xlabel('Q1')
axes[1].set_xlabel('Q2')
axes[2].set_xlabel('Q3')
axes[0].set_ylabel('Frequency')
plt.tight_layout()
plt.show()
```

Exercise 3.48 Generate the EBD of size $M = 1,000$, for the sample correlation ρ_{XY} between ISC-t1 and ISC-t2 in data file **SOCELL.csv**. Compute the bootstrap confidence interval for ρ_{XY}, at confidence level of 0.95.

Solution 3.48 In Python:

```
socell = mistat.load_data('SOCELL')
t1 = socell['t1']
t2 = socell['t2']

# use the index
idx = list(range(len(socell)))
```

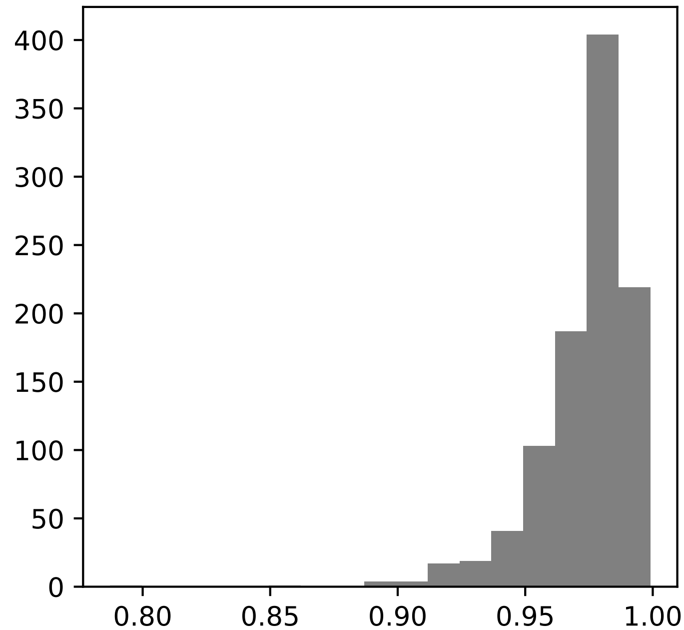


Fig. 3.8 Histograms of EBD for correlation for **SOCELL.csv** data {fig:histEBD_SOCELL_correlation}

```
def sample_correlation(x):
  return stats.pearsonr(t1[x], t2[x])[0]

B = pg.compute_bootci(idx, func=sample_correlation,
                      n_boot=1000, confidence=0.95, return_dist=True, seed=1)
ci, dist = B
print(f'rho_XY: {np.mean(dist):.3f} 95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
```

```
rho_XY: 0.974 95%-CI: 0.930 - 1.000
```

Histogram of bootstrap correlations, see Figure 3.8. {fig:histEBD_SOCELL_correlation}

```
fig, ax = plt.subplots(figsize=[4, 4])
ax.hist(dist, color='grey', bins=17)
plt.show()
```

Exercise 3.49 Generate the EBD of the regression coefficients (a, b) of Miles per Gallon/City, Y, versus Horsepower, X, in data file **CAR.csv**. For each of the $M = 100$ bootstrap samples, run a simple regression with the `scipy` command `stats.linregress`. The result (e.g. called `result`) of this command contains the slope (b: `result.slope`) and the intercept (a: `result.intercept`).

(i) Determine a bootstrap confidence interval for the intercept, at level 0.95.
(ii) Determine a bootstrap confidence interval for the slope, at level 0.95.

(iii) Compare the bootstrap standard errors of intercept and slope to those obtained from the formulae of Section 4.3.2.1. {sec:least-squares-single}

Solution 3.49 **(i)** and **(ii)**

```
car = mistat.load_data('CAR')
mpg = car['mpg']
hp = car['hp']

idx = list(range(len(mpg)))
sample_intercept = []
sample_slope = []
for _ in range(1000):
  x = random.choices(idx, k=len(idx))
  result = stats.linregress(hp[x], mpg[x])
  sample_intercept.append(result.intercept)
  sample_slope.append(result.slope)

ci = np.quantile(sample_intercept, [0.025, 0.975])
print(f'intercept (a): {np.mean(sample_intercept):.3f} ' +
      f'95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
ci = np.quantile(sample_slope, [0.025, 0.975])
print(f'slope (b): {np.mean(sample_slope):.4f} ' +
      f'95%-CI: {ci[0]:.4f} - {ci[1]:.4f}')

reg = stats.linregress(hp, mpg)
hm = np.mean(hp)

print(np.std(sample_intercept))
print(np.std(sample_slope))
```

```
intercept (a): 30.724 95%-CI: 28.766 - 32.691
slope (b): -0.0741 95%-CI: -0.0891 - -0.0599
1.0170449375724464
0.0074732552885114645
```

(iii) The bootstrap S.E. of *slope* and *intercept* are 1.017 and 0.00747, respectively. The standard errors of a and b, according to the formulas of Section 4.3.2.1 are 0.8099 and 0.00619, respectively. The bootstrap estimates are quite close to the correct values. {sec:least-squares-single}

Exercise 3.50 Test the hypothesis that the data in **CYCLT.csv** comes from a distribution with mean $\mu_0 = 0.55$ sec.

(i) Calculate and compare the t-test P-value and the boostrapped P^*-value?
(ii) Does the confidence interval derived in Exercise 3.46 include $\mu_0 = 0.55$? {exc:bootstrap_mean_sd_cyclt}
(iii) Could we have guessed the answer of part (ii) after completing part (i)?

Solution 3.50 In Python:

```
cyclt = mistat.load_data('CYCLT')
X50 = np.mean(cyclt)
SD50 = np.std(cyclt)
result = stats.ttest_1samp(cyclt, 0.55)
print(f'Xmean_50 = {X50:.3f}')
print(result)

B = pg.compute_bootci(cyclt, func=np.mean, n_boot=1000,
                      confidence=0.95, return_dist=True, seed=1)
```

```
ci_mean, dist = B
pstar = sum(dist < 0.55) / len(dist)
print(f'p*-value: {pstar}')
```

```
Xmean_50 = 0.652
Ttest_1sampResult(statistic=1.9425149510299369,
pvalue=0.057833259176805)
p*-value: 0.025
```

The mean of the sample is $\bar{X}_{50} = 0.652$. The studentized difference from $\mu_0 = 0.55$ is $t = 1.943$.

(i) The t-test obtained a P-level of 0.058 and the bootstrap resulted in $P^* = 0.025$.

(ii) Yes, but μ is very close to the lower bootstrap confidence limit (0.540). The null hypothesis $H_0 : \mu = 0.55$ is accepted.

(iii) No, but since P^* is close to 0.05, we expect that the bootstrap confidence interval will be very close to μ_0.

Exercise 3.51 Compare the variances of the two measurements recorded in data file **ALMPIN2.csv**

(i) What is the P-value?
(ii) Draw box plots of the two measurements.

Solution 3.51 In Python:

```
almpin = mistat.load_data('ALMPIN')
diam1 = almpin['diam1']
diam2 = almpin['diam2']

# calculate the ratio of the two variances:
var_diam1 = np.var(diam1)
var_diam2 = np.var(diam2)
F = var_diam2 / var_diam1
print(f'Variance diam1: {var_diam1:.5f}')
print(f'Variance diam2: {var_diam2:.5f}')
print(f'Ratio: {F:.4f}')

# Calculate the p-value
p_value = stats.f.cdf(F, len(diam1) - 1, len(diam2) - 1)
print(f'p-value: {p_value:.3f}')
```

```
Variance diam1: 0.00027
Variance diam2: 0.00032
Ratio: 1.2016
p-value: 0.776
```

The variances are therefore not significantly different.

(ii) The box plots of the two measurements are shown in Figure 3.9. {fig:boxplot-almpin}

```
almpin.boxplot(column=['diam1', 'diam2'])
plt.show()
```

Exercise 3.52 Compare the means of the two measurements on the two variables Diameter1 and Diameter2 in **ALMPIN2.csv**. What is the bootstrap estimate of the P-values for the means and variances?

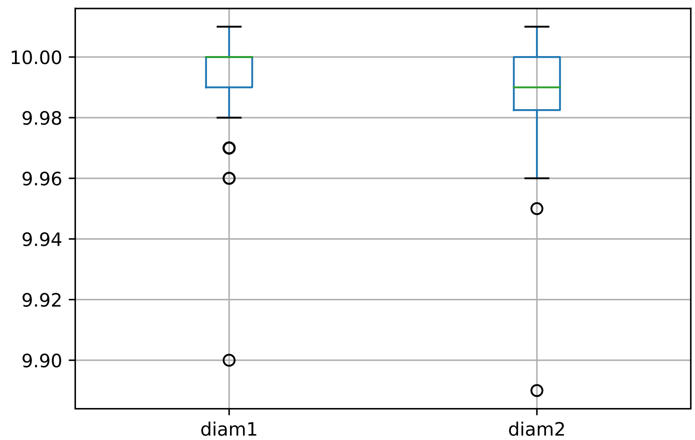


Fig. 3.9 Box plots of `diam1` and `diam2` measurements of the ALMPIN dataset {fig:boxplot-almpin}

Solution 3.52 The variances were already compared in the previous exercise. To compare the means use:

```
almpin = mistat.load_data('ALMPIN')
diam1 = almpin['diam1']
diam2 = almpin['diam2']

# Compare means
mean_diam1 = np.mean(diam1)
mean_diam2 = np.mean(diam2)
print(f'Mean diam1: {mean_diam1:.5f}')
print(f'Mean diam2: {mean_diam2:.5f}')

# calculate studentized difference and p-value
se1, se2 = stats.sem(diam1), stats.sem(diam2)
sed = np.sqrt(se1**2.0 + se2**2.0)
t_stat = (mean_diam1 - mean_diam2) / sed
print(f'Studentized difference: {t_stat:.3f}')
df = len(diam1) + len(diam2) - 2
p = (1 - stats.t.cdf(abs(t_stat), df)) * 2
print(f'p-value: {p:.3f}')

# or use any of the available implementations of the t-test
print(stats.ttest_ind(diam1, diam2))
```

```
Mean diam1: 9.99286
Mean diam2: 9.98729
Studentized difference: 1.912
p-value: 0.058
Ttest_indResult(statistic=1.9119658005133064,
pvalue=0.05795318184124417)
```

The bootstrap based p-value for the comparison of the means is:

```
random.seed(1)

# return studentized distance between random samples from diam1 and diam2
def stat_func():
```

```
    d1 = random.choices(diam1, k=len(diam1))
    d2 = random.choices(diam2, k=len(diam2))
    return stats.ttest_ind(d1, d2).statistic

dist = np.array([stat_func() for _ in range(1000)])

pstar = sum(dist < 0) / len(dist)
print(f'p*-value: {pstar}')
```

```
p*-value: 0.014
```

The bootstrap based p-value for the comparison of the variances is:

```
columns = ['diam1', 'diam2']
# variance for each column
S2 = almpin[columns].var(axis=0, ddof=0)
F0 = max(S2) / min(S2)
print('S2', S2)
print('F0', F0)

# Step 1: sample variances of bootstrapped samples for each column
seed = 1
B = {}
for column in columns:
    ci = pg.compute_bootci(almpin[column], func='var', n_boot=500,
                           confidence=0.95, seed=seed, return_dist=True)
    B[column] = ci[1]
Bt = pd.DataFrame(B)

# Step 2: compute Wi
Wi = Bt / S2

# Step 3: compute F*
FBoot = Wi.max(axis=1) / Wi.min(axis=1)
FBoot95 = np.quantile(FBoot, 0.95)
print('FBoot 95%', FBoot95)
pstar = sum(FBoot >= F0)/len(FBoot)
print(f'p*-value: {pstar}')
```

```
S2 diam1    0.000266
diam2    0.000320
dtype: float64
F0 1.2016104294478573
FBoot 95% 1.21577695826553
p*-value: 0.058
```

The variance of Sample 1 is $S_1^2 = 0.00027$. The variance of Sample 2 is $S_2^2 = 0.00032$. The variance ratio is $F = S_2^2/S_1^2 = 1.202$. The bootstrap level for variance ratios is $P^* = 0.058$.

Exercise 3.53 Compare the variances of the gasoline consumption (MPG/City) of cars by origin. The data is saved in file **MPG.csv**. There are $k = 3$ samples of sizes $n_1 = 58$, $n_2 = 14$ and $n_3 = 37$. Do you accept the null hypothesis of equal variances?

Solution 3.53 In Python:

```
mpg = mistat.load_data('MPG')
columns = ['origin1', 'origin2', 'origin3']
# variance for each column
S2 = mpg[columns].var(axis=0, ddof=1)
```

```
F0 = max(S2) / min(S2)
print('S2', S2)
print('F0', F0)

# Step 1: sample variances of bootstrapped samples for each column
seed = 1
B = {}
for column in columns:
    ci = pg.compute_bootci(mpg[column].dropna(), func='var', n_boot=500,
                           confidence=0.95, seed=seed, return_dist=True)
    B[column] = ci[1]
Bt = pd.DataFrame(B)

# Step 2: compute Wi
Wi = Bt / S2

# Step 3: compute F*
FBoot = Wi.max(axis=1) / Wi.min(axis=1)
FBoot95 = np.quantile(FBoot, 0.95)
print('FBoot 95%', FBoot95)
pstar = sum(FBoot >= F0)/len(FBoot)
print(f'p*-value: {pstar}')
```

```
S2 origin1    12.942529
origin2     6.884615
origin3    18.321321
dtype: float64
F0 2.6611975103595213
FBoot 95% 3.025515191724999
p*-value: 0.068
```

With $M = 500$ we obtained the following results:

- 1st sample variance = 12.9425,
- 2nd sample variance = 6.8846,
- 3rd sample variance = 18.3213,

$F_{\max/\min} = 2.6612$ and the bootstrap P value is $P^* = 0.068$. The bootstrap test does not reject the hypothesis of equal variances at the 0.05 significance level.

{exc:mpg-equal-mean}

Exercise 3.54 Test the equality of mean gas consumption (MPG/City) of cars by origin. The data file to use is **MPG.csv**. The sample sizes are $n_1 = 58$, $n_2 = 14$ and $n_3 = 37$. The number of samples is $k = 3$. Do you accept the null hypothesis of equal means using a bootstrap approach?

Solution 3.54 With $M = 500$ we obtained

$$\begin{aligned} \bar{X}_1 &= 20.931 & S_1^2 &= 12.9425 \\ \bar{X}_2 &= 19.5 & S_2^2 &= 6.8846 \\ \bar{X}_3 &= 23.1081 & S_3^2 &= 18.3213 \end{aligned}$$

{sec:comp-means-one-way-anova}

Using the approach shown in Section 3.11.5.2, we get:

```
mpg = mistat.load_data('MPG.csv')
samples = [mpg[key].dropna() for key in ['origin1', 'origin2', 'origin3']]

def test_statistic_F(samples):
```

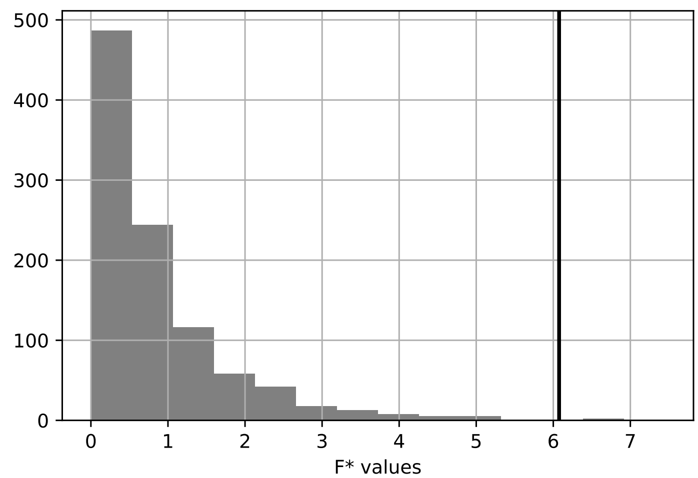


Fig. 3.10 Distribution of EBD for exercise 3.54. exc:mpg-equal-mean {fig:mpg-equal-mean}

```
    return stats.f_oneway(*samples).statistic

# Calculate sample shifts
Ni = np.array([len(sample) for sample in samples])
N = np.sum(Ni)
XBni = np.array([np.mean(sample) for sample in samples])
XBB = np.sum(Ni * XBni) / N
DB = XBni - XBB

F0 = test_statistic_F(samples)
Ns = 1000
Fstar = []
for _ in range(Ns):
    Ysamples  = []
    for sample, DBi in zip(samples, DB):
        Xstar = np.array(random.choices(sample, k=len(sample)))
        Ysamples.append(Xstar - DBi)
    Fs = test_statistic_F(Ysamples)
    Fstar.append(Fs)
Fstar = np.array(Fstar)

print(f'F = {F0:.3f}')
print('ratio', sum(Fstar > F0)/len(Fstar))

ax = pd.Series(Fstar).hist(bins=14, color='grey')
ax.axvline(F0, color='black', lw=2)
ax.set_xlabel('F* values')
plt.show()
```

```
F = 6.076
ratio 0.003
```

$F = 6.076$, $P^* = 0.003$ and the hypothesis of equal means is rejected. See Figure 3.10 for the calculated EBD. {fig:mpg-equal-mean}

Exercise 3.55 Use Python to generate 50 random Bernoulli numbers, with $p = 0.2$. Use these numbers to obtain tolerance limits with $\alpha = 0.05$ and $\beta = 0.05$, for the number of non-conforming items in future batches of 50 items, when the process proportion defectives is $p = 0.2$. Repeat this for $p = 0.1$ and $p = 0.05$.

Solution 3.55 In Python:

```
np.random.seed(1)

def qbinomBoot(x, p):
    return stats.binom.ppf(p, 50, p=x.mean())

for p_real in (0.2, 0.1, 0.05):
    defects = stats.bernoulli.rvs(p_real, size=50)
    B_025 = pg.compute_bootci(defects, func=lambda x: qbinomBoot(x, p=0.025),
                              n_boot=500, seed=1, return_dist=True)
    B_975 = pg.compute_bootci(defects, func=lambda x: qbinomBoot(x, p=0.975),
                              n_boot=500, seed=1, return_dist=True)
    tol_int = [np.quantile(B_025[1], 0.025),np.quantile(B_975[1], 0.975)]
    print(f'Tolerance interval p={p_real}: ({tol_int[0]}, {tol_int[1]})')
```

```
Tolerance interval p=0.2: (1.0, 23.0)
Tolerance interval p=0.1: (0.0, 17.0)
Tolerance interval p=0.05: (0.0, 9.0)
```

The tolerance intervals of the number of defective items in future batches of size $N = 50$, with $\alpha = 0.05$ and $\beta = 0.05$ are

	Limits	
p	Lower	Upper
0.2	1	23
0.1	0	17
0.05	0	9

Exercise 3.56 Use Python to calculate a (.95, .95) tolerance interval for the piston cycle time from the data in **OTURB.csv**.

Solution 3.56 In Python:

```
np.random.seed(1)

def getQuantile(x, p):
    return np.quantile(x, p)

oturb = mistat.load_data('OTURB.csv')
B_025 = pg.compute_bootci(oturb, func=lambda x: getQuantile(x, p=0.025),
                          n_boot=500, seed=1, return_dist=True)
B_975 = pg.compute_bootci(oturb, func=lambda x: getQuantile(x, p=0.975),
                          n_boot=500, seed=1, return_dist=True)
tol_int = [np.quantile(B_025[1], 0.025),np.quantile(B_975[1], 0.975)]
print(f'Tolerance interval ({tol_int[0]}, {tol_int[1]})')
```

```
Tolerance interval (0.2399, 0.68305)
```

A $(0.95, 0.95)$ tolerance interval for OTURB.csv is (0.24, 0.683).

Exercise 3.57 Using the sign test, test the hypothesis that the median, $\xi_{.5}$, of the distribution of cycle time of the piston, is not exceeding $\xi^* = .7$ [min]. The sample data is in file **CYCLT.csv**. Use $\alpha = 0.10$ for level of significance.

Solution 3.57 In Python:

```
cyclt = mistat.load_data('CYCLT.csv')
# make use of the fact that a True value is interpreted as 1 and False as 0
print('Values greater 0.7:', sum(cyclt>0.7))
```

```
Values greater 0.7: 20
```

We find that in the sample of $n = 50$ cycle times, there are $X = 20$ values greater than 0.7. If the hypothesis is $H_0 : \xi_{.5} \leq 0.7$, the probability of observing a value smaller than 0.7 is $p \geq \frac{1}{2}$. Thus, the sign test rejects H_0 if $X < B^{-1}(\alpha; 50, \frac{1}{2})$. For $\alpha = 0.10$ the critical value is $k_\alpha = 20$. H_0 is not rejected.

Exercise 3.58 Use the WSR Test on the data of file **OELECT.csv** to test whether the median of the distribution $\xi_{.5} = 220$ [Volt].

Solution 3.58 We apply the `wilcoxon` test from `scipy` on the differences of `oelect` from 220.

```
oelect = mistat.load_data('OELECT.csv')
print(stats.wilcoxon(oelect-220))
```

```
WilcoxonResult(statistic=1916.0, pvalue=0.051047599707252124)
```

The null hypothesis is rejected with P value equal to 0.051.

Exercise 3.59 Apply the randomization test on the **CAR.csv** file to test whether the turn diameter of foreign cars, having four cylinders, is different from that of US made cars with four cylinders.

Solution 3.59 In Python

```
car = mistat.load_data('CAR.csv')
fourCylinder = car[car['cyl'] == 4]
uscars = fourCylinder[fourCylinder['origin'] == 1]
foreign = fourCylinder[fourCylinder['origin'] != 1]

print(f'Mean of Sample 1 (U.S. made) {np.mean(uscars["turn"]):.3f}')
print(f'Mean of Sample 2 (foreign) {np.mean(foreign["turn"]):.3f}')

_ = mistat.randomizationTest(uscars['turn'], foreign['turn'], np.mean,
                             aggregate_stats=lambda x: x[0] - x[1],
                             n_boot=1000, seed=1)
```

```
Mean of Sample 1 (U.S. made) 36.255
Mean of Sample 2 (foreign) 33.179
Original stat is 3.075758
Original stat is at quantile 1001 of 1001 (100.00%)
Distribution of bootstrap samples:
 min: -2.12, median: 0.01,  max: 2.68
```

The original stat 3.08 is outside of the distribution of the bootstrap samples. The difference between the means of the turn diameters is therefore significant. Foreign cars have on the average a smaller turn diameter.

Chapter 4
Variability in Several Dimensions and Regression Models

Import required modules and define required functions

```
import random
import numpy as np
import pandas as pd
import pingouin as pg
from scipy import stats
import statsmodels.api as sm
import statsmodels.formula.api as smf
import statsmodels.stats as sms
import seaborn as sns
import matplotlib.pyplot as plt

import mistat
```

Exercise 4.1 Use file **CAR.csv** to prepare multiple or matrix scatter plots of Turn Diameter versus Horsepower versus Miles per Gallon. What can you learn from these plots?

Solution 4.1 In Figure 4.1 one sees that horsepower and miles per gallon are inversely proportional. Turn diameter seems to increase with horsepower. {fig:ex_car_pairplot}

```
car = mistat.load_data('CAR')
sns.pairplot(car[['turn', 'hp', 'mpg']])
plt.show()
```

Exercise 4.2 Make a multiple (side by side) box plots of the Turn Diameter by Car Origin, for the data in file **CAR.csv**. Can you infer that Turn Diameter depends on the Car Origin?

Solution 4.2 The box plots in Figure 4.2 show that cars from Asia generally have the smallest turn diameter. The maximal turn diameter of cars from Asia is smaller than the median turn diameter of U.S. cars. European cars tend to have larger turn diameter than those from Asia, but smaller than those from the U.S. {fig:ex_car_boxplots}

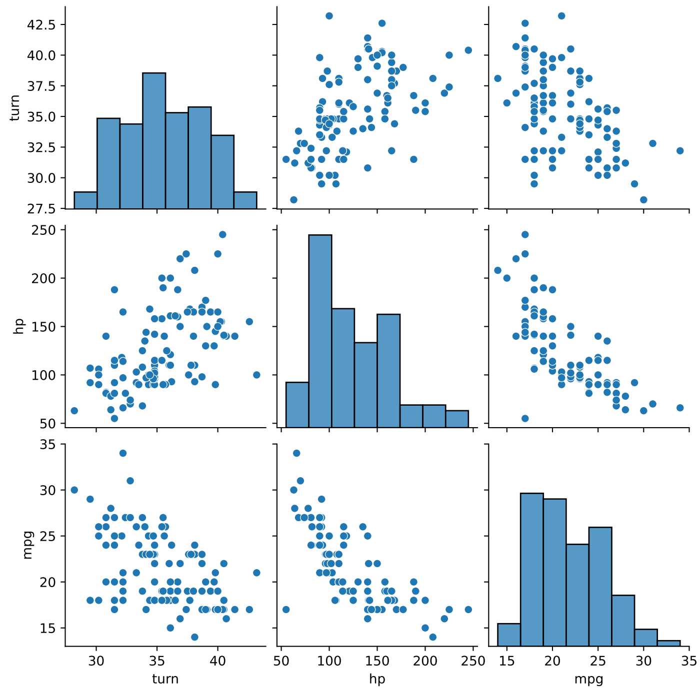


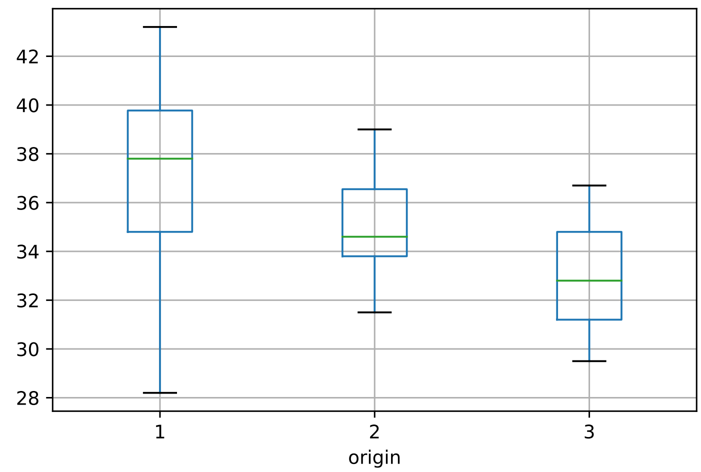
Fig. 4.1 Scatterplot matrix for CAR dataset {fig:ex_car_pairplot}

```
car = mistat.load_data('CAR')

ax = car.boxplot(column='turn', by='origin')
ax.set_title('')
ax.get_figure().suptitle('')
ax.set_xlabel('origin')
plt.show()
```

Exercise 4.3 Data file **HADPAS.csv** contains the resistance values (Ohms) of five resistors placed in six hybrids on 32 ceramic substrates. The file contains eight columns. The variables in these columns are:

1. Record Number
2. Substrate Number
3. Hybrid Number
4. Res 3.
5. Res 18.
6. Res 14.
7. Res 7.
8. Res 20.

(i) Make a multiple box plot of the resistance in Res 3, by hybrid.



{fig:ex_car_boxplots} **Fig. 4.2** Boxplots of turn diameter by origin for CAR dataset

(ii) Make a matrix plot of all the Res variables. What can you learn from the plots?

Solution 4.3 **(i)** The multiple box plots (see Figure 4.3) show that the conditional distributions of `res3` at different hybrids are different. {fig:ex_hadpas_plot_i}

```
hadpas = mistat.load_data('HADPAS')
ax = hadpas.boxplot(column='res3', by='hyb')
ax.set_title('')
ax.get_figure().suptitle('')
ax.set_xlabel('Hybrid number')
ax.set_ylabel('Res 3')
plt.show()
```

(ii) The matrix plot of all the Res variables (see Figure 4.4) reveals that Res 3 and Res 7 are positively correlated. Res 20 is generally larger than the corresponding Res 14. Res 18 and Res 20 seem to be negatively associated. {fig:ex_hadpas_plot_ii}

```
sns.pairplot(hadpas[['res3', 'res7', 'res18', 'res14', 'res20']])
plt.show()
```

Exercise 4.4 Construct a joint frequency distribution of the variables Horsepower and MPG/City for the data in file **CAR.csv**.

Solution 4.4 The joint frequency distribution of horsepower versus miles per gallon is

```
car = mistat.load_data('CAR')
binned_car = pd.DataFrame({
  'hp': pd.cut(car['hp'], bins=np.arange(50, 275, 25)),
  'mpg': pd.cut(car['mpg'], bins=np.arange(10, 40, 5)),
```

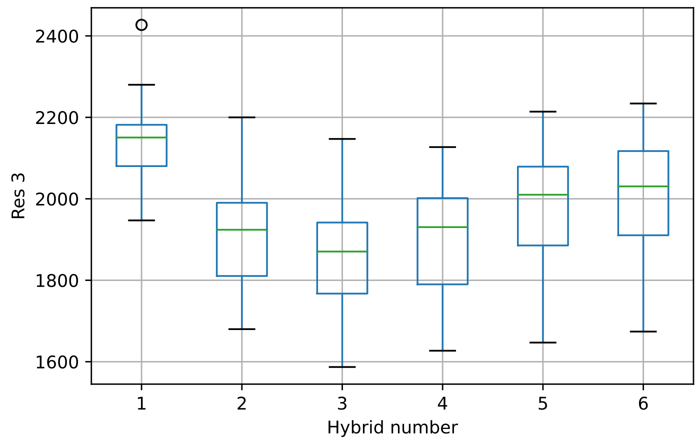


Fig. 4.3 Multiple box plots of Res 3 grouped by hybrid number

{fig:ex_hadpas_plot_i}

```
})
freqDist = pd.crosstab(binned_car['hp'], binned_car['mpg'])
print(freqDist)
# You can get distributions for hp and mpg by summing along an axis
print(freqDist.sum(axis=0))
print(freqDist.sum(axis=1))
```

```
mpg          (10, 15]  (15, 20]  (20, 25]  (25, 30]  (30, 35]
hp
(50, 75]            0         1         0         4         2
(75, 100]           0         0        23        11         0
(100, 125]          0        10        11         1         0
(125, 150]          0        14         3         1         0
(150, 175]          0        17         0         0         0
(175, 200]          1         5         0         0         0
(200, 225]          1         3         0         0         0
(225, 250]          0         1         0         0         0
mpg
(10, 15]     2
(15, 20]    51
(20, 25]    37
(25, 30]    17
(30, 35]     2
dtype: int64
hp
(50, 75]       7
(75, 100]     34
(100, 125]    22
(125, 150]    18
(150, 175]    17
(175, 200]     6
(200, 225]     4
(225, 250]     1
dtype: int64
```

The intervals for HP are from 50 to 250 at fixed length of 25. The intervals for MPG are from 10 to 35 at length 5. Students may get different results by defining the intervals differently.

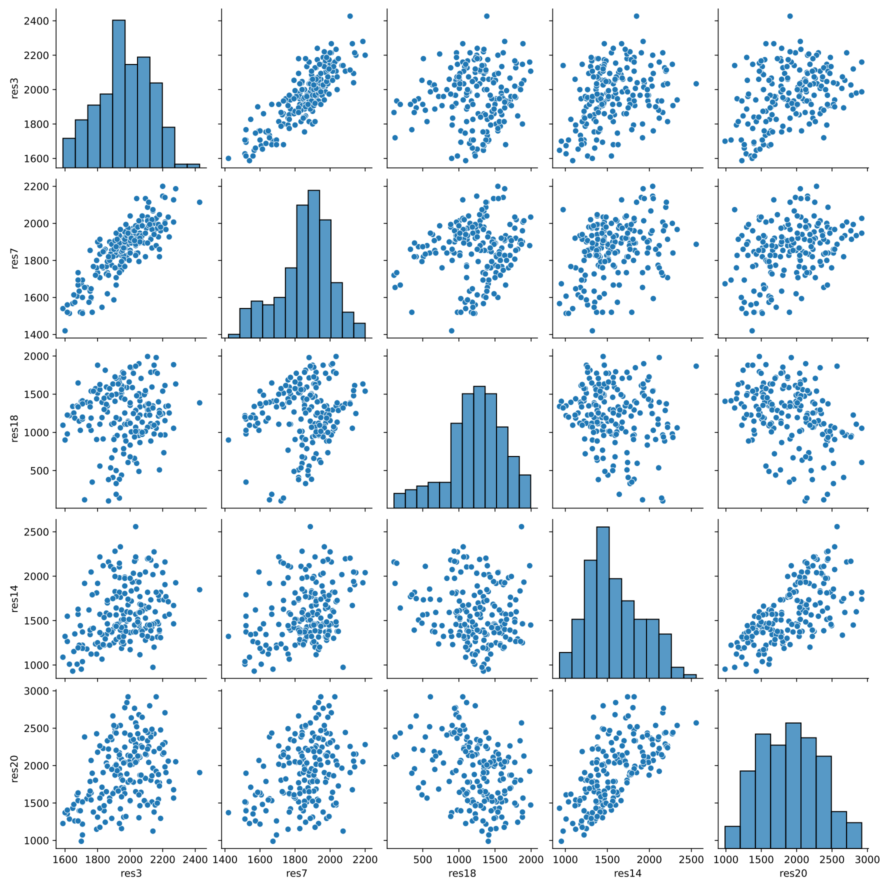


Fig. 4.4 Scatterplot matrix Res variables for HADPAS dataset {fig:ex_hadpas_plot_ii}

Exercise 4.5 Construct a joint frequency distribution for the resistance values of `res3` and `res14`, in data file **HADPAS.csv**. [Code the variables first, see instructions in Example 4.3.] {ex:joint-freq-dist}

Solution 4.5 The joint frequency distribution of Res 3 and Res 14 is given in the following table:

```
hadpas = mistat.load_data('HADPAS')
binned_hadpas = pd.DataFrame({
  'res3': pd.cut(hadpas['res3'], bins=np.arange(1580, 2780, 200)),
  'res14': pd.cut(hadpas['res14'], bins=np.arange(900, 3000, 300)),
})
pd.crosstab(binned_hadpas['res14'], binned_hadpas['res3'])
```

```
res3          (1580, 1780]  (1780, 1980]  (1980, 2180]  (2180, 2380]  \
res14
```

```
(900, 1200]           11         3          3          0
(1200, 1500]          11        33         28          2
(1500, 1800]           5        16         24          6
(1800, 2100]           2        11         12          5
(2100, 2400]           0         9          8          1
(2400, 2700]           0         0          1          0

res3         (2380, 2580]
res14
(900, 1200]            0
(1200, 1500]           0
(1500, 1800]           0
(1800, 2100]           1
(2100, 2400]           0
(2400, 2700]           0
```

The intervals for Res 3 start at 1580 and end at 2580 with length of 200. The intervals of Res 14 start at 900 and end at 2700 with length of 300.

Exercise 4.6 Construct the conditional frequency distribution of `res3`, given that the resistance values of `res14` is between 1300 and 1500 (Ohms).

Solution 4.6 The following is the conditional frequency distribution of Res 3, given that Res 14 is between 1300 and 1500 ohms:

```
hadpas = mistat.load_data('HADPAS')
in_range = hadpas[hadpas['res14'].between(1300, 1500)]
pd.cut(in_range['res3'], bins=np.arange(1580, 2780, 200)).value_counts(sort=False)
```

```
(1580, 1780]     8
(1780, 1980]    21
(1980, 2180]    25
(2180, 2380]     2
(2380, 2580]     0
Name: res3, dtype: int64
```

Exercise 4.7 In the present exercise we compute the **conditional** means and standard deviations of one variable given another one. Use file **HADPAS.csv**.

We classify the data according to the values of Res 14 (`res14`) to 5 subgroups. Bin the values for Res 14 using bin edges at [900, 1200, 1500, 1800, 2100, 3000] and use the `groupby` method to split the hadpas data set by these bins. For each group, determine the mean and standard deviation of the Res 3 (`res3`) column. Collect the results and combine into a data frame for presentation.

Solution 4.7 Following the instructions in the question we obtained the following results:

```
hadpas = mistat.load_data('HADPAS')
bins = [900, 1200, 1500, 1800, 2100, 3000]
binned_res14 = pd.cut(hadpas['res14'], bins=bins)

results = []
for group, df in hadpas.groupby(binned_res14):
  res3 = df['res3']
  results.append({
    'res3': group,
```

```
    'N': len(res3),
    'mean': res3.mean(),
    'std': res3.std(),
  })
pd.DataFrame(results)
```

```
          res3   N         mean         std
0   (900, 1200]  17  1779.117647  162.348730
1  (1200, 1500]  74  1952.175676  154.728251
2  (1500, 1800]  51  1997.196078  151.608841
3  (1800, 2100]  31  2024.774194  156.749845
4  (2100, 3000]  19  1999.736842  121.505758
```

Exercise 4.8 Given below are four data sets of (X, Y) observations

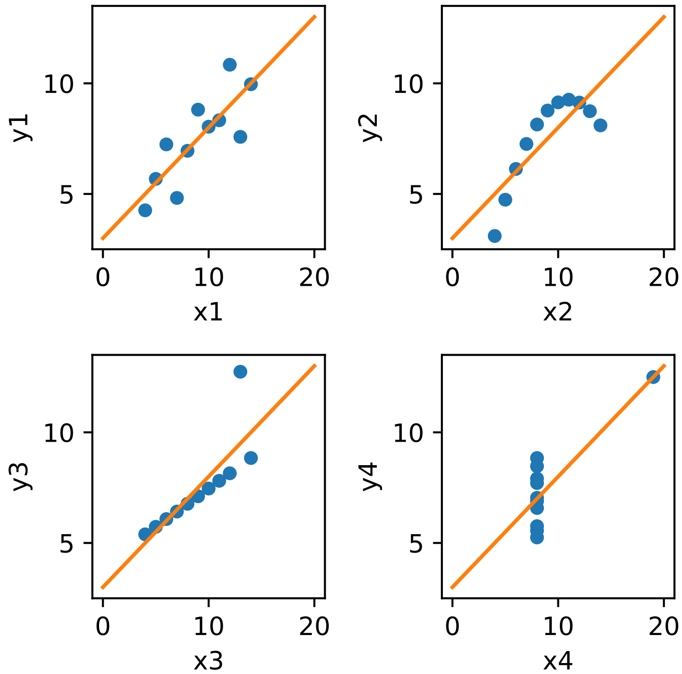
(i) Compute the least squares regression coefficients of Y on X, for the four data sets.
(ii) Compute the coefficient of determination, R^2, for

each set.

Data Set 1		Data Set 2		Data Set 3		Data Set 4	
$X^{(1)}$	$Y^{(1)}$	$X^{(2)}$	$Y^{(2)}$	$X^{(3)}$	$Y^{(3)}$	$X^{(4)}$	$Y^{(4)}$
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.68
8.0	6.95	8.0	8.14	8.0	6.67	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	11.0	9.13	12.0	8.16	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

Solution 4.8 In Python

```
df = pd.DataFrame([
  [10.0, 8.04, 10.0, 9.14, 10.0, 7.46, 8.0, 6.58],
  [8.0, 6.95, 8.0, 8.14, 8.0, 6.77, 8.0, 5.76],
  [13.0, 7.58, 13.0, 8.74, 13.0, 12.74, 8.0, 7.71],
  [9.0, 8.81, 9.0, 8.77, 9.0, 7.11, 8.0, 8.84],
  [11.0, 8.33, 11.0, 9.26, 11.0, 7.81, 8.0, 8.47],
  [14.0, 9.96, 14.0, 8.10, 14.0, 8.84, 8.0, 7.04],
  [6.0, 7.24, 6.0, 6.13, 6.0, 6.08, 8.0, 5.25],
  [4.0, 4.26, 4.0, 3.10, 4.0, 5.39, 19.0, 12.50],
  [12.0, 10.84, 12.0, 9.13, 12.0, 8.15, 8.0, 5.56],
  [7.0, 4.82, 7.0, 7.26, 7.0, 6.42, 8.0, 7.91],
  [5.0, 5.68, 5.0, 4.74, 5.0, 5.73, 8.0, 6.89],
], columns=['x1', 'y1', 'x2', 'y2', 'x3', 'y3', 'x4', 'y4'])

results = []
for i in (1, 2, 3, 4):
  x = df[f'x{i}']
  y = df[f'y{i}']
```



{fig:AnscombeQuartet}

Fig. 4.5 Anscombe's quartet

```
  model = smf.ols(formula=f'y{i} ~ 1 + x{i}', data=df).fit()
  results.append({
    'Data Set': i,
    'Intercept': model.params['Intercept'],
    'Slope': model.params[f'x{i}'],
    'R2': model.rsquared,
  })
pd.DataFrame(results)
```

	Data Set	Intercept	Slope	R2
0	1	3.000091	0.500091	0.666542
1	2	3.000909	0.500000	0.666242
2	3	3.002455	0.499727	0.666324
3	4	3.001727	0.499909	0.666707

Notice the influence of the point (19,12.5) on the regression in Data Set 4. Without this point the correlation between x and y is zero.

{fig:AnscombeQuartet}

The dataset is known as Anscombe's quartet (see Figure 4.5). It not only has identical linear regression, but it has also identical means and variances of x and y, and correlation between x and y. The dataset clearly demonstrates the importance of visualization in data analysis.

Exercise 4.9 Compute the correlation matrix of the variables Turn Diameter, Horsepower and Miles per Gallon/City for the data in file **CAR.csv**.

Solution 4.9 The correlation matrix:

```
car = mistat.load_data('CAR')
car[['turn', 'hp', 'mpg']].corr()
```

```
          turn        hp       mpg
turn  1.000000  0.507610 -0.541061
hp    0.507610  1.000000 -0.754716
mpg  -0.541061 -0.754716  1.000000
```

Exercise 4.10 (i) Differentiate partially the quadratic function

$$\mathrm{SSE} = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})^2$$

with respect to β_0, β_1 and β_2 to obtain the linear equations in the least squares estimates b_0, b_1, b_2. These linear equations are called **the normal equations**.

(ii) Obtain the formulae for b_0, b_1 and b_2 from the normal equations.

Solution 4.10 $SSE = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})^2$

(i)

$$\begin{aligned}
\frac{\partial}{\partial \beta_0} SSE &= -2\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2}) \\
\frac{\partial}{\partial \beta_1} SSE &= -2\sum_{i=1}^{n} X_{i1}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2}) \\
\frac{\partial}{\partial \beta_2} SSE &= -2\sum_{i=1}^{n} X_{i2}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2}).
\end{aligned}$$

Equating these partial derivatives to zero and arranging terms, we arrive at the following set of linear equations:

$$\begin{bmatrix} n & \sum_{i=1}^{n} X_{i1} & \sum_{i=1}^{n} X_{i2} \\ \sum_{i=1}^{n} X_{i1} & \sum_{i=1}^{n} X_{i1}^2 & \sum_{i=1}^{n} X_{i1}X_{i2} \\ \sum_{i=1}^{n} X_{i2} & \sum_{i=1}^{n} X_{i1}X_{i2} & \sum_{i=1}^{n} X_{i2}^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} Y_i \\ \sum_{i=1}^{n} X_{i1}Y_i \\ \sum_{i=1}^{n} X_{i2}Y_i \end{bmatrix}$$

(ii) Let b_0, b_1, b_2 be the (unique) solution. From the first equation we get, after dividing by n, $b_0 = \bar{Y} - \bar{X}_1 b_1 - \bar{X}_2 b_2$, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{i1}$, $\bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_{i2}$. Substituting b_0 in the second and third equations and arranging terms, we obtain the reduced system of equations:

$$\begin{bmatrix} (Q_1 - n\bar{X}_1^2) & (P_{12} - n\bar{X}_1\bar{X}_2) \\ (P_{12} - n\bar{X}_1\bar{X}_2) & (Q_2 - n\bar{X}_2^2) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} P_{1y} - n\bar{X}_1\bar{Y} \\ P_{2y} - n\bar{X}_2\bar{Y} \end{bmatrix}$$

where $Q_1 = \sum X_{i1}^2$, $Q_2 = \sum X_{i2}^2$, $P_{12} = \sum X_{i1}X_{i2}$ and $P_{1y} = \sum X_{i1}Y_i$, $P_{2y} = \sum X_{i2}Y_i$. Dividing both sides by $(n-1)$ we obtain Eq. (4.4.3), and solving we get b_1 and b_2.

{exc:two-var-reg-car}

Exercise 4.11 Consider the variables Miles per Gallon, Horsepower, and Turn Diameter in the data set **CAR.csv**. Find the least squares regression line of MPG (y) on Horsepower (x_1) and Turn Diameter (x_2). For this purpose use first the equations in Section 4.4 and then verify your computations by using `statsmodels` `ols` method. {sec:multiple-regression}

Solution 4.11 We get the following result using `statsmodels`

```
car = mistat.load_data('CAR')

model = smf.ols(formula='mpg ~ 1 + hp + turn', data=car).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.596
Dependent Variable: mpg              AIC:                511.2247
Date:               2022-01-29 23:50 BIC:                519.2988
No. Observations:   109              Log-Likelihood:     -252.61
Df Model:           2                F-statistic:        80.57
Df Residuals:       106              Prob (F-statistic): 5.29e-22
R-squared:          0.603            Scale:              6.2035
------------------------------------------------------------------
               Coef.   Std.Err.     t     P>|t|    [0.025   0.975]
------------------------------------------------------------------
Intercept     38.2642    2.6541  14.4167  0.0000  33.0020  43.5263
hp            -0.0631    0.0069  -9.1070  0.0000  -0.0768  -0.0493
turn          -0.2510    0.0838  -2.9965  0.0034  -0.4171  -0.0849
-----------------------------------------------------------------
Omnibus:               12.000       Durbin-Watson:          2.021
Prob(Omnibus):         0.002        Jarque-Bera (JB):       25.976
Skew:                  -0.335       Prob(JB):               0.000
Kurtosis:              5.296        Condition No.:          1507
=================================================================
* The condition number is large (2e+03). This might indicate
strong multicollinearity or other numerical problems.
```

The regression equation is MPG = 38.3 − 0.251 × turn − 0.0631 × hp.

We see that only 60% of the variability in MPG is explained by the linear relationship with Turn and HP. Both variables contribute significantly to the regression.

Exercise 4.12 Compute the partial correlation between Miles per Gallon and Horsepower, given the Number of Cylinders, in data file **CAR.csv**.

Solution 4.12 The partial correlation is −0.70378.

```
car = mistat.load_data('CAR')

# y: mpg, x1: cyl, x2: hp
model_1 = smf.ols(formula='mpg ~ cyl + 1', data=car).fit()
e_1 = model_1.resid

model_2 = smf.ols(formula='hp ~ cyl + 1', data=car).fit()
e_2 = model_2.resid

print(f'Partial correlation {stats.pearsonr(e_1, e_2)[0]:.5f}')
```

```
Partial correlation -0.70378
```

{exc:two-var-part-reg-car}

Exercise 4.13 Compute the partial regression of Miles per Gallon and Turn Diameter, Given Horsepower, in data file **CAR.csv**.

Solution 4.13 In Python:

```
car = mistat.load_data('CAR')

# y: mpg, x1: hp, x2: turn
model_1 = smf.ols(formula='mpg ~ hp + 1', data=car).fit()
e_1 = model_1.resid
print('Model mpg ~ hp + 1:\n', model_1.params)

model_2 = smf.ols(formula='turn ~ hp + 1', data=car).fit()
e_2 = model_2.resid
print('Model turn ~ hp + 1:\n', model_2.params)

print('Partial correlation', stats.pearsonr(e_1, e_2)[0])
df = pd.DataFrame({'e1': e_1, 'e2': e_2})
model_partial = smf.ols(formula='e1 ~ e2 - 1', data=df).fit()
# print(model_partial.summary2())
print('Model e1 ~ e2:\n', model_partial.params)
```

```
Model mpg ~ hp + 1:
 Intercept    30.663308
hp           -0.073611
dtype: float64
Model turn ~ hp + 1:
 Intercept    30.281255
hp            0.041971
dtype: float64
Partial correlation -0.27945246615045016
Model e1 ~ e2:
 e2   -0.251008
dtype: float64
```

The partial regression equation is $\hat{e}_1 = -0.251\hat{e}_2$.

Exercise 4.14 Use the three stage algorithm of Section 4.4.2 to obtain the multiple regression of Exercise 4.11 from the results of 4.13. {sec:part-regr-corr} {exc:two-var-part-reg-car}

Solution 4.14 The regression of MPG on HP is MPG = 30.6633 − 0.07361 HP. The regression of TurnD on HP is TurnD = 30.2813 + 0.041971 HP. The regression of the residuals $\hat{e}_1$ on $\hat{e}_2$ is $\hat{e}_1 = -0.251 \cdot \hat{e}_2$. Thus,

$$
\begin{aligned}
\text{Const.}: &\quad b_0 = 30.6633 + 30.2813 \times 0.251 = 38.2639 \\
\text{HP}: &\quad b_1 = -0.07361 + 0.041971 \times 0.251 = -0.063076 \\
\text{TurnD}: &\quad b_2 = -0.251.
\end{aligned}
$$

Exercise 4.15 Consider Example 4.10. From the calculation output we see that, when regression Cap Diam on Diam1, Diam2 and Diam3, the regression coefficient of Diam2 is not significant (P value = .925), and this variable can be omitted. Perform a regression of Cap Diam on Diam2 and Diam3. Is the regression coefficient for Diam2 significant? How can you explain the difference between the results of the two regressions? {ex:lin-regression-almpin}

Solution 4.15 The regression of Cap Diameter on Diam2 and Diam3 is

```
almpin = mistat.load_data('ALMPIN')
model = smf.ols('capDiam ~ 1 + diam2 + diam3', data=almpin).fit()
model.summary2()
```

```
<class 'statsmodels.iolib.summary2.Summary'>
"""
                  Results: Ordinary least squares
===================================================================
Model:              OLS              Adj. R-squared:     0.842
Dependent Variable: capDiam          AIC:                -482.1542
Date:               2022-01-29 23:50 BIC:                -475.4087
No. Observations:   70               Log-Likelihood:     244.08
Df Model:           2                F-statistic:        184.2
Df Residuals:       67               Prob (F-statistic): 5.89e-28
R-squared:          0.846            Scale:              5.7272e-05
---------------------------------------------------------------------
                Coef.    Std.Err.     t      P>|t|    [0.025   0.975]
---------------------------------------------------------------------
Intercept       4.7565     0.5501   8.6467   0.0000   3.6585   5.8544
diam2           0.5040     0.1607   3.1359   0.0025   0.1832   0.8248
diam3           0.5203     0.1744   2.9830   0.0040   0.1722   0.8684
-------------------------------------------------------------------
Omnibus:                1.078        Durbin-Watson:           2.350
Prob(Omnibus):          0.583        Jarque-Bera (JB):        0.976
Skew:                   -0.071       Prob(JB):                0.614
Kurtosis:               2.439        Condition No.:           8689
===================================================================
* The condition number is large (9e+03). This might indicate
strong multicollinearity or other numerical problems.
"""
```

The dependence of CapDiam on Diam2, without Diam1 is significant. This is due to the fact that Diam1 and Diam2 are highly correlated ($\rho = 0.957$). If Diam1 is in the regression, then Diam2 does not furnish additional information on CapDiam. If Diam1 is not included then Diam2 is very informative.

Exercise 4.16 Regress the yield in **GASOL.csv** on all the four variables $x_1, x_2, astm$, $endPt$.

(i) What is the regression equation?
(ii) What is the value of R^2?
(iii) Which regression coefficient(s) is (are) non-significant?
(iv) Which factors are important to control the yield?
(v) Are the residuals from the regression distributed normally? Make a graphical test.

Solution 4.16 The regression of yield ($Yield$) on the four variables is:

```
gasol = mistat.load_data('GASOL')
# rename column 'yield' to 'Yield' as 'yield' is a special keyword in Python
gasol = gasol.rename(columns={'yield': 'Yield'})
model = smf.ols(formula='Yield ~ x1 + x2 + astm + endPt',
                data=gasol).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.957
Dependent Variable: Yield            AIC:                146.8308
Date:               2022-01-29 23:50 BIC:                154.1595
No. Observations:   32               Log-Likelihood:     -68.415
Df Model:           4                F-statistic:        171.7
Df Residuals:       27               Prob (F-statistic): 8.82e-19
R-squared:          0.962            Scale:              4.9927
------------------------------------------------------------------
              Coef.   Std.Err.     t     P>|t|    [0.025    0.975]
------------------------------------------------------------------
Intercept    -6.8208   10.1232  -0.6738  0.5062  -27.5918  13.9502
x1            0.2272    0.0999   2.2739  0.0311    0.0222   0.4323
x2            0.5537    0.3698   1.4976  0.1458   -0.2049   1.3124
astm         -0.1495    0.0292  -5.1160  0.0000   -0.2095  -0.0896
endPt         0.1547    0.0064  23.9922  0.0000    0.1414   0.1679
------------------------------------------------------------------
Omnibus:               0.635       Durbin-Watson:          1.402
Prob(Omnibus):         0.728       Jarque-Bera (JB):       0.719
Skew:                  0.190       Prob(JB):               0.698
Kurtosis:              2.371       Condition No.:          10714
=================================================================
* The condition number is large (1e+04). This might indicate
strong multicollinearity or other numerical problems.
```

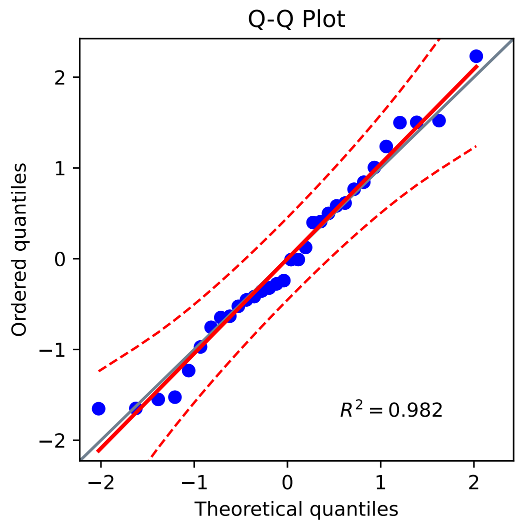
(i) The regression equation is

$$\hat{y} = -6.8 + 0.227x_1 + 0.554x_2 - 0.150\,\text{astm} + 0.155\,\text{endPt}.$$

(ii) $R^2 = 0.962$.
(iii) The regression coefficient of x_2 is not significant.
(iv) Running the multiple regression again, without x_2, we obtain the equation

```
model = smf.ols(formula='Yield ~ x1 + astm + endPt', data=gasol).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.955
Dependent Variable: Yield            AIC:                147.3842
Date:               2022-01-29 23:50 BIC:                153.2471
No. Observations:   32               Log-Likelihood:     -69.692
Df Model:           3                F-statistic:        218.5
Df Residuals:       28               Prob (F-statistic): 1.59e-19
R-squared:          0.959            Scale:              5.2143
------------------------------------------------------------------
             Coef.   Std.Err.     t      P>|t|    [0.025    0.975]
------------------------------------------------------------------
Intercept    4.0320    7.2233    0.5582  0.5811  -10.7643  18.8284
x1           0.2217    0.1021    2.1725  0.0384    0.0127   0.4308
astm        -0.1866    0.0159  -11.7177  0.0000   -0.2192  -0.1540
endPt        0.1565    0.0065   24.2238  0.0000    0.1433   0.1698
------------------------------------------------------------------
Omnibus:               0.679       Durbin-Watson:          1.150
Prob(Omnibus):         0.712       Jarque-Bera (JB):       0.738
Skew:                  0.174       Prob(JB):               0.692
Kurtosis:              2.343       Condition No.:          7479
=================================================================
* The condition number is large (7e+03). This might indicate
strong multicollinearity or other numerical problems.
```



{fig:qqPlotGasolRegResid} **Fig. 4.6** Q-Q plot of gasol regression residuals.

$$\hat{y} = 4.03 + 0.222x_1 + 0.554x_2 - 0.187\,\text{astm} + 0.157\,\text{endPt},$$

with $R^2 = 0.959$. Variables x_1, astm and endPt are important.

(v) Normal probability plotting of the residuals $\hat{e}$ from the equation of **(iv)** shows that they are normally distributed. (see Figure 4.6) {fig:qqPlotGasolRegResid}

Exercise 4.17 (i) Show that the matrix $(H) = (X)(B)$ is idempotent, i.e., $(H)^2 = (H)$.

(ii) Show that the matrix $(Q) = (I - H)$ is idempotent, and therefore $s_e^2 = \mathbf{y}'(Q)\mathbf{y}/(n-k-1)$.

Solution 4.17 (i)

$$\begin{aligned}
(H) &= (X)(B) = (X)[(X)'(X)]^{-1}(X)' \\
H^2 &= (X)[(X)'(X)]^{-1}(X)'(X)[(X)'(X)]^{-1}(X)' \\
&= (X)[(X)'(X)]^{-1}(X)' \\
&= H.
\end{aligned}$$

(ii)

$$\begin{aligned}(Q) &= I - (H)\\ (Q)^2 &= (I-(H))(I-(H))\\ &= I - (H) - (H) + (H)^2\\ &= I - (H)\\ &= Q.\\ s_e^2 &= \mathbf{y}'(Q)(Q)\mathbf{y}/(n-k-1)\\ &= \mathbf{y}'(Q)\mathbf{y}/(n-k-1).\end{aligned}$$

Exercise 4.18 Show that the vectors of fitted values, $\hat{\mathbf{y}}$, and of the residuals, $\hat{\mathbf{e}}$, are orthogonal, i.e., $\hat{\mathbf{y}}'\hat{\mathbf{e}} = 0$.

Solution 4.18 We have $\hat{\mathbf{y}} = (X)\hat{\boldsymbol{\beta}} = (X)(B)\mathbf{y} = (H)\mathbf{y}$ and $\hat{\mathbf{e}} = Q\mathbf{y} = (I-(H))\mathbf{y}$.

$$\begin{aligned}\hat{\mathbf{y}}'\hat{\mathbf{e}} &= \mathbf{y}'(H)(I-(H))\mathbf{y}\\ &= \mathbf{y}'(H)\mathbf{y} - \mathbf{y}'(H)^2\mathbf{y}\\ &= 0.\end{aligned}$$

Exercise 4.19 Show that the $1 - R^2_{y|(x)}$ is proportional to $||\hat{\mathbf{e}}||^2$, which is the squared Euclidean norm of $\hat{\mathbf{e}}$.

Solution 4.19 $1 - R^2_{y(x)} = \dfrac{SSE}{SSD_y}$ where $SSE = \hat{\mathbf{e}}'\hat{\mathbf{e}} = ||\hat{\mathbf{e}}||^2$.

Exercise 4.20 In Section 2.5.2 we presented properties of the $\mathrm{cov}(X, Y)$ operator. {sec:covariance-correlation} Prove the following generalization of property (iv). Let $\mathbf{X}' = (X_1, \ldots, X_n)$ be a vector of n random variables. Let $(\boldsymbol{\Sigma})$ be an $n \times n$ matrix whose (i, j)-th element is $\boldsymbol{\Sigma}_{ij} = \mathrm{cov}(X_i, X_j)$, $i, j = 1, \ldots, n$. Notice that the diagonal elements of $(\boldsymbol{\Sigma})$ are the variances of the components of $\mathbf{X}$. Let $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be two n-dimensional vectors. Prove that $\mathrm{cov}(\boldsymbol{\beta}'\mathbf{X}, \boldsymbol{\gamma}'\mathbf{X}) = \boldsymbol{\beta}'(\boldsymbol{\Sigma})\boldsymbol{\gamma}$. [The matrix $\boldsymbol{\Sigma}$ is called the variance-covariance matrix of $\mathbf{X}$.]

Solution 4.20 From the basic properties of the $\mathrm{cov}(X, Y)$ operator,

$$\begin{aligned}\mathrm{cov}\left(\sum_{i=1}^n \beta_i X_i, \sum_{j=1}^n \gamma_j X_j\right) &= \sum_{i=1}^n \sum_{j=1}^n \beta_i\gamma_j \mathrm{cov}(X_i, X_j)\\ &= \sum_{i=1}^n \sum_{j=1}^n \beta_i\gamma_j\, \boldsymbol{\Sigma}_{ij}\\ &= \boldsymbol{\beta}'(\boldsymbol{\Sigma})\boldsymbol{\gamma}.\end{aligned}$$

Exercise 4.21 Let $\mathbf{X}$ be an n-dimensional random vector, having a variance-covariance matrix $(\boldsymbol{\Sigma})$. Let $\mathbf{W} = (\mathbf{B})\mathbf{X}$, where (B) is an $m \times n$ matrix. Show that the variance-covariance matrix of $\mathbf{W}$ is $(\mathbf{B})(\boldsymbol{\Sigma})(\mathbf{B})'$.

Solution 4.21 $\mathbf{W} = (W_1, \ldots, W_m)'$ where $W_i = \mathbf{b}_i'\mathbf{X}$ $(i = 1, \ldots, m)$. $\mathbf{b}_i'$ is the i-th row vector of B. Thus, by the previous exercise $\text{cov}(W_i, W_j) = \mathbf{b}_i'(\boldsymbol{\Sigma})\mathbf{b}_j$. This is the (i, j) element of the covariance matrix of $\mathbf{W}$. Hence, the covariance matrix of $\mathbf{W}$ is $C(\mathbf{W}) = (B)(\boldsymbol{\Sigma})(B)'$.

Exercise 4.22 Consider the linear regression model $\mathbf{y} = (X)\boldsymbol{\beta} + \mathbf{e}$. $\mathbf{e}$ is a vector of random variables, such that $E\{e_i\} = 0$ for all$i = 1, \ldots, n$ and

$$\text{cov}(e_i, e_j) = \begin{cases} \sigma^2, & \text{if } i = j \\ \\ 0, & \text{if } i \neq j \end{cases}$$

$i, j = 1, \ldots, n$. Show that the variance-covariance matrix of the LSE $\mathbf{b} = (\mathbf{B})\mathbf{y}$ is $\sigma^2[(\mathbf{X})'(\mathbf{X})]^{-1}$.

Solution 4.22 From the model, $\boldsymbol{\Sigma}(\mathbf{Y}) = \sigma^2 I$ and $\mathbf{b} = (B)\mathbf{Y}$.

$$\begin{aligned} \boldsymbol{\Sigma}(\mathbf{b}) &= (B)\boldsymbol{\Sigma}(\mathbf{Y})(B)' = \sigma^2(B)(B)' \\ &= \sigma^2[(\mathbf{X})'(\mathbf{X})]^{-1} \cdot \mathbf{X}'\mathbf{X}[(\mathbf{X})'(\mathbf{X})]^{-1} \\ &= \sigma^2[(\mathbf{X})'(\mathbf{X})]^{-1}. \end{aligned}$$

Exercise 4.23 Consider **SOCELL.csv** data file. Compare the slopes and intercepts of the two simple regressions of ISC at time t_3 on that at time t_1, and ISC at t_3 on that at t_2.

Solution 4.23 Rearrange the dataset into the format suitable for the test outlines in the multiple linear regression section.

```
socell = mistat.load_data('SOCELL')

# combine the two datasets and add the additional columns z and w
socell_1 = socell[['t3', 't1']].copy()
socell_1.columns = ['t3', 't']
socell_1['z'] = 0
socell_2 = socell[['t3', 't2']].copy()
socell_2.columns = ['t3', 't']
socell_2['z'] = 1
combined = pd.concat([socell_1, socell_2])
combined['w'] = combined['z'] * combined['t']

# multiple linear regression model
model_test  = smf.ols(formula='t3 ~ t + z + w + 1',
                      data=combined).fit()
print(model_test.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.952
Dependent Variable: t3               AIC:                -58.2308
Date:               2022-01-29 23:50 BIC:                -52.3678
No. Observations:   32               Log-Likelihood:     33.115
Df Model:           3                F-statistic:        205.8
Df Residuals:       28               Prob (F-statistic): 3.55e-19
R-squared:          0.957            Scale:              0.0084460
-------------------------------------------------------------------
```

```
             Coef.    Std.Err.      t      P>|t|     [0.025   0.975]
--------------------------------------------------------------------
Intercept    0.5187     0.2144    2.4196   0.0223    0.0796   0.9578
t            0.9411     0.0539   17.4664   0.0000    0.8307   1.0515
z           -0.5052     0.3220   -1.5688   0.1279   -1.1648   0.1545
w            0.0633     0.0783    0.8081   0.4259   -0.0971   0.2237
------------------------------------------------------------------
Omnibus:               1.002         Durbin-Watson:           1.528
Prob(Omnibus):         0.606         Jarque-Bera (JB):        0.852
Skew:                  -0.378        Prob(JB):                0.653
Kurtosis:              2.739         Condition No.:           111
==================================================================
```

Neither of the z nor w The P-values corresponding to z and w are 0.128 and 0.426 respectively. Accordingly, we can conclude that the slopes and intercepts of the two simple linear regressions given above are not significantly different. Combining the data we have the following regression line for the combined dataset:

```
model_combined  = smf.ols(formula='t3 ~ t + 1',
                          data=combined).fit()
print(model_combined.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.870
Dependent Variable: t3               AIC:                -28.1379
Date:               2022-01-29 23:50 BIC:                -25.2064
No. Observations:   32               Log-Likelihood:     16.069
Df Model:           1                F-statistic:        208.3
Df Residuals:       30               Prob (F-statistic): 4.86e-15
R-squared:          0.874            Scale:              0.022876
-------------------------------------------------------------------
             Coef.    Std.Err.      t      P>|t|    [0.025   0.975]
-------------------------------------------------------------------
Intercept    0.6151     0.2527    2.4343   0.0211   0.0990   1.1311
t            0.8882     0.0615   14.4327   0.0000   0.7625   1.0139
-----------------------------------------------------------------
Omnibus:               1.072        Durbin-Watson:           0.667
Prob(Omnibus):         0.585        Jarque-Bera (JB):        0.883
Skew:                  0.114        Prob(JB):                0.643
Kurtosis:              2.219        Condition No.:           41
=================================================================
```

{exc:prog-lin-reg-cement}

Exercise 4.24 The following data (see Draper and Smith, 1998) gives the amount of heat evolved in hardening of element (in calories per gram of cement), and the percentage of four various chemicals in the cement (relative to the weight of clinkers from which the cement was made). The four regressors are

x_1 : amount of tricalcium aluminate;

x_2 : amount of tricalcium silicate;

x_3 : amount of tetracalcium alumino ferrite;

x_4 : amount of dicalcium silicate.

The regressant Y is the amount of heat evolved. The data are given in the following table and as dataset **CEMENT.csv**.

X_1	X_2	X_3	X_4	Y
7	26	6	60	78.5
1	29	15	52	74.3
11	56	8	20	104.3
11	31	8	47	87.6
7	52	6	33	95.9
11	55	9	22	109.2
3	71	17	6	102.7
1	31	22	44	72.5
2	54	18	22	93.1
21	47	4	26	115.9
1	40	23	34	83.8
11	66	9	12	113.3
10	68	8	12	109.4

Compute in a sequence the regressions of Y on X_1; of Y on X_1, X_2; of Y on X_1, X_2, X_3; of Y on X_1, X_2, X_3, X_4. For each regression compute the partial-F of the new regression added, the corresponding partial correlation with Y, and the sequential SS.

Solution 4.24 Load the data frame

```
df = mistat.load_data('CEMENT.csv')
```

(a) Regression of Y on X_1 is

```
model1 = smf.ols('y ~ x1 + 1', data=df).fit()
print(model1.summary().tables[1])
r2 = model1.rsquared
print(f'R-sq: {r2:.3f}')

anova = sms.anova.anova_lm(model1)
print('Analysis of Variance\n', anova)

F = anova.F['x1']
SSE_1 = anova.sum_sq['Residual']
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     81.4793      4.927     16.536      0.000      70.634      92.324
x1             1.8687      0.526      3.550      0.005       0.710       3.027
==============================================================================
R-sq: 0.534
Analysis of Variance
             df       sum_sq      mean_sq          F    PR(>F)
x1         1.0  1450.076328  1450.076328  12.602518  0.004552
Residual  11.0  1265.686749   115.062432        NaN       NaN
```

- $R^2_{Y|(X_1)}$ =0.534.
- $SSE_1 =$ 1265.7,
- $F =$ 12.60 (In the 1st stage F is equal to the partial$-F$.)

(b) The regression of Y on X_1 and X_2 is

```
model2 = smf.ols('y ~ x1 + x2 + 1', data=df).fit()
r2 = model2.rsquared
print(model2.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model2)
print('Analysis of Variance\n', anova)
SEQ_SS_X2 = anova.sum_sq['x2']
SSE_2 = anova.sum_sq['Residual']
s2e2 = anova.mean_sq['Residual']
partialF = np.sum(anova.sum_sq) * (model2.rsquared - model1.rsquared) / s2e2

anova = sms.anova.anova_lm(model1, model2)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     52.5773      2.286     22.998      0.000      47.483      57.671
x1             1.4683      0.121     12.105      0.000       1.198       1.739
x2             0.6623      0.046     14.442      0.000       0.560       0.764
==============================================================================
R-sq: 0.979
Analysis of Variance
             df       sum_sq      mean_sq           F        PR(>F)
x1         1.0  1450.076328  1450.076328  250.425571  2.088092e-08
x2         1.0  1207.782266  1207.782266  208.581823  5.028960e-08
Residual  10.0    57.904483     5.790448         NaN           NaN
Comparing models
    df_resid          ssr  df_diff      ss_diff           F        Pr(>F)
0      11.0  1265.686749      0.0          NaN         NaN           NaN
1      10.0    57.904483      1.0  1207.782266  208.581823  5.028960e-08
```

- $R^2_{Y|(X_1,X_2)}$ =0.979.
- $SSE_2 =$ 57.9,
- $s^2_{e_2} =$ 5.79,
- $F =$ 12.60
- Partial-$F =$ 208.582

Notice that SEQ SS for $X_2 = 2716.9(0.974 - 0.529) =$ 1207.782.

(c) The regression of Y on X_1, X_2, and X_3 is

```
model3 = smf.ols('y ~ x1 + x2 + x3 + 1', data=df).fit()
r2 = model3.rsquared
print(model3.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model3)
print('Analysis of Variance\n', anova)
SEQ_SS_X3 = anova.sum_sq['x3']
SSE_3 = anova.sum_sq['Residual']
s2e3 = anova.mean_sq['Residual']

anova = sms.anova.anova_lm(model2, model3)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
```

```
------------------------------------------------------------------------------
Intercept     48.1936      3.913     12.315      0.000      39.341      57.046
x1             1.6959      0.205      8.290      0.000       1.233       2.159
x2             0.6569      0.044     14.851      0.000       0.557       0.757
x3             0.2500      0.185      1.354      0.209      -0.168       0.668
==============================================================================
R-sq: 0.982
Analysis of Variance
            df       sum_sq      mean_sq           F        PR(>F)
x1        1.0  1450.076328  1450.076328  271.264194  4.995767e-08
x2        1.0  1207.782266  1207.782266  225.938509  1.107893e-07
x3        1.0     9.793869     9.793869    1.832128  2.088895e-01
Residual  9.0    48.110614     5.345624         NaN           NaN
Comparing models
    df_resid        ssr  df_diff   ss_diff         F    Pr(>F)
0       10.0  57.904483      0.0       NaN       NaN       NaN
1        9.0  48.110614      1.0  9.793869  1.832128  0.208889
```

- $R^2_{Y|(X_1,X_2,X_3)}$ =0.982.
- Partial-F = 1.832

The SEQ SS of X_3 is 9.79. The .95-quantile of $F[1, 9]$ is 5.117. Thus, the contribution of X_3 is not significant.

(d) The regression of Y on X_1, X_2, X_3, and X_4 is

```
model4 = smf.ols('y ~ x1 + x2 + x3 + x4 + 1', data=df).fit()
r2 = model4.rsquared
print(model4.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model4)
print('Analysis of Variance\n', anova)
SEQ_SS_X4 = anova.sum_sq['x4']
SSE_4 = anova.sum_sq['Residual']
s2e4 = anova.mean_sq['Residual']

anova = sms.anova.anova_lm(model3, model4)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     62.4054     70.071      0.891      0.399     -99.179     223.989
x1             1.5511      0.745      2.083      0.071      -0.166       3.269
x2             0.5102      0.724      0.705      0.501      -1.159       2.179
x3             0.1019      0.755      0.135      0.896      -1.638       1.842
x4            -0.1441      0.709     -0.203      0.844      -1.779       1.491
==============================================================================
R-sq: 0.982
Analysis of Variance
            df       sum_sq      mean_sq           F        PR(>F)
x1        1.0  1450.076328  1450.076328  242.367918  2.887559e-07
x2        1.0  1207.782266  1207.782266  201.870528  5.863323e-07
x3        1.0     9.793869     9.793869    1.636962  2.366003e-01
x4        1.0     0.246975     0.246975    0.041280  8.440715e-01
Residual  8.0    47.863639     5.982955         NaN           NaN
Comparing models
    df_resid        ssr  df_diff   ss_diff        F    Pr(>F)
0        9.0  48.110614      0.0       NaN      NaN       NaN
1        8.0  47.863639      1.0  0.246975  0.04128  0.844071
```

- $R^2_{Y|(X_1,X_2,X_3)}$ =0.982.

• Partial-F = 0.041

The effect of X_4 is not significant.

Exercise 4.25 For the data of Exercise 4.24, construct a linear model of the relationship between Y and $X_1, \ldots, X_4$, by the forward step-wise regression method. {exc:prog-lin-reg-cement}

Solution 4.25 Using the step-wise regression method from the `mistat` package, we get:

```
outcome = 'y'
all_vars = ['x1', 'x2', 'x3', 'x4']

included, model = mistat.stepwise_regression(outcome, all_vars, df)

formula = ' + '.join(included)
formula = f'{outcome} ~ 1 + {formula}'
print()
print('Final model')
print(formula)
print(model.params)
```

```
Step 1 add - (F: 22.80)  x4
Step 2 add - (F: 108.22)  x1 x4
Step 3 add - (F: 5.03)  x1 x2 x4

Final model
y ~ 1 + x1 + x4 + x2
Intercept    71.648307
x1            1.451938
x4           -0.236540
x2            0.416110
dtype: float64
```

Exercise 4.26 Consider the linear regression of Miles per Gallon on Horsepower for the cars in data file **CAR.csv**, with Origin = 3. Compute for each car the residuals, RESI, the standardized residuals, SRES, the leverage HI and the Cook distance, D.

Solution 4.26 Build the regression model using `statsmodels`.

```
car = mistat.load_data('CAR')
car_3 = car[car['origin'] == 3]
print('Full dataset shape', car.shape)
print('Origin 3 dataset shape', car_3.shape)
model = smf.ols(formula='mpg ~ hp + 1', data=car_3).fit()
print(model.summary2())
```

```
Full dataset shape (109, 5)
Origin 3 dataset shape (37, 5)
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.400
Dependent Variable: mpg              AIC:                195.6458
Date:               2022-01-29 23:50 BIC:                198.8676
No. Observations:   37               Log-Likelihood:     -95.823
Df Model:           1                F-statistic:        25.00
Df Residuals:       35               Prob (F-statistic): 1.61e-05
R-squared:          0.417            Scale:              10.994
------------------------------------------------------------------
```

```
                Coef.   Std.Err.     t     P>|t|    [0.025   0.975]
-------------------------------------------------------------------
Intercept     31.8328    1.8282  17.4117  0.0000  28.1213  35.5444
hp            -0.0799    0.0160  -4.9996  0.0000  -0.1123  -0.0474
------------------------------------------------------------------
Omnibus:               7.375       Durbin-Watson:          1.688
Prob(Omnibus):         0.025       Jarque-Bera (JB):       6.684
Skew:                  -0.675      Prob(JB):               0.035
Kurtosis:              4.584       Condition No.:          384
==================================================================
```

Compute the additional properties

```
influence = model.get_influence()
df = pd.DataFrame({
  'hp': car_3['hp'],
  'mpg': car_3['mpg'],
  'resi': model.resid,
  'sres': influence.resid_studentized_internal,
  'hi': influence.hat_matrix_diag,
  'D': influence.cooks_distance[0],
})
print(df.round(4))
```

```
      hp  mpg     resi    sres      hi       D
0    118   25   2.5936  0.7937  0.0288  0.0093
1    161   18  -0.9714 -0.3070  0.0893  0.0046
51    55   17 -10.4392 -3.3101  0.0953  0.5770
52    98   23  -1.0041 -0.3075  0.0299  0.0015
53    92   27   2.5166  0.7722  0.0339  0.0105
54    92   29   4.5166  1.3859  0.0339  0.0337
55   104   20  -3.5248 -1.0781  0.0277  0.0165
56    68   27   0.5993  0.1871  0.0665  0.0012
57    70   31   4.7591  1.4826  0.0627  0.0736
58   110   20  -3.0455 -0.9312  0.0270  0.0120
62   121   19  -3.1668 -0.9699  0.0303  0.0147
63    82   24  -1.2823 -0.3956  0.0442  0.0036
64   110   22  -1.0455 -0.3197  0.0270  0.0014
65   158   19  -0.2110 -0.0664  0.0823  0.0002
71    92   26   1.5166  0.4654  0.0339  0.0038
72   102   22  -1.6846 -0.5154  0.0282  0.0039
73    81   27   1.6378  0.5056  0.0455  0.0061
74   142   18  -2.4892 -0.7710  0.0520  0.0163
75   107   18  -5.2852 -1.6161  0.0271  0.0364
76   160   19  -0.0513 -0.0162  0.0869  0.0000
77    90   24  -0.6432 -0.1975  0.0356  0.0007
78    90   26   1.3568  0.4167  0.0356  0.0032
79    97   21  -3.0840 -0.9446  0.0305  0.0140
80   106   18  -5.3650 -1.6406  0.0273  0.0377
81   140   20  -0.6489 -0.2007  0.0490  0.0010
82   165   18  -0.6518 -0.2071  0.0993  0.0024
94    66   34   7.4396  2.3272  0.0704  0.2051
95    97   23  -1.0840 -0.3320  0.0305  0.0017
96   100   25   1.1557  0.3537  0.0290  0.0019
97   115   24   1.3539  0.4141  0.0278  0.0025
98   115   26   3.3539  1.0259  0.0278  0.0151
99    90   27   2.3568  0.7238  0.0356  0.0097
100  190   19   2.3453  0.7805  0.1786  0.0662
101  115   25   2.3539  0.7200  0.0278  0.0074
102  200   18   2.1441  0.7315  0.2184  0.0748
103   78   28   2.3982  0.7419  0.0497  0.0144
108   64   28   1.2798  0.4012  0.0745  0.0065
```

Notice that points 51 and 94 have residuals with large magnitude (-10.4, 7.4). Points 51 and 94 have also the largest Cook's distance (0.58, 0.21) Points 100 and 102 have high HI values (leverage; 0.18, 0.22).

Exercise 4.27 A simulation of the operation of a piston is available as the piston simulator function *pistonSimulation*. In order to test whether changing the piston weight from 30 to 60 [kg] effects the cycle time significantly, run the simulation program four times at weight 30, 40, 50, 60 [kg], keeping all other factors at their low level. In each run make $n = 5$ observations. Perform a one-way ANOVA of the results, and state your conclusions.

Solution 4.27 Run piston simulation for different piston weights and visualize variation of times (see Figure 4.7). {fig:anovaWeightPiston}

```
np.random.seed(1)
settings = {'s': 0.005, 'v0': 0.002, 'k': 1000, 'p0': 90_000,
            't': 290, 't0': 340}
results = []
n_simulation = 5
for m in [30, 40, 50, 60]:
  simulator = mistat.PistonSimulator(m=m, n_simulation=n_simulation,
                                     **settings)
  sim_result = simulator.simulate()
  results.extend([m, s] for s in sim_result['seconds'])
results = pd.DataFrame(results, columns=['m', 'seconds'])

group_std = results.groupby('m').std()
pooled_std = np.sqrt(np.sum(group_std**2) / len(group_std))[0]
print('Pooled standard deviation', pooled_std)

group_mean = results.groupby('m').mean()
ax = results.plot.scatter(x='m', y='seconds', color='black')
ax.errorbar(group_mean.index, results.groupby('m').mean().values.flatten(),
            yerr=[pooled_std] * 4, color='grey')
plt.show()
```

```
Pooled standard deviation 0.4948439561427665
```

Perform ANOVA of data.

```
model = smf.ols(formula='seconds ~ C(m)', data=results).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

	df	sum_sq	mean_sq	F	PR(>F)
C(m)	3.0	0.076379	0.025460	0.103972	0.956549
Residual	16.0	3.917929	0.244871	NaN	NaN

We see that the differences between the sample means are not significant in spite of the apparent upward trend in cycle times.

{exc:integ-circuits}

Exercise 4.28 In experiments performed for studying the effects of some factors on the integrated circuits fabrication process, the following results were obtained, on the pre-etch line width (μ_m)

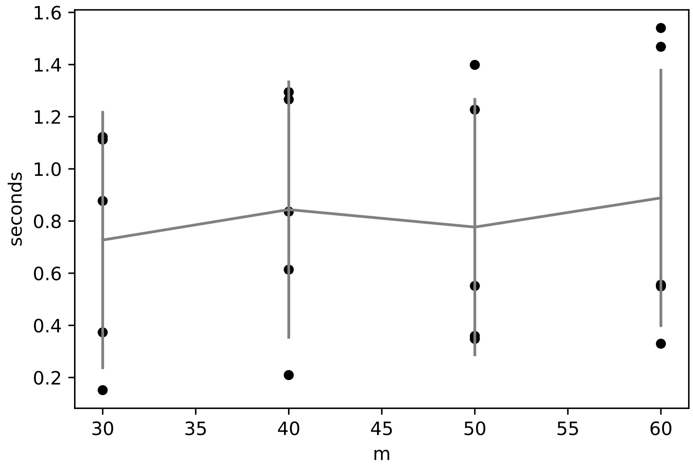


Fig. 4.7 ANOVA of effect of changing weight in piston simulation {fig:anovaWeightPiston}

Exp. 1	Exp. 2	Exp. 3
2.58	2.62	2.22
2.48	2.77	1.73
2.52	2.69	2.00
2.50	2.80	1.86
2.53	2.87	2.04
2.46	2.67	2.15
2.52	2.71	2.18
2.49	2.77	1.86
2.58	2.87	1.84
2.51	2.97	1.86

Perform an ANOVA to find whether the results of the three experiments are significantly different by using Python. Do the two test procedures (normal and bootstrap ANOVA) yield similar results?

{fig:boxplotIntegratedCircuits} **Solution 4.28** Prepare dataset and visualize distributions (see Figure 4.8).

```
df = pd.DataFrame([
  [2.58, 2.62, 2.22],
  [2.48, 2.77, 1.73],
  [2.52, 2.69, 2.00],
  [2.50, 2.80, 1.86],
  [2.53, 2.87, 2.04],
  [2.46, 2.67, 2.15],
  [2.52, 2.71, 2.18],
  [2.49, 2.77, 1.86],
  [2.58, 2.87, 1.84],
  [2.51, 2.97, 1.86]
], columns=['Exp. 1', 'Exp. 2', 'Exp. 3'])
df.boxplot()
```

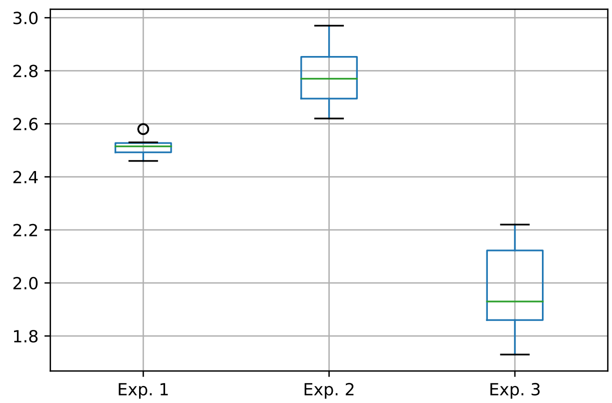


Fig. 4.8 Box plot of pre-etch line width from integrated circuits fabrication process {fig:boxplotIntegratedCircuits}

```
# Convert data frame to long format using melt
df = df.melt(var_name='Experiment', value_name='mu')
```

Analysis using ANOVA:

```
model = smf.ols(formula='mu ~ C(Experiment)', data=df).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
                 df    sum_sq   mean_sq           F        PR(>F)
C(Experiment)   2.0  3.336327  1.668163  120.917098  3.352509e-14
Residual       27.0  0.372490  0.013796         NaN           NaN
```

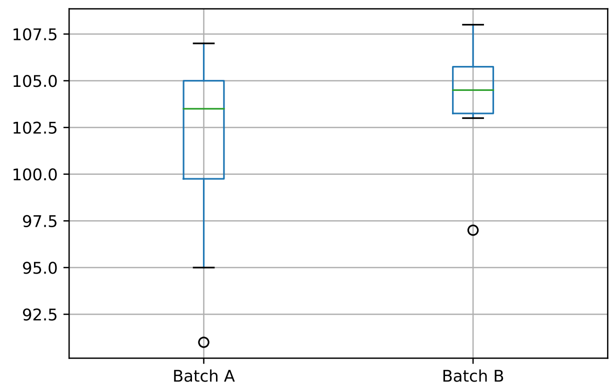
The difference in the experiments is significant.

Bootstrap test:

```
experiment = df['Experiment']
mu = df['mu']
def onewayTest(x, verbose=False):
    df = pd.DataFrame({
        'value': x,
        'variable': experiment,
    })
    aov = pg.anova(dv='value', between='variable', data=df)
    return aov['F'].values[0]

B = pg.compute_bootci(mu, func=onewayTest, n_boot=1000,
    seed=1, return_dist=True)

Bt0 = onewayTest(mu)
print('Bt0', Bt0)
print('ratio', sum(B[1] >= Bt0)/len(B[1]))
```

```
Bt0 120.91709844559576
ratio 0.0
```

The bootstrap test also shows that the difference in means is significant.



{fig:filmSpeedData} **Fig. 4.9** Box plot of film speed data

Exercise 4.29 In manufacturing film for industrial use, samples from two different batches gave the following film speed:

Batch A: 103, 107, 104, 102, 95, 91, 107, 99, 105, 105

Batch B: 104, 103, 106, 103, 107, 108, 104, 105, 105, 97

Test whether the differences between the two batches are significant, by using (i) a randomization test; (ii) an ANOVA.

{fig:filmSpeedData} **Solution 4.29** Create dataset and visualize distribution (see Figure 4.9).

```
df = pd.DataFrame({
  'Batch A': [103, 107, 104, 102, 95, 91, 107, 99, 105, 105],
  'Batch B': [104, 103, 106, 103, 107, 108, 104, 105, 105, 97],
})
df.boxplot()
plt.show()
```

{sec:randomizaton-test} **(i)** Randomization test (see Section 3.13.2)

```
dist = mistat.randomizationTest(df['Batch A'], df['Batch B'], np.mean,
                                aggregate_stats=lambda x: x[0] - x[1],
                                n_boot=10000, seed=1)
# ax = sns.distplot(dist)
# ax.axvline(np.mean(df['Batch A']) - np.mean(df['Batch B']))
```

```
Original stat is -2.400000
Original stat is at quantile 1062 of 10001 (10.62%)
Distribution of bootstrap samples:
 min: -5.40, median: 0.00,  max: 5.60
```

The randomization test gave a P value of 0.106. The difference between the means is not significant.

(ii)

```
# Convert data frame to long format using melt
df = df.melt(var_name='Batch', value_name='film_speed')

model = smf.ols(formula='film_speed ~ C(Batch)', data=df).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
            df  sum_sq    mean_sq         F    PR(>F)
C(Batch)   1.0    28.8  28.800000  1.555822  0.228263
Residual  18.0   333.2  18.511111       NaN       NaN
```

The ANOVA also shows no significant difference in the means. The P value is 0.228. Remember that the F-test in the ANOVA is based on the assumption of normality and equal variances. The randomization test is nonparametric.

Exercise 4.30 Use a randomization test to test the significance of the differences between the results of the three experiments in Exercise 4.28. {exc:integ-circuits}

Use this statistic:

$$\delta = \frac{\sum_{k=1}^{3} n_k \bar{x}_k^2 - n\bar{x}^2}{S_x^2}$$

with $n = n_1 + n_2 + n_3$ and x the combined set of all results.

Solution 4.30 Define function that calculates the statistic and execute bootstrap.

```
def func_stats(x):
    m = pd.Series(x).groupby(df['Experiment']).agg(['mean', 'count'])
    top = np.sum(m['count'] * m['mean'] ** 2) - len(x)*np.mean(x)**2
    return top / np.std(x) ** 2

Bt = []
mu = list(df['mu'])
for _ in range(1000):
    mu_star = random.sample(mu, len(mu))
    Bt.append(func_stats(mu_star))

Bt0 = func_stats(mu)
print('Bt0', Bt0)
print('ratio', sum(Bt >= Bt0)/len(Bt))
```

```
Bt0 26.986990459670288
ratio 0.0
```

The result demonstrates that the differences between the results is significant.

Exercise 4.31 In data file **PLACE.csv** we have 26 samples, each one of size n = 16, of x-, y-, θ-deviations of components placements. Make an ANOVA, to test the significance of the sample means in the x-deviation. Classify the samples into homogeneous groups such that the differences between sample means in the same group are not significant, and those in different groups are significant. Use the Scheffé coefficient S_α for $\alpha = .05$.

Solution 4.31 Load the data and visualize the distributions (see Figure 4.10). {fig:boxplotXdevCrcBrdPlace}

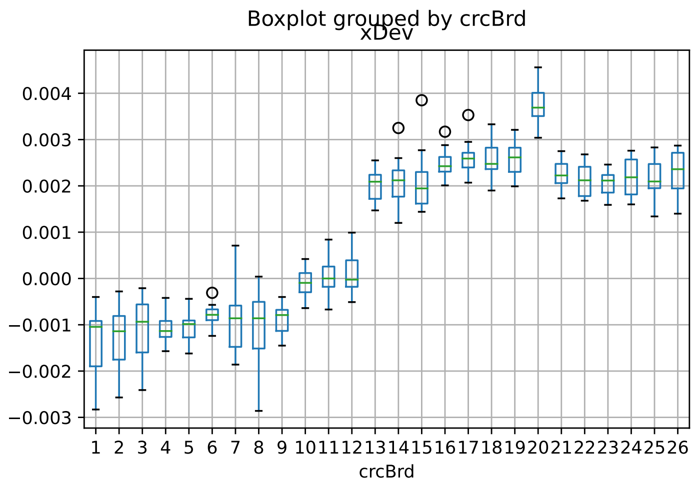


Fig. 4.10 Box plot visualisation of xDev distribution by crcBrd for the PLACE dataset {fig:boxplotXdevCrcBrdPlace}

```
place = mistat.load_data('PLACE')
place.boxplot('xDev', by='crcBrd')
plt.show()
```

(a) ANOVA for the fulldataset

```
model = smf.ols(formula='xDev ~ C(crcBrd)', data=place).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
              df    sum_sq       mean_sq           F          PR(>F)
C(crcBrd)   25.0  0.001128  4.512471e-05  203.292511  2.009252e-206
Residual   390.0  0.000087  2.219694e-07         NaN            NaN
```

(b) There seem to be four homogeneous groups: $G_1 = \{1, 2, \ldots, 9\}$, $G_2 = \{10, 11, 12\}$, $G_3 = \{13, \ldots, 19, 21, \ldots, 26\}$, $G_4 = \{20\}$.

In multiple comparisons we use the Scheffé coefficient $S_{.05} = (25 \times F_{.95}[25, 390])^{1/2} = (25 \times 1.534)^{1/2} = 6.193$. The group means and standard errors are:

```
G1 = [1, 2, 3, 4, 5, 6, 7, 8, 9]
G2 = [10, 11, 12]
G3 = [13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26]
G4 = [20]
place['group'] = 'G1'
place.loc[place['crcBrd'].isin(G2), 'group'] = 'G2'
place.loc[place['crcBrd'].isin(G3), 'group'] = 'G3'
place.loc[place['crcBrd'].isin(G4), 'group'] = 'G4'

statistics = place['xDev'].groupby(place['group']).agg(['mean', 'sem', 'count'])
statistics = statistics.sort_values(['mean'], ascending=False)
print(statistics.round(8))
statistics['Diff'] = 0
n = len(statistics)
```

```
print(statistics['mean'][:-1].values - statistics['mean'][1:].values)
print(statistics['mean'][:(n-1)].values - statistics['mean'][1:].values)
statistics.loc[1:, 'Diff'] = (statistics['mean'][:-1].values -
                              statistics['mean'][1:].values)
statistics['CR'] = 6.193 * statistics['Diff']
print(statistics.round(8))
# 0.001510 0.0022614 0.0010683
# 0.000757 0.000467 0.000486

sem = statistics['sem'].values
sem = sem**2
sem = np.sqrt(sem[:-1] + sem[1:])
print(sem * 6.193)
print(757/644, 467/387, 486/459)
```

```
           mean       sem  count
group
G4     0.003778  0.000100     16
G3     0.002268  0.000030    208
G2     0.000006  0.000055     48
G1    -0.001062  0.000050    144
[0.00151029 0.00226138 0.00106826]
[0.00151029 0.00226138 0.00106826]
           mean       sem  count      Diff        CR
group
G4     0.003778  0.000100     16  0.000000  0.000000
G3     0.002268  0.000030    208  0.001510  0.009353
G2     0.000006  0.000055     48  0.002261  0.014005
G1    -0.001062  0.000050    144  0.001068  0.006616
[0.00064435 0.00038718 0.00045929]
1.1754658385093169 1.20671834625323 1.0588235294117647
```

The differences between the means of the groups are all significant.

```
from statsmodels.stats.multicomp import pairwise_tukeyhsd
m_comp = pairwise_tukeyhsd(endog=place['xDev'], groups=place['group'],
                           alpha=0.05)
print(m_comp)
```

```
Multiple Comparison of Means - Tukey HSD, FWER=0.05
=================================================
group1 group2 meandiff p-adj lower  upper  reject
-------------------------------------------------
    G1     G2   0.0011 0.001 0.0009 0.0013   True
    G1     G3   0.0033 0.001 0.0032 0.0035   True
    G1     G4   0.0048 0.001 0.0045 0.0052   True
    G2     G3   0.0023 0.001 0.0021 0.0025   True
    G2     G4   0.0038 0.001 0.0034 0.0041   True
    G3     G4   0.0015 0.001 0.0012 0.0018   True
-------------------------------------------------
```

Exercise 4.32 The frequency distribution of cars by origin and number of cylinders is given in the following table.

Num. Cylinders	US	Europe	Asia	Total
4	33	7	26	66
6 or more	25	7	11	43
Total	58	14	37	109

Perform a chi-square test of the dependence of number of cylinders and the origin of car.

Solution 4.32

```
df = pd.DataFrame({
    'US': [33, 25],
    'Europe': [7, 7],
    'Asia': [26, 11],
  })

print(df)

col_sums = df.sum(axis=0)
row_sums = df.sum(axis=1)
total = df.to_numpy().sum()

expected_frequencies = np.outer(row_sums, col_sums) / total

chi2 = (df - expected_frequencies) ** 2 / expected_frequencies
chi2 = chi2.to_numpy().sum()
print(f'chi2: {chi2:.3f}')
print(f'p-value: {1 - stats.chi2.cdf(chi2, 2):.3f}')
```

```
   US  Europe  Asia
0  33       7    26
1  25       7    11
chi2: 2.440
p-value: 0.295
```

The chi-square test statistic is $X^2 = 2.440$ with d.f. = 2 and P value = 0.295. The null hypothesis that the number of cylinders a car has is independent of the origin of the car is not rejected.

We can also use the `scipy` function `chi2_contingency`.

```
chi2 = stats.chi2_contingency(df)
print(f'chi2-statistic: {chi2[0]:.3f}')
print(f'p-value: {chi2[1]:.3f}')
print(f'd.f.: {chi2[2]}')
```

```
chi2-statistic: 2.440
p-value: 0.295
d.f.: 2
```

Exercise 4.33 Perform a chi-squared test of the association between turn diameter and miles/gallon based on Table 4.17.

{tbl:cont-table-turn-mpg}

Solution 4.33 In Python:

```
car = mistat.load_data('CAR')
binned_car = pd.DataFrame({
  'turn': pd.cut(car['turn'], bins=[27, 30.6, 34.2, 37.8, 45]), #np.arange(27, 50, 3.6)),
  'mpg': pd.cut(car['mpg'], bins=[12, 18, 24, 100]),
})
freqDist = pd.crosstab(binned_car['mpg'], binned_car['turn'])
print(freqDist)

chi2 = stats.chi2_contingency(freqDist)
print(f'chi2-statistic: {chi2[0]:.3f}')
print(f'p-value: {chi2[1]:.3f}')
print(f'd.f.: {chi2[2]}')
```

```
turn       (27.0, 30.6]  (30.6, 34.2]  (34.2, 37.8]  (37.8, 45.0]
mpg
(12, 18]              2             4            10            15
(18, 24]              0            12            26            15
(24, 100]             4            15             6             0
chi2-statistic: 34.990
p-value: 0.000
d.f.: 6
```

The dependence between turn diameter and miles per gallon is significant.

Exercise 4.34 In a customer satisfaction survey several questions were asked regarding specific services and products provided to customers. The answers were on a 1-5 scale, where 5 means "very satisfied with the service or product" and 1 means "very dissatisfied". Compute the Mean Squared Contingency, Tschuprow's Index and Cramer's Index for both contingency tables.

	Question 1				
Question 3	1	2	3	4	5
1	0	0	0	1	0
2	1	0	2	0	0
3	1	2	6	5	1
4	2	1	10	23	13
5	0	1	1	15	100

	Question 2				
Question 3	1	2	3	4	5
1	1	0	0	3	1
2	2	0	1	0	0
3	0	4	2	3	0
4	1	1	10	7	5
5	0	0	1	30	134

Solution 4.34 In Python:

```
question_13 = pd.DataFrame({
  '1': [0,0,0,1,0],
  '2': [1,0,2,0,0],
  '3': [1,2,6,5,1],
  '4': [2,1,10,23,13],
  '5': [0,1,1,15,100],
  }, index = ['1', '2', '3', '4', '5']).transpose()
question_23 = pd.DataFrame({
  '1': [1,0,0,3,1],
  '2': [2,0,1,0,0],
  '3': [0,4,2,3,0],
  '4': [1,1,10,7,5],
  '5': [0,0,1,30,134],
  }, index = ['1', '2', '3', '4', '5']).transpose()

chi2_13 = stats.chi2_contingency(question_13)
chi2_23 = stats.chi2_contingency(question_23)
```

```
msc_13 = chi2_13[0] / question_13.to_numpy().sum()
tschuprov_13 = np.sqrt(msc_13 / (2 * 2)) # (4 * 4))
cramer_13 = np.sqrt(msc_13 / 2) # min(4, 4))

msc_23 = chi2_23[0] / question_23.to_numpy().sum()
tschuprov_23 = np.sqrt(msc_23 / 4) # (4 * 4))
cramer_23 = np.sqrt(msc_23 / 2) # min(4, 4))

print('Question 1 vs 3')
print(f'  Mean squared contingency : {msc_13:.3f}')
print(f'  Tschuprov : {tschuprov_13:.3f}')
print(f"  Cramer's index : {cramer_13:.3f}")
print('Question 2 vs 3')
print(f'  Mean squared contingency : {msc_23:.3f}')
print(f'  Tschuprov : {tschuprov_23:.3f}')
print(f"  Cramer's index : {cramer_23:.3f}")
```

```
Question 1 vs 3
  Mean squared contingency : 0.629
  Tschuprov : 0.397
  Cramer's index : 0.561
Question 2 vs 3
  Mean squared contingency : 1.137
  Tschuprov : 0.533
  Cramer's index : 0.754
```

Chapter 5
Sampling for Estimation of Finite Population Quantities

Import required modules and define required functions

```
import random
import numpy as np
import pandas as pd
import pingouin as pg
from scipy import stats
import matplotlib.pyplot as plt
import mistat
```

Exercise 5.1 Consider a finite population of size N, whose elements have values $x_1, \cdots, x_N$. Let $\hat{F}_N(x)$ be the c.d.f., i.e.,

$$\hat{F}_N(x) = \frac{1}{N} \sum_{i=1}^{N} I\{x_i \leq x\}.$$

Let $X_1, \cdots, X_n$ be the values of a RSWR. Show that $X_1, \cdots, X_n$ are independent having a common distribution $\hat{F}_N(x)$.

Solution 5.1 Define the binary random variables

$$I_{ij} = \begin{cases} 1, & \text{if the } j\text{-th element is selected at the } i\text{-th sampling} \\ \\ 0, & \text{otherwise.} \end{cases}$$

The random variables $X_1, \cdots, X_n$ are given by $X_i = \sum_{j=1}^{N} x_j I_{ij}$, $i = 1, \cdots, n$. Since sampling is RSWR, $\Pr\{X_i = x_j\} = \dfrac{1}{N}$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, N$. Hence, $\Pr\{X_i \leq x\} = F_N(x)$ for all x, and all $i = 1, \ldots, n$. Moreover, by definition of RSWR, the vectors $\mathbf{I}_i = (I_{i1}, \ldots, I_{iN})$, $i = 1, \ldots, n$ are mutually independent. Therefore $X_1, \ldots, X_n$ are i.i.d., having a common c.d.f. $F_N(x)$.

Exercise 5.2 Show that if $\bar{X}_n$ is the mean of a RSWR then, $\bar{X}_n \to \mu_N$ as $n \to \infty$ in probability (WLLN).

Solution 5.2 In continuation of the previous exercise, $E\{X_i\} = \dfrac{1}{N}\sum_{i=1}^{N} x_i = \mu_N$. Therefore, by the weak law of large numbers, $\lim_{n\to\infty} P\{|\bar{X}_n - \mu_N| < \epsilon\} = 1$.

Exercise 5.3 What is the large sample approximation to $\Pr\{\sqrt{n} \mid \bar{X}_n - \mu_N \mid < \delta\}$ in RSWR?

Solution 5.3 By the CLT ($0 < \sigma_N^2 < \infty$),

$$\Pr\{\sqrt{n}|\bar{X}_n - \mu_N| < \delta\} \approx 2\Phi\left(\frac{\delta}{\sigma_N}\right) - 1,$$

as $n \longrightarrow \infty$.

Exercise 5.4 Use Python to draw random samples with or without replacement from data file **PLACE.csv**. Write a function which computes the sample correlation between the x-dev and y-dev in the sample values. Execute this function 100 times and make a histogram of the sample correlations.

Solution 5.4 We create samples of size $k = 20$ with and without replacement, determine the correlation coefficient and finally create the two histograms (see Figure 5.1).

{fig:exDistCorrPlace}

```
random.seed(1)
place = mistat.load_data('PLACE')

# calculate correlation coefficient based on a sample of rows
def stat_func(idx):
    return stats.pearsonr(place['xDev'][idx], place['yDev'][idx])[0]

rswr = []
rswor = []
idx = list(range(len(place)))
for _ in range(100):
    rswr.append(stat_func(random.choices(idx, k=20)))
    rswor.append(stat_func(random.sample(idx, k=20)))

corr_range = (min(*rswr, *rswor), max(*rswr, *rswor))

def makeHistogram(title, ax, data, xrange):
  ax = pd.Series(data).hist(color='grey', ax=ax, bins=20)
  ax.set_title(title)
  ax.set_xlabel('Sample correlation')
  ax.set_ylabel('Frequency')
  ax.set_xlim(*xrange)

fig, axes = plt.subplots(figsize=[5, 3], ncols=2)
makeHistogram('RSWR', axes[0], rswr, corr_range)
makeHistogram('RSWOR', axes[1], rswor, corr_range)
plt.tight_layout()
plt.show()
```

Exercise 5.5 Use file **CAR.csv** and Python. Construct samples of 50 records at random, without replacement (RSWOR). For each sample, calculate the median of the variables turn-diameter, horsepower and mpg. Repeat this 200 times and present the histograms of the sampling distributions of the medians.

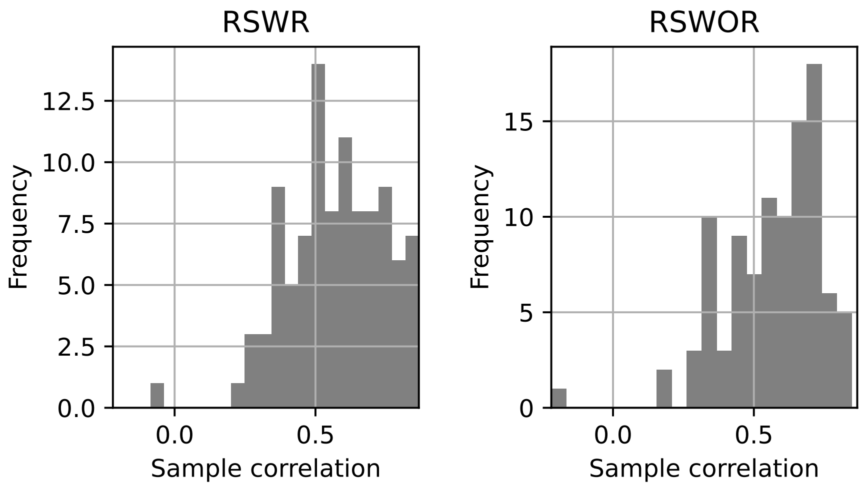


Fig. 5.1 Distribution of correlation between xDev and yDev for sampling with and without distribution. {fig:exDistCorrPlace}

{fig:exDistMedianCAR} **Solution 5.5** The Python code creates the histograms shown in Figure 5.2.

```
random.seed(1)
car = mistat.load_data('CAR')
columns = ['turn', 'hp', 'mpg']

# calculate correlation coefficient based on a sample of rows
def stat_func(idx):
    sample = car[columns].loc[idx,]
    return sample.median()

idx = list(range(len(car)))
result = []
for _ in range(200):
    result.append(stat_func(random.sample(idx, k=50)))
result = pd.DataFrame(result)

fig, axes = plt.subplots(figsize=[8, 3], ncols=3)
for ax, column in zip(axes, columns):
    result[column].hist(color='grey', ax=ax)
    ax.set_xlabel(f'Median {column}')
    ax.set_ylabel('Frequency')
plt.tight_layout()
plt.show()
```

Exercise 5.6 In continuation of Example 5.5, how large should the sample be from the three strata, so that the S.E. $\{\bar{X}_i\}$ $(i = 1, \ldots, 3)$ will be smaller than $\delta = 0.05$? {ex:placeStratSample}

Solution 5.6 For RSWOR,

$$\text{S.E.}\{\bar{X}_i\} = \frac{\sigma}{\sqrt{n}}\left(1 - \frac{n-1}{N-1}\right)^{1/2}$$

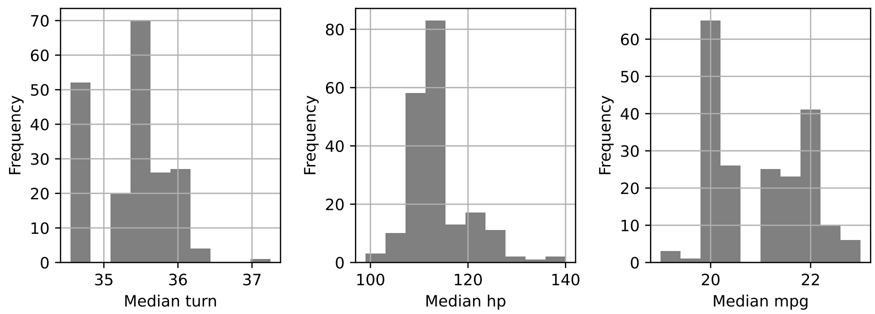


Fig. 5.2 Distribution of median of turn-diameter, horsepower, and mpg of the CAR dataset using random sampling without replacement. {fig:exDistMedianCAR}

Equating the standard error to δ we get $n_1 = 30$, $n_2 = 116$, $n_3 = 84$.

Exercise 5.7 The proportion of defective chips in a lot of $N = 10,000$ chips is $\mathbf{P} = 5 \times 10^{-4}$. How large should a RSWOR be so that, the width of the confidence interval for P, with coverage probability $1 - \alpha = .95$, will be 0.002?

Solution 5.7 The required sample size is a solution of the equation $0.002 = 2 \cdot 1.96 \cdot \sqrt{\frac{P(1-P)}{n}\left(1 - \frac{n-1}{N}\right)}$. The solution is $n = 1611$.

{ex:placeStratSample} **Exercise 5.8** Use Python to perform stratified random samples from the three strata of the data file **PLACE.csv** (see Example 5.5). Allocate 500 observations to the three samples proportionally. Estimate the population mean (of x-dev). Repeat this 100 times and estimate the standard-error or your estimates. Compare the estimated standard error to the exact one.

Solution 5.8 The following are Python commands to estimate the mean of all $N = 416$ x-dev values by stratified sampling with proportional allocation. The total sample size is $n = 200$ and the weights are $W_1 = 0.385$, $W_2 = 0.115$, $W_3 = 0.5$. Thus, $n_1 = 77$, $n_2 = 23$, and $n_3 = 100$.

```
# load dataset and split into strata
place = mistat.load_data('PLACE')
strata_1 = list(place['xDev'][:160])
strata_2 = list(place['xDev'][160:208])
strata_3 = list(place['xDev'][208:])
N = len(place)
w_1 = 0.385
w_2 = 0.115
w_3 = 0.5
n_1 = int(w_1 * 200)
n_2 = int(w_2 * 200)
n_3 = int(w_3 * 200)

sample_means = []
```

```
for _ in range(500):
    m_1 = np.mean(random.sample(strata_1, k=n_1))
    m_2 = np.mean(random.sample(strata_2, k=n_2))
    m_3 = np.mean(random.sample(strata_3, k=n_3))
    sample_means.append(w_1*m_1 + w_2*m_2 + w_3*m_3)
std_dev_sample_means = np.std(sample_means)
print(std_dev_sample_means)
print(stats.sem(place['xDev'], ddof=0))
```

```
3.442839155174113e-05
8.377967188860638e-05
```

The standard deviation of the estimated means is an estimate of S.E.$(\hat{\mu}_N)$. The true value of this S.E. is 0.000034442.

Exercise 5.9 Derive the formula for n_i^0 $(i = 1, \cdots, k)$ in the optimal allocation, by differentiating $L(n_1, \cdots, n_k, \lambda)$ and solving the equations.

Solution 5.9 $L(n_1, \ldots, n_k; \lambda) = \sum_{i=1}^k W_i^2 \dfrac{\tilde{\sigma}_{N_i}^2}{n_i} - \lambda \left(n - \sum_{i=1}^k n_i\right)$. Partial differentiation of L w.r.t. $n_1, \ldots, n_k$ and λ and equating the result to zero yields the following equations:

$$\frac{W_i^2 \tilde{\sigma}_{N_i}^2}{n_i^2} = \lambda, \quad i = 1, \ldots, k$$

$$\sum_{i=1}^k n_i = n.$$

Equivalently, $n_i = \dfrac{1}{\sqrt{\lambda}} W_i \tilde{\sigma}_{N_i}$, for $i = 1, \ldots, k$ and $n = \dfrac{1}{\sqrt{\lambda}} \sum_{i=1}^k W_i \tilde{\sigma}_{N_i}$. Thus $n_i^0 = n \dfrac{W_i \tilde{\sigma}_{N_i}}{\sum_{j=1}^k W_j \tilde{\sigma}_{N_j}}, \quad i = 1, \ldots, k.$

Exercise 5.10 Consider the prediction model

$$y_i = \beta + e_i, \quad i = 1, \ldots, N$$

where $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2$ and COV$(e_i, e_j) = 0$ for $i \neq j$. We wish to predict the population mean $\mu_N = \dfrac{1}{N} \sum_{i=1}^N y_i$. Show that the sample mean $\bar{Y}_n$ is prediction unbiased. What is the prediction MSE of $\bar{Y}_n$?

Solution 5.10 The prediction model is $y_i = \beta + e_i$, $i = 1, \ldots, N$, $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2$, cov$(e_i, e_j) = 0$ for all $i \neq j$.

$$\begin{aligned} E\{\bar{Y}_n - \mu_N\} &= E\left\{\beta + \frac{1}{n} \sum_{i=1}^N I_i e_i - \beta - \frac{1}{N} \sum_{i=1}^N e_i\right\} \\ &= \frac{1}{n} \sum_{i=1}^N E\{I_i e_i\}, \end{aligned}$$

where $I_i = \begin{cases} 1, & \text{if } i\text{-th population element is sampled} \\ 0, & \text{otherwise.} \end{cases}$

Notice that $I_1, \ldots, I_N$ are independent of $e_1, \ldots, e_N$. Hence, $E\{I_i e_i\} = 0$ for all $i = 1, \ldots, N$. This proves that $\bar{Y}_n$ is prediction unbiased, irrespective of the sample strategy. The prediction MSE of $\bar{Y}_n$ is

$$\begin{aligned} PMSE\{\bar{Y}_n\} &= E\{(\bar{Y}_n - \mu_N)^2\} \\ &= V\left\{\frac{1}{n}\sum_{i=1}^{N} I_i e_i - \frac{1}{N}\sum_{i=1}^{N} e_i\right\} \\ &= V\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\sum_{i=1}^{N} I_i e_i - \frac{1}{N}\sum_{i=1}^{N}(1 - I_i)e_i\right\}. \end{aligned}$$

Let $\mathbf{s}$ denote the set of units in the sample. Then

$$\begin{aligned} PMSE\{\bar{Y}_n \mid \mathbf{s}\} &= \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right)^2 + \frac{1}{N}\left(1 - \frac{n}{N}\right)\sigma^2 \\ &= \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right). \end{aligned}$$

Notice that $PMSE\{\bar{Y}_n \mid \mathbf{s}\}$ is independent of $\mathbf{s}$, and is equal for all samples.

Exercise 5.11 Consider the prediction model

$$y_i = \beta_0 + \beta_1 x_i + e_i, \quad i = 1, \ldots, N,$$

where $e_1, \ldots, e_N$ are independent r.v.'s with $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2 x_i$ $(i = 1, \ldots, n)$. We wish to predict $\mu_N = \dfrac{1}{N}\sum_{i=1}^{N} y_i$. What should be a good predictor for μ_N?

Solution 5.11 The model is $y_i = \beta_0 + \beta_1 x_i + e_i$, $i = 1, \ldots, N$. $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2 x_i$, $i = 1, \ldots, N$. Given a sample $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, we estimate β_0 and β_1 by the weighted LSE because the variances of y_i depend on x_i, $i = 1, \ldots, N$. These weighted LSE values $\hat{\beta}_0$ and $\hat{\beta}_1$ minimizing $Q = \sum_{i=1}^{N} \dfrac{1}{X_i}(Y_i - \beta_0 - \beta_1 X_i)^2$, are given by

$$\hat{\beta}_1 = \frac{\bar{Y}_n \cdot \frac{1}{n}\sum_{i=1}^{N}\frac{1}{X_i} - \frac{1}{n}\sum_{i=1}^{N}\frac{Y_i}{X_i}}{\bar{X}_n \cdot \frac{1}{n}\sum_{i=1}^{N}\frac{1}{X_i} - 1} \quad \text{and} \quad \hat{\beta}_0 = \frac{1}{\sum_{i=1}^{N}\frac{1}{X_i}}\left(\sum_{i=1}^{N}\frac{Y_i}{X_i} - n\hat{\beta}_1\right).$$

It is straightforward to show that $E\{\hat{\beta}_1\} = \beta_1$ and $E\{\hat{\beta}_0\} = \beta_0$. Thus, an unbiased predictor of μ_N is $\hat{\mu}_N = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}_N$.

Exercise 5.12 Prove that $\hat{Y}_{RA}$ and $\hat{Y}_{RG}$ are unbiased predictors and derive their prediction variances.

Solution 5.12 The predictor $\hat{Y}_{RA}$ can be written as $\hat{Y}_{RA} = \bar{x}_N \cdot \dfrac{1}{n} \sum_{i=1}^{N} I_i \frac{y_i}{x_i}$, where

$$I_i = \begin{cases} 1, & \text{if } i\text{-th population element is in the sample } \mathbf{s} \\ \\ 0, & \text{otherwise.} \end{cases}$$

Recall that $y_i = \beta x_i + e_i$, $i = 1, \ldots, N$, and that for any sampling strategy, $e_1, \ldots, e_N$ are independent of $I_1, \ldots, I_N$. Hence, since $\sum_{i=1}^{N} I_i = n$,

$$\begin{aligned} E\{\hat{Y}_{RA}\} &= \bar{x}_N \cdot \frac{1}{n} \sum_{i=1}^{N} E\left\{ I_i \frac{\beta x_i + e_i}{x_i} \right\} \\ &= \bar{x}_N \left(\beta + \frac{1}{n} \sum_{i=1}^{N} E\left\{ I_i \frac{e_i}{x_i} \right\} \right) \\ &= \bar{x}_N \beta, \end{aligned}$$

because $E\left\{ I_i \dfrac{e_i}{x_i} \right\} = E\left\{ \dfrac{I_i}{x_i} \right\} E\{e_i\} = 0$, $i = 1, \ldots, N$. Thus, $E\{\hat{Y}_{RA} - \mu_N\} = 0$ and $\hat{Y}_{RA}$ is an unbiased predictor.

$$\begin{aligned} PMSE\{\hat{Y}_{RA}\} &= E\left\{ \left(\frac{\bar{x}_N}{n} \sum_{i=1}^{N} I_i \frac{e_i}{x_i} - \frac{1}{N} \sum_{i=1}^{N} e_i \right)^2 \right\} \\ &= V\left\{ \sum_{i \in \mathbf{s}_n} \left(\frac{\bar{x}_N}{n x_i} - \frac{1}{N} \right) e_i - \sum_{i' \in \mathbf{r}_n} \frac{e_{i'}}{N} \right\} \end{aligned}$$

where $\mathbf{s}_n$ is the set of elements in the sample and $\mathbf{r}_n$ is the set of elements in $\mathcal{P}$ but not in $\mathbf{s}_n$, $\mathbf{r}_n = \mathcal{P} - \mathbf{s}_n$. Since $e_1, \ldots, e_N$ are uncorrelated,

$$\begin{aligned} PMSE\{\hat{Y}_{RA} \mid \mathbf{s}_n\} &= \sigma^2 \sum_{i \in \mathbf{s}_n} \left(\frac{\bar{x}_N}{n x_i} - \frac{1}{N} \right)^2 + \sigma^2 \frac{N - n}{N^2} \\ &= \frac{\sigma^2}{N^2} \left[(N - n) + \sum_{i \in \mathbf{s}_n} \left(\frac{N \bar{x}_N}{n x_i} - 1 \right)^2 \right]. \end{aligned}$$

A sample $\mathbf{s}_n$ which minimizes $\sum_{i \in \mathbf{s}_n} \left(\frac{N \bar{x}_N}{n x_i} - 1 \right)^2$ is optimal.

The predictor $\hat{Y}_{RG}$ can be written as

$$\hat{Y}_{RG} = \bar{x}_N \frac{\sum_{i=1}^{N} I_i x_i y_i}{\sum_{i=1}^{N} I_i x_i^2} = \bar{x}_N \left(\frac{\sum_{i=1}^{N} I_i x_i (\beta x_i + e_i)}{\sum_{i=1}^{N} I_i x_i^2} \right) = \beta \bar{x}_N + \bar{x}_N \frac{\sum_{i=1}^{N} I_i x_i e_i}{\sum_{i=1}^{N} I_i x_i^2}.$$

Hence, $E\{\hat{Y}_{RG}\} = \beta \bar{x}_N$ and $\hat{Y}_{RG}$ is an unbiased predictor of μ_N. The conditional prediction MSE, given $\mathbf{s}_n$, is

$$PMSE\{\hat{Y}_{RG} \mid \mathbf{s}_n\} = \frac{\sigma^2}{N^2} \left[N + \frac{N^2 \bar{x}_N^2}{\sum_{i \in \mathbf{s}_n} x_i^2} - 2N\bar{x}_N \frac{n\bar{X}_n}{\sum_{i \in \mathbf{s}_n} x_i^2} \right].$$

Chapter 6
Time Series Analysis and Prediction

Import required modules and define required functions

```
import math
import mistat
import numpy as np
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import acf, pacf
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
from statsmodels.tsa import tsatools
import statsmodels.formula.api as smf
```

Exercise 6.1 Evaluate trends and peaks in the data on COVID19 related mortality available in `https://www.euromomo.eu/graphs-and-maps/`. Evaluate the impact of the time window on the line chart pattern. Identify periods with changes in mortality and periods with stability in mortality.

Solution 6.1 TODO: Provide a sample solution

{exc:seascom-1}

Exercise 6.2 The data set "SeasCom" provides the monthly demand for a seasonal commodity during 102 months.

(i) Plot the data to see the general growth of the demand;
(ii) Fit to the data the trend function

$$f(t) = \beta_1 + \beta_2((t-51)/102) + \beta_3 \cos(\pi t/6) + \beta_4 \sin(\pi t/6);$$

(iii) Plot the deviations of the data from the fitted trend, i.e. $\hat{U}_t = X_t - \hat{f}(t)$;
(iv) Compute the correlations between $(\hat{U}_t, \hat{U}_{t+1})$ and $(\hat{U}_t, \hat{U}_{t+2})$;
(v) What can you infer from these results?

Solution 6.2 (i) Figure 6.1 shows the change of demand over time {fig:seascom-timeline}

(ii) We are first fitting the seasonal trend to the data in SeasCom data set. We use the linear model$Y = X\beta + \varepsilon$, where the Y vector has 102 elements, which are in data set. X is a 102x4 matrix of four column vectors. We combined X and Y in a data frame. In addition to the SeasCom data, we add a column of 1's (`const`), a column with numbers 1 to 102 (`trend`) and two columns to describe the seasonal pattern. The column `season_1` consists of $\cos(\pi \times \text{trend}/6)$, and the column `season_2` is $\sin(\pi \times \text{trend}/6)$.

```
seascom = mistat.load_data('SEASCOM.csv')
df = tsatools.add_trend(seascom, trend='ct')
df['season_1'] = [np.cos(math.pi * tx/6) for tx in df['trend']]
df['season_2'] = [np.sin(math.pi * tx/6) for tx in df['trend']]
print(df.head())

model = smf.ols(formula='SeasCom ~ trend + 1 + season_1 + season_2',
                data=df).fit()
print(model.params)
print(f'r2-adj: {model.rsquared_adj:.3f}')
```

```
    SeasCom  const  trend      season_1  season_2
0  71.95623    1.0    1.0  8.660254e-01  0.500000
1  56.36048    1.0    2.0  5.000000e-01  0.866025
2  64.85331    1.0    3.0  6.123234e-17  1.000000
3  59.93460    1.0    4.0 -5.000000e-01  0.866025
4  51.62297    1.0    5.0 -8.660254e-01  0.500000
Intercept    47.673469
trend         1.047236
season_1     10.653968
season_2     10.130145
dtype: float64
r2-adj: 0.981
```

The least squares estimates of β is

$$b = (47.67347, 1.04724, 10.65397, 10.13015)'$$

{fig:seascom-timeline} The fitted trend is $Y_t = Xb$. (see Figure 6.1).

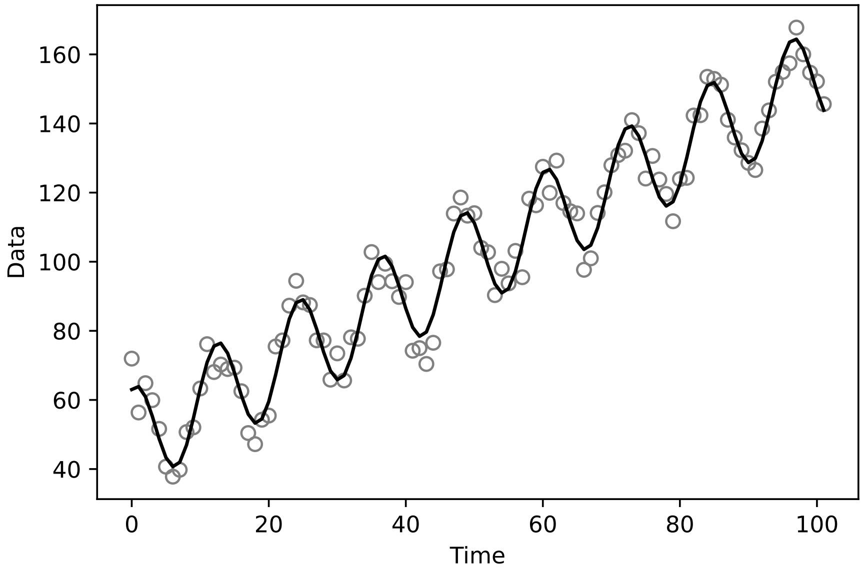
```
seascom = mistat.load_data('SEASCOM.csv')
fig, ax = plt.subplots()
ax.scatter(seascom.index, seascom, facecolors='none', edgecolors='grey')
model.predict(df).plot(ax=ax, color='black')
ax.set_xlabel('Time')
ax.set_ylabel('Data')
plt.show()
```

{fig:seascom-model-deviation} **(iii)** Calculate the residuals and plot them (see Figure 6.2).

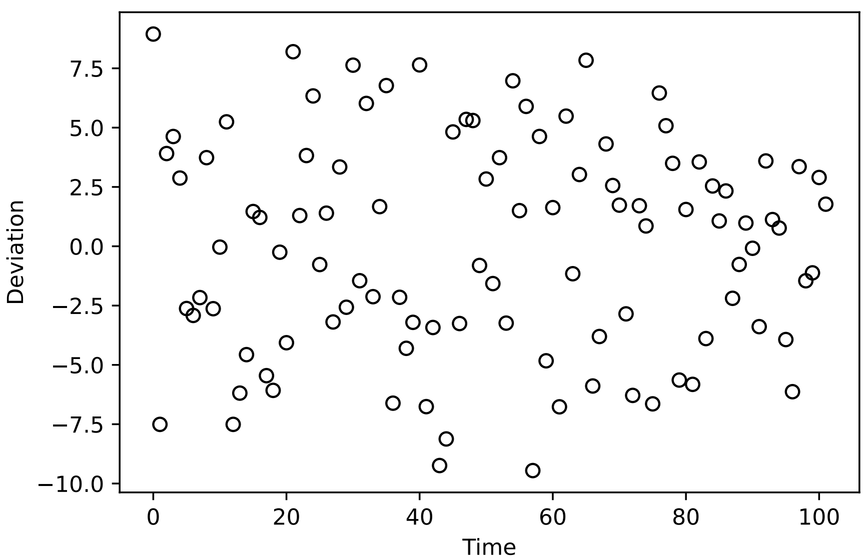
```
U = df['SeasCom'] - model.predict(df)
fig, ax = plt.subplots()
ax.scatter(U.index, U, facecolors='none', edgecolors='black')
ax.set_xlabel('Time')
ax.set_ylabel('Deviation')
plt.show()
```

(iv) Calculate the correlations

```
# use slices to get sublists
corr_1 = np.corrcoef(U[:-1], U[1:])[0][1]
corr_2 = np.corrcoef(U[:-2], U[2:])[0][1]
print(f'Corr(Ut,Ut-1) = {corr_1:.3f}')
print(f'Corr(Ut,Ut-2) = {corr_2:.3f}')
```



{fig:seascom-timeline} **Fig. 6.1** Seasonal trend model of SeasCom data set



{fig:seascom-model-deviation} **Fig. 6.2** Deviation of SeasCom data from cyclical trend.

```
Corr(Ut,Ut-1) = -0.191
Corr(Ut,Ut-2) = 0.132
```

Indeed the correlations between adjacent data points are $corr(X_t, X_{t-1}) = -0.191$, and $corr(X_t, X_{t-2}) = 0.132$.

(iv) A plot of the deviations and the low autocorrelations shows something like randomness.

```
# keep some information for later exercises
seascom_model = model
seascom_df = df
```

Exercise 6.3 Write the formula for the lag-correlation $\rho(h)$ for the case of stationary $MA(q)$.

{eqn:ma-covariance} **Solution 6.3** According to Equation 6.2.2, the auto-correlation in a stationary MA(q) is

$$\rho(h) = \frac{K(h)}{K(0)} = \frac{\sum_{j=0}^{q-h} \beta_j \beta_{j+h}}{\sum_{j=0}^{q} \beta_j^2}$$

for $0 \le h \le q$).

Notice that $\rho(h) = \rho(-h)$, and $\rho(h) = 0$ for all $h > q$.

Exercise 6.4 For a stationary $MA(5)$, with coefficients $\beta' = (1, 1.05, .76, -.35, .45, .55)$ make a table of the covariances $K(h)$ and lag correlations $\rho(h)$.

Solution 6.4 In Python

```
beta = np.array([1, 1.05, 0.76, -0.35, 0.45, 0.55])
data = []
n = len(beta)
sum_0 = np.sum(beta * beta)
for h in range(6):
  sum_h = np.sum(beta[:n-h] * beta[h:])
  data.append({
    'h': h,
    'K(h)': sum_h,
    'rho(h)': sum_h / sum_0,
  })
```

	0	1	2	3	4	5
K(h)	3.308	1.672	0.542	0.541	1.028	0.550
rho(h)	1.000	0.506	0.164	0.163	0.311	0.166

Exercise 6.5 Consider the infinite moving average $X_t = \sum_{j=0}^{\infty} q^j e_{t-j}$ where $e_t \sim WN(0, \sigma^2)$, where $0 < q < 1$. Compute

(i) $E\{X_t\}$,
(ii) $V\{X_t\}$.

Solution 6.5 We consider the $AQ(\infty)$, given by coefficients $\beta_j = q^j$, with $0 < q < 1$. In this case,

(i) $E\{X_t\} = 0$, and

(ii) $V\{X_t\} = \sigma^2 \sum_{j=0}^{\infty} q^{2j} = \sigma^2/(1-q^2)$.

Exercise 6.6 Consider the AR(1) given by $X_t = 0.75X_{t-1} + e_t$, where $e_t \sim WN(0, \sigma^2)$, and $\sigma^2 = 5$. Answer the following,

(i) Is this sequence covariance stationary?
(ii) Find $E\{X_t\}$,
(iii) Determine $K(0)$ and $K(1)$.

Solution 6.6 We consider the AR(1), $X_t = 0.75X_{t-1} + \varepsilon_t$.

(i) This time-series is equivalent to $X_t = \sum_{j=0}^{\infty}(-0.75)^j \varepsilon_{t-j}$, hence it is covariance stationary.

(ii) $E\{X_t\} = 0$

(iii) According to the Yule-Walker equations,

$$\begin{bmatrix} 1 & -0.75 \\ -0.75 & 1 \end{bmatrix} \begin{bmatrix} K(0) \\ K(1) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

It follows that $K(0) = 2.285714\,\sigma^2$ and $K(1) = 1.714286\,\sigma^2$.

{exc:ar2-series}

Exercise 6.7 Consider the auto-regressive series AR(2) namely,

$$X_t = 0.5X_{t-1} - 0.3X_{t-2} + e_t,$$

where $e_t \sim WN(0, \sigma^2)$.

(i) Is this series covariance stationary?
(ii) Express this AR(2) in the form $(1 - \phi_1 z^{-1})(1 - \phi_2 z^{-1})X_t = e_t$, and find the values of ϕ_1 and ϕ_2.
(iii) Write this AR(2) as an $MA(\infty)$. (Hint: write $(1 - \phi z^{-1})^{-1} = \sum_{j=0}^{\infty} \phi^j z^{-j}$)

Solution 6.7 The given AR(2) can be written as $\mathbf{X}_t - 0.5\mathbf{X}_{t-1} + 0.3\mathbf{X}_{t-2} = \boldsymbol{\varepsilon}_t$.

(i) The corresponding characteristic polynomial is $\mathbf{P}_2(\mathbf{z}) = 0.3 - 0.5\mathbf{z} + \mathbf{z}^2$. The two characteristic roots are $\mathbf{z}_{1,2} = \frac{1}{4} \pm \mathrm{i}\frac{\sqrt{95}}{20}$. These two roots belong to the unit circle. Hence this AR(2) is covariance stationary.

(ii) We can write $A_2(z)X_t = \varepsilon_t$, where $A_2(z) = 1 - 0.5z^{-1} + 0.3z^{-2}$. Furthermore, $\phi_{1,2}$ are the two roots of $A_2(z) = 0$.

It follows that

$$\begin{aligned} X_t &= (A_2(z))^{-1}\varepsilon_t \\ &= \varepsilon_t + 0.5\varepsilon_{t-1} - 0.08\varepsilon_{t-2} - 0.205\varepsilon_{t-3} - 0.0761\varepsilon_{t-4} + 0.0296\varepsilon_{t-5} + \ldots \end{aligned}$$

Exercise 6.8 Consider the AR(3) given by $X_t - 0.5X_{t-1} + 0.3X_{t-2} - 0.2X_{t-3} = e_t$, where $e_t \sim WN(0, 1)$. Use the Yule-Walker equations to determine $K(h)$, $|h| = 0, 1, 2, 3$.

Solution 6.8 The Yule-Walker equations are:

$$\begin{bmatrix} 1 & -0.5 & 0.3 & -0.2 \\ -0.5 & 1.3 & -0.2 & 0 \\ 0.3 & -0.7 & 1 & 0 \\ -0.2 & 0.3 & -0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} K(0) \\ K(1) \\ K(2) \\ K(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $K(0) = 1.2719$, $K(1) = 0.4825$, $K(2) = -0.0439$, $K(3) = 0.0877$.

{exc:ar2-series} **Exercise 6.9** Write the Toeplitz matrix R_4 corresponding to the series in exercise 6.7.

Solution 6.9 The Toeplitz matrix is

$$R_4 = \begin{bmatrix} 1.0000 & 0.3794 & -0.0235 & 0.0690 \\ 0.3794 & 1.0000 & 0.3794 & -0.0235 \\ -0.0235 & 0.3794 & 1.0000 & 0.3794 \\ 0.0690 & -0.0235 & 0.3794 & 1.0000 \end{bmatrix}$$

Exercise 6.10 Consider the series $X_t - X_{t-1} + 0.25X_{t-2} = e_t + .4e_{t-1} - .45e_{t-2}$,

(i) Is this an ARMA(2,2) series?
(ii) Write the process as an MA(∞) series.

Solution 6.10 This series is an ARMA(2,2), given by the equation

$$(1 - z^{-1} + 0.25z^{-2})X_t = (1 + 0.4z^{-1} - 0.45z^{-2})\varepsilon_t.$$

Accordingly,

$$\begin{aligned} X_t &= (1 + 0.4z^{-1} - 0.45z^{-2})(1 - z^{-1} + 0.25z^{-2})^{-1}\varepsilon_t \\ &= \varepsilon_t + 1.4\varepsilon_{t-1} + 0.7\varepsilon_{t-2} + 0.35\varepsilon_{t-3} + 0.175\varepsilon_{t-4} \\ &\quad + 0.0875\varepsilon_{t-5} + 0.0438\varepsilon_{t-6} + 0.0219\varepsilon_{t-7} + 0.0109\varepsilon_{t-8} + \dots \end{aligned}$$

Exercise 6.11 Consider the 2-nd order difference, $\Delta^2 X_t$ of the DOW1941 series.

(i) Plot the acf and the pacf of these differences.
(ii) Can we infer that the DOW1941 series is an integrated ARIMA(1,2,2)?

Solution 6.11 Let X denote the DOW1941 data set. We create a new set, Y of second order difference, i.e. $Y_t = X_t - 2X_{t-1} + X_{t-2}$.

```
dow1941 = mistat.load_data('DOW1941.csv')

X = dow1941.values # extract values to remove index for calculations
Y = X[2:] - 2 * X[1:-1] + X[:-2]
```

(i) In the following table we present the autocorrelations, acf, and the partial autocorrelations, pacf, of Y. For a visualisation see Figure 6.3. {fig:acf-pacf-dow-second-order}

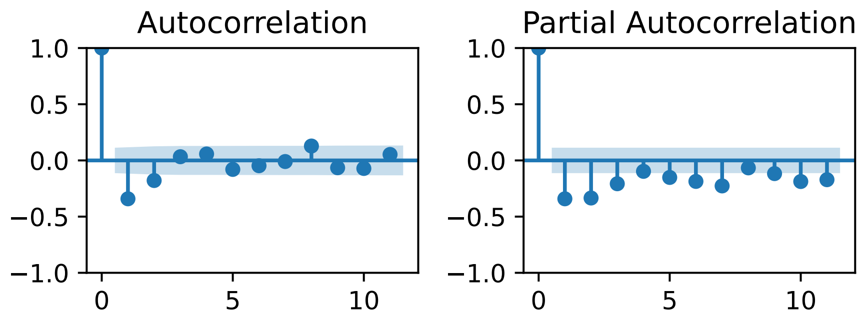


Fig. 6.3 Autocorrelations, acf, and the partial autocorrelations, pacf, of the second order differences of the DOW1941 dataset. {fig:acf-pacf-dow-second-order}

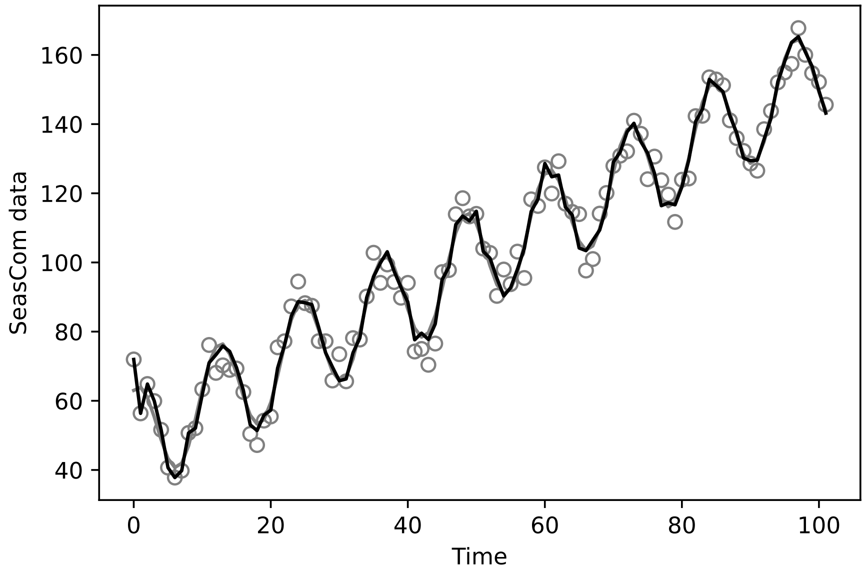
```
# use argument alpha to return confidence intervals
y_acf, ci_acf = acf(Y, nlags=11, fft=True, alpha=0.05)
y_pacf, ci_pacf = pacf(Y, nlags=11, alpha=0.05)

# determine if values are significantly different from zero
def is_significant(y, ci):
    return not (ci[0] < 0 < ci[1])

s_acf = [is_significant(y, ci) for y, ci in zip(y_acf, ci_acf)]
s_pacf = [is_significant(y, ci) for y, ci in zip(y_pacf, ci_pacf)]
```

h	acf	S/NS	pacf	S/NS
1	-0.342	S	-0.343	S
2	-0.179	S	-0.337	S
3	0.033	NS	-0.210	S
4	0.057	NS	-0.100	NS
5	-0.080	NS	-0.155	S
6	-0.047	NS	-0.193	S
7	-0.010	NS	-0.237	S
8	0.128	NS	-0.074	NS
9	-0.065	NS	-0.127	S
10	-0.071	NS	-0.204	S
11	0.053	NS	-0.193	S

S denotes significantly different from 0. NS denotes not significantly different from 0.

(ii) All other correlations are not significant. It seems that the ARIMA(1,2,2) is a good approximation.

Exercise 6.12 Consider again the data set SeasCom and the trend function $f(t)$, which was determined in Ex. 6.2. Apply the function `optimalLinearPredictor` to the deviations of the data from its trend function. Add the results to the trend function to obtain a one day ahead prediction of the demand. {exc:seascom-1}



{fig:seascom-one-day-ahead-model} **Fig. 6.4** One-day ahead prediction model of SeasCom data set

Solution 6.12 In Fig. 6.4 we present the seasonal data SeasCom, and the one-day ahead predictions, We see an excellent prediction. {fig:seascom-one-day-ahead-model}

```
predictedError = mistat.optimalLinearPredictor(seascom_model.resid,
                      10, nlags=9)
predictedTrend = seascom_model.predict(seascom_df)
correctedTrend = predictedTrend + predictedError

fig, ax = plt.subplots()
ax.scatter(seascom_df.index, seascom_df['SeasCom'],
           facecolors='none', edgecolors='grey')
predictedTrend.plot(ax=ax, color='grey')
correctedTrend.plot(ax=ax, color='black')
ax.set_xlabel('Time')
ax.set_ylabel('SeasCom data')
plt.show()
```

Chapter 7
Modern analytic methods: Part I

Import required modules and define required functions

```
import warnings
import mistat
import random
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from sklearn.metrics import confusion_matrix, ConfusionMatrixDisplay
from sklearn.metrics import mean_squared_error
from sklearn.tree import DecisionTreeClassifier, DecisionTreeRegressor
from sklearn.ensemble import RandomForestClassifier
from sklearn.preprocessing import KBinsDiscretizer
from sklearn.preprocessing import OneHotEncoder
from sklearn.preprocessing import LabelEncoder
from sklearn.model_selection import train_test_split
from sklearn.naive_bayes import MultinomialNB
from sklearn.metrics import accuracy_score
from xgboost import XGBClassifier
from sklearn.cluster import KMeans
from sklearn.pipeline import make_pipeline

# Uncomment the following if xgboost crashes
import os
os.environ['KMP_DUPLICATE_LIB_OK'] = 'TRUE'
```

Exercise 7.1 Make up a list of supervised and unsupervised applications mentioned in COVID19 related applications.

Solution 7.1 Articles reviewing the application of supervised and unsupervised methods can be found online (see e.g. doi:10.1016/j.chaos.2020.110059)

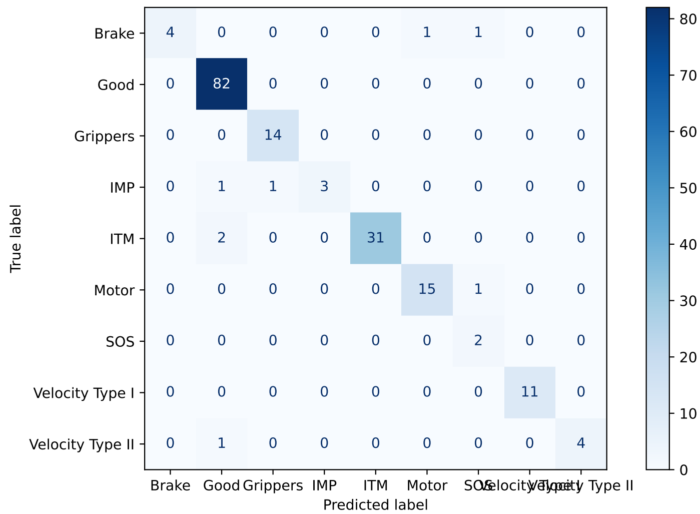
An example for supervised applications is the classification of a COVID-19 based on diagnostic data (see e.g. doi:10.1155/2021/4733167 or doi:10.3390/s21103322)

An example of unsupervised learning is hierarchical clustering to evaluate COVID-19 pandemic preparedness and performance in 180 countries (see doi:10.1016/j.rinp.2021.104639)

{exc:sensors-test-result}

Exercise 7.2 Create a pruned decision tree model for the `testResult` column in the **SENSORS.csv** data set using `scikit-learn`. Compare the results to the `status` model from example 7.2.

{ex:decision-tree-sensors}



{fig:ex-confusion-matrix-testResult} **Fig. 7.1** Decision tree model to predict testResult from sensor data (Exc. 7.2) exc:sensors-test-result

Solution 7.2 The decision tree model for `testResult` results in the confusion matrix shown in Figure 7.1. {fig:ex-confusion-matrix-testResult}

```
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'testResult'
X = sensors[predictors]
y = sensors[outcome]

# Train the model
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
clf.fit(X, y)

fig, ax = plt.subplots(figsize=(10, 6))
ConfusionMatrixDisplay.from_estimator(clf, X, y, ax=ax, cmap=plt.cm.Blues)

plt.show()
```

The models classification performance is very good. The test result ‘Good’, which corresponds to status ‘Pass’ is correctly predicted. Most of the other individual test results have also low missclassification rates. The likely reason for this is that each test result is associated with a specific subset of the sensors.

Exercise 7.3 Fit a gradient boosting model to the sensors data to predict status as the outcome. Use the property `feature_importances_` to identify important predictors and compare to the results from the decision tree model in section 7.5. {sec:decision-trees}

Solution 7.3 In Python

```
# convert the status information into numerical labels
outcome = 'status'
```

```
y = sensors[outcome].replace({'Pass': 1, 'Fail': 0})

# Train the model
xgb = XGBClassifier(objective='binary:logistic', subsample=.63,
                    eval_metric='logloss', use_label_encoder=False,
                    seed=1)
xgb.fit(X, y)

# actual in rows / predicted in columns
print('Confusion matrix')
print(confusion_matrix(y, xgb.predict(X)))
```

```
Confusion matrix
[[92  0]
 [ 0 82]]
```

The models confusion matrix is perfect.

```
var_imp = pd.DataFrame({
  'importance': xgb.feature_importances_,
  }, index=predictors)
var_imp = var_imp.sort_values(by='importance', ascending=False)
var_imp['order'] = range(1, len(var_imp) + 1)
print(var_imp.head(10))
var_imp.loc[var_imp.index.isin(['sensor18', 'sensor07', 'sensor21'])]
```

```
          importance  order
sensor18    0.290473      1
sensor54    0.288680      2
sensor53    0.105831      3
sensor55    0.062423      4
sensor61    0.058112      5
sensor48    0.040433      6
sensor07    0.026944      7
sensor12    0.015288      8
sensor03    0.013340      9
sensor52    0.013160     10
```

```
          importance  order
sensor18    0.290473      1
sensor07    0.026944      7
sensor21    0.000000     50
```

The decision tree model uses sensors 18, 7, and 21. The xgboost model identifies sensor 18 as the most important variable. Sensor 7 is ranked 7th, sensor 21 has no importance.

Exercise 7.4 Fit a random forest model to the sensors data to predict status as the outcome. Use the property `feature_importances_` to identify important predictors and compare to the results from the decision tree model in section 7.5. {sec:decision-trees}

Solution 7.4 Create the random forest classifier model.

```
y = sensors['status']

# Train the model
model = RandomForestClassifier(ccp_alpha=0.012, random_state=0)
model.fit(X, y)

# actual in rows / predicted in columns
print('Confusion matrix')
print(confusion_matrix(y, model.predict(X)))
```

```
Confusion matrix
[[92  0]
 [ 0 82]]
```

The models confusion matrix is perfect.

```
var_imp = pd.DataFrame({
  'importance': model.feature_importances_,
  }, index=predictors)
var_imp = var_imp.sort_values(by='importance', ascending=False)
var_imp['order'] = range(1, len(var_imp) + 1)
print(var_imp.head(10))
var_imp.loc[var_imp.index.isin(['sensor18', 'sensor07', 'sensor21'])]
```

```
          importance  order
sensor61    0.138663      1
sensor18    0.100477      2
sensor53    0.079890      3
sensor52    0.076854      4
sensor46    0.052957      5
sensor50    0.051970      6
sensor44    0.042771      7
sensor48    0.037087      8
sensor24    0.036825      9
sensor21    0.035014     10
```

```
          importance  order
sensor18    0.100477      2
sensor21    0.035014     10
sensor07    0.023162     17
```

The decision tree model uses sensors 18, 7, and 21. Sensor 18 has the second largest importance value, sensor 21 ranks 10th, and sensor 7 is on rank 17.

Exercise 7.5 Build decision tree, gradient boosting, and random forest models for the sensors data using status as a target variable.

Use `LabelEncoder` from the `scikit-learn` package to convert the outcome variable into numerical values prior to model building. Split the dataset into a 60% training and 40% validation set using `sklearn.model_selection.train_test_split`

Solution 7.5 In Python:

```
# convert outcome values from strings into numerical labels
# use le.inverse_transform to convert the predictions to strings
le = LabelEncoder()
y = le.fit_transform(sensors['status'])

train_X, valid_X, train_y, valid_y = train_test_split(X, y,
  test_size=0.4, random_state=2)

dt_model = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
dt_model.fit(train_X, train_y)

xgb_model = XGBClassifier(objective='binary:logistic', subsample=.63,
                  eval_metric='logloss', use_label_encoder=False,
                  seed=1)
xgb_model.fit(train_X, train_y)

rf_model = RandomForestClassifier(ccp_alpha=0.014, random_state=0)
rf_model.fit(train_X, train_y)
```

```
print('Decision tree model')
print(f'Accuracy {accuracy_score(valid_y, dt_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, dt_model.predict(valid_X)))

print('Gradient boosting model')
print(f'Accuracy {accuracy_score(valid_y, xgb_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, xgb_model.predict(valid_X)))

print('Random forest model')
print(f'Accuracy {accuracy_score(valid_y, rf_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, rf_model.predict(valid_X)))
```

```
Decision tree model
Accuracy 0.900
[[37  2]
 [ 5 26]]
Gradient boosting model
Accuracy 0.957
[[36  3]
 [ 0 31]]
Random forest model
Accuracy 0.986
[[38  1]
 [ 0 31]]
```

The accuracy for predicting the validation set is very good for all three models, with random forest giving the best model with an accuracy of 0.986. The xgboost model has a slightly lower accuracy of 0.957 and the accuracy for the decision tree model is 0.900.

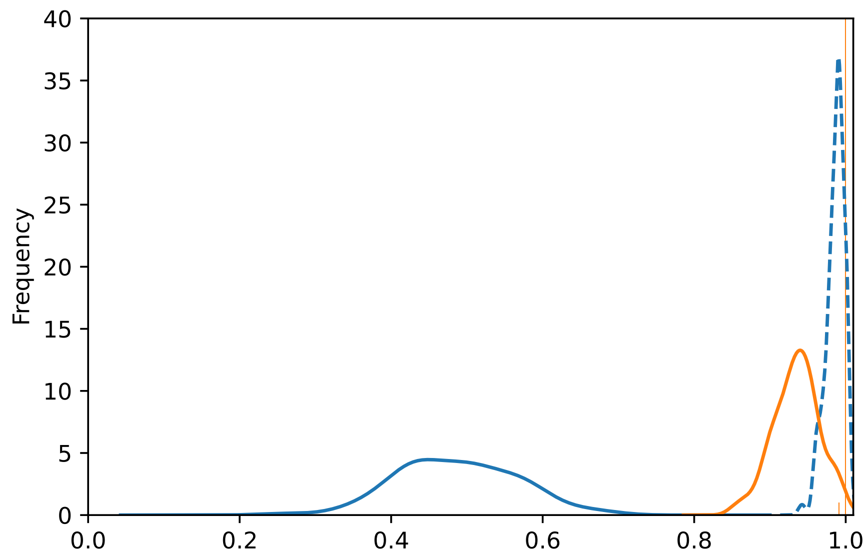
If you change the random seeds or remove them for the various commands, you will see that the accuracies vary and that the order can change.

Exercise 7.6 One way of assessing overfitting in models is assess model performance by repeated randomization of the outcome variable. Build a decision tree model for the sensors data using status as a target variable. Repeat the model training 100 times with randomized outcome.

Solution 7.6 In Python:

```
dt_model = DecisionTreeClassifier(ccp_alpha=0.012)

random_valid_acc = []
random_train_acc = []
org_valid_acc = []
org_train_acc = []
for _ in range(100):
  train_X, valid_X, train_y, valid_y = train_test_split(X, y,
    test_size=0.4)
  dt_model.fit(train_X, train_y)
  org_train_acc.append(accuracy_score(train_y, dt_model.predict(train_X)))
  org_valid_acc.append(accuracy_score(valid_y, dt_model.predict(valid_X)))

  random_y = random.sample(list(train_y), len(train_y))
  dt_model.fit(train_X, random_y)
  random_train_acc.append(accuracy_score(random_y, dt_model.predict(train_X)))
  random_valid_acc.append(accuracy_score(valid_y, dt_model.predict(valid_X)))

ax = pd.Series(random_valid_acc).plot.density(color='C0')
pd.Series(random_train_acc).plot.density(color='C0', linestyle='--',
```

```
                                          ax=ax)
pd.Series(org_valid_acc).plot.density(color='C1', ax=ax)
pd.Series(org_train_acc).plot.hist(color='C1', linestyle='--',
                                   ax=ax)
ax.set_ylim(0, 40)
ax.set_xlim(0, 1.01)
plt.show()
```



{exc:dec-tree-reg-distillation}

Exercise 7.7 The dataset `DISTILLATION-TOWER.csv` contains a number of sensor data from a distillation tower measured at regular intervals. Use the temperature data measured at different locations in the tower (`TEMP#`) to create a decision tree regressor to predict the resulting vapor pressure (`VapourPressure`).

(i) Split the dataset into training and validation set using a 80-20 ratio
(ii) For each `ccp_alpha` value of the decision tree regressor model use the test set to estimate the MSE (`mean_squared_error`) of the resulting model. Select a value of `ccp_alpha` to build the final model. The `ccp_alpha` values are returned using the `cost_complexity_pruning_path` method.
(iii) Visualize the final model using any of the available methods

Solution 7.7 Load data

```
distTower = mistat.load_data('DISTILLATION-TOWER.csv')
predictors = ['Temp1', 'Temp2', 'Temp3', 'Temp4', 'Temp5', 'Temp6',
              'Temp7', 'Temp8', 'Temp9', 'Temp10', 'Temp11', 'Temp12']
outcome = 'VapourPressure'
Xr = distTower[predictors]
yr = distTower[outcome]
```

(i) Split dataset into train and validation set

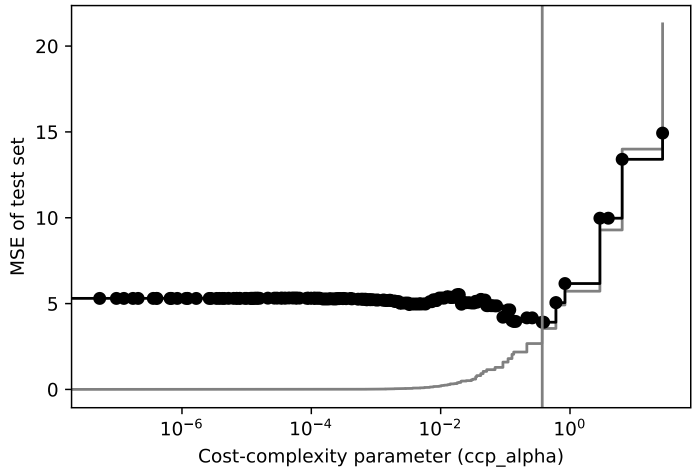


Fig. 7.2 Decision tree regressor complexity as a function of `ccp_alpha`. The validation set error is shown in black, the training set error in grey (Exercise exc:dec-tree-reg-distillation 7.7) {fig:exc-cpp-pruning}

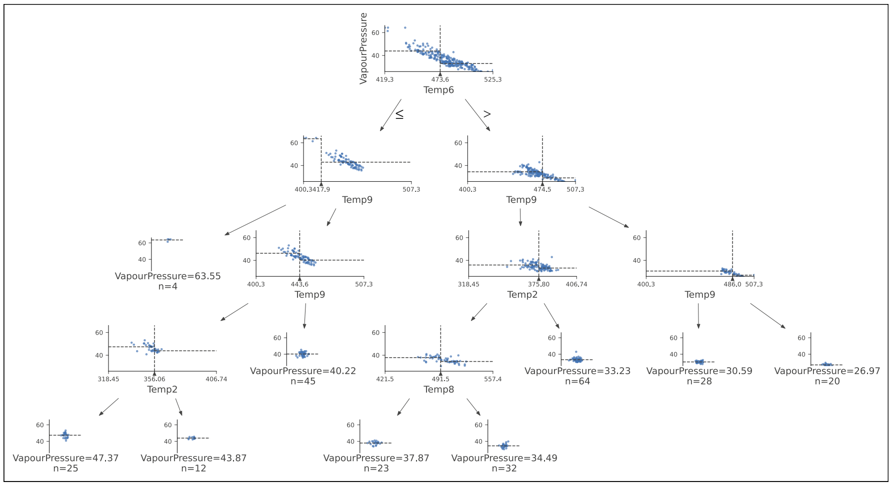
```
train_X, valid_X, train_y, valid_y = train_test_split(Xr, yr,
  test_size=0.2, random_state=2)
```

(ii) Determine model performance for different tree complexity along the dependence of tree depth on ccp parameter; see Figure 7.2. {fig:exc-cpp-pruning}

```
# Code to analyze tree depth vs alpha
model = DecisionTreeRegressor(random_state=0)
path = model.cost_complexity_pruning_path(Xr, yr)
ccp_alphas, impurities = path.ccp_alphas, path.impurities
mse = []
mse_train = []
for ccp_alpha in ccp_alphas:
    model = DecisionTreeRegressor(random_state=0, ccp_alpha=ccp_alpha)
    model.fit(train_X, train_y)
    mse.append(mean_squared_error(valid_y, model.predict(valid_X)))
    mse_train.append(mean_squared_error(train_y, model.predict(train_X)))
ccp_alpha = ccp_alphas[np.argmin(mse)]
```

The smallest validation set error is obtained for `ccp_alpha` = 0.372. The dependence of training and validation error on `ccp_alpha` is shown in Figure 7.2. {fig:exc-cpp-pruning}
(iii) The final model is visualized using `dtreeviz` in Figure 7.3. {fig:exc-dtreeviz-visualization}

```
# the dtreeviz methods requires the classifier to be trained with a numerical
# representation of the classes
# Train the model
model = DecisionTreeRegressor(ccp_alpha=ccp_alpha, random_state=0)
model.fit(Xr, yr)

viz = dtreeviz(model, Xr, yr,
               target_name=outcome,
```



{fig:exc-dtreeviz-visualization}

Fig. 7.3 Decision tree visualization of regression tree (Exercise 7.7) exc:dec-tree-reg-distillation

```
                  feature_names=Xr.columns)
viz2pdf(viz, 'compiled/figures/Chap008_ex_dtreeviz_regressor.pdf')
```

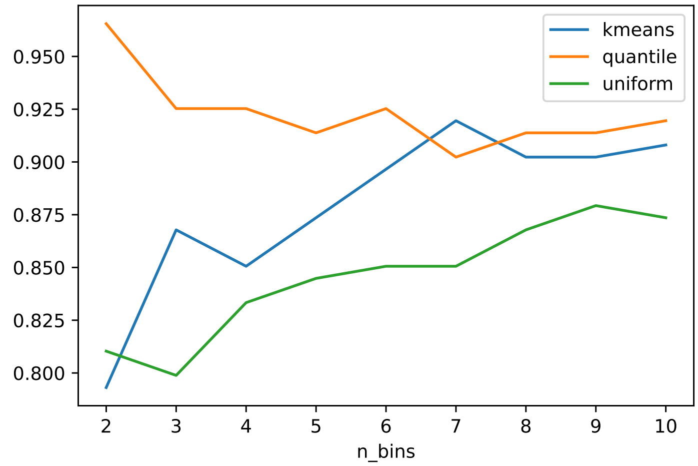
Exercise 7.8 Create a Naïve Bayes classifier for the sensors data using status as a target. Compare the confusion matrix to the decision tree model (see Figure 7.9). {fig:dt-conf-matrix}

Hint: Use the `scikit-learn` method `KBinsDiscretizer` to bin the sensors data and encode them as `ordinal` data. Try different number of bins and binning strategies

Solution 7.8 Vary the number of bins and binning strategy. The influence of the two model parameter on model performance is shown in Figure 7.4. {fig:nb-binning-performance}

```
y = sensors['status']

results = []
for strategy in 'uniform', 'quantile', 'kmeans':
  for n_bins in range(2, 11):
    kbinsDiscretizer = KBinsDiscretizer(encode='ordinal',
      strategy=strategy, n_bins=n_bins)
    X_binned = kbinsDiscretizer.fit_transform(X)
    nb_model = MultinomialNB()
    nb_model.fit(X_binned, y)
    results.append({'strategy': strategy, 'n_bins': n_bins,
       'accuracy': accuracy_score(y, nb_model.predict(X_binned))})
results = pd.DataFrame(results)
fig, ax = plt.subplots()
for key, group in results.groupby('strategy'):
  group.plot(x='n_bins', y='accuracy', label=key, ax=ax)
```

The `quantile` binning strategy (for each feature, each bin has the same number of data points) with splitting each column into two bins leads to the best performing



{fig:nb-binning-performance} **Fig. 7.4** Influence of number of bins and binning strategy on model performance for the sensors data with status as outcome

model. With this strategy, performance declines with increasing number of bins. The `uniform` (for each feature, each bin has the same width) and the `kmeans` (for each feature, a k-means clustering is used to bin the feature) strategies on the other hand, show increasing performance with increasing number of bins.

The confusion matrix for the best performing models is:

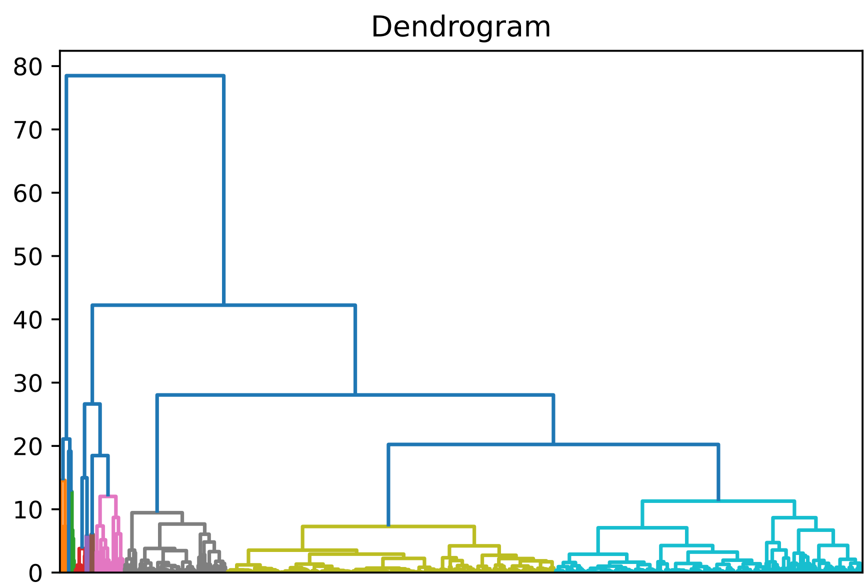
```
kbinsDiscretizer = KBinsDiscretizer(encode='ordinal',
  strategy='quantile', n_bins=2)
X_binned = kbinsDiscretizer.fit_transform(X)
nb_model = MultinomialNB()
nb_model.fit(X_binned, y)
print('Confusion matrix')
print(confusion_matrix(y, nb_model.predict(X_binned)))
```

```
Confusion matrix
[[87  5]
 [ 1 81]]
```

The decision tree model missclassified three of the ‘Pass’ data points as ‘Fail’. The Naïve Bayes model on the other hand missclassifies six data points. However, five of these are ‘Pass’ and predicted as ‘Fail’. Depending on your use case, you may prefer a model with more false negatives or false positives.

{exc:ward-cluster-nutritional}

Exercise 7.9 Nutritional data from 961 different food items is given in the file **FOOD.csv**. For each food item, there are 7 variables: fat (grams), food energy (calories), carbohydrates (grams), protein (grams), cholesterol (milligrams), weight (grams), and saturatedfat (grams). Use Ward’s distance to construct 10 clusters of food items with similarity in the 7 recorded variables using cluster analysis of variables.



{fig:food-ward-10-clusters} **Fig. 7.5** Hierarchical clustering of food data set using Ward clustering

Solution 7.9 In Python:

```
from sklearn.cluster import AgglomerativeClustering
from sklearn.preprocessing import StandardScaler
from mistat import plot_dendrogram

food = mistat.load_data('FOOD.csv')

scaler = StandardScaler()
model = AgglomerativeClustering(n_clusters=10, compute_distances=True)

X = scaler.fit_transform(food)
model = model.fit(X)
fig, ax = plt.subplots()
plot_dendrogram(model, ax=ax)
ax.set_title('Dendrogram')
ax.get_xaxis().set_ticks([])
plt.show()
```

{exc:ward-cluster-nutritional} **Exercise 7.10** Repeat exercise 7.9 with different linkage methods and compare the results.

Solution 7.10 In Python:

```
food = mistat.load_data('FOOD.csv')
scaler = StandardScaler()
X = scaler.fit_transform(food)
```

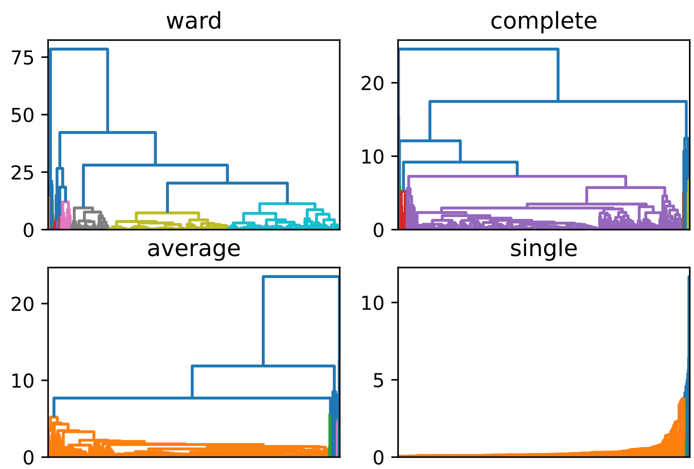


Fig. 7.6 Comparison of different linkage methods for hierarchical clustering of food data set {fig:food-compare-linkage}

```
fig, axes = plt.subplots(ncols=2, nrows=2)

for linkage, ax in zip(['ward', 'complete', 'average', 'single'], axes.flatten()):
  model = AgglomerativeClustering(n_clusters=10, compute_distances=True,
    linkage=linkage)
  model = model.fit(X)
  plot_dendrogram(model, ax=ax)
  ax.set_title(linkage)
  ax.get_xaxis().set_ticks([])
plt.show()
```

The comparison of the different linkage methods is shown in Figure 7.6. We can see that Ward's clustering gives the most balanced clusters; three bigger clusters and seven small clusters. Complete, average, and single linkage lead to one big cluster. {fig:food-compare-linkage}

Exercise 7.11 Apply the K-means cluster feature to the sensor variables in SENSORS.cvs and interpret the clusters using the test result and status label.

Solution 7.11 We determine 10 clusters using K-means clustering.

```
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'status'
X = sensors[predictors]

scaler = StandardScaler()
X = scaler.fit_transform(X)
model = KMeans(n_clusters=10, random_state=1).fit(X)
```

Combine the information and analyse cluster membership by status.

```
df = pd.DataFrame({
  'status': sensors['status'],
```

```
  'testResult': sensors['testResult'],
  'cluster': model.predict(X),
})

for status, group in df.groupby('status'):
  print(f'Status {status}')
  print(group['cluster'].value_counts())
```

```
Status Fail
8    19
0    15
4    13
1    13
9    13
3    10
5     5
7     2
6     1
2     1
Name: cluster, dtype: int64
Status Pass
1    48
8    34
Name: cluster, dtype: int64
```

There are several clusters that contain only 'Fail' data points. They correspond to specific sensor value combinations that are very distinct to the sensor values during normal operation. The 'Pass' data points are found in two clusters. Both of these clusters contain also 'Fail' data points.

Analyse cluster membership by testResult.

```
print('Number of clusters by testResult')
for cluster, group in df.groupby('cluster'):
  print(f'Cluster {cluster}')
  print(group['testResult'].value_counts())
  print()
```

```
Number of clusters by testResult
Cluster 0
ITM    15
Name: testResult, dtype: int64

Cluster 1
Good        48
Brake        6
IMP          4
Grippers     1
Motor        1
ITM          1
Name: testResult, dtype: int64

Cluster 2
SOS    1
Name: testResult, dtype: int64

Cluster 3
Velocity Type I    10
Name: testResult, dtype: int64

Cluster 4
Grippers    10
ITM          3
Name: testResult, dtype: int64
```

```
Cluster 5
Velocity Type II    5
Name: testResult, dtype: int64

Cluster 6
SOS    1
Name: testResult, dtype: int64

Cluster 7
Grippers    2
Name: testResult, dtype: int64

Cluster 8
Good                34
Motor               15
Grippers             1
IMP                  1
ITM                  1
Velocity Type I      1
Name: testResult, dtype: int64

Cluster 9
ITM    13
Name: testResult, dtype: int64
```

We can see that some of the test results are only found in one or two clusters.

{exc:kmeans-classifier}

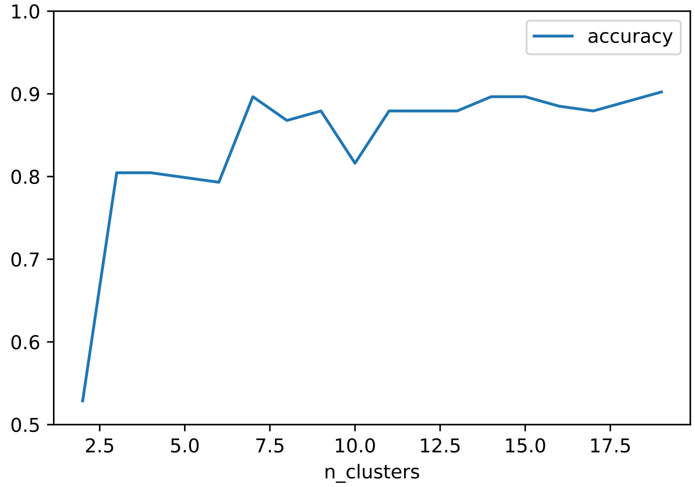
Exercise 7.12 Develop a procedure based on K-means for quality control using the SENSORS.cvs data. Derive its confusion matrix

Solution 7.12 The `scikit-learn` K-means clustering method can return either the cluster centers or the distances of a data point to all the cluster centers. We evaluate both as features for classification.

```
# Data preparation
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'status'
X = sensors[predictors]
scaler = StandardScaler()
X = scaler.fit_transform(X)
y = sensors[outcome]
```

First, classifying data points based on the cluster center. In order to use that information in a classifier, we transform the cluster center information using on-hot-encoding.

```
# Iterate over increasing number of clusters
results = []
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
for n_clusters in range(2, 20):
  # fit a model and assign the data to clusters
  model = KMeans(n_clusters=n_clusters, random_state=1)
  model.fit(X)
  Xcluster = model.predict(X)
  # to use the cluster number in a classifier, use one-hot encoding
  # it's necessary to reshape the vector of cluster numbers into a column vector
  Xcluster = OneHotEncoder().fit_transform(Xcluster.reshape(-1, 1))
```



{fig:cluster-number-model} **Fig. 7.7** Dependence of accuracy on number of clusters using cluster membership as feature (Exc. exc:kmeans-classifier 7.12)

```
  # create a decision tree model and determine the accuracy
  clf.fit(Xcluster, y)
  results.append({'n_clusters': n_clusters,
                  'accuracy': accuracy_score(y, clf.predict(Xcluster))})
ax = pd.DataFrame(results).plot(x='n_clusters', y='accuracy')
ax.set_ylim(0.5, 1)
plt.show()
pd.DataFrame(results).round(3)
```

```
    n_clusters  accuracy
0            2     0.529
1            3     0.805
2            4     0.805
3            5     0.799
4            6     0.793
5            7     0.897
6            8     0.868
7            9     0.879
8           10     0.816
9           11     0.879
10          12     0.879
11          13     0.879
12          14     0.897
13          15     0.897
14          16     0.885
15          17     0.879
16          18     0.891
17          19     0.902
```

{fig:cluster-number-model} The accuracies are visualized in Figure 7.7. We see that splitting the dataset into 7 clusters gives a classification model with an accuracy of about 0.9.

Next we use the distance to the cluster centers as variable in the classifier.

```
results = []
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
```

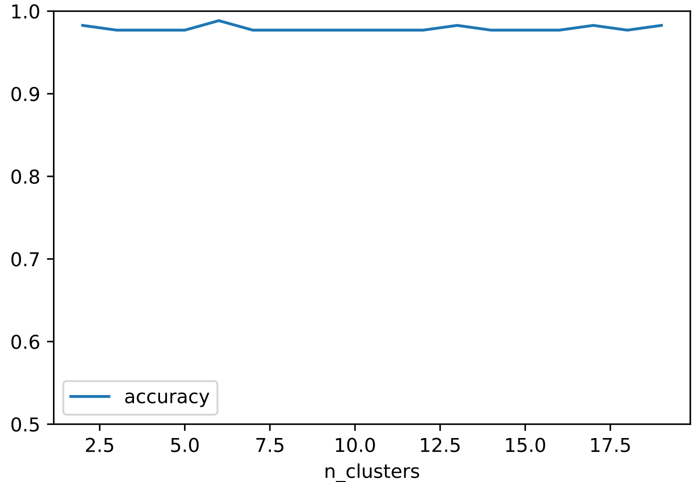


Fig. 7.8 Dependence of accuracy on number of clusters using distance to cluster center as feature (Exc. 7.12) {fig:cluster-distance-model}

```
for n_clusters in range(2, 20):
  # fit a model and convert data to distances
  model = KMeans(n_clusters=n_clusters, random_state=1)
  model.fit(X)
  Xcluster = model.transform(X)

  # create a decision tree model and determine the accuracy
  clf.fit(Xcluster, y)
  results.append({'n_clusters': n_clusters,
                  'accuracy': accuracy_score(y, clf.predict(Xcluster))})
ax = pd.DataFrame(results).plot(x='n_clusters', y='accuracy')
ax.set_ylim(0.5, 1)
plt.show()
pd.DataFrame(results).round(3)
```

	n_clusters	accuracy
0	2	0.983
1	3	0.977
2	4	0.977
3	5	0.977
4	6	0.989
5	7	0.977
6	8	0.977
7	9	0.977
8	10	0.977
9	11	0.977
10	12	0.977
11	13	0.983
12	14	0.977
13	15	0.977
14	16	0.977
15	17	0.983
16	18	0.977
17	19	0.983

The accuracies of all models are very high. The largest accuracy is achived for 6 clusters.

Based on these results, we would design the procedure using the decision tree classifier combined with K-means clustering into six clusters. Using `scikit-learn`, we can define the full procedure as a single pipeline as follows:

```
pipeline = make_pipeline(
  StandardScaler(),
  KMeans(n_clusters=6, random_state=1),
  DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
)
X = sensors[predictors]
y = sensors[outcome]

process = pipeline.fit(X, y)
print('accuracy', accuracy_score(y, process.predict(X)))
print('Confusion matrix')
print(confusion_matrix(y, process.predict(X)))
```

```
accuracy 0.9885057471264368
Confusion matrix
[[91  1]
 [ 1 81]]
```

The final model has two missclassified data points.

Chapter 8
Modern analytic methods: Part II

Import required modules and define required functions

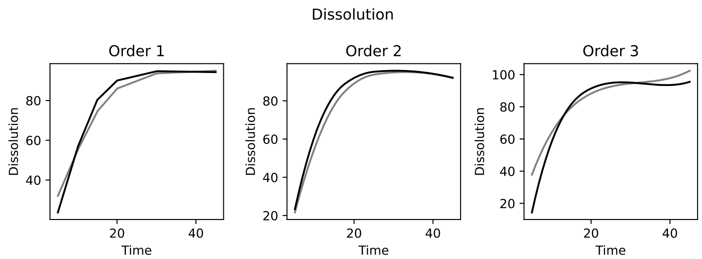
```
import mistat
import networkx as nx
from pgmpy.estimators import HillClimbSearch
import pandas as pd
import numpy as np
from matplotlib import pyplot as plt
```

Exercise 8.1 Use Functional Data Analysis to analyse the Dissolution data of reference and test tablets. Use shift registration with split interpolation of order 1, 2, and 3 to align the curves. Determine the mean dissolution curves for the reference and test tablets and compare the result for the different interpolation methods. Compare the curves and discuss the differences.

Solution 8.1 Load the data and convert to `FDataGrid`.

```
from skfda import FDataGrid
from skfda.representation.interpolation import SplineInterpolation

dissolution = mistat.load_data('DISSOLUTION.csv')

# convert the data to FDataGrid
data = []
labels = []
names = []
for label, group in dissolution.groupby('Label'):
  data.append(group['Data'].values)
  labels.append('Reference' if label.endswith('R') else 'Test')
  names.append(label)
labels = np.array(labels)
grid_points = np.array(sorted(dissolution['Time'].unique()))
fd = FDataGrid(np.array(data), grid_points,
       dataset_name='Dissolution',
       argument_names=['Time'],
       coordinate_names=['Dissolution'])
```

Use shift registration to align the dissolution data with spline interpolation of order 1, 2, and 3.



{fig:fdaDissolutionSplineComparison} **Fig. 8.1** Mean dissolution curves for reference and test tablets derived using spline interpolation of order 1, 2, and 3.

```
from skfda.preprocessing.registration import ShiftRegistration
shift_registration = ShiftRegistration()

fd_registered = {}
for order in (1, 2, 3):
    fd.interpolation = SplineInterpolation(interpolation_order=order)
    fd_registered[order] = shift_registration.fit_transform(fd)
```

For each of the three registered datasets, calculate the mean dissolution curves for reference and test tablets and plot the results.

```
from skfda.exploratory import stats

group_colors = {'Reference': 'grey', 'Test': 'black'}

fig, axes = plt.subplots(ncols=3, figsize=(8, 3))
for ax, order in zip(axes, (1, 2, 3)):
    mean_ref = stats.mean(fd_registered[order][labels=='Reference'])
    mean_test = stats.mean(fd_registered[order][labels=='Test'])
    means = mean_ref.concatenate(mean_test)
    means.plot(axes=[ax], group=['Reference', 'Test'], group_colors=group_colors)
    ax.set_title(f'Order {order}')
plt.tight_layout()
```

{fig:fdaDissolutionSplineComparison} The dissolution curves are shown in Figure 8.1. We can see in all three graphs, that the test tablets show a slightly faster dissolution than the reference tablets. If we compare the shape of the curves, the curves for the linear splines interpolation shows a levelling off with time. In the quadratic spline interpolation result, the dissolution curves go through a maximum. This behaviour is unrealistic. The cubic spline interpolation also leads to unrealistic curves that first level of and then start to increase again.

Exercise 8.2 The **Pinch** dataset contains measurements of pinch force for 20 replications from start of measurement. The pinch force is measured every 2 milliseconds over a 300 milliseconds interval

(i) Load the data. The data are available in the `fda` R-package as datasets `pinchraw` and `pinchtime`. Load the two datasets using the command `fetch_cran` com-

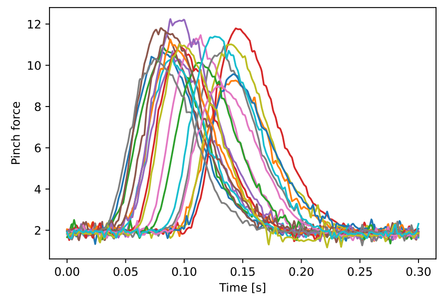


Fig. 8.2 Twenty measurements of pinch force {fig:fdaPinchforceOriginal}

mand anc combine in a `FDataGrid`.
(`skfda.datasets.fetch_cran(name, package_name)`)
(ii) Plot the dataset and discuss the graph
(iii) Use smoothing to focus on the shape of the curve. You can use `skfda.preprocessing.smoothing.kernel_smoothers.NadarayaWatsonSmoother`. Explore various values for the `smoothing_parameter` and discuss its effect. Select a suitable `smoothing_parameter` to create a smoothed version of the dataset for further processing.
(iv) Use landmark registration to align the smoothed measurements by their maximum value. As a first step identify the times at which each measurements had it maximum (use `fd.data_matrix.argmax(axis=1)` to identify the index of the measurement and use `pinchtime` to get the time to get the landmark values). Next use `skfda.preprocessing.registration.landmark_shift` to register the smoothed curves.
(v) Plot the registered curves and discuss the graph.

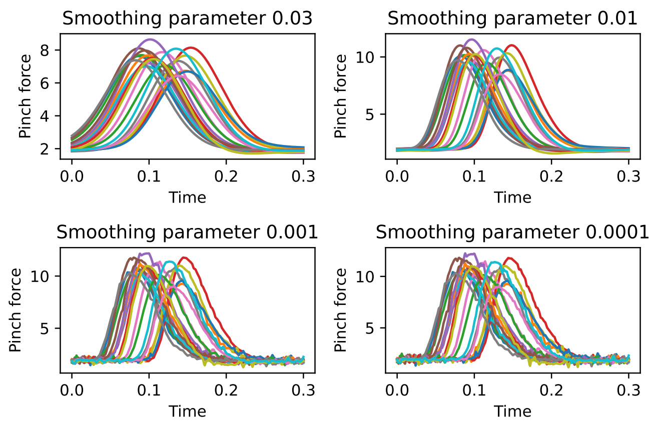
Solution 8.2 (i)

```
import skfda
from skfda import FDataGrid

pinchraw = skfda.datasets.fetch_cran('pinchraw', 'fda')['pinchraw']
pinchtime = skfda.datasets.fetch_cran('pinch', 'fda')['pinchtime']

fd = FDataGrid(pinchraw.transpose(), pinchtime)
```

Note that the measurement data need to be transposed.
(ii)

```
fig = fd.plot()
ax = fig.axes[0]
ax.set_xlabel('Time [s]')
ax.set_ylabel('Pinch force')
plt.show()
```

Figure 8.2 shows the measure pinch forces. We can see that the start of the force varies from 0.025 to 0.1 seconds. This makes it difficult to compare the shape of the {fig:fdaPinchforceOriginal}



{fig:fdaPinchforceSmoothing} **Fig. 8.3** Effect of smoothing parameter on measurement curves

curves. The shapes of the individual curves is not symmetric with a faster onset of the force and slower decline.

(iii) We create a variety of smoothed version of the dataset to explore the effect of varying the `smoothing_parameter`.

```
import itertools
from skfda.preprocessing.smoothing.kernel_smoothers import NadarayaWatsonSmoother

def plotSmoothData(fd, smoothing_parameter, ax):
    smoother = NadarayaWatsonSmoother(smoothing_parameter=smoothing_parameter)
    fd_smooth = smoother.fit_transform(fd)
    _ = fd_smooth.plot(axes=[ax])
    ax.set_title(f'Smoothing parameter {smoothing_parameter}')
    ax.set_xlabel('Time')
    ax.set_ylabel('Pinch force')

fig, axes = plt.subplots(ncols=2, nrows=2)
axes = list(itertools.chain(*axes))  # flatten list of lists
for i, sp in enumerate([0.03, 0.01, 0.001, 0.0001]):
    plotSmoothData(fd, sp, axes[i])
plt.tight_layout()
```

{fig:fdaPinchforceSmoothing} Figure 8.3 shows smoothed measurement curves for a variety of `smoothing_parameter` values. If values are too large, the data are oversmoothed and the asymmetric shape of the curves is lost. With decreasing values, the shape is reproduced better but the curves are getting noisier again. We select 0.005 as the smoothing parameter.

```
smoother = NadarayaWatsonSmoother(smoothing_parameter=0.005)
fd_smooth = smoother.fit_transform(fd)
```

(iii) We first determine the maxima of the smoothed curves:

```
max_idx = fd_smooth.data_matrix.argmax(axis=1)
landmarks = [pinchtime[idx] for idx in max_idx]
```

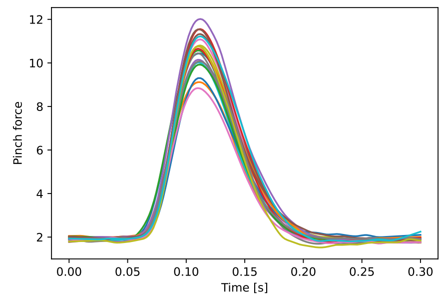


Fig. 8.4 Registered measurement curves of the **Pinch** dataset {fig:fdaPinchforceRegistered}

Use the landmarks to shift register the measurements:

```
from skfda.preprocessing.registration import landmark_shift
fd_landmark = landmark_shift(fd_smooth, landmarks)
```

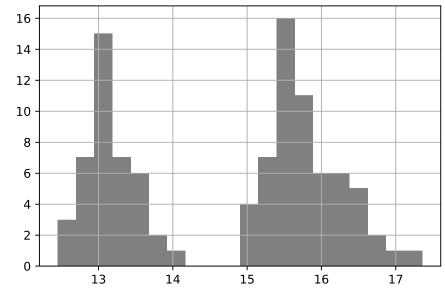
(iv) Figure 8.4 shows the measurements after smoothing and landmark shift registration. {fig:fdaPinchforceRegistered}

```
fig = fd_landmark.plot()
ax = fig.axes[0]
ax.set_xlabel('Time [s]')
ax.set_ylabel('Pinch force')
plt.show()
```

{exc:fda-moisture-classification}

Exercise 8.3 The **Moisture** dataset contains near-infrared reflectance spectra of 100 wheat samples together with the samples' moisture content. Convert the moisture values into two classes and develop a classification model to predict the moisture content of the sample.

(i) Load the data. The data are available in the `fds` R-package as datasets `Moisturespectrum` and `Moisturevalues`. Load the two datasets using the `skfda.datasets.fetch_cran(name, package_name)` command.
(ii) Determine a threshold value to split the moisture values in `high` and `low` moisture content.
(iii) Convert the spectrum information into the `FDataGrid` representation of the `scikit-fda` package and plot the spectra. What do you observe?
(iv) Normalize the sample spectra so that the differences in intensities are less influential. This is in general achived using the Standard Normal Variate (SVN) method. For each spectrum, substract the mean of the intensities and divide by their standard deviation. As before plot the spectra and discuss the observed difference.
(v) Create k-nearest neighbor classification models to predict the moisture content class from the raw and normalized spectra.
(use `skfda.ml.classification.KNeighborsClassifier`)



{fig:fdaMoistureDistribution} **Fig. 8.5** Histogram of moisture content

Solution 8.3 **(i)** Load the data

```
import skfda

moisturespectrum = skfda.datasets.fetch_cran('Moisturespectrum', 'fds')
moisturevalues = skfda.datasets.fetch_cran('Moisturevalues', 'fds')

frequencies = moisturespectrum['Moisturespectrum']['x']
spectra = moisturespectrum['Moisturespectrum']['y']
moisture = moisturevalues['Moisturevalues']
```

(ii) We can use a histogram to look at the distribution of the moisture values.

```
_ = pd.Series(moisture).hist(bins=20, color='grey', label='Moisture content')
```

{fig:fdaMoistureDistribution} Figure 8.5 shows a bimodal distribution of the moisture content with a clear separation of the two peaks. Based on this, we select 14.5 as the threshold to separate into high and low moisture content.

```
moisture_class = ['high' if m > 14.5 else 'low' for m in moisture]
```

(iii - iv) The `spectrum` information is already in the array format required for the `FDataGrid` class. In order to do this, we need to transpose the spectrum information. {fig:fdaMoistureSpectrum} As we can see in the left graph of Figure 8.6, the spectra are not well aligned but show a large spread in the intensities. This is likely due to the difficulty in having a clearly defined concentration between samples. In order to reduce this variation, we can transform the intensities by dividing the intensities by the mean of each sample.

```
intensities = spectra.transpose()
fd = skfda.FDataGrid(intensities, frequencies)

# divide each sample spectrum by it's mean intensities
intensities_normalized = (intensities - intensities.mean(dim='dim_0')) / intensities.std(dim='dim_0')
fd_normalized = skfda.FDataGrid(intensities_normalized, frequencies)
```

Code for plotting the spectra:

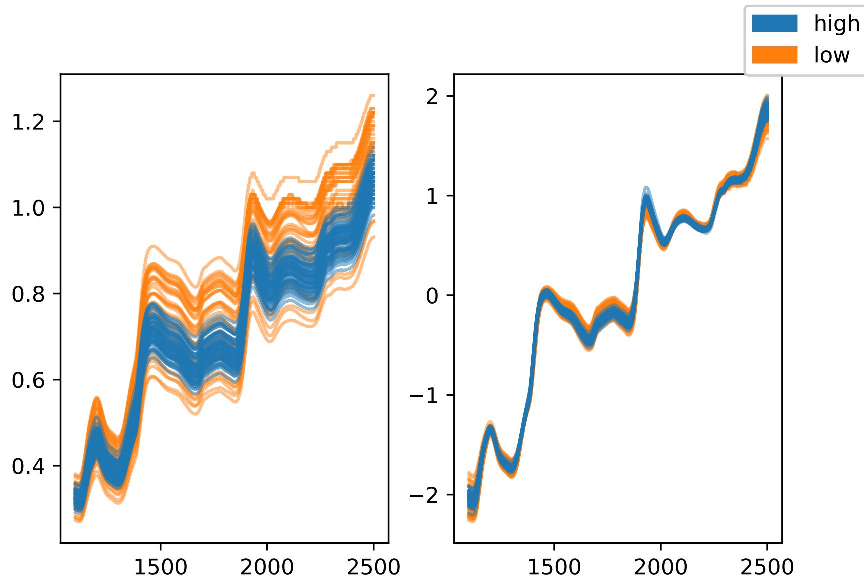


Fig. 8.6 Near-infrared spectra of the moisture dataset. Left: raw spectra, Right: normalized spectra {fig:fdaMoistureSpectrum}

```
fig, axes = plt.subplots(ncols=2)
_ = fd.plot(axes=axes[0], alpha=0.5,
            # color lines by moisture class
            group=moisture_class, group_names={'high': 'high', 'low': 'low'})
_ = fd_normalized.plot(axes=axes[1], alpha=0.5,
            group=moisture_class, group_names={'high': 'high', 'low': 'low'})
```

As we can see in right graph of Figure 8.6, the normalized spectra are now better aligned. We also see that the overall shape of the spectra is fairly consistent between samples. {fig:fdaMoistureSpectrum}

(v) We repeat the model building both for the original and normalized spectra 50 times. At each iteration, we split the data set into training and test sets (50-50), build the model with the training set and measure accuracy using the test set. By using the same random seed for splitting the original and the normalized dataset, we can better compare the models. The accuracies from the 50 iteration are compared in Figure 8.7. {fig:fdaMoistureAccuracies}

```
from skfda.ml.classification import KNeighborsClassifier
from sklearn.model_selection import train_test_split
from sklearn.metrics import accuracy_score, confusion_matrix

accuracies = []
for rs in range(50):
    X_train, X_test, y_train, y_test = train_test_split(fd,
        moisture_class, random_state=rs, test_size=0.5)
    knn_original = KNeighborsClassifier()
```

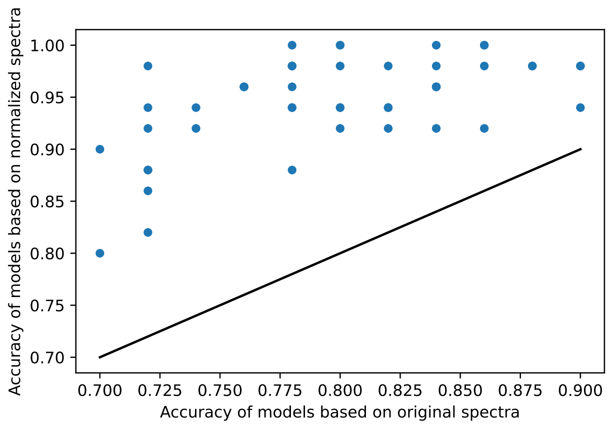


Fig. 8.7 Accuracies of classification models based original and normalized spectra. The line indicates equal performance. {fig:fdaMoistureAccuracies}

```
    knn_original.fit(X_train, y_train)
    acc_original = accuracy_score(y_test, knn_original.predict(X_test))

    X_train, X_test, y_train, y_test = train_test_split(fd_normalized,
        moisture_class, random_state=rs, test_size=0.5)
    knn_normalized = KNeighborsClassifier()
    knn_normalized.fit(X_train, y_train)
    acc_normalized = accuracy_score(y_test, knn_normalized.predict(X_test))
    accuracies.append({
        'original': acc_original,
        'normalized': acc_normalized,
    })
accuracies = pd.DataFrame(accuracies)
ax  = accuracies.plot.scatter(x='original', y='normalized')
_ = ax.plot([0.7, 0.9], [0.7, 0.9], color='black')
ax.set_xlabel('Accuracy of models based on original spectra')
ax.set_ylabel('Accuracy of models based on normalized spectra')
plt.show()

# mean of accuracies
mean_accuracy = accuracies.mean()
mean_accuracy
```

```
original      0.7976
normalized    0.9468
dtype: float64
```

{fig:fdaMoistureAccuracies} Figure 8.7 clearly shows that classification models based on the normalized spectra achieve better accuracies. The mean accuracy increases from 0.8 to 0.95.

{exc:fda-moisture-classification} **Exercise 8.4** Repeat the previous Exercise 8.3 creating K-nearest neighbor regression models to predict the moisture content of the samples.

{exc:fda-moisture-classification}
(i) Load and preprocess the **Moisture** data as described in Exercise 8.3.
(ii) Create k-nearest neighbor regression models to predict the moisture content from the raw and normalized spectra (use `skfda.ml.regression.KNeighborsRegressor`). Discuss the results.

(iii) Using one of the regression models based on the normalized spectra, plot predicted versus actual moisture content. Discuss the result. Does a regression model add additional information compared to a classification model?

Solution 8.4 **(i)** See solution for Exercise 8.3. {exc:fda-moisture-classification}

(ii) We use the method `skfda.ml.regression.KNeighborsRegressor` to build the regression models.

```
from skfda.ml.regression import KNeighborsRegressor
from sklearn.model_selection import train_test_split
from sklearn.metrics import mean_absolute_error

mae = []
for rs in range(50):
    X_train, X_test, y_train, y_test = train_test_split(fd,
        moisture, random_state=rs, test_size=0.5)
    knn_original = KNeighborsRegressor()
    knn_original.fit(X_train, y_train)
    mae_original = mean_absolute_error(y_test, knn_original.predict(X_test))

    X_train, X_test, y_train, y_test = train_test_split(fd_normalized,
        moisture, random_state=rs, test_size=0.5)
    knn_normalized = KNeighborsRegressor()
    knn_normalized.fit(X_train, y_train)
    mae_normalized = mean_absolute_error(y_test, knn_normalized.predict(X_test))
    mae.append({
        'original': mae_original,
        'normalized': mae_normalized,
    })
mae = pd.DataFrame(mae)
ax  = mae.plot.scatter(x='original', y='normalized')
ax.plot([0.3, 1.0], [0.3, 1.0], color='black')
ax.set_xlabel('MAE of models based on original spectra')
ax.set_ylabel('MAE of models based on normalized spectra')
plt.show()

# mean of MAE
mean_mae = mae.mean()
mean_mae
```

```
original      0.817016
normalized    0.433026
dtype: float64
```

Figure 8.8 clearly shows that regression models based on the normalized spectra achieve better performance. The mean absolute error descrease from 0.82 to 0.43. {fig:fdaMoistureMAE}

(iii) We use the last regression model from **(ii)** to create a plot of actual versus predicted moisture content for the test data.

```
y_pred = knn_normalized.predict(X_test)
predictions = pd.DataFrame({'actual': y_test, 'predicted': y_pred})
minmax = [min(*y_test, *y_pred), max(*y_test, *y_pred)]

ax = predictions.plot.scatter(x='actual', y='predicted')
ax.set_xlabel('Moisture content')
ax.set_ylabel('Predicted moisture content')
ax.plot(minmax, minmax, color='grey')
plt.show()
```

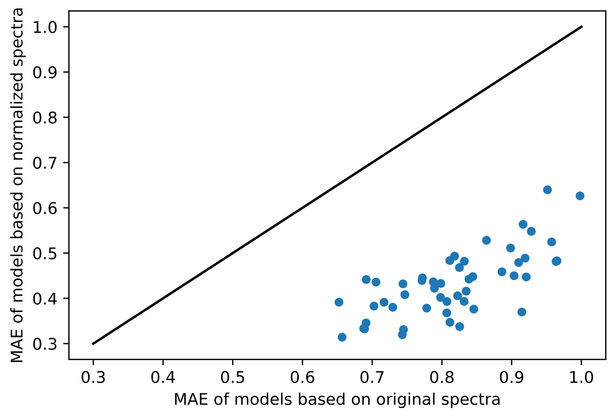


Fig. 8.8 Mean absolute error of regression models using original and normalized spectra. The line indicates equal performance. {fig:fdaMoistureMAE}

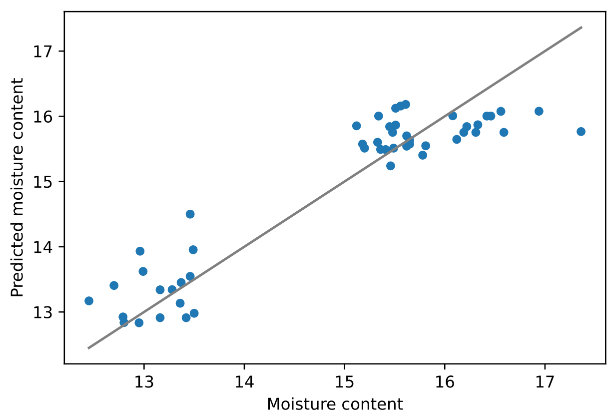


Fig. 8.9 Actual versus predicted moisture content {fig:fdaMoisturePredictions}

Figure 8.9 shows two clusters of points. One cluster contains the samples with the high moisture content and the other cluster the samples with low moisture content. The clusters are well separated and the predictions are in the typical range for each cluster. However, within a cluster, predictions and actual values are not highly correlated. In other words, while the regression model can distinguish between samples with a high and low moisture content, the moisture content is otherwise not well predicted. There is therefore no advantage of using the regression model compared to the classification model. {fig:fdaMoisturePredictions}

Exercise 8.5 In this exercise, we look at the result of a functional PCA using the **Moisture** dataset from Exercise 8.3. {exc:fda-moisture-classification}

(i) Load and preprocess the **Moisture** data as described in Exercise 8.3. {exc:fda-moisture-classification}

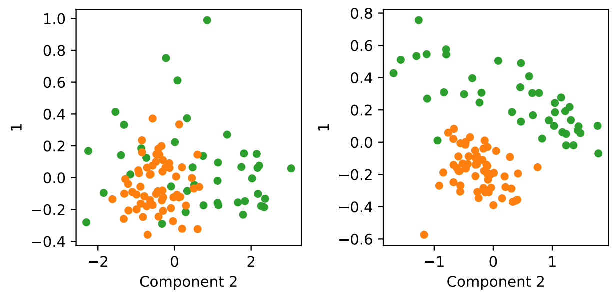


Fig. 8.10 Projection of spectra onto first two principal components. Left: original spectra, right: normalized spectra {fig:fdaMoisturePCA}

(ii) Carry out a functional principal component analysis of the raw and normalized spectra with two components. Plot the projection of the spectra onto the two components and color by moisture class. Discuss the results. (use `skfda.preprocessing.dim_reduction.projection.FPCA`)

Solution 8.5 **(i)** See solution for Exercise 8.3. {exc:fda-moisture-classification}
(ii) In Python:

```
from skfda.preprocessing.dim_reduction.projection import FPCA

fpca_original = FPCA(n_components=2)
_ = fpca_original.fit(fd)

fpca_normalized = FPCA(n_components=2)
_ = fpca_normalized.fit(fd_normalized)
```

The projections of the spectra can now be visualized:

```
def plotFPCA(fpca, fd, ax):
  fpca_df = pd.DataFrame(fpca.transform(fd))
  fpca_df.plot.scatter(x=0, y=1,
      c=['C1' if mc == 'high' else 'C2' for mc in moisture_class], ax=ax)
  ax.set_xlabel('Component 1')
  ax.set_xlabel('Component 2')

fig, axes = plt.subplots(ncols=2, figsize=[6, 3])
plotFPCA(fpca_original, fd, axes[0])
plotFPCA(fpca_normalized, fd_normalized, axes[1])
plt.tight_layout()
```

Figure 8.10 compares the PCA projections for the original and normalized data. We can see that the second principal component clearly separates the two moisture content classes for the normalized spectra. This is not the case for the original spectra. {fig:fdaMoisturePCA}

{exc:nlp-topic-1}

Exercise 8.6 Pick articles on global warming from two journals on the web. Use the same procedure for identifying stop words, phrases and other data preparation steps. Compare the topics in these two articles using 5 topics. Repeat the analysis using 10 topics. Report on the differences.

(i) Convert the two documents into a list of paragraphs and labels
(ii) Treating each paragraph as an individual document, create a a document term matrix (DTM). Ignore numerical values as terms. Which terms occur most frequently in the two articles.
(iii) Use TF-IDF to convert the DTM
(iv) Use latent semantic analysis (LSA) to find 5 topics

Solution 8.6 (i) We demonstrate a solution to this exercise using two blog posts preprocessed and included in the `mistat` package. The content of the posts was converted text with each paragraph on a line. The two blog posts can be loaded as follows:

```
from mistat.nlp import globalWarmingBlogs
blogs = globalWarmingBlogs()
```

The variable `blogs` is a dictionary with labels as keys and text as values. We next split the data into a list of labels and a list of non-empty paragraphs.

```
paragraphs = []
labels = []
for blog, text in blogs.items():
  for paragraph in text.split('\n'):
    paragraph = paragraph.strip()
    if not paragraph: # ignore empty paragraphs
      continue
    paragraphs.append(paragraph)
    labels.append(blog)
```

(ii) Using `CountVectorizer`, transform the list of paragraphs into a document-term matrix (DTM).

```
import re
from sklearn.feature_extraction.text import CountVectorizer

def preprocessor(text):
    text = text.lower()
    text = re.sub(r'\d[\d,]*', '', text)
    text = '\n'.join(line for line in text.split('\n')
                     if not line.startswith('ntsb'))
    return text

vectorizer = CountVectorizer(preprocessor=preprocessor,
                             stop_words='english')
counts = vectorizer.fit_transform(paragraphs)

print('shape of DTM', counts.shape)
print('total number of terms', np.sum(counts))
```

```
shape of DTM (123, 1025)
total number of terms 2946
```

The ten most frequenly occurring terms are:

```
termCounts = np.array(counts.sum(axis=0)).flatten()
topCounts = termCounts.argsort()
```

```
terms = vectorizer.get_feature_names_out()
for n in reversed(topCounts[-10:]):
  print(f'{terms[n]} & {termCounts[n]} \\\\')
```

```
global & 63 \\
climate & 59 \\
warming & 57 \\
change & 55 \\
ice & 35 \\
sea & 34 \\
earth & 33 \\
ocean & 29 \\
temperatures & 28 \\
heat & 25 \\
```

(iii) Conversion of counts using TF-IDF.

```
from sklearn.feature_extraction.text import TfidfTransformer

tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

(iv)

```
from sklearn.decomposition import TruncatedSVD
from sklearn.preprocessing import Normalizer
svd = TruncatedSVD(5)
norm_tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(norm_tfidf)
```

We can analyze the loadings to get an idea of topics.

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

```
print("{\\tiny")
print(df.style.format(precision=2).hide(axis='index').to_latex(hrules=True))
print("}")
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
change	0.24	ice	0.39	sea	0.25	extreme	0.54	snow	0.39
climate	0.24	sea	0.35	earth	0.24	events	0.31	cover	0.23
global	0.23	sheets	0.27	energy	0.21	heat	0.23	sea	0.17
sea	0.22	shrinking	0.19	light	0.21	precipitation	0.20	level	0.13
warming	0.21	level	0.17	gases	0.19	light	0.13	temperatures	0.12
ice	0.20	arctic	0.15	ice	0.18	earth	0.13	climate	0.11
temperature	0.18	ocean	0.13	infrared	0.17	energy	0.13	decreased	0.11
ocean	0.16	declining	0.10	greenhouse	0.16	gases	0.12	temperature	0.10
earth	0.16	levels	0.08	level	0.15	greenhouse	0.11	increase	0.10
extreme	0.15	glaciers	0.07	arctic	0.12	infrared	0.11	rise	0.10

We can identify topics related to sea warming, ice sheets melting, greenhouse effect, extreme weather events, and hurricanes.

(v) Repeat the analysis requesting 10 components in the SVD.

```
svd = TruncatedSVD(10)
norm_tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(norm_tfidf)
```

We now get the following topics.

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
change	0.24	ice	0.39	sea	0.25	extreme	0.54	snow	0.39
climate	0.24	sea	0.34	earth	0.24	events	0.31	cover	0.23
global	0.23	sheets	0.27	energy	0.21	heat	0.23	sea	0.17
sea	0.22	shrinking	0.19	light	0.21	precipitation	0.20	level	0.13
warming	0.21	level	0.17	gases	0.19	light	0.13	climate	0.12
ice	0.20	arctic	0.15	ice	0.18	earth	0.12	temperatures	0.12
temperature	0.18	ocean	0.13	infrared	0.17	energy	0.12	hurricanes	0.11
ocean	0.16	declining	0.10	greenhouse	0.16	gases	0.11	decreased	0.11
earth	0.16	levels	0.08	level	0.15	greenhouse	0.11	rise	0.10
extreme	0.15	glaciers	0.07	atmosphere	0.12	infrared	0.10	temperature	0.10

Topic 6	Loading 6	Topic 7	Loading 7	Topic 8	Loading 8	Topic 9	Loading 9	Topic 10	Loading 10
sea	0.37	ocean	0.32	ocean	0.38	glaciers	0.37	responsibility	0.34
level	0.35	snow	0.30	hurricanes	0.22	retreat	0.25	authorities	0.27
rise	0.17	cover	0.21	acidification	0.19	glacial	0.22	heat	0.18
extreme	0.14	acidification	0.20	glaciers	0.16	water	0.21	pollution	0.16
hurricanes	0.12	extreme	0.14	water	0.14	months	0.16	wildfires	0.15
events	0.11	carbon	0.13	waters	0.11	summer	0.14	personal	0.15
global	0.11	pollution	0.12	temperatures	0.10	going	0.13	arctic	0.12
impacts	0.10	waters	0.11	glacial	0.09	plants	0.11	percent	0.11
temperature	0.09	decreased	0.11	retreat	0.08	stream	0.11	carbon	0.11
coastal	0.08	point	0.11	reduce	0.08	animals	0.11	fossil	0.08

The first five topics are identical to the result in **(iv)**. This is an expected property of the SVD.

{fig:nlpSVD} **(vi)** Figure 8.11 shows the individual documents projected onto the first two singular values of the LSA. Based on this visualization, we can say that the two documents discuss different aspects of global warming and that blog post 1 contains more details about the area.

```
fig, ax = plt.subplots()
blog1 = [label == 'blog-1' for label in labels]
blog2 = [label == 'blog-2' for label in labels]
ax.plot(lsa_tfidf[blog1, 0], lsa_tfidf[blog1, 1], 'ro')
ax.plot(lsa_tfidf[blog2, 0], lsa_tfidf[blog2, 1], 'go')
ax.set_xlabel('First component')
ax.set_xlabel('Second component')
plt.show()
```

Exercise 8.7 Pick three articles on COVID 19 economic impact from the same author. Use the same procedure for identifying stop words, phrases and other data preparation steps. Compare the topics in these three articles using 10 topics.

{exc:nlp-topic-1} **Solution 8.7** We follow the same procedure as in Exercise 8.6 using a set of three articles preprocessed and included in the `mistat` package.

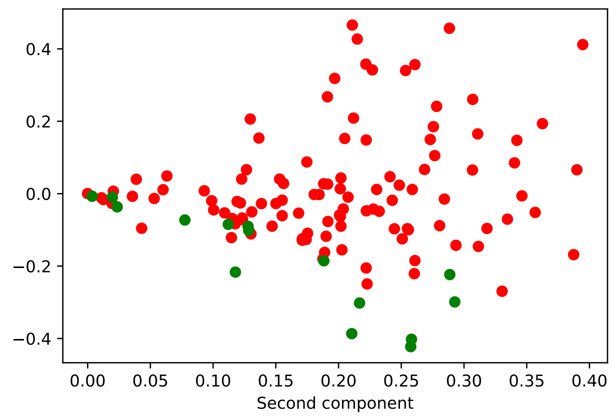


Fig. 8.11 Projection of the documents onto first two singular values. Red: blog post 1, green: blog post 2 {fig:nlpSVD}

```
from mistat.nlp import covid19Blogs
blogs = covid19Blogs()
```

Determine DTM using paragraphs as documents:

```
paragraphs = []
labels = []
for blog, text in blogs.items():
  for paragraph in text.split('\n'):
    paragraph = paragraph.strip()
    if not paragraph:
      continue
    paragraphs.append(paragraph)
    labels.append(blog)

def preprocessor(text):
  text = text.lower()
  text = re.sub(r'\d[\d,]*', '', text)
  text = '\n'.join(line for line in text.split('\n')
                   if not line.startswith('ntsb'))
  return text

vectorizer = CountVectorizer(preprocessor=preprocessor, stop_words='english')
counts = vectorizer.fit_transform(paragraphs)
```

TF-IDF transformation

```
tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

Latent semantic analysis (LSA)

```
svd = TruncatedSVD(10)
tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(tfidf)
```

Topics:

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
labour	0.29	labour	0.28	percent	0.23	capacity	0.23	covid	0.23
covid	0.22	south	0.27	economic	0.23	financial	0.19	america	0.21
impact	0.19	north	0.22	covid	0.19	firms	0.18	reset	0.20
market	0.19	differences	0.19	gdp	0.18	household	0.18	latin	0.18
south	0.18	americas	0.17	impact	0.15	international	0.17	economic	0.17
america	0.16	channel	0.16	imf	0.14	state	0.15	needs	0.17
pandemic	0.15	agenda	0.14	pre	0.12	largely	0.14	social	0.17
channel	0.15	covid	0.13	social	0.12	depends	0.14	asymmetric	0.14
economic	0.14	post	0.11	growth	0.11	access	0.13	consequences	0.13
north	0.14	welfare	0.09	world	0.10	support	0.12	labor	0.11

Topic 6	Loading 6	Topic 7	Loading 7	Topic 8	Loading 8	Topic 9	Loading 9	Topic 10	Loading 10
economic	0.25	covid	0.23	poverty	0.16	self	0.19	lightness	0.22
channel	0.21	economic	0.22	crisis	0.15	employed	0.17	unbearable	0.22
market	0.15	occupations	0.20	labor	0.14	poverty	0.17	self	0.21
social	0.15	consequences	0.16	inequality	0.13	lightness	0.16	employed	0.21
recovery	0.14	asymmetric	0.16	south	0.13	unbearable	0.16	capacity	0.13
labour	0.12	social	0.13	north	0.12	informal	0.15	state	0.12
governments	0.11	differences	0.13	deepen	0.12	social	0.14	informal	0.10
additionally	0.08	transition	0.12	differences	0.11	international	0.13	workers	0.10
vaccines	0.08	north	0.12	levels	0.10	economic	0.12	support	0.09
crucially	0.08	americas	0.12	structure	0.09	financial	0.11	infrastructure	0.08

Looking at the differnt loadings, we can see different topics emerging.

{fig:nlpSVD-covid} In Figure 8.12, we can see that the paragraphs in the article show more overlap compared to what we've observed in the Exercise 8.6. {exc:nlp-topic-1}

```
fig, ax = plt.subplots()
for blog in blogs:
    match = [label == blog for label in labels]
    ax.plot(lsa_tfidf[match, 0], lsa_tfidf[match, 1], 'o', label=blog)
ax.legend()
ax.set_xlabel('First component')
ax.set_xlabel('Second component')
plt.show()
```

Exercise 8.8 Use the **LAPTOP_REVIEWS.csv** dataset to analyze reviews and build a model to predict positive and negative reviews.

(i) Load the **LAPTOP_REVIEWS** data using the `mistat` package. Preprocess the dataset by combining the values of the columns `Review title` and `Review content` into a new column `Review` and remove missing rows with missing values in these two columns.
(ii) Convert the `Reviews` into a document-term matrix (DTM) using a count vectorizer. Split the reviews into words and remove English stopwords. Use a custom preprocessor to remove numbers from each word.
(iii) Convert the counts in the DTM into TF-IDF scores.
(iv) Normalize the TF-IDF scores and apply partial singular value decomposition (SVD) to convert the sparse document representation into a dense representation. Keep 20 components from the SVD.

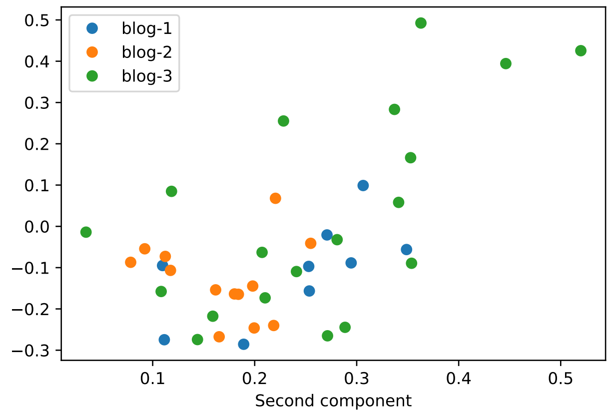


Fig. 8.12 Projection of the documents onto first two singular values. Red: blog post 1, green: blog post 2 {fig:nlpSVD-covid}

(v) Build a logistic regression model to predict positive and negative reviews. A review is positive if the `User rating` is five. Determine the predictive accuracy of the model by splitting the dataset into 60% training and 40% test sets.

Solution 8.8 (i) Load and preprocess the data

```
data = mistat.load_data('LAPTOP_REVIEWS')
data['Review'] = data['Review title'] + ' ' + data['Review content']
reviews = data.dropna(subset=['User rating', 'Review title', 'Review content'])
```

(ii) Convert the text representation into a document term matrix (DTM).

```
import re
from sklearn.feature_extraction.text import CountVectorizer
def preprocessor(text):
    text = text.lower()
    text = re.sub(r'\d[\d,]*', '', text)
    return text

vectorizer = CountVectorizer(preprocessor=preprocessor,
                             stop_words='english')
counts = vectorizer.fit_transform(reviews['Review'])
print('shape of DTM', counts.shape)
print('total number of terms', np.sum(counts))
```

```
shape of DTM (7433, 12823)
total number of terms 251566
```

(iii) Convert the counts in the document term matrix (DTM) using TF-IDF.

```
from sklearn.feature_extraction.text import TfidfTransformer

tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

(iv) Using scikit-learn's `TruncatedSVD` method, we convert the sparse tfidf matrix to a denser representation.

```
from sklearn.decomposition import TruncatedSVD
from sklearn.preprocessing import Normalizer
svd = TruncatedSVD(20)
tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(tfidf)
print(lsa_tfidf.shape)
```

```
(7433, 20)
```

(v) We use logistic regression to classify reviews with a user rating of five as positive and negative otherwise.

```
from sklearn.linear_model import LogisticRegression
from sklearn.metrics import accuracy_score, confusion_matrix
from sklearn.model_selection import train_test_split

outcome = ['positive' if rating == 5 else 'negative'
           for rating in reviews['User rating']]

# split dataset into 60% training and 40% test set
Xtrain, Xtest, ytrain, ytest = train_test_split(lsa_tfidf, outcome,
                                                test_size=0.4, random_state=1)

# run logistic regression model on training
logit_reg = LogisticRegression(solver='lbfgs')
logit_reg.fit(Xtrain, ytrain)

# print confusion matrix and accuracty
accuracy = accuracy_score(ytest, logit_reg.predict(Xtest))
print(accuracy)
confusion_matrix(ytest, logit_reg.predict(Xtest))
```

```
0.761600537995965
```

```
array([[ 839,  407],
       [ 302, 1426]])
```

The predicted accuracy of the classification model is 0.76.

Chapter 9
Bibliography