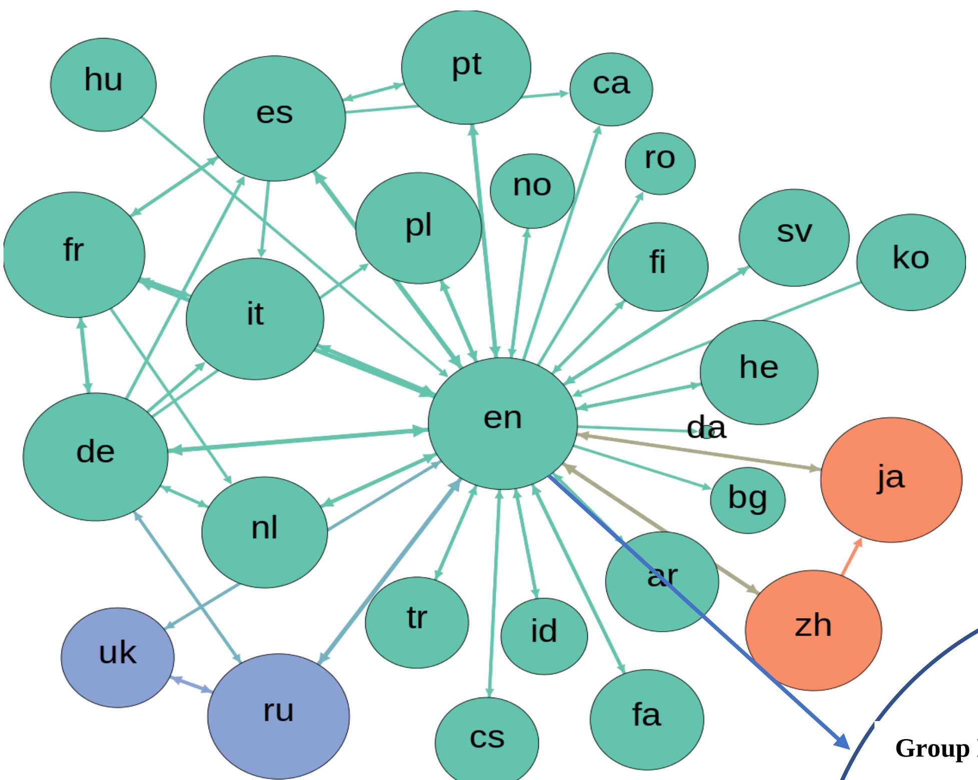
Research Cycle II

Graph Navigation



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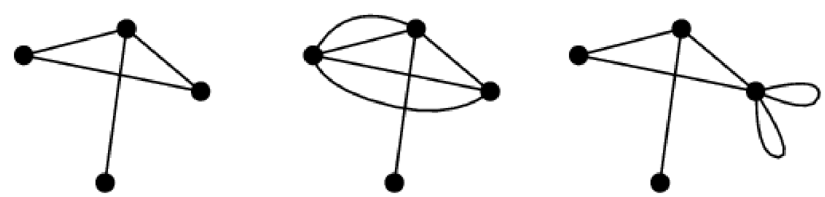
❖ Graph Navigation:

- Proposition 1.1: Let G = (V, E) be a simple graph of order n. Any path in G has length at most n – 1.
- Proposition 1.2: Every u-v walk in a graph contains a u-v path.
- Proposition 1.3: Suppose that exactly 2 vertices in a graph have odd degree. Then those two vertices are connected by a path

❖ Definitions and theorems:

➢ Simple graph:
A simple graph is an unweighted, undirected graph containing no graph loops or multiple edges



simple graph *nonsimple graph with multiple edges* *nonsimple graph with loops*

➢ Walk: -
A walk is a sequence v_0, e_1, v_1, ..., v_k of graph vertices v_i and graph edges e_i such that for $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i

➢ Path: -
A path is a walk v_0, e_1, v_1, ..., v_k with no repeated vertices

➢ Trail: -
A trail is a walk v_0, e_1, v_1, ..., v_k with no repeated edge.

➢ Handshake Lemma: -
If G= (V, E) is an undirected graph with n edges, 2 $E(G)$ = deg(v) $v \in V(G)$

⇨ Corollary: From Handshake Lemma we can prove that, every finite undirected graph has an even number of odd degree vertices.

➢ Euler trail:
A Eulerian path, also called a Euler chain, Euler trail, Euler walk, or "Eulerian" version of any of these variants, is a walk on the graph edges of a graph which uses each graph edge in the original graph exactly once.

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A Eulerian path, also called an Euler chain, Euler trail, Euler walk, or "Eulerian" version of any of these variants, is a walk on the graph edges of a graph which uses each graph edge in the original graph exactly once. A connected graph has a Eulerian path iff it has at most two graph vertices of odd degree.

➢ Euler Theorem:

For a connected multi-graph G, G contains a Euler circuit if and only if every vertex has even degree

Or

A graph G that is connected is semi-Eulerian if and only if two vertices in the graph have odd degree

❖ **<u>Case Study</u>**: -

➢ **<u>Proposition1.1:</u>**

Let G = (V, E) be a simple (di) graph of order n . Any path in G has length at most n – 1.

Proof by Induction

Let assume G=(V,E) is a simple directed graph and n is the number of vertices in G.

● n=1   Graph G has 1 vertex and 0 edges. Here there are no paths and maximum path length is 0 which is (n-1).

Vertex=1
Edges=0
(1-1) edges at most

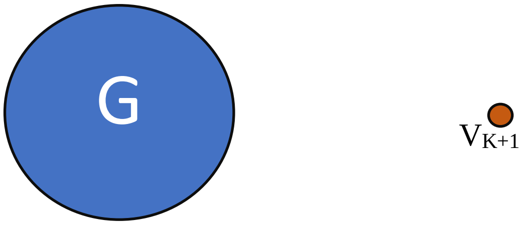
● n=2   Here graph G has 2 vertices and there may be a path exists between 2 vertices at length 1.So maximum path length is 1(n-1).(Shown below)

● Now assume G is an order n graph with maximum path length of size n-1.

● By adding one more vertex to graph G, it becomes a graph of order n+1.

Case 1:

Let the new node V_{K+1} is added as unconnected ➔ Since unconnected, it will not add new node to the connected graph. Hence no change in the highest length of n nodes.



Case 2:

- Connect the new vertex V_{K+1} to the node of the graph G.
- So as to obtain the longest path we have to add the node V_{K+1} to the terminals of graph Hence any path in G had n-1 as the length at most
- Now With the addition of the node:

⇨ (n-1)+1 = n

So,from this we can conclude the **length at most is n.**

By the principle of mathematical induction we can prove for any simple graph with node n, the length of a path be n-1 at the most.

- **Proposition1.2:**
  - ✓ Proof by Cases:
- Let's assume G= (V, E) and there exists a walk W between node U and node V.

  W= {V1, V2, ...Vn}.

  Here V1 is U and Vn is V.

- CASE 1: - There is no repeating nodes in the walk W. ○ So walk W has no repeating edges so by definition W is a path.
- CASE 2: -There is some repeating nodes in the walk W.

- Let's consider a repeating a cycle in the walk which is described below.

  W={V1, V2,......Vx-1,Vx,Vx+1.......Vy,Vy+1.....Vn}.

  Here Vx and Vy are repeating nodes.

- As Vx and Vy are same nodes there is an edge exists between Vx and Vy+1. So we can take out {Vx+1,…………,Vy} from the walk W without disconnecting the walk from U to V.

- So new walk between U and V is {V1,V2,......Vx-1,Vx,Vy+1.....Vn}.
- We can iterate Case 2 again and again until there are no repeated nodes.
- At last we can gain a walk between U and V which is a path

❖ . **<u>Preposition 1.3</u>**: Suppose that exactly 2 vertices in a graph have odd degree. Then those two vertices are connected by a path

⇨ <u>PROOF1:</u>

Case 1: The graph is a connected graph

By definition, for a connected graph, there exist a path between every pair of vertex. So obviously the two vertices with the odd degree are connected.

Case 2: The graph is a disconnected graph

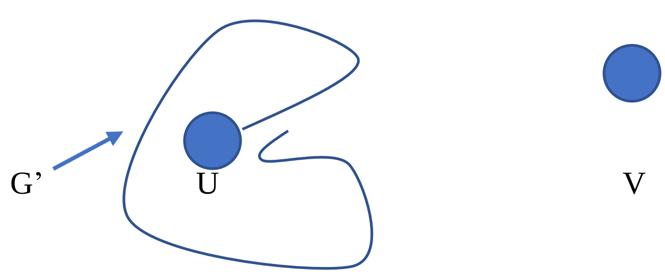
- Proof by contraposition:

  p: There are exactly two vertices in the graph with odd degree.
  q: The only two odd degree vertices in the graph are connected by a path

- Original preposition: p => q
- By contraposition, p => q is same as! q =>! p. That is,

  If the only two odd degree vertices in the graph are not connected by a path, then the number of odd degree vertices in the graph is not equal to two.

- Let the two odd degree vertices be u and v. Consider the connected graph (which is a part of graph G) which includes u (but not v) and let us name it as graph G'. According to the corollary of hand shaking lemma, every finite undirected graph has an even number of vertices with odd.



- In G’ u is a vertex with odd degree. For G’ to be a valid graph, another vertex with odd degree should exist. Therefore, the number of odd degree vertices is not equal to two but greater than two.
- The preposition, q => !p is true.
- By contraposition p => q is true.
- Therefore, from case 1 and case 2, suppose that exactly 2 vertices in a graph have odd degree, then those 2 vertices are connected by a path.

⇨ PROOF2:

- A connected graph has exactly two vertices with odd degree ←→
  Graph has a Eulerian path between them.

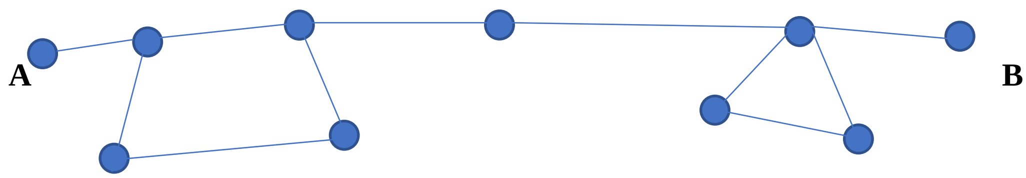
  -Since initial point != final point

  Proof:
  ←→By Eulerian theorem there exist a walk between the odd degree vertices
  ←→ There exist a path between those vertices (From proven 2nd preposition)

  Hence a graph with exactly 2 vertices with odd degree, then, those two vertices are connected by a path.

❖ **Questions:**

1. **In the “KOH” city, there are 10 bus stops with equal distance in between them. The map of the bus stops is given below. A bus takes 10 minutes to go from one bus stop to another and spends a minute at each bus stop. One day the driver got bored and decided to take the longest path from stop A to stop B. But he can’t go to a bus stop to which he is already been to. He left stop A at 7.00 am. At what time will he reach stop B? (Assume that there will be no unexpected delays)**



2. **Length of the longest path of the following walk sequence of a graph is 2. v1, e1, v2, e2, v3, e3, v4, e4, v2, e5, v5 (True/False)**

3. **Every path is a trail, but every trail is not a path. (True/False)**

Answers:

1. Time spent at each bus stop = number of vertices x 1

= 8 x 1

= 8 minutes

Time taken to travel = number of edges in the path x 10

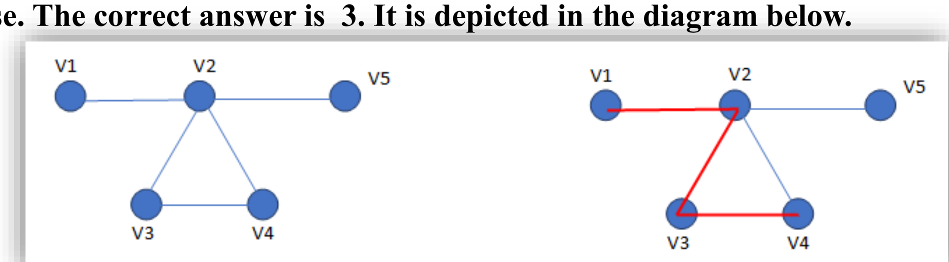
= (number of vertices in the path – 1) x 10

= 7 x 10

= 70 minutes

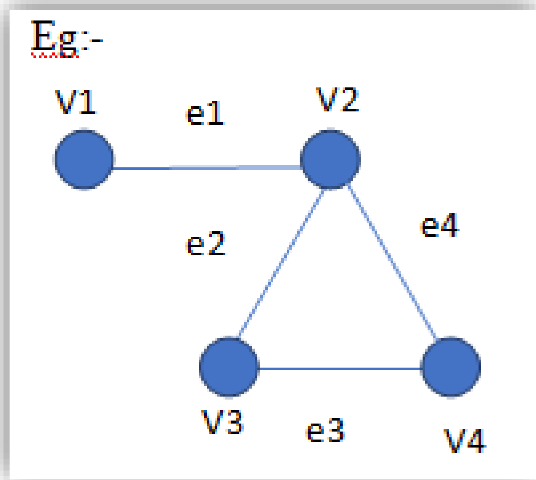
Reaching time = 8.18 am

2. False. The correct answer is 3. It is depicted in the diagram below.



3. True. In a path, no vertex can be repeated. When we write down a walk as a sequence of alternating vertices and edges, on either side of each edge, are the two vertices it is connected to (incident on). If either of these vertices are not repeated, the edge cannot repeat. Thus each path is a trail.

But not each trail is a path.



V1, e1, V2, e2, V3, e3, V4, e4 is a trail but not a path since V2 is repeating and it's not the starting vertex.

❖ References:

1. Walk: http://mathworld.wolfram.com/Walk.html
2. Path: http://mathworld.wolfram.com/Path.html
3. Trail: http://mathworld.wolfram.com/Trail.html
4. Hand ShakeLemma: https://en.wikipedia.org/wiki/Handshaking_lemma
5. Vertex degree: http://mathworld.wolfram.com/VertexDegree.html
6. Euler Graph: http://mathworld.wolfram.com/EulerGraph.html
7. Euler's Theorem: http://mathworld.wolfram.com/EulersTheorem.html