

Up to six planes can be flown:

$$n \leq 6.$$

### 13.24.3 Objective

$$\text{Maximize} \quad \sum_{i,c,h} Q_i r_{1ich} + \sum_{i,j,c,h} Q_i Q_j r_{2ijch} + \sum_{i,j,k,c,h} Q_i Q_j Q_k r_{3ijkch} - 50n$$

(measuring in £1000) where  $Q_i$  is the probability of scenario  $i$ .

This model has 1200 constraints, one bound and 982 variables, of which 117 are 0–1 integer and one general integer.

In defining the data, it is desirable that the demands in the scenarios cover the *extremes* as well as the most likely cases. The recommended sales will not exceed those of the most extreme scenario in the solution to this model. In this example, it can be seen that final demands (known with hindsight) exceed those of all scenarios in a few cases. As a result, the solution will not result in sales to meet all of these demands.

Models for subsequent weeks (with recourse) are obtained from this model by fixing prices and sales for weeks that have elapsed.

## 13.25 Car rental 1

We model this problem as a deterministic linear programme. There would be advantage to be gained from modelling it as a stochastic programme. In order to do this, we would need to obtain data to quantify the uncertain demand.

### 13.25.1 Indices

$i, j$  = Glasgow, Manchester, Birmingham, Plymouth

$t$  = Monday, Tuesday, Wednesday, Thursday, Friday, Saturday

$k$  = 1, 2, 3 (days hired)

Although this is a ‘dynamic’ problem, we seek a steady-state solution and can therefore regard the set of days as ‘circular’, that is, Monday of a week ‘follows’ after the subsequent Saturday; that is, if  $t$  = Monday then  $t - 1$  = Saturday.

### 13.25.2 Given data

$D_{it}$  = estimated rental demand at depot  $i$  on day  $t$  as given in Table 12.19

$P_{ij}$  = proportion of cars rented at depot  $i$  to be returned to depot  $j$  as given in Table 12.21

$C_{ij}$  = cost of transfer of a car from depot  $i$  to depot  $j$  as given in Table 12.22

$Q_k$  = proportion of cars hired for  $k$  days

- $R_i$  = repair capacity of depot  $i$   
 $RCA_k$  = rental cost for  $k$  days with return to same depot as given in Table 12.23  
 $RCB_k$  = rental cost for  $k$  days with return to another depot as given in Table 12.23  
 $RCC$  = rental cost for Saturday with return to same depot on Monday  
 $RCD$  = rental cost for Saturday with return to another depot on Monday  
 $CS_k$  = marginal cost to company of  $k$  day hire of a car for all  $i, t$

### 13.25.3 Variables

- $n$  = total number of cars owned  
 $nu_{it}$  = number of undamaged cars at depot  $i$  at the beginning of day  $t$  for all  $i, t$   
 $nd_{it}$  = number of damaged cars at depot  $i$  at the beginning of day  $t$  for all  $i, t$   
 $tr_{it}$  = number of cars rented out from depot  $i$  at beginning of day  $t$  for all  $i, t$   
 $eu_{it}$  = undamaged cars left at depot  $i$  during day  $t$  for all  $i, t$   
 $ed_{it}$  = damaged cars left at depot  $i$  at the beginning of day  $t$  for all  $i, t$   
 $tu_{ijt}$  = number of undamaged cars at depot  $i$  at the beginning of day  $t$ , to be transferred to depot  $j$  for all  $i, j, t$   
 $td_{ijt}$  = number of damaged cars at depot  $i$  at the beginning of day  $t$ , transferred to depot  $j$  for all  $i, j, t$   
 $rp_{it}$  = number of damaged cars to be repaired at depot  $i$  on day  $t$  for all  $i, t$

### 13.25.4 Constraints

Total number of undamaged cars into depot  $i$  on day  $t$

$$\sum_{jk} 0.9P_{ji} Q_k tr_{jt-k} + \sum_j tu_{jit-1} + rp_{it-1} + eu_{it-1} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars into depot  $i$  on day  $t$

$$\sum_{jk} 0.1P_{ji} Q_k tr_{jt-k} + \sum_j td_{jit-1} + ed_{it-1} = nd_{it} \quad \text{for all } i, t.$$

Total number of undamaged cars out of depot  $i$  on day  $t$

$$tr_{it} + \sum_j tu_{ijt} + rp_{it-1} + eu_{it} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars out of depot  $i$  on day  $t$

$$rp_{it-1} + \sum_j td_{ijt} + ed_{it} = nd_{it} \quad \text{for all } i, t.$$

Repair capacity of depot  $i$  on all days

$$rp_{it} \leq R_i \quad \text{for all } i, t.$$

Demand at depot  $i$  on day  $t$

$$tr_{it} \leq D_{it} \quad \text{for all } i, t.$$

Total number of cars equals number hired out from all depots on Monday for 3 days, plus those on Tuesday for 2 or 3 days, plus all damaged and undamaged cars in depots at the beginning of Wednesday.

$$\sum_i (0.25tr_{i1} + 0.45tr_{i2} + nu_{i2} + nd_{i2}) = n \quad \text{for all } i.$$

### 13.25.5 Objective

Note that £10 has been added to the profit on each rented car to reflect the surcharge of £100 charged on the 10% of cars that are returned damaged.

$$\begin{aligned} \text{Profit} = & \sum_{itk, t \neq \text{SATURDAY}} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} \\ & + \sum_{ijt, t \neq \text{SATURDAY}} P_{ij} Q_k (RCB_k - CS_k + 10) tr_{it} \\ & + \sum_{i \text{ SATURDAY}} P_{ii} Q_1 (RCC - CS_1 + 10) tr_{it} \\ & + \sum_{ij \text{ SATURDAY}} P_{ij} Q_1 (RCD - CS + 10) tr_{it} \\ & + \sum_{i \text{ SATURDAY}k} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} + \sum_{i \text{ SATURDAY}k} P_{ij} Q_k (RCB_k - CS_k \\ & + 10) tr_{it} \\ & - \sum_{ijt} C_{ij} tu_{ijt} - \sum_{ijt} C_{ij} td_{ijt} - 15n. \end{aligned}$$

This model has a total of 84 constraints and 337 variables.

It is not necessary to constrain the  $w_{ijkl}$  to be non-negative. They can be regarded as ‘free’ variables. Avoiding these two stipulations helps the model to solve more easily.

## 13.26 Car rental 2

We introduce the following 0–1 integer variables with the following interpretations:

- $\delta_{B1} = 1$ , if the Birmingham repair capacity is expanded by 5 cars per day.  
 $\delta_{B2} = 1$ , if the Birmingham repair capacity is expanded by a further 5 cars per day.  
 $\delta_{M1} = 1$ , if the Manchester repair capacity is expanded by 5 cars per day.  
 $\delta_{M2} = 1$ , if the Manchester repair capacity is expanded by a further 5 cars per day.  
 $\delta_P = 1$ , if the Plymouth repair capacity is expanded by 5 cars per day.

The following expression is added to the objective function:

$$18\,000\delta_{B1} + 8\,000\delta_{B2} + 20\,000\delta_{M1} + 5\,000\delta_{M2} + 19\,000\delta_P$$

together with the following extra constraints:

$$\delta_{B1} \geq \delta_{B2}, \quad \delta_{M1} \geq \delta_{M2}, \quad \delta_{B1} + \delta_{B2} + \delta_{M1} + \delta_{M2} + \delta_P \leq 3$$

and the repair capacity constraints for Birmingham, Manchester and Plymouth have  $5\delta_{B1} + 5\delta_{B2}$ ,  $5\delta_{M1} + 5\delta_{M2}$  and  $5\delta_P$ , respectively added.

## 13.27 Lost baggage distribution

We formulate this problem as an integer programming model. It is an extension of the travelling salesman problem, discussed in section 9.5. It can be regarded as a (simple) case of the vehicle routing problem. In the problem here, there are no vehicle capacities and no time windows for the delivery to each location (apart from the overall 2 hours guarantee). Nevertheless, it is potentially a very difficult problem to solve, for more than a modest number of locations. Also it can give rise to a very large model. There are a number of different ways of formulating a model, which vary greatly in their size and difficulty of solution. We suggest a model that is practicable in both size and computational tractability. Although the problem is 'symmetric' in the sense that the distance from  $X$  to  $Y$  is the same as that from  $Y$  to  $X$ , we extend the asymmetric formulation of the travelling salesman problem in order to give the 'time' to return to Heathrow, after a van has completed all its deliveries, as 0. In this way, only the total time taken to reach the last delivery is counted within the 2 hours time limit. We are therefore seeking a 'Hamiltonian path' (as opposed to a circuit) for each van, starting from Heathrow.

### 13.27.1 Variables

All variables are 0–1 and integer

$x_{ijk} = 1$ , iff van  $k$  visits, and goes directly from location  $i$  to  $j$  for all  $i, j$ , and  $k$ .

$y_{ik} = 1$ , iff location  $i$  is visited by van  $k$  for all  $i, k$ .

$\delta_k = 1$ , iff van  $k$  is used.