

Chapter 6 Orthogonality and Least Squares

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Inner Product, Length, and Orthogonality

Theorem 1

Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in \mathbb{R}^n , and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

The Length of a Vector

The length (or norm) of \mathbf{v} is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

Distance in \mathbb{R}^n

For \mathbf{u} and \mathbf{v} in \mathbb{R}^n , the **distance between \mathbf{u} and \mathbf{v}** , written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$.

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Orthogonal Vectors

Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 2 — The Pythagorean Theorem

Two vectors \mathbf{u} and \mathbf{v} are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A , and the orthogonal complement of the column space of A is the null space of A^T

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

Key Points

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \times \mathbf{v}$
2. A unit vector in the direction of a vector can be determined by dividing that vector by its length.
3. $\|c\mathbf{v}\|$ is not always equal to $c\|\mathbf{v}\|$. Since length is always positive, the value of $\|c\mathbf{v}\|$ is positive for all values of c . However, $c\|\mathbf{v}\|$ is negative if c is negative.