Chapter 6 Orthogonality and Least Squares

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Inner Product, Length, and Orthogonality

Theorem 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = u \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

The Length of a Vector

The length (or norm) of $\mathbf{v}$ is the nonnegative scalar $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad ||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$$

Distance in $\mathbb{R}^n$

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$.

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\ \mathbf{u} - \mathbf{v}\ ||$$

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 2 — The Pythagorean Theorem

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $||\ \mathbf{u} + \mathbf{v}\ ||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

Key Points

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \times \mathbf{v}$
2. A unit vector in the direction of a vector can be determined by dividing that vector by its length.
3. $||c\mathbf{v}||$ is not always equal to $c||\mathbf{v}||$. Since lenght is always positive, the value of $||c\mathbf{v}||$ is positive for all values of c. However, $c||\mathbf{v}||$ is negative if c is negative.

Orthogonal Sets

Theorem 4

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Orthogonal Basis

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 5

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination are

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{c}_p \quad \text{given by} \quad c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \ldots, p)$$

Theorem 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Theorem 7

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $||U\mathbf{x}|| = ||\mathbf{x}||$
2. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

Key Points

1. A set of vectors is orthogonal if each pair of distinct vectors from the set is orthogonal.
2. The vector $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$

3. $\mathbf{y}$ can be writtena s the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

4. An orthonormal set is an orthogonal set where all of the vectors are unit vectors.
5. If A is a matrix with orthonormal columns, then $||A\mathbf{x}|| = ||\mathbf{x}||$.
6. If U is an orthogonal matrix, $U^T = U^{-1}$.