Chapter 4 Vector Spaces

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Vector Spaces and Subspaces

A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and mulitplication by scalars (real numbers), subject to the ten rules listed below. The rules must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

A **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Theorem 1

If $\mathbf{v}_1,\dots,\mathbf{v}_p$ are in a vector space V, then $\mathrm{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ is a subspace of V.

Key Points

1. If $\mathbf{v}$ is in $\mathbb{R}^3$, $H=\mathrm{Span}\{\mathbf{v}\}$ is a subspace of $\mathbb{R}^3$.

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Null Spaces

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

To find the vectors that span the null space:

1. Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to reduced echelon form.
2. Write the solution in terms of the free variables.
3. The column vectors that are multiplied by the free variables in the previous step form the spanning set for Nul A.

Theorem 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Column Spaces

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span}\{a_1, \ldots, a_n\}$$

Theorem 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Row Spaces

The **row space** of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A. If $A = \begin{bmatrix} r_1 \\ \cdots \\ r_n \end{bmatrix}$, then

$$\text{Row } A = \text{Span}\{r_1, \ldots, r_n\}$$

Linear Transformations

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

$$
\begin{aligned}
(i) \quad & T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) && \text{for all } \mathbf{u}, \mathbf{v} \text{ in } V \\
(ii) \quad & T(c\mathbf{u}) = cT(\mathbf{u}) && \text{for all } \mathbf{u} \text{ in } V text and all scalar s c.
\end{aligned}
$$

Linear Independent Sets; Bases

Theorem 4

An indexed set $\{v_1, \ldots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linear dependent if and only if some v_j (with $j > 1$) is a linear comibnation of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Bases

Let H be a subspace of a vector space V. A set of vectors