Lines and Planes

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Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through point $P = (x_0, y_0, z_0)$ can be described by the following parametric equations.

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$$

Equations for a Plane

1. **Vector Equation**

   Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying the following equation forms a plane.

   $$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

2. **Scalar Equation**

   Given a plane containing $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$.

   $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. **General Form of the Equation**

   $$ax + by + cz + d = 0$$

   where $d = -ax_0 - by_0 - cz_0$.

The Distance between a Plane and a Point

Suppose a plane with normal vector $\mathbf{n}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_n \overrightarrow{QP}\| = |\text{comp}_n \overrightarrow{QP}| = \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Let $P = (x_0, y_0, z_0)$ be a point. The distance from P to plane $ax + by + cz + k = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}$$

Line of Intersection

The direction vector of the line of intersection of two planes is given by the cross product of their normal vectors.

Angle between Two Intersecting Planes

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \quad \theta = \cos^{-1}\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right)$$

Area of a Triangle

Let the traingle have vertices P, Q, and R.

$$A = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{QR}\|$$