Chapter 4 Vector Spaces

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Vector Spaces and Subspaces

A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and mulitplication by scalars (real numbers), subject to the ten rules listed below. The rules must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

A **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Theorem 1

If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then $\mathrm{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

If $\mathbf{v}$ is in $\mathbb{R}^3$, $H=\mathrm{Span}\{\mathbf{v}\}$ is a subspace of $\mathbb{R}^3$.

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Null Spaces

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

To find the vectors that span the null space:

1. Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to reduced echelon form.
2. Write the solution in terms of the free variables.
3. The column vectors that are multiplied by the free variables in the previous step form the spanning set for Nul A.

Theorem 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Column Spaces

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span}\{a_1, \ldots, a_n\}$$

To find the vectors that span the column space:

1. Determine the pivot columns in the matrix.
2. The pivot columns in the original matrix form the spanning set for Col A.

Theorem 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Row Spaces

The **row space** of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A. If $A = \begin{bmatrix} r_1 \\ \cdots \\ r_n \end{bmatrix}$, then

$$\text{Row } A = \text{Span}\{r_1, \dots, r_n\}$$

To find the vectors that span the row space:

1. Row reduce the matrix to echelon form.
2. The pivot rows in the resulting matrix form the the spanning set for Row A.

Linear Transformations

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

$$\begin{aligned} (i) \quad & T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) && \text{for all } \mathbf{u}, \mathbf{v} \text{ in } V \\ (ii) \quad & T(c\mathbf{u}) = cT(\mathbf{u}) && \text{for all } \mathbf{u} \text{ in } V text and all scalars c. \end{aligned}$$

Linear Independent Sets; Bases

Theorem 4

An indexed set $\{v_1, \dots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linear dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

Bases

Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span } \mathcal{B}$$

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V

- By the definition of a basis, $\mathbf{b}_1, \dots, \mathbf{b}_n$ are in V.
- By the Unique Representation Theorem, for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.
- $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for some unique set of scalars $c_1, \dots, c_n$.
- Thus, the coordinate vector $[\mathbf{b}_k]_{\mathcal{B}}$ of $\mathbf{b}_k$ is $\mathbf{e}_k$, or the kth column of the $n \times n$ identity matrix.

Theorem 5 — The Spanning Set Theorem

Let $S = \{v_1, \dots, v_p\}$ be a set in a vector space V, and let $H = \text{Span}\{v_1, \dots, v_p\}$.

1. If one of the vectors in S, v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.
2. If $H \neq 0$, some subset of S is a basis for H.

Theorem 6

The pivot columns of a matrix A form a basis for Col A.

Theorem 7

If two matrices A and B are row equivalent, then their row spacesare the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Coordinate Systems

Theorem 8 — The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Coordinates

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The coordinates of x relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of x) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Theorem 9

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

$$\mathcal{B}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$$

Key Points

- If $B = [\mathbf{b}_1, \dots, \mathbf{b}_n]$, then the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is $\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$.
- If a set of coordinate vectors of each polynomial has a pivot position in each row, by isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the set of polynomials spans $\mathbb{P}_2$.

Isomorphism

Let V and W be vector spaces. A linear mapping $T : V \to W$ that is one-to-one and onto is called an **isomorphism** from V to W.

- Isomorphisms are invertible.
- If T is an isomorphism from V to W, then T^{-1} is an isomorphism from W to V.
- If there are is an isomorphism from V to W, then the spaces V and W are **isomorphic**.

The Dimension of a Vector Space

If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

The **rank** of an $m \times n$ matrix A is the dimension of the column space and the **nullity** of A is the dimension of the null space.

Theorem 10

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 11

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Theorem 12

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and dim $H \leq$ dim V.

Theorem 13 — The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Theorem 14 — The Rank Theorem

The dimensions of the column space and the null space of an $m \times n$ matrix A satisfy the equation

$$\text{rank } A + \text{nullity } A = \text{number of columns in } A$$

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.
2. Col $A = \mathbb{R}^n$
3. rank $A = n$
4. nullity $A = 0$
5. Nul $A = \{\mathbf{0}\}$

Dimensions

The dimension of the:

- Nul A is the number of free variables in A.
- Col A is the number of pivot columns in A.
- Row A is the number of pivot rows in A.

Theorem 15 — Change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors in the basis $\mathcal{B}$.
That is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

The columns of $\underset{C \leftarrow B}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set B.

Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. To find the change-of-coordinates matrix from B to C.

1. Reduce the matrix $\begin{bmatrix} c_1 & c_2 & b_1 & b_2 \end{bmatrix}$ to reduced echelon form.
2. The last two columns form the change-of-coordinates matrix from B to C, $\underset{C \leftarrow B}{P}$.

Extra Notes

1. $\sin x \cos x = \frac{1}{2} \sin 2x$
2. If $\mathbb{P}_n$ is the space of all polynomials of degree at most n, its dimension is $n + 1$.