Cross Product

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Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then, the **cross product** $\mathbf{w}$ is a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{w} = \langle u_2 v_3 - u_3 v_2,\ -(u_1 v_3 - u_3 v_1),\ u_1 v_2 - u_2 v_1 \rangle$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

$$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = 0$$

$$\mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

Magnitude of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta$$

Cross Product using Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{w}$ is

$$\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Area of a Parallelogram

If we locate vectors $\mathbf{u}$ and $\mathbf{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$

The Triple Scalar Product

The **triple scalar product** of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

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Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of the triple scalar product

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Torque

Torque, τ, measures the tendency of a force to produce rotation about an axis of rotation. Let $\mathbf{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\mathbf{F}$ represent the force. Then torque is equal to the cross product of $\mathbf{r}$ and $\mathbf{F}$.

$$\tau = \mathbf{r} \times \mathbf{F}$$