

Linear Equations in Linear Algebra

A matrix is in **echelon form** if:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

A matrix is in **reduced echelon form** if:

1. It is in echelon form.
2. The leading entry in each nonzero row is 1.

Properties

- Two matrices are row equivalent if there exists a sequence of elementary row operations that transforms one matrix into the other.
- Each matrix is row equivalent to only one reduced echelon matrix.
- The echelon form of a matrix is not unique, but the reduced echelon form is unique.

Existence and Uniqueness Theorem

A linear system is consistent if the rightmost column of echelon form of the augmented matrix is not a pivot column.

Row Reduction Algorithm

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system, leading to a general solution of a system.

1. Forward Phase (reducing a matrix to echelon form)
 - (a) Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
 - (b) Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
 - (c) Use row replacement operations to create zeros in all positions below the pivot.
 - (d) Ignore the row containing the pivot position and all rows above it.

- (e) Repeat until there are no more nonzero rows to modify.
- 2. Backward Phase (reducing a matrix to reduced echelon form)
 - (a) Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Span (Linear Combination)

- The span of two vectors, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, represents all vectors that can be reached by scaling and adding the two vectors.
- If the system consisting of vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{b} is consistent, then \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- A matrix can only span \mathbb{R}^n if it has pivot positions in n rows.

Matrix Equation $A\mathbf{x} = \mathbf{b}$

$A\mathbf{x} = \mathbf{b}$ can be represented as a vector or matrix equation.

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= d \\ ex_1 + fx_2 + gx_3 &= h \end{aligned}$$

Vector Equation:

$$x_1 \begin{bmatrix} a \\ e \end{bmatrix} + x_2 \begin{bmatrix} b \\ f \end{bmatrix} + x_3 \begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} d \\ h \end{bmatrix}$$

Matrix Equation:

$$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d \\ h \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.

1. For each \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
3. The columns of A span \mathbb{R}^m .
4. A has a pivot position in every row.

Homogeneous Equation

A homogeneous equation is a linear equation in the form $Ax = 0$

1. The homogeneous equation always has at least one solution (the trivial solution), where $\mathbf{x} = \mathbf{0}$.
2. The homogeneous equation has a nontrivial solution if the equation has at least one free variable.
3. If the matrix \mathbf{A} has more columns than rows ($n > m$), the system often has infinitely many solutions.
4. If \mathbf{A} has n pivot columns, the columns of \mathbf{A} are linearly independent, since every variable is a basic variable.

Parametric Vector Equation

The equation can be represented in parametric vector form if there is a free variable so that all of the other variables are represented in terms of the free variable. For example, if x_3 is a free variable in \mathbb{R}^3 ,

$$x = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

where $x_1 = ax_3 + c$ and $x_2 = bx_3 + d$.

$$x = \begin{bmatrix} a + bx_3 \\ c + dx_3 \\ e + fx_3 \end{bmatrix}, \text{ geometrically describes a line through } \begin{bmatrix} a \\ c \\ e \end{bmatrix} \text{ parallel to } \begin{bmatrix} b \\ d \\ f \end{bmatrix}$$

The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where \mathbf{v}_h is any solution of the equation $A\mathbf{x} = \mathbf{0}$, only when the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given \mathbf{b} , and there exists a vector \mathbf{p} such that \mathbf{p} is a solution.

Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be linearly independent if the vector equation $x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution.

- If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n contains the zero vector, then the set is linearly dependent.
- Two vectors are linearly dependent if they live on a line through the origin.

Linear Transformations

A linear transformation is a function from \mathbb{R}^n to \mathbb{R}^m that assigns to each vector \mathbf{x} in \mathbb{R}^n a vector $T(\mathbf{x})$ in \mathbb{R}^m . For a transformation to be linear, then:

- $T(\mathbf{0}) = \mathbf{0}$.
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

Key Points

- If A is a $m \times n$ matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is \mathbb{R}^n and the range of T is the solution set.
- A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if each vector in \mathbb{R}^n maps onto a unique vector in \mathbb{R}^m . Also, T is one-to-one if and only if the columns of A are linearly independent.
- A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto \mathbb{R}^m if every vector in \mathbb{R}^m is mapped onto by some vector \mathbf{x} in \mathbb{R}^n . Also, T is onto \mathbb{R}^m if there are n pivot columns.

Standard matrix of T to rotate points about the origin by θ : $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then there exists a unique matrix A such that the following equation is true.

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j th column is the vector, $T(e_j)$, where e_j is the j th column of the identity matrix in \mathbb{R}^n , as shown in the following equation.

$$A = [T(e_1) \quad \cdots \quad T(e_n)]$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \end{bmatrix} = \begin{bmatrix} ag + bi & ah + bj \\ cg + di & ch + dj \\ eg + fi & eh + fj \end{bmatrix}$$

The (i, j) entry in AB is

$$\sum_{k=1}^n a_{ik}b_{kj} \quad \text{or} \quad a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

$$(AB)^T = B^T A^T$$

Key Points

- Matrix multiplication is only allowed for matrices where the number of columns in the first matrix equal the number of rows in the second matrix.
- If A is a $a \times b$ matrix and B is a $b \times c$ matrix, then AB is a $a \times c$ matrix.
- Given a square $n \times n$ matrix A , there are infinite matrices for B in $AB = BA$ because any multiple of I_n will satisfy the equation.