

# Chapter 5 Eigenvalues and Eigenvectors

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## Eigenvalues and Eigenvectors

An **eigenvector** of an  $n \times n$  matrix  $A$  is a nonzero vector  $\mathbf{x}$  such that  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . A scalar  $\lambda$  is called an **eigenvalue** of  $A$  if there is a nontrivial solution  $\mathbf{x}$  of  $A\mathbf{x} = \lambda\mathbf{x}$ ; such an  $\mathbf{x}$  is called an *eigenvector corresponding to  $\lambda$* .

### Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

### Theorem 2

If  $\mathbf{v}_1, \dots, \mathbf{v}_r$  are eigenvectors that correspond to distinct eigenvalues  $\lambda_1, \dots, \lambda_r$  of an  $n \times n$  matrix  $A$ , then the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$  is linearly independent.

### Validating an Eigenvalue

1. Start with the equation  $A\mathbf{x} = \lambda\mathbf{x}$
2. Form the matrix  $A - \lambda I$
3. If the columns are linearly dependent,  $\lambda$  is an eigenvalue
4. Reduce the matrix to reduced echelon form and each column vector in terms of the free variables is a corresponding eigenvector and a part of the basis for the eigenspace

### Validating an Eigenvector

1. Start with the equation  $A\mathbf{x} = \lambda\mathbf{x}$
2. Compute the product of  $A\mathbf{x}$
3. If  $A\mathbf{x}$  is proportional to  $\mathbf{x}$ , then  $\mathbf{x}$  is an eigenvector and the scaling factor is the eigenvalue

## Key Points

- If the columns of  $A$  are linearly dependent, one eigenvalue of  $A$  is  $\lambda = 0$
- If  $A$  is the zero matrix, then the only eigenvalue of  $A$  is 0

## The Characteristic Equation

The scalar equation  $\det(A - \lambda I) = 0$  is called the characteristic equation. The characteristic polynomial is the simplified polynomial in the characteristic equation and the eigenvalues are the values for  $\lambda$ .

### Theorem 3 — Properties of Determinants

Let  $A$  and  $B$  be  $n \times n$  matrices.

1.  $A$  is invertible if and only if  $\det A \neq 0$ .
2.  $\det AB = (\det A)(\det B)$ .
3.  $\det A^T = \det A$ .
4. If  $A$  is triangular, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .
5. A row replacement operation on  $A$  does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

### The Invertible Matrix Theorem (continued)

Let  $A$  be an  $n \times n$  matrix. Then  $A$  is invertible if and only if, the number 0 is **not** an eigenvalue of  $A$ .

### Theorem 4

If  $n \times n$  matrices  $A$  and  $B$  are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).