Chapter 4 Vector Spaces

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Vector Spaces and Subspaces

A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and mulitplication by scalars (real numbers), subject to the ten rules listed below. The rules must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

A **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Theorem 1

If $\mathbf{v}_1,\dots,\mathbf{v}_p$ are in a vector space V, then $\text{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ is a subspace of V.

Key Points

1. If $\mathbf{v}$ is in $\mathbb{R}^3$, $H=\text{Span}\{\mathbf{v}\}$ is a subspace of $\mathbb{R}^3$.

Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Null Spaces

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

To find the vectors that span the null space:

1. Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to reduced echelon form.
2. Write the solution in terms of the free variables.
3. The column vectors that are multiplied by the free variables in the previous step form the spanning set for Nul A.

Theorem 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Column Spaces

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span}\{a_1, \dots, a_n\}$$

To find the vectors that span the column space:

1. Determine the pivot columns in the matrix.
2. The pivot columns in the original matrix form the spanning set for Col A.

Theorem 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Row Spaces

The **row space** of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A. If $A = \begin{bmatrix} r_1 \\ \cdots \\ r_n \end{bmatrix}$, then

$$\text{Row } A = \text{Span}\{r_1, \dots, r_n\}$$

To find the vectors that span the row space:

1. Row reduce the matrix to echelon form.
2. The pivot rows in the resulting matrix form the the spanning set for Row A.

Linear Transformations

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

$$\begin{aligned} (i) \quad & T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) && \text{for all } \mathbf{u}, \mathbf{v} \text{ in } V \\ (ii) \quad & T(c\mathbf{u}) = cT(\mathbf{u}) && \text{for all } \mathbf{u} \text{ in } V textandallscalarsc. \end{aligned}$$

Linear Independent Sets; Bases

Theorem 4

An indexed set $\{v_1, \ldots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linear dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Bases

Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span } \mathcal{B}$$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V

- By the definition of a basis, $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are in V.
- By the Unique Representation Theorem, for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.
- $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for some unique set of scalars $c_1, \ldots, c_n$.
- Thus, the coordinate vector $[\mathbf{b}_k]_{\mathcal{B}}$ of $\mathbf{b}_k$ is $\mathbf{e}_k$, or the kth column of the $n \times n$ identity matrix.

Theorem 5 — The Spanning Set Theorem

Let $S = \{v_1, \ldots, v_p\}$ be a set in a vector space V, and let $H = \text{Span}\{v_1, \ldots, v_p\}$.

1. If one of the vectors in S, v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.
2. If $H \neq 0$, some subset of S is a basis for H.

Theorem 6

The pivot columns of a matrix A form a basis for Col A.

Theorem 7

If two matrices A and B are row equivalent, then their row spacesare the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Coordinate Systems

Theorem 8 — The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Coordinates

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The coordinates of x relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of x) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Theorem 9

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

$$\mathcal{B}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$$

Key Points

- If $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, then the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is $\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$.
- If a set of coordinate vectors of each polynomial has a pivot position in each row, by isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the set of polynomials spans $\mathbb{P}_2$.

Extra Notes

1. $\sin x \cos x = \frac{1}{2}\sin 2x$