Matrix and Linear Algebra

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Linear Equations in Linear Algebra

A matrix is in **echelon form** if:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

A matrix is in **reduced echelon form** if:

1. It is in echelon form.
2. The leading entry in each nonzero row is 1.

Properties

- Two matrices are row equivalent if there exists a sequence of elementary row operations that transforms one matrix into the other.
- Each matrix is row equivalent to only one reduced echelon matrix.
- The echelon form of a matrix is not unique, but the reduced echelon form is unique.

Existence and Uniqueness Theorem

A linear system is consistent if the rightmost column of echelon form of the augmented matrix is not a pivot column.

Row Reduction Algorithm

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system, leading to a general solution of a system.

1. Forward Phase (reducing a matrix to echelon form)

(a) Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
(b) Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
(c) Use row replacement operations to create zeros in all positions below the pivot.
(d) Ignore the row containing the pivot position and all rows above it.
(e) Repeat until there are no more nonzero rows to modify.

2. Backward Phase (reducing a matrix to reduced echelon form)

   (a) Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Span (Linear Combination)

- The span of two vectors, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, represents all vectors that can be reached by scaling and adding the two vectors.
- If the system consisting of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{b}$ is consistent, then $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- A matrix can only span $\mathbb{R}^n$ if it has pivot positions in n rows.

Matrix Equation $A\mathbf{x} = b$

$A\mathbf{x} = b$ can be represented as a vector or matrix equation.

$$ax_1 + bx_2 + cx_3 = d$$
$$ex_1 + fx_2 + gx_3 = h$$

Vector Equation:

$$x_1 \begin{bmatrix} a \\ e \end{bmatrix} + x_2 \begin{bmatrix} b \\ f \end{bmatrix} + x_3 \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} d \\ h \end{bmatrix}$$

Matrix Equation:

$$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d \\ h \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Homogeneous Equation

A homogeneous equation is a linear equation in the form $Ax = 0$

1. The homogeneous equation always has at least one solution (the trivial solution), where $\mathbf{x} = \mathbf{0}$.
2. The homogeneous equation has a nontrivial solution if the equation has at least one free variable.
3. If the matrix $\mathbf{A}$ has more columns than rows ($n > m$), the system often has infinitely many solutions.
4. If $\mathbf{A}$ has n pivot columns, the columns of $\mathbf{A}$ are linearly independent, since every variable is a basic variable.

Parametric Vector Equation

The equation can be represented in parametric vector form if there is a free variable so that all of the other variables are represented in terms of the free variable. For example, if x_3 is a free variable in $\mathbb{R}^3$,

$$x = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

where $x_1 = ax_3 + c$ and $x_2 = bx_3 + d$.

$x = \begin{bmatrix} a + bx_3 \\ c + dx_3 \\ e + fx_3 \end{bmatrix}$, geometrically describes a line through $\begin{bmatrix} a \\ c \\ e \end{bmatrix}$ parallel to $\begin{bmatrix} b \\ d \\ f \end{bmatrix}$

The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$, only when the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and there exists a vector $\mathbf{p}$ such that $\mathbf{p}$ is a solution.

Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = 0$ has only the trivial solution.

- If a set $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.
- Two vectors are linearly dependent if they live on a line through the origin.

Linear Transformations

A linear transformation is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. For a transformation to be linear, then:

- $T(\mathbf{0}) = 0$.
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

Key Points

- If A is a $m \times n$ matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^n$ and the range of T is the solution set.
- A mapping T: $\mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unqiue vector in $\mathbb{R}^m$. Also, T is one-to-one if and only if the columns of A are linearly independent.
- A mapping T: $\mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is mapped onto by some vector $\mathbf{x}$ in $\mathbb{R}^n$. Also, T is onto $\mathbb{R}^m$ if there are n pivot columns.

Standard matrix of T to rotate points about the origin by θ: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
If T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique matrix A such that the following equation is true.

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector, $T(e_j)$, where e_j is the jth column of the identity matrix in $\mathbb{R}^n$, as shown in the following equation.

$$A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$$

Matrix Multiplication

The definition of matrix multiplication states that if A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \cdots, \mathbf{b}_p$, then $AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \end{bmatrix} = \begin{bmatrix} ag+bi & ah+bj \\ cg+di & ch+dj \\ eg+fi & eh+fj \end{bmatrix}$$

The (i, j) entry in AB is

$$\sum_{k=1}^{n} a_{\text{ik}} b_{\text{kj}} \quad \text{or} \quad a_{\text{i1}} b_{\text{1j}} + \cdots + a_{\text{in}} b_{\text{nj}}$$

$$(AB)^T = B^T A^T$$

Key Points

- Matrix multiplication is only allowed for matrices where the number of columns in the first matrix equal the number of rows in the second matrix.
- If A is a $a \times b$ matrix and B is a $b \times c$ matrix, then AB is a $a \times c$ matrix.
- Given a square $n \times n$ matrix A, there are infinite matrices for B in $AB = BA$ because any multiple of I_n will satisfy the equation.
- The definition of AB states that each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Inverse Matrices

$$A \times A^{-1} = I$$

where A is an invertible matrix, A^{-1} is the inverse of A, and I is the identity matrix with the same shape as A. Calculating the inverse of a 2×2 square matrix.

Calculating the Inverse of a Matrix

1. Start with a square matrix A.
2. Reduce the matrix $\begin{bmatrix} A & I \end{bmatrix}$ (where I is the identity matrix with the same shape as A) to reduced echelon form which will result in $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

If A is a 2×2 matrix, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Invertible Matrix Theorem

The following statements are either all true or all false

1. A is an invertible matrix.
2. A has n pivot positions.
3. The columns of A form a linearly independent set.
4. The columns of A span $\mathbb{R}^n$.
5. There are $n \times n$ matrices C and D such that $CA = I$ and $AD = I$.

The deteminant of a standard $n \times n$ matrix is $ad - bc$.

Key Points

- $(AB)^{-1}$ is $B^{-1}A^{-1}$.
- If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^n$ because $A^{-1}\mathbf{b}$ exists for all $\mathbf{b}$ in $\mathbb{R}^n$ and $\mathbf{x} = A^{-1}\mathbf{b}$.
- The columns of an $n \times n$ matrix A are linearly independent when A is invertible because the equation $A\mathbf{x} = 0$ has the unique solution $\mathbf{x} = 0$.
- The product of invertible matrices is also invertible.
- If A and B are $n \times n$ matrices and AB is invertible, then B is also invertible

Other

1. It is not possible that $CA = I_4$ for some 4×2 matrix C when A is a 2×4 matrix because if it were true, then $CA\mathbf{x}$ would equal $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. Since the columns of A are linearly dependent as there are more columns than rows, $A\mathbf{x} = 0$ for some nonzero vector $\mathbf{x}$, so $\mathbf{x} = I\mathbf{x} \neq CA\mathbf{x} = 0$.