# Iterated Integration

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## Volumes and Double Integrals

The **double integral** of the function $f(x, y)$ over the rectangular region $R$ in the $xy$-plane is defined as

$$\iint\limits_R f(x, y)\, dA = \lim_{m,n \to \inf} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

### Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region $R$; $S$ and $T$ are subregions of $R$; and assume that $m$ and $M$ are real numbers.

1. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint\limits_R [f(x, y) + g(x, y)]\, dA = \iint\limits_R f(x, y)\, dA + \iint\limits_R g(x, y)\, dA$$

2. If $c$ is a constant, then $cf(x, y)$ is integrable and

$$\iint\limits_R cf(x, y)\, dA = c \iint\limits_R f(x, y)\, dA$$

3. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x, y)\, dA + \iint\limits_T f(x, y)\, dA$$

4. If $f(x, y) \geq g(x, y)$ for $(x, y)$ in $R$, then

$$\iint\limits_R f(x, y)\, dA \geq \iint\limits_R g(x, y)\, dA$$

5. If $m \leq f(x, y) \leq M$, then

$$m \times A(R) \leq \iint\limits_R f(x, y)\, dA \leq M \times A(R)$$

6. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of $x$ only and a function $h(y)$ of $y$ only, then over the region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, the double integral can be written as

$$\iint\limits_R f(x, y)\, dA = \left( \int_a^b g(x)\, dx \right) \left( \int_c^d h(y)\, dy \right)$$

## Iterated Integrals

Assume $a$, $b$, $c$, and $d$ are real numbers. We define an **iterated integral** for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

1. $$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[ \int_c^d f(x,y)\,dy \right] dx$$

2. $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[ \int_a^b f(x,y)\,dx \right] dy$$

## Tubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$. Then the double integral of $f$ over the regions is

$$\iint\limits_R f(x,y)\,dA = \iint\limits_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

## Applications of Double Integrals

The area of the region $R$ is given by $A(R) = \iint\limits_R 1\,dA$.

The average value of a function of two variables over a region $R$ is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint\limits_R f(x,y)\,dA$$