Calculus of Vector-Valued Functions

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Derivatives of Vector-Valued Functions

The **derivative of a vector-valued function** $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

provided the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is differentiable at t. If $\mathbf{r}'(t)$ exists for all t in an open interval (a, b), then $\mathbf{r}$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0^+} \frac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t} \quad \text{and} \quad \mathbf{r}'(b) = \lim_{\Delta t \to 0^-} \frac{\mathbf{r}(b+\Delta t) - \mathbf{r}(b)}{\Delta t}$$

Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}$.
2. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$.

Properties of the Derivative of Vector-Valued Functions

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

$$\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t) \qquad \text{Scalar Multiple}$$

$$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t) \qquad \text{Sum and Difference}$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \qquad \text{Scalar Product}$$

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t) \qquad \text{Dot Product}$$

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t) \qquad \text{Cross Product}$$

$$\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t)) \cdot f'(t) \qquad \text{Chain Rule}$$

$$\text{If } \mathbf{r}(t) \cdot \mathbf{r}(t) = c, \text{ then } \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

Tangent Vectors

Let C be a curve defined by a vector-valued function $\mathbf{r}$, and assume that $\mathbf{r}'(t)$ exists when $t = t_0$. A tangent vector $\mathbf{v}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector $\mathbf{v}$ is tangent to curve C. Vector $\mathbf{r}'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\mathbf{r}'(t) \neq \mathbf{0}$. The **principal unit tangent vector** at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \text{if} \quad \|\mathbf{r}'(t)\| \neq \mathbf{0}$$

Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}$$

2. The **definite integral of a vector-valued function** is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}$$