Chapter 6 Symmetric Matrices and Quadratic Forms

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Diagonalization of Symmetric Matrices

A **symmetric** matrix is a matrix A such that $A^T = A$.

Theorem 1

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Theorem 2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

Theorem 3 — The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicities.
2. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
4. A is orthogonally diagonalizable.

Spectral Decomposition

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

Key Points

1. A matrix U is orthogonal if $U^T U = I$, and if so, $U^T = U^{-1}$.
2. A matrix A can be orthogonally diagonalized by finding the n eigenvalues and forming D as a diagonal matrix of the eigenvalues and P as the normalized orthogonal eigenvectors for the eigenvalues. (Use Gram-Schmidt Process to form orthogonal basis from eigenvectors).

3. Multiplying a column vector u of $\mathbb{R}^n$ on the right by $u^T x$ is the same as multiplying the column vector by the scalar $u \cdot x$.