Limits and Continuity

David Robinson

Limit of a Function of Two Variables

A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b).

$$\{(x, y) \in \mathbb{R}^2 | (x-a)^2 + (y-b)^2 < \delta^2\}$$

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L.

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

Limit Laws

Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$, and let c be a constant.

1. **Constant Law**
$$\lim_{(x,y)\to(a,b)} c = c$$
2. **Identity Laws**
$$\lim_{(x,y)\to(a,b)} x = a$$
$$\lim_{(x,y)\to(a,b)} y = b$$
3. **Sum Law**
$$\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$$
4. **Difference Law**
$$\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$$
5. **Constant Multiple Law**
$$\lim_{(x,y)\to(a,b)} (cf(x, y)) = cL$$
6. **Product Law**
$$\lim_{(x,y)\to(a,b)} (f(x, y)g(x, y)) = LM$$
7. **Quotient Law**
$$\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad \text{for} \quad M \neq 0$$
8. **Power Law**
$$\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$$
for any positive integer n.
9. **Root Law**
$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x, y)} = \sqrt[n]{L}$$
for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive provided that $f(x, y) \geq 0$ for all $(x, y) \neq (a, b)$ in neighborhood of (a, b).

Interior and Boundary Points

Let S be a subset of $\mathbb{R}^2$.

A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S. A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.

S is called an **open set** if every point of S is an interior point. S is called a **closed set** if it contains all its boundary points.

An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets. A set S is a **region** if it is open, connected, and nonempty.

Continuity of Functions of Two Variables

A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists.
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Continuity Laws

1. If $f(x, y)$ is continuous at (x_0, y_0), and $g(x, y)$ is continuous at (x_0, y_0), then $f(x, y)+g(x, y)$ is continuous at (x_0, y_0).
2. If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).
3. Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq \mathbb{R}$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and defined $z_0 = g(x_0, y_0)$. Let f be a function that maps $\mathbb{R}$ to $\mathbb{R}$ such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0).

Functions of Three or More Variables

Let (x_0, y_0, z_0) be a point in $\mathbb{R}^3$. Then a δ **ball** in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0).

$$\left\{(x, y, z) \in \mathbb{R}^3 \middle| \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\}$$

To define a δ ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension.

Key Points

1. If the limit along different paths through a point have different values, then the limit does not exist at that point.