Chain Rule

David Robinson

Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule for Two Independent Variables

Suppose that $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(u, b), y(u, b))$ is a differentiable function of u and v, and

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{dx}{du} + \frac{\partial z}{\partial y} \cdot \frac{dy}{du} \quad \text{and} \quad \frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dv}$$

Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x, y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x, y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} \quad \text{if } f_y(x, y) \neq 0$$

If the equation $f(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z} \quad \text{if } f_z(x, y, z) \neq 0$$