# Chapter 3 Determinants

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## Introduction to Determinants

A $n \times n$ matrix is invertible if and only if its determinant is nonzero.
For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is the sum of $n$ terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of $A$.

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

**Theorem 1**

The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column. The expansion across the $i$th row using the cofactor, $C_{ij} = (-1)^{i+j} \det A_{ij}$, is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the $j$th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

**Theorem 2**

If $A$ is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of $A$.

# Properties of Determinants

**Theorem 3 — Row Operations**

Let $A$ be a square matrix.

- If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det B = \det A$.
- If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.
- If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \det A$.

**Theorem 4**

A square matrix $A$ is invertible if and only if $\det A \neq 0$.

**Theorem 5**

If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.

**Theorem 6 — Multiplicative Property**

If $A$ and $B$ are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

## Key Notes

1. $\det A^n = (\det A)^n$
2. $\det(rA) = r^n \cdot \det A$