Vector-Valued Functions and Space Curves

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A **vector-valued function** is a function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where the **component functions** f, g, and h, are real-valued functions of the parameter g. Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

A two-dimensional vector-valued function traces a **plane curve**, while a three-dimensional vector-valued function traces a **space curve**.

Limits and Continuity of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L} \quad \text{provided} \quad \lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$$

Let f, g, and h be functions of t. Then, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at point $t = a$ if the following three conditions hold:

1. $\mathbf{r}(a)$ exists
2. $\lim_{t \to a} \mathbf{r}(t)$ exists
3. $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

The curve defined by the vector-valued function $\mathbf{r}(t) = (at + b)\mathbf{i} + (ct + d)\mathbf{j} + (et + f)\mathbf{k}$ is the line in space with the direction vector $\langle a, c, e \rangle$.