Directional Derivatives and Gradients

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Directional Derivatives

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist and $f(x, y)$ is differentiable everywhere. Then, the directional derivative of f in the direction of $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is given by

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Let $f(x, y, z)$ be a differentiable function of three variables and let $\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of $\mathbf{u}$ is given by

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u} = f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma$$

Gradients

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. The vector $\nabla f(x, y)$ is called the **gradient** of f and is defined as

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0).

1. If $\nabla f(x_0, y_0) = 0$, then $D_u f(x_0, y_0) = 0$ for any unit vector u.
2. If $\nabla f(x_0, y_0) \neq 0$, then $D_u f(x_0, y_0)$ is maximized when u points in the same direction as $\nabla f(x_0, y_0)$. The maximum value of $D_u f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$.
3. If $\nabla f(x_0, y_0) \neq 0$, then $D_u f(x_0, y_0)$ is minimized when u points in the opposite direction from $\nabla f(x_0, y_0)$. The minimum value of $D_u f(x_0, y_0)$ is $-\|\nabla f(x_0, y_0)\|$.

Level Curves

Suppose the function $z = f(x, y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0, y_0). If $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level curve of f at (x_0, y_0).