Double Integrals

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General Regions of Integration

A region D in the (x, y)-plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$.

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

A region D in the (x, y)-plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Double Integrals over Nonrectangular Regions

Suppose $g(x, y)$ is the extension to the rectangle R of the integrable function $f(x, y)$ defined on the region D, where D is inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by

$$\iint\limits_D f(x, y)\, dA = \iint\limits_R g(x, y)\, dA$$

Fubini's Theorem

For a function $f(x, y)$ that is continuous on a region D of Type I, we have

$$\iint\limits_D f(x, y)\, dA = \iint\limits_D f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx$$

Similarly, for a function $f(x, y)$ that is continuous on a region D of Type II, we have

$$\iint\limits_D f(x, y)\, dA = \iint\limits_D f(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy$$

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint\limits_D f(x, y)\, dA = \iint\limits_{D_1} f(x, y)\, dA + \iint\limits_{D_2} f(x, y)\, dA$$

The area of a plane-bounded region D is defined as the double integral $\iint\limits_D 1\, dA$.

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{\text{ave}} = \frac{1}{A(D)} \iint\limits_D f(x, y)\, dA$$

Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by $\{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$ and also by $\{(x, y) : c \leq y \leq d, j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D, then

$$\iint\limits_D f\, dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x, y)\, dy\, dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x, y)\, dx\, dy$$

Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x, y) : a \leq x < \inf, c \leq y < \inf\}$, then when the limit exists, we have

$$\iint\limits_R f(x,y)\,dA = \lim_{(b,d)\to(\inf,\inf)} \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \lim_{(b,d)\to(\inf,\inf)} \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Consider a pair of continuous random variables X and Y, such as the birthdays of two people or the number of sunny and rainy days in a month. The joint density function f of X and Y satisfies the probability that (X, y) lies in a certain region D:

$$P((X,Y) \in D) = \iint\limits_D f(x,y)\,dA$$

Since the probabilities can never be negative and must lie between 0 and 1, the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \quad \text{and} \quad \iint\limits_{R^2} f(x,y)\,dA = 1$$

The variables X and Y are said to be independent random variables if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x) f_2(y)$$

In probability theory, we denote the ex pected values $E(X)$ and $E(Y)$, respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint\limits_S x f(x,y)\,dA \quad \text{and} \quad E(Y) = \iint\limits_S y f(x,y)\,dA$$

where S is the sample sapce of the random variables X and Y.