Chapter 5 Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Validating an Eigenvalue

1. Start with the equation $A\mathbf{x} = \lambda\mathbf{x}$
2. Form the matrix $A - \lambda I$
3. If the columns are linearly dependent, λ is an eigenvalue
4. Reduce the matrix to reduced echelon form and each column vector in terms of the free variables is a corresponding eigenvector and a part of the basis for the eigenspace

Validating an Eigenvector

1. Start with the equation $A\mathbf{x} = \lambda\mathbf{x}$
2. Compute the product of $A\mathbf{x}$
3. If $A\mathbf{x}$ is proportional to $\mathbf{x}$, then $\mathbf{x}$ is an eigenvector and the scaling factor is the eigenvalue

Key Points

- If the columns of A are linearly dependent, one eigenvalue of A is $\lambda = 0$
- If A is the zero matrix, then the only eigenvalue of A is 0

The Characteristic Equation

The scalar equation $\det(A - \lambda I) = 0$ is called the characteristic equation. The characteristic polynomial is the simplified polynomial in the characteristic equation and the eigenvalues are the values for λ.

Theorem 3 — Properties of Determinants

Let A and B be $n \times n$ matrices.

1. A is invertible if and only if $\det A \neq 0$.
2. $\det AB = (\det A)(\det B)$.
3. $\det A^T = \det A$.
4. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.
5. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if, the number 0 is **not** an eigenvalue of A.

Theorem 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Diagonalization

Theorem 5 — The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$, known as an eigenvector basis of $\mathbb{R}^n$.

Theorem 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
2. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if the characteristic polynomial factors completely into linear factors and the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.
3. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Key Points

- $A = PDP^{-1}, A^2 = PD^2P^{-1}, \ldots, A^k = PD^kP^{-1}$
- If A is both diagonalizable and invertible, then so is A^{-1}.
    1. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D.
    2. Zero is not an eigenvalue of A, so the diagonal entries in D are not zero, so D is invertible.
    3. $A^{-1} = PD^{-1}P^{-1}$, therefore A^{-1} is also diagonalizable.