Chapter 3 Determinants

David Robinson

Introduction to Determinants

A $n \times n$ matrix is invertible if and only if its determinant is nonzero.
For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Theorem 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactor, $C_{ij} = (-1)^{i+j} \det A_{ij}$, is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Properties of Determinants

Theorem 3 — Row Operations

Let A be a square matrix.

- If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
- If two rows of A are interchanged to produce B, then $\det B = -\det A$.
- If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Theorem 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 5

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 6 — Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Key Notes

1. $\det A^n = (\det A)^n$
2. $\det(rA) = r^n \cdot \det A$
3. If A is a $n \times n$ matrix with values in the range $[-p, p]$, the range of the possible determinants is $[-np^n, np^n]$
4. If the determinant of a matrix of a set of vectors is 0, the set of vectors is linearly dependent