Chapter 7 Symmetric Matrices and Quadratic Forms

David Robinson

Diagonalization of Symmetric Matrices

A **symmetric** matrix is a matrix A such that $A^T = A$.

Theorem 1

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Theorem 2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

Theorem 3 — The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicities.
2. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
4. A is orthogonally diagonalizable.

Spectral Decomposition

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

Key Points

1. A matrix U is orthogonal if $U^T U = I$, and if so, $U^T = U^{-1}$.

2. A matrix A can be orthogonally diagonalized by finding the n eigenvalues and forming D as a diagonal matrix of the eigenvalues and P as the normalized orthogonal eigenvectors for the eigenvalues. (Use Gram-Schmidt Process to form orthogonal basis from eigenvectors).

3. Multiplying a column vector u of $\mathbb{R}^n$ on the right by $u^T x$ is the same as multiplying the column vector by the scalar $u \cdot x$.

The Singular Value Decomposition

Let A be an $m \times n$ matrix. The singular values of A are the square roots of the eigenvalues of $A^T A$, and they are arranged in decreasing order.

Theorem 9

Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy $\lambda_1 \geq \cdots \geq \lambda_n$, and suppose A has r nonzero singular values. Then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for Col A, and rank $A = r$.

Theorem 10 — The Singular Value Decomposition

Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ as in

$$\Sigma = \begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T$$

1. $U = \{u_1, \ldots, u_r\}$ where $u_i = \frac{1}{\sigma_i} A v_i$.
2. V contains the unit eigenvectors for $A^T A$.
3. $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq n$

The Invertible Matrix Theorem (concluded)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:

1. $(\text{Col } A)^\perp = \{\mathbf{0}\}$.
2. $(\text{Nul } A)^\perp = \mathbb{R}^n$.
3. Row $A = \mathbb{R}^n$.
4. A has n nonzero singular values.

Key Points

1. Let A be an $m \times n$ matrix. If there are more columns than rows, $m < n$, then the singular value decomposition can be solved by first calculating the singular value decomposition of A^T. $A^T = U\Sigma V^T \rightarrow (A^T)^T = (U\Sigma V^T)^T = V\Sigma^T U^T$.

2. $\det A = (\det U)(\det \Sigma)(\det V^T)$.