Matrices

A matrix is in **echelon form** if:

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

A matrix is in **reduced echelon form** if:

1. It is in echelon form
2. The leading entry in each nonzero row is 1

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is, if and only if an echelon form of the augmented matrix has no row of the form $\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$ with b being nonzero.

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbb{R}^m$
4. A has a pivot position in every row

Homogeneous Equation

A linear equation in the form $Ax = 0$ where:

- **A** is an $m \times n$ matrix
- $\mathbf{x}$ is a vector in $\mathbb{R}^n$
- $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$

Properties

1. The homogeneous equation always has at least one solution (the trivial solution), where $\mathbf{x} = \mathbf{0}$
2. If the matrix $\mathbf{A}$ has more columns than rows ($n > m$), the system often has infinitely many solutions
3. If $\mathbf{A}$ has n pivot columns, the columns of $\mathbf{A}$ are linearly independent, since every variable is a basic variable

If T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique matrix A such that the following equation is true.

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$, as shown in the equation, $A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$

The mapping T: $\mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

The mapping T: $\mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.