Chapter 4 Vector Spaces

David Robinson

Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Validating an Eigenvalue

1. Start with the equation $A\mathbf{x} = \lambda\mathbf{x}$
2. Form the matrix $A - \lambda I$
3. If the columns are linearly dependent, λ is an eigenvalue
4. Reduce the matrix to reduced echelon form and each column vector in terms of the free variables is a corresponding eigenvector and a part of the basis for the eigenspace

Validating an Eigenvector

1. Start with the equation $A\mathbf{x} = \lambda\mathbf{x}$
2. Compute the product of $A\mathbf{x}$
3. If $A\mathbf{x}$ is proportional to $\mathbf{x}$, then $\mathbf{x}$ is an eigenvector and the scaling factor is the eigenvalue

Key Points

- If the columns of A are linearly dependent, one eigenvalue of A is $\lambda = 0$
- If A is the zero matrix, then the only eigenvalue of A is 0