Arc Length and Curvature

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Arc-Length Formulas

1. **Plane curve**: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt$$

2. **Space curve**: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt$$

Arc-Length Function

Let $\mathbf{r}(t)$ describes a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du$$

$\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$. If $\|\mathbf{r}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

Curvature

Let C be a smooth curve in the plane or in space given by $\mathbf{r}(s)$, where s is the arc-length parameter. The **curvature** κ at s is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|$$

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature at t is

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

If C is a three-dimensional curve, then the curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

If C is a graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature at point (x, y) is

$$\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{\frac{3}{2}}}$$

If C is a curve described by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the curvature is

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}$$

Normal and Binormal Vectors

Let C be a three-dimensional **smooth** curve represented by $\mathbf{r}$ over an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the principal unit normal vector at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

The binormal vector at t is defined as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

where $\mathbf{T}(t)$ is the unit tangent vector.

Equations of the Curve Planes

1. The orthogonal vector to the normal plane is $\mathbf{T}(t)$
2. The orthogonal vector to the osculating plane is $\mathbf{B}(t)$