Line Integrals

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A **line integral** integrates multivariable functions and vector fields over arbitrary curves in a plane or in space.

Scalar Line Integrals

A **scalar line integral** is an integral of a scalar function over a curve in a plane or in space. Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, a \leq t \leq b$. The scalar line integral of f along C is

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i$$

if that limit exists, where $n + 1$ points $P_0, \ldots, P_n$ divide curve C into n pieces, with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.

If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$,$y = y(t)$, and $a \leq t \leq b$. If C is smooth and $f(x, y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i$$

if that limit exists.

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t), a \leq t \leq b$. Then

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt$$

Vector Line Integrals

A **vector line integral** is an integral of a vector field over a curve in a place or in space. The vector line integral of vector field F along oriented smooth curve C is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*)\, \Delta s_i$$

if that limit exists.

The work required to move an object along a curve C in a force field F is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{r}(t), a \leq t \leq b$, is a parameterization of C.

Let $\mathbf{F}$ and $\mathbf{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$
2. $\int_C k\mathbf{F} \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$, where k is a constant

3. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$

4. Suppose instead that C is a piecewise smooth curve in the domains of $\mathbf{F}$ and $\mathbf{F}$ and $\mathbf{G}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}$$

Flux and Circulation

The **flux** of $\mathbf{F}$ across C is line integral $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\, ds$.

Let $\mathbf{F}$ be a vector field and let C be a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle, a \leq t \leq b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)\, dt$$

The **circulation** of $\mathbf{F}$ along C is the line integral of $\mathbf{F}$ along an oriented closed curve.