Extrema

David Robinson

Critical Points

Let $z = f(x, y)$ be a function of two variables that is defined on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a **critical point of a function of two variables** f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a global maximum at (x_0, y_0).

The function f has a **local minimum** at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a global minimum at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a local extremum.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
3. If $D < 0$, then f has a saddle point at (x_0, y_0).
4. If $D = 0$, then the test is inconclusive.

Absolute Maxima and Minima

A continuous function $f(x, y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed and bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

Finding Critical Points on Boundary

Consider a line segment connecting (a, c) and (b, c). It can be parameterized by the equations $x(t) = t$, $y(t) = c$ for $a \leq t \leq b$. Define $g(t) = f(x(t), y(t))$. The critical points are where $g'(t) = 0$.