Triple Integrals in Cylindrical and Spherical Coordinates

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Integration in Cylindrical Coordinates

Consider the cylindrical box

$$B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k]$, then we can define the **triple integral in cylindrical coordinates** as the limit of a triple Riemann sum, provided the following limit exists.

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z$$

0.1 Fubini's Theorem for Cylindrical Coordinates

Suppose that $g(x, y, z)$ is continuous on a portion of a circular cylinder B, which when described in cylindrical coordinates looks like $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$.

Then $g(x, y, z) = g(r \cos\theta, r \sin\theta, z) = f(r, \theta, z)$ and

$$\iiint\limits_{B} g(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(r, \theta, z) r\, dr\, d\theta\, dz$$

1 Integration in Spherical Coordinates

$$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$$

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum if the limit exists.

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk}) (\rho^*_{ijk})^2 \sin\varphi^*_{ijk} \Delta\rho \Delta\theta \Delta\varphi$$

1.1 Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint\limits_{B} f(\rho, \theta, \varphi) \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi) \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$