Vector Fields

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Vector Field

A **vector field F** in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

In a **radial field**, all vectors either point directly toward or directly away from the origin. In a **rotational field**, the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$.

Gradient Fields

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a **gradient field**, also called a conservative field, if there exists a scalar function f such that $\nabla f = F$.

Uniqueness of Potential Functions

Let $\mathbf{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \mathbf{F}$ and $\nabla g = \mathbf{G}$. Then, there is a constant C such that $f = g + C$.

The Cross-Partial Property of Conservative Vector Fields

Let $\mathbf{F}$ be a vector field in two or three dimensions such that the component functions of $\mathbf{F}$ have continuous first-order partial derivatives on the domain of $\mathbf{F}$.

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$