

Line Integrals

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A **line integral** integrates multivariable functions and vector fields over arbitrary curves in a plane or in space.

Scalar Line Integrals

A **scalar line integral** is an integral of a scalar function over a curve in a plane or in space. Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. The scalar line integral of f along C is

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(P_i^*) \Delta s_i$$

if that limit exists, where $n + 1$ points P_0, \dots, P_n divide curve C into n pieces, with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.

If C is a planar curve, then C can be represented by the parametric equations $x = x(t), y = y(t)$, and $a \leq t \leq b$. If C is smooth and $f(x, y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(P_i^*) \Delta s_i$$

if that limit exists.

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t)$, $a \leq t \leq b$. Then

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$

Vector Line Integrals

A **vector line integral** is an integral of a vector field over a curve in a plane or in space. The vector line integral of vector field \mathbf{F} along oriented smooth curve C is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \Delta s_i$$

if that limit exists.

Let \mathbf{F} and \mathbf{G} be continuous vector fields with domains that include the oriented smooth curve C . Then

1. $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$
2. $\int_C k\mathbf{F} \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$, where k is a constant
3. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}$
4. Suppose instead that C is a piecewise smooth curve in the domains of \mathbf{F} and \mathbf{F} and \mathbf{G} , where $C = C_1 + C_2 + \dots + C_n$ and C_1, C_2, \dots, C_n are smooth curves such that the endpoint of C_i is the starting point of C_{i+1} . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \dots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}$$

Flux and Circulation

The **flux** of \mathbf{F} across C is line integral $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|} ds$.

Let \mathbf{F} be a vector field and let C be a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of \mathbf{F} across C is

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t) dt$$

The **circulation** of \mathbf{F} along C is the line integral of \mathbf{F} along an oriented closed curve.