Informal Derivation of Euler's Identity

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> *"Who would have thought that π which enters as the ratio of circumference to diameter, e, as the natural base for logarithms, i, as the fundamental imaginary unit and 0 and 1 (which we know all about from infancy) would all be tied together in any way, not to mention such a simple and compact way? I hope I never stumble into anything like this formula, for nothing I do after that in life would have any significance."*
>
> — Ramamurti Shankar

The Nobel Prize winning physicist Richard Feynman called equation (1), known as Euler's identity, "one of the most remarkable, almost astounding, formulas in all of mathematics.

$$e^{i\pi} + 1 = 0 \tag{1}$$

Euler's identity can be derived easily from the Taylor series...

$$f\left(x\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(0\right)}{n!} x^n \tag{2}$$

...as well as defining e^{ax} as

$$e^{ax} = \sum_{n=0}^{\infty} \frac{\left(ax\right)^n}{n!} \tag{3}$$

In equation (2), $f^(n)$ represents the n-th derivative of the function f, with the zeroth derivative of f being equal to f.

The Taylor series of $cos(x)$ can be easily derived from equation (2).

$$\begin{aligned}
&f^0\left(x\right) = f\left(x\right) = \cos x\\
&f^0\left(0\right) = f\left(x\right) = \cos 0 = 1\\
&f^1\left(x\right) = \tfrac{df(x)}{dx} = \tfrac{d}{dx}\cos x = -\sin x\\
&f^1\left(0\right) = -\sin 0 = 0\\
&f^2\left(x\right) = \tfrac{d^2f(x)}{dx} = -\tfrac{d}{dx}\sin x = -\cos x\\
&f^2\left(0\right) = -\cos 0 = -1\\
&f^3\left(x\right) = -\tfrac{d}{dx}\cos x = \sin x\\
&f^3\left(0\right) = \sin 0 = 0\\
&f^4\left(x\right) = \tfrac{d}{dx}\sin x = \cos x\\
&f^4\left(0\right) = \cos 0 = 1
\end{aligned}$$

From the equations above, as well at equation (2), it follows that:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = x^{2n} \tag{4}$$

The Taylor series of $\sin x$ can also be derived equation from equation (2).

$$\begin{aligned}
&f^0(x) = f(x) = \sin x \\
&f^0(0) = \sin 0 = 0 \\
&f^1(x) = \tfrac{d}{dx}\sin x = \cos x \\
&f^1(0) = \cos 0 = 1 \\
&f^2(x) = \tfrac{d}{dx}\cos x = -\sin x \\
&f^2(0) = -\sin 0 = 0 \\
&f^3(x) = -\tfrac{d}{dx}\sin x = -\cos x \\
&f^3(0) = -\cos 0 = -1 \\
&f^4(x) = -\tfrac{d}{dx}\cos x = \sin x \\
&f^4(0) = \sin 0 = 0 \\
&f^5(x) = \tfrac{d}{dx}\sin x = \cos x \\
&f^5(0) = \cos 0 = 1
\end{aligned}$$

From the equations above and equation (2), it follows that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{5}$$

From equations (3), (4), and (5), if θ is a real number:

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \tfrac{(i\theta)^n}{n!} = 1 + i\theta + \underbrace{i^2}_{-1} \tfrac{\theta^2}{2!} + \underbrace{i^3}_{-i} \tfrac{\theta^3}{3!} + \underbrace{i^4}_{1} \tfrac{\theta^4}{4!} + \underbrace{i^5}_{i} \tfrac{\theta^5}{5!} - \dots \\
e^{i\theta} &= 1 + i\theta - \tfrac{\theta^2}{2!} - \tfrac{i\theta^3}{3!} + \tfrac{\theta^4}{4!} + \tfrac{i\theta^5}{5!} - \dots \\
e^{i\theta} &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos\theta} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin\theta} = \cos\theta + i\sin\theta
\end{aligned} \tag{6}$$

Finally, let $\theta = \pi$. Equation (6) becomes:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

And finally:

$$\boxed{e^{i\pi} + 1 = 0}$$

References

1. Ramamurti Shankar, *Principles of Quantum Mechanics*. Springer, 1994.
2. James Stewart, *Calculus: Early Transcendentals*. Cengage Learning, 2012.