Intuitive "Proof" Of The Associative Law of Matrix Multiplication

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Letting A be a $m \times n$ matrix, it follows that B must have n rows. So letting B be a $n \times p$ matrix, (AB) will be an $m \times p$ matrix. For (AB)C to be allowed, C must have p rows, so let C be a $p \times r$ matrix.

C is a $p \times r$ matrix, B must have p columns. So (again) letting B be an $n \times p$ matrix, BC will be an $n \times p$ matrix. For A(BC) to be allowed A must have n columns. So (again) letting A be an $m \times n$ matrix, A(BC) will be an $m \times p$ matrix.

For (AB)C and A(BC) to have the same shape, A must have the same number of columns as B has rows, and C must have the same number of rows as B has columns. When this is the case it will be true (as shown below) that $A(BC) = (AB)C$. **All we have to do is** prove that an arbitrary column of (AB)C will be equal to the same arbitrary column in A(BC) (I'll call this column the jth column of both matrices.)

Notation: K_i represents the ith column of the matrix K.

$$B = \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \underbrace{c_{p1}}_{C_1} & \vdots & \underbrace{c_{pj}}_{C_j} & \cdots & \underbrace{c_{pr}}_{C_R} \end{bmatrix}$$

$$AB = A \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix} = \begin{bmatrix} AB_1 & \cdots & AB_p \end{bmatrix}$$

$$(BC)_j = BC_j = \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix} \begin{bmatrix} c_{1j} \\ \vdots \\ c_{pj} \end{bmatrix} = \begin{bmatrix} c_{1j}B_1 & \cdots & c_{pj}B_p \end{bmatrix}$$

$$((AB)\,C)_j = (AB)_{C_j} = AB \begin{bmatrix} c_{1j} \\ \vdots \\ c_{pj} \end{bmatrix} = \begin{bmatrix} AB_1 & \cdots & AB_p \end{bmatrix} \begin{bmatrix} c_{1j} \\ \vdots \\ c_{pj} \end{bmatrix} = c_{1j}AB_1 + \cdots + c_{pj}AB_p$$

$$\boxed{(A\,(BC))_j} = A\,(BC_j) = A \begin{bmatrix} c_{1j}B_1 & \cdots & c_{pj}B_p \end{bmatrix} = c_{1j}AB_1 + \cdots + c_{pj}AB_p \boxed{= ((AB)C)_j}$$