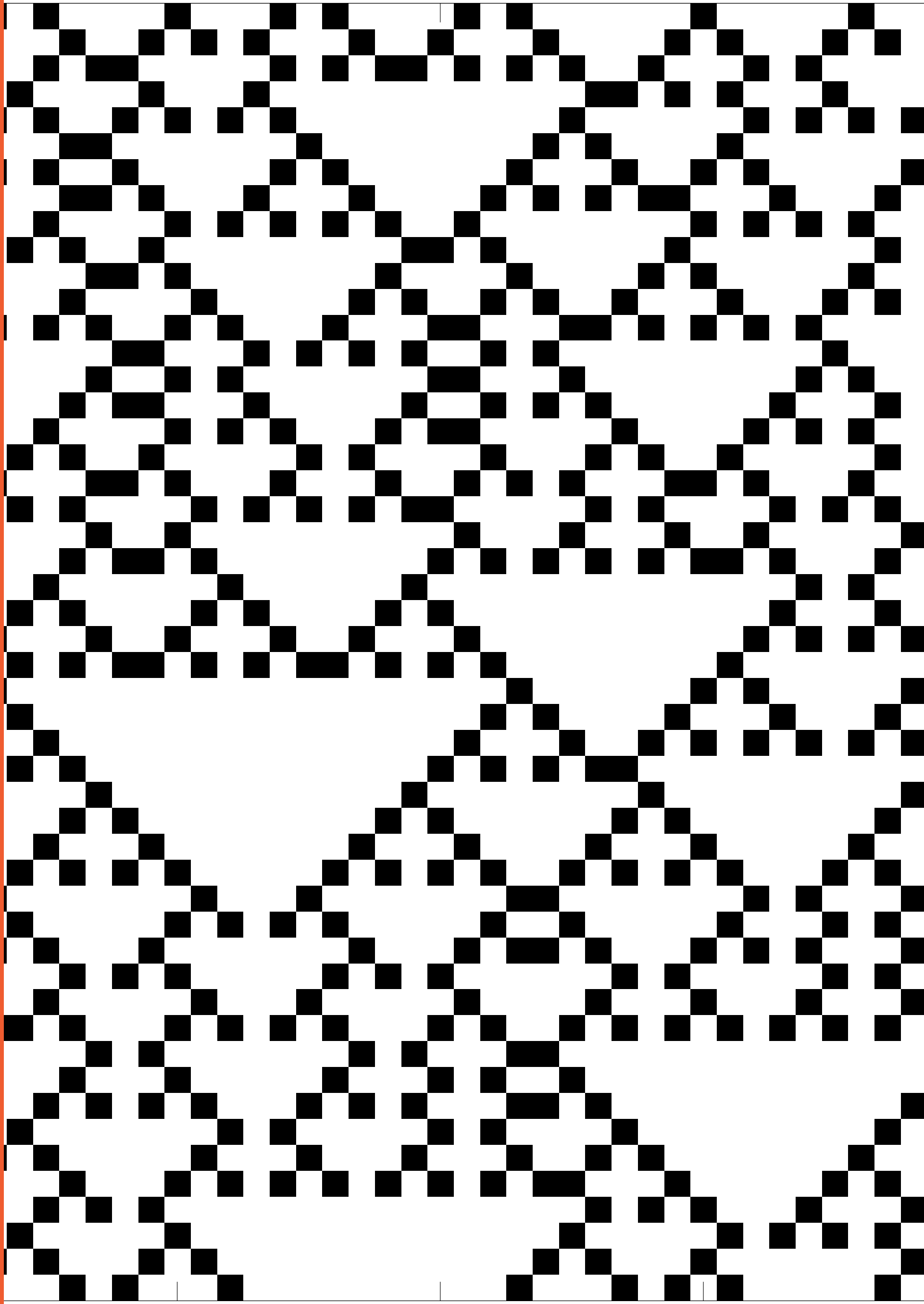
Lecture 2 – What is information?



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What is Information: session outcomes

- Ability to express ideas about information and relate that to uncertainty.
- Understand further fundamental measures of information theory including: entropy, joint entropy, conditional entropy, mutual information, conditional mutual information.
- Ability to partially construct Matlab code to compute such measures, and apply that code to examples.

- Primary references:
  - Cover and Thomas, "Elements of Information Theory", Hoboken, New Jersey: John Wiley and Sons, Inc., 2006 (2nd ed.); section 2.2-2.5, 2.6, 2.8
  - Mackay, "Information Theory, Inference, and Learning Algorithms", Cambridge: Cambridge University Press, 2003; sections 2.6, chapter 8.
  - Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.1-3.2.4.
  - Lizier, "JIDT: An information-theoretic toolkit for studying the dynamics of complex systems", Frontiers in Robotics and AI, 1:11, 2014; Appendix A.1 and A.3

Joint Entropy

- We can consider joint entropy of a multivariate, e.g. $\{X, Y\}$:

$$\mathrm{H}(X,Y) = \sum_{x \in A_x} \sum_{y \in A_y} p(x,y) \log_2 \frac{1}{p(x,y)}$$

$$\mathrm{H}(X,Y) = \sum_{x \in A_x} \sum_{y \in A_y} -p(x,y) \log_2 p(x,y)$$

$$\mathrm{H}(X,Y) = \langle h(x,y) \rangle$$

- Surprise $h(x,y)$ / Uncertainty $H(X,Y)$ for the joint sample $\{x,y\}$
- Same properties as for $H(X)$, only now X is multivariate!
- $H(X,Y) \geq H(X)$
- We refer to H(X) or H(Y) as *marginal* entropies to distinguish them
- Is $H(X, Y) = H(X) + H(Y)$?
  - Only for *independent* variables where p(x, y) = p(x)p(y) !

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.2
D. J. C. MacKay. Information Theory, Inference, and Learning Algorithms. Cambridge University Press, Cambridge, 2003. Section 8.1

Joint Entropy

– We can consider joint entropy of a multivariate, e.g. $\{X, Y\}$:

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$$\mathrm{H}(X,Y) = \sum_{x \in A_x} \sum_{y \in A_y} -p(x,y) \log_2 p(x,y)$$

$$\mathrm{H}(X,Y) = \langle h(x,y) \rangle$$

– Exercise: How to code $H(X,Y)$?

1. Edit the Matlab function `jointentropy(p)` to return the joint Shannon entropy for the joint probability p.
   a. You can assume p is 2D (`p(x,y)`) for now.
   b. Can you simply alter your `entropy(p)` code?
   c. Try some test cases that you come up with yourself.
   d. *Challenge*: try dropping the assumption that p is 2D. Does your code change?

Joint Entropy

- We can consider joint entropy of a multivariate, e.g. $\{X, Y\}$:

$$\mathrm{H}(X,Y) = \sum_{x \in A_x} \sum_{y \in A_y} -p(x,y) \log_2 p(x,y)$$

- Exercise: Let's code it assuming we don't have $P(X,Y)$ given, but are given empirical data to compute $P(X,Y)$ from:
  1. Edit the Matlab function `jointentropyempirical(xn)` to return the Shannon entropy for the given samples x_n of X (n is sample index)
     a. Input is x_n as a matrix of samples, where each row is a vector of samples of each variable (e.g. `[0,1]`)
     b. Trick: can we use our existing `entropyempirical()` by combining {x,y} into a joint variable?

Aside: Shannon entropy – derivation

- Shannon entropy of a random variable X:

$$\mathrm{H}(X) = \sum_{x \in A_X} -p(x)\log_2 p(x)$$

- Is a unique form that satisfies three axioms (Ash, 1965; Shannon, 1948):
  - Continuity w.r.t. $p(x)$
  - Monotony – $H(X)\uparrow$ as $|A_X|\uparrow$, for $p(x) = 1/|A_X|$
  - Grouping – For independent variables X and Y, $H(X, Y) = H(X) + H(Y)$

C. E. Shannon. A mathematical theory of communication. Bell System Technical Journal, 27(3–4):379–423, 623–656, 1948.
R. B. Ash. Information Theory. Dover Publications Inc., New York, 1965. p. 5-12.

Conditional entropy

– What if we already know something that may pertain to X? Does this change our surprise/uncertainty?
– Conditional entropy: (average) surprise remaining about sample *x* of *X* if we already know the sample *y* of *Y*

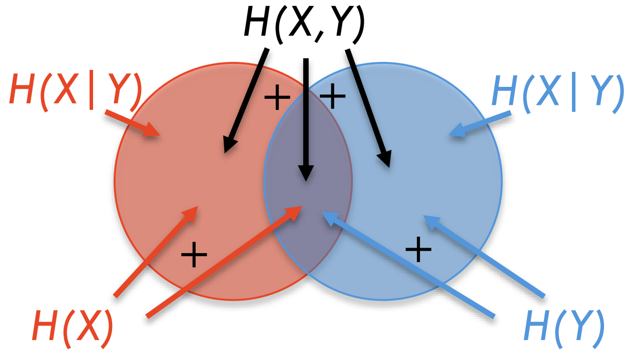
$$\mathrm{H}(X|y) = \sum_{x \in A_x} p(x|y) \log_2 \frac{1}{p(x|y)}$$

$$\mathrm{H}(X|Y) = \sum_{y \in A_y} p(x) \mathrm{H}(X|y)$$

$$\mathrm{H}(X|Y) = \sum_{x \in A_x} \sum_{y \in A_y} p(x,y) \log_2 \frac{1}{p(x|y)}$$

$$\mathrm{H}(X|Y) = \mathrm{H}(X,Y) - H(Y)$$

$$h(x|y) = h(x,y) - h(y)$$
$$h(x|y) = -\log_2 p(x|y)$$



Venn diagram

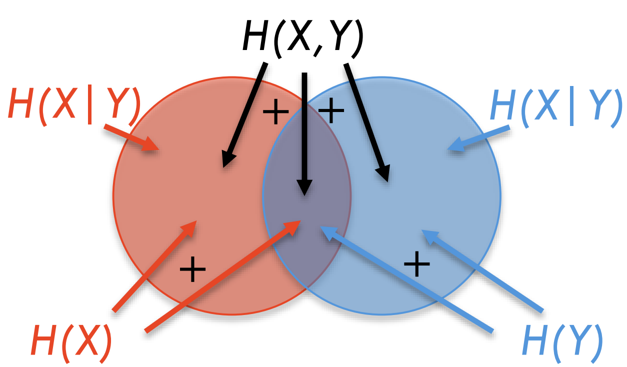
Properties:
– $0 \le H(X|Y) \le H(X)$
– $H(X|Y) = H(X)$ iff *X* and *Y* are independent
– $H(X|Y) = 0$ means there is no surprise left in *X* once we know *Y*.

Conditional entropy

– Conditional entropy: (average) surprise remaining about sample x of X if we already know the sample y of Y

$$h(x|y) = h(x,y) - h(y)$$

$$\mathrm{H}(X|Y) = \mathrm{H}(X,Y) - H(Y)$$



Venn diagram

Coding interpretation of $H(X|Y)$?

Example 1:

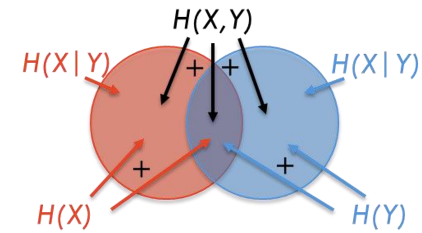
– Coding characters in English text –what variable Y would drop $H(X)$ to some $H(X|Y)$ and therefore the code length for a conditional encoding of incoming character X?

– Context of previous character(s) Y changes the probability of the next character X – Markov chains

i	a_i	p_i	$h(p_i)$
1	a	.0575	4.1
2	b	.0128	6.3
3	c	.0263	5.2
4	d	.0285	5.1
5	e	.0913	3.5
6	f	.0173	5.9
7	g	.0133	6.2
8	h	.0313	5.0
9	i	.0599	4.1
10	j	.0006	10.7
11	k	.0084	6.9
12	l	.0335	4.9
13	m	.0235	5.4
14	n	.0596	4.1
15	o	.0689	3.9
16	p	.0192	5.7
17	q	.0008	10.3
18	r	.0508	4.3
19	s	.0567	4.1
20	t	.0706	3.8
21	u	.0334	4.9
22	v	.0069	7.2
23	w	.0119	6.4
24	x	.0073	7.1
25	y	.0164	5.9
26	z	.0007	10.4
27	-	.1928	2.4
$\sum_i p_i \log_2 \frac{1}{p_i}$			4.1

Table 2.9. Shannon information contents of the outcomes a–z.

Conditional entropy



– Conditional entropy: (average) surprise remaining about sample x of X if we already know the sample y of Y

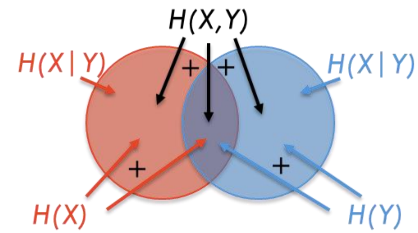
$$h(x|y) = h(x,y) - h(y)$$

$$\mathrm{H}(X|Y) = \mathrm{H}(X,Y) - H(Y)$$

– Exercise: Let's code it!

1. Edit the Matlab function `conditionalentropy(p)` to return the conditional entropy for X *given* Y for the joint probability p.
   a. You can assume p is 2D (`p(x,y)`); this is the input.
   b. Trick: can we use our existing `entropy()` and `jointentropy()`?
   c. Test: `conditionalentropy([0.5,0;0,0.5])`=0
   d. Test: `conditionalentropy([0.25,0.25;0.25,0.25])` = 1
   e. Guess Who? H(`sex|earings`)? Construct p(`sex,earings`) first. Is p(`earings|sex`) the same?

Conditional entropy



– Conditional entropy: (average) surprise remaining about sample x of X if we already know the sample y of Y

$$h(x|y) = h(x,y) - h(y)$$

$$\mathrm{H}(X|Y) = \mathrm{H}(X,Y) - H(Y)$$

– Exercise: Let's code it!

1. Edit the Matlab function `conditionalentropyempirical(xn,yn)` to return the conditional entropy for X *given* Y from empirical samples x_n,y_n:
   a. Input is samples x_n,y_n.
   b. Trick: can we use our existing `entropyempirical()` and `jointentropyempirical()`?
   c. Test: `conditionalentropyempirical([0,0,1,1], [0,1,0,1])` =1
   d. Test: `conditionalentropyempirical([0,0,1,1], [0,0,1,1])` =0

Chain rule for entropy and information content

- Chain rule for entropy:
  - $H(X,Y) = H(X) + H(Y|X)$
  - $H(X,Y) = H(Y) + H(X|Y)$
  - $H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i | X_1 .. X_{i-1})$
  - Same applies for $h(x,y)$, $H(X,Y|Z)$ and $h(x,y|z)$.

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.5
D. J. C. MacKay. Information Theory, Inference, and Learning Algorithms. Cambridge University Press, Cambridge, 2003. Section 8.1

Cross entropy and Kullback-Leibler divergence

- Cross-entropy:

$$G(p||q) = \sum_{x \in A_x} p(x) \log_2 \frac{1}{q(x)}$$

  - Average code length if using the PDF *q(x)* to optimally encode x, which has actual PDF *p(x)*
- Kullback-Leibler (KL) divergence:

$$D(p||q) = \sum_{x \in A_x} p(x) \log_2 \frac{p(x)}{q(x)}$$

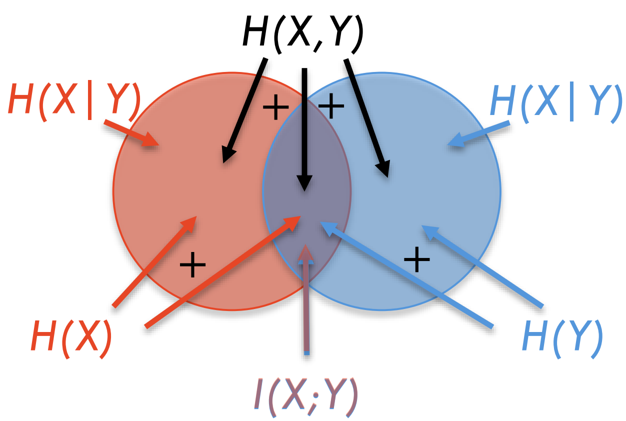
  - Average coding penalty from using the PDF *q(x)* to optimally encode x, which has actual PDF *p(x)*
  - $D(p||q) \geq 0$ with equality iff *q*=*p*
    - You always incur a cost for using incorrect PDF!

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.3
Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.4

Mutual information (MI)

– Mutual information *I(X;Y)* is the reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*

– Interpretation 1: from Venn diagrams –



$$I(X;Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X,Y)$$
$$I(X;Y) = \mathrm{H}(X) - \mathrm{H}(X|Y)$$
$$I(X;Y) = \mathrm{H}(Y) - \mathrm{H}(Y|X)$$
$$I(X;Y) = I(Y;X)$$

Properties:

– $0 \leq I(X;Y) \leq \min(\mathrm{H}(X), \mathrm{H}(Y))$

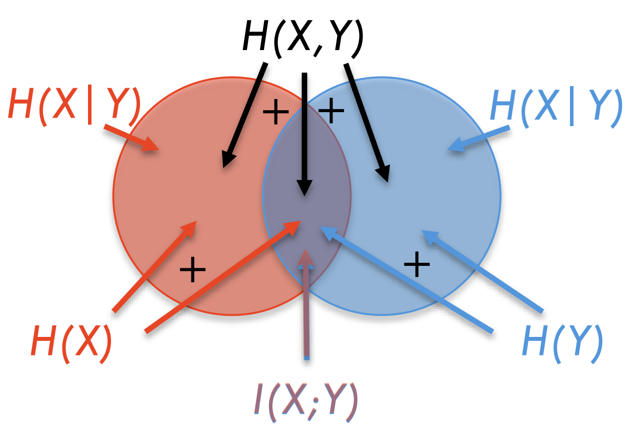
– Is symmetric in *X* and *Y*

– $I(X;Y) = \mathrm{H}(X) \rightarrow \mathrm{H}(X|Y) = 0$

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.4

Mutual information (MI)

- Mutual information *I(X;Y)* is the reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*
- Interpretation 2: KL divergence / Bayesian view –



$$I(X;Y) = \sum_{x\in A_x, y\in A_y} p(x,y)\log_2\frac{p(x,y)}{p(x)p(y)}$$

$$I(X;Y) = \sum_{x\in A_x, y\in A_y} p(x,y)\log_2\frac{p(x|y)}{p(x)}$$

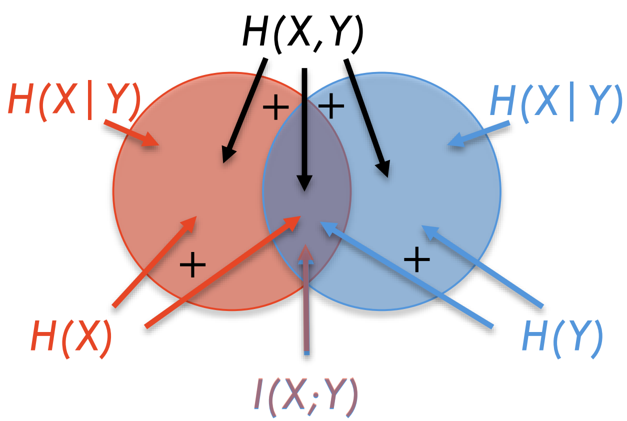
Properties:

- $I(X;Y) = D(p(x,y)||p(x)p(y))$
- MI is code length penalty for coding {*x*,*y*} assuming *x* and *y* are independent, or for coding *x* without using knowledge of *y*.

Mutual information (MI)

– Mutual information *I(X;Y)* is the reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*

– Interpretation 3: statistical view –



$$\mathrm{I}(X;Y) = \sum_{x \in A_x, y \in A_y} p(x,y) \log_2 \frac{p(x|y)}{p(x)}$$

Properties:

– $\mathrm{I}(X;Y) = 0 \; \leftrightarrow$ *X* is independent of *Y*

– MI is a non-linear form of correlation

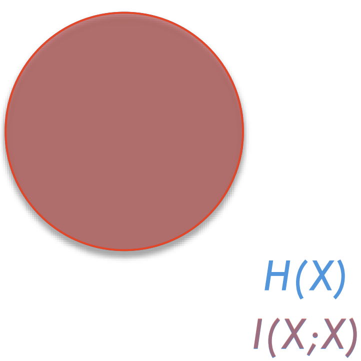
T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.3

Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.2

Mutual information (MI)

- Mutual information $I(X;Y)$ is the reduction in uncertainty or surprise about one variable X that we obtain from another variable Y
- Interpretation 4: self-information and uncertainty –

$$\mathrm{I}(X;X) = \mathrm{H}(X) + \mathrm{H}(X) - \mathrm{H}(X,X)$$
$$\mathrm{I}(X;X) = \mathrm{H}(X)$$



Properties:

- Entropy $H(X)$ (uncertainty) is equivalent to the self-information I(X;X) (uncertainty reduction) obtained from that variable about itself.
- Entropy and information are complementary quantities!

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.4

Local or pointwise Mutual information

- Mutual information $i(x;y)$ is the reduction in uncertainty or surprise about one sample x of variable X that we obtain from one sample y of another variable Y

$$\mathrm{i}(x;y) = \mathrm{h}(x) + \mathrm{h}(y) - \mathrm{h}(x,y)$$
$$\mathrm{i}(x;y) = \mathrm{h}(x) - \mathrm{h}(x|y)$$
$$\mathrm{i}(x;y) = \mathrm{h}(y) - \mathrm{h}(y|x)$$
$$\mathrm{i}(x;y) = \log_2 \frac{p(x|y)}{p(x)}$$
$$\mathrm{I}(X;Y) = \langle \mathrm{i}(x;y) \rangle$$

- $i(x;y) > 0$ means $p(x|y) > p(x)$, so y increased our expectation that x would occur, positively informing us.
- $i(x;y) < 0$ means $p(x|y) < p(x)$, so y reduced our expectation that x would occur, misinforming us.
  - e.g. when the weather report says 'sunshine' but it actually rains, we may have p(`rain|sunny_forecast`) = 0.05 whilst p(`rain`)=0.2.
  - But: *on average* over all samples Y provides $I(X;Y) \geq 0$.

Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.2/3.2.2.1

Local or pointwise Mutual information

- Mutual information $i(x;y)$ is the reduction in uncertainty or surprise about one sample x of variable X that we obtain from one sample y of another variable Y
- Interpretation 5: information comes from effect of *exclusions* –

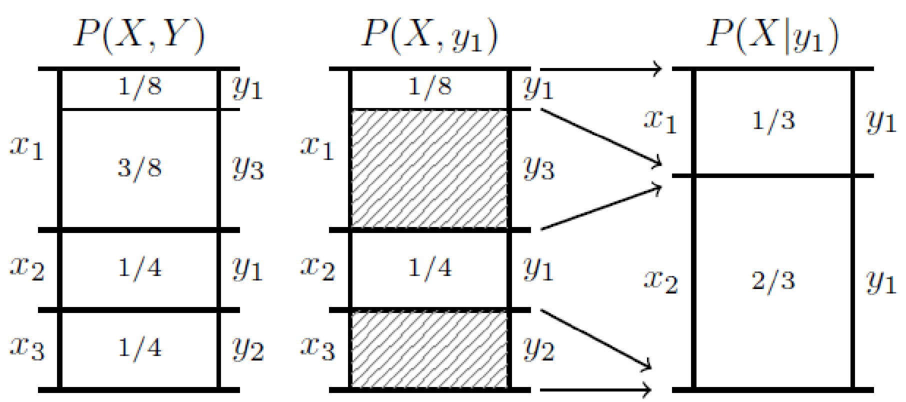


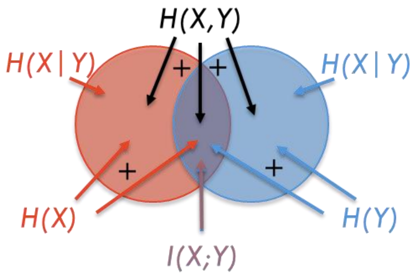
Fig. 1. Probability mass diagrams, which use length to represent the probability mass of each joint event (x, y) (summing to 1 over all (x, y) of course). These illustrate: a. (left) starting from the joint distribution $P(X, Y)$; b. (middle) the occurrence of the event $Y = y_1$ leads to exclusions of $\bar{y}_1 = \{\mathcal{Y} \setminus y_1\} = \{y_2, y_3\}$ to leave $P(X, y_1)$; c. (right) and the remaining space is then normalised into $P(X|y_1)$.

1. Learning the value $Y=y_1$ leads to exclusions in the joint space $P(X,y_1)$. The potential "value" of the exclusions is $h(y_1)$.
2. Renormalise the probability space to get $P(X|y_1)$.
3. Compare $P(x_1)$ and $P(x_1|y_1)$ for the event x_1 which occurred.

- Consider how exclusions in Guess Who provide information in this way …

Finn, Prokopenko, Lizier, "Decomposing Local Information into Directed Positive and Negative Components", 2017.

Mutual information (MI) – code

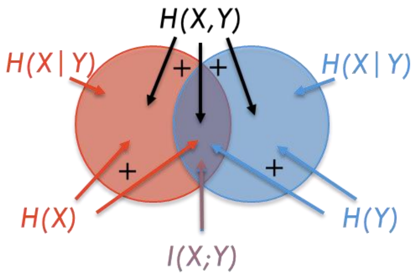


- The reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*

$$\mathrm{i}(x;y) = \mathrm{h}(x) + \mathrm{h}(y) - \mathrm{h}(x,y)$$
$$\mathrm{I}(X;Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X,Y)$$

- Exercise: Let's code it!
  1. Edit the Matlab function `mutualinformation(p)` to return the MI between *X* and *Y* for the joint probability *p*.
     a. You can assume *p* is 2D (`p(x,y)`); this is the input.
     b. Trick: can we use our existing `entropy()` and `jointentropy()`?
     c. Test: `mutualinformation([0.5,0;0,0.5])`=1
     d. Test: `mutualinformation([0.25,0.25;0.25,0.25])` = 0
     e. Guess Who? I(`sex;earings`)? Construct p(`sex,earings`) first. Why is there MI here?

Mutual information (MI) – code



– The reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*

$$\mathrm{i}(x;y) = \mathrm{h}(x) + \mathrm{h}(y) - \mathrm{h}(x,y)$$
$$\mathrm{I}(X;Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X,Y)$$

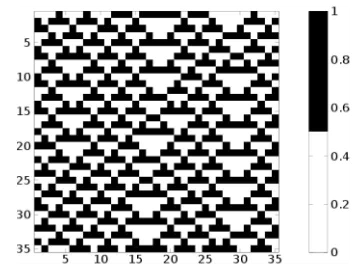
– Exercise: Let's code it!

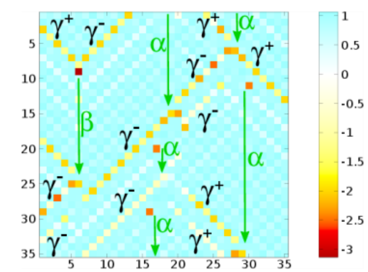
1. Edit the Matlab function `mutualinformationempirical(xn,yn)` to return the MI between *X and Y* from empirical samples x_n, y_n:
   a. Input is samples x_n, y_n.
   b. Trick: can we use our existing `jointentropyempirical()`?
   c. Test: `mutualinformationempirical([0,0,1,1], [0,1,0,1])` =0
   d. Test: `mutualinformationempirical([0,0,1,1], [0,0,1,1])` =1

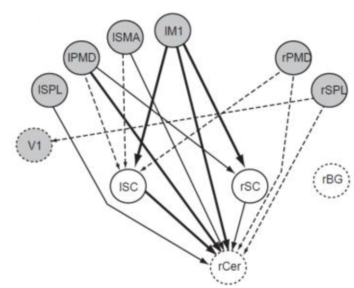
Mutual information (MI)

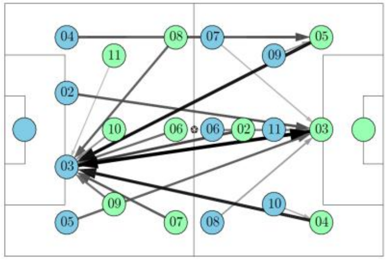
- Is a great model-free tool to:
  - detect relationships between variables;
  - reveal patterns;
  - show how such relationships and patterns fluctuate in time.

- Example uses:
  - Feature selection in machine learning
  - Space-time characterisation of information processing in complex systems – lectures 4,5
  - Inferring relationships (i.e. networks) in multivariate time-series data (e.g. brain imaging) – lecture 6









J. T. Lizier. "Measuring the dynamics of information processing on a local scale in time and space". In M. Wibral, R. Vicente, and J. T. Lizier, editors, "Directed Information Measures in Neuroscience", Springer, Berlin/Heidelberg, 2014; pp. 161–193.
J. T. Lizier, J. Heinzle, A. Horstmann, J.-D. Haynes, & M. Prokopenko. "Multivariate information-theoretic measures reveal directed information structure and task relevant changes in fMRI connectivity". J. Computational Neuroscience, 30 (1):85–107, 2011.
O.M. Cliff, J.T. Lizier, P. Wang, X.R. Wang, O. Obst, M. Prokopenko, "Quantifying Long-Range Interactions and Coherent Structure in Multi-Agent Dynamics", Artificial Life, vol. 23, no. 1, pp. 34-57, 2017.

Conditional mutual information (CMI)

- *I(X;Y|Z)* is the reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*, given the value of another variable Z.
- Interpretation 1: in the context of Z –

$$\mathrm{I}(X;Y|Z) = \mathrm{H}(X|Z) + \mathrm{H}(Y|Z) - \mathrm{H}(X,Y|Z)$$
$$\mathrm{I}(X;Y|Z) = \mathrm{H}(X|Z) - \mathrm{H}(X|Y,Z)$$
$$\mathrm{I}(X;Y|Z) = \mathrm{H}(Y|Z) - \mathrm{H}(Y|X,Z)$$
$$\mathrm{I}(X;Y|Z) = \mathrm{I}(Y;X|Z)$$
$$\mathrm{I}(X;Y|Z) = \mathrm{I}(X;Y,Z) - \mathrm{I}(X;Z)$$

All maths changes to be conditional on Z

Ok this one is new!

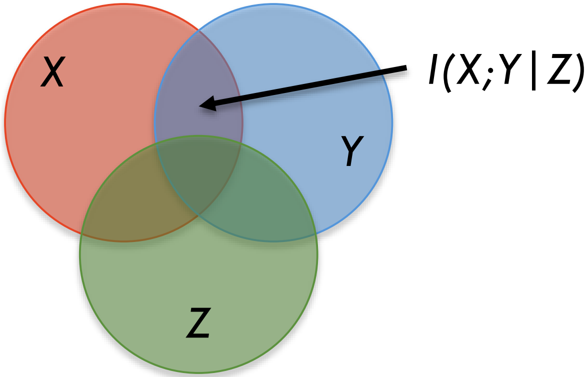
Properties:

- $0 \leq \mathrm{I}(X;Y|Z) \leq \min\big(\mathrm{H}(X|Z), \mathrm{H}(Y|Z)\big)$
  - e.g. if *Z* explains *X* (*H(X|Z)*=0), then $\mathrm{I}(X;Y|Z)$=0
- Is symmetric in *X* and *Y*
- $\mathrm{I}(X;Y|Z) = \mathrm{H}(X|Z) \rightarrow \mathrm{H}(X|Y,Z) = 0$

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.5

Conditional mutual information (CMI)

- $I(X;Y|Z)$ is the reduction in uncertainty or surprise about one variable X that we obtain from another variable Y, given the value of another variable Z.
- Be warned against using Venn diagrams to interpret 3-term entropies!
  - Areas in the diagram add up correctly **but** the diagram gives the misleading impression that all areas are positive! (They aren't!)



  - Mackay emphasises that there are no other well-defined "3-term entropies"

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Problem 2.25
D. J. C. MacKay. Information Theory, Inference, and Learning Algorithms. Cambridge University Press, Cambridge, 2003. Exercise 8.8 (sol'n p. 143) & p. 139

Conditional mutual information (CMI)

- *I(X;Y|Z)* is the reduction in uncertainty or surprise about one variable *X* that we obtain from another variable *Y*, given the value of another variable Z.
- Interpretation 2: KL divergence / Bayesian view –

$$\mathrm{I}(X;Y|Z) = \sum_{x \in A_x, y \in A_y, z \in A_z} p(x,y,z) \log_2 \frac{p(x,y|z)}{p(x|z)p(y|z)}$$

$$\mathrm{I}(X;Y|Z) = \sum_{x \in A_x, y \in A_y, z \in A_z} p(x,y,z) \log_2 \frac{p(x|y,z)}{p(x|z)}$$

Properties:

- $\mathrm{I}(X;Y|Z) = \mathrm{D}(p(x,y|z)||p(x|z)p(y|z))$
- CMI is code length penalty for coding {*x*,*y*} assuming *x* and *y* are conditionally independent (on z), *or* for coding *x* without using knowledge of *y in addition* to knowledge of *z*.

Conditional mutual information (CMI)

- $I(X;Y|Z)$ is the reduction in uncertainty or surprise about one variable X that we obtain from another variable Y, given the value of another variable Z.
- Interpretation 3: statistical view –

$$\mathrm{I}(X;Y|Z) = \sum_{x\in A_x, y\in A_y, z\in A_z} p(x,y,z) \log_2 \frac{p(x|y,z)}{p(x|z)}$$

Properties:

- $\mathrm{I}(X;Y|Z) = 0 \leftrightarrow$ X, conditional on Z, is independent of Y
- CMI is a non-linear form of partial correlation

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.5
Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.3

Local or pointwise Conditional Mutual information

- *i(x;y|z)* is the reduction in uncertainty or surprise about one sample *x* of variable *X* that we obtain from one sample *y* another variable *Y*, given the sample *z* of another variable Z.

$$\mathrm{i}(x;y|z) = \mathrm{h}(x|z) + \mathrm{h}(y|z) - \mathrm{h}(x,y|z)$$
$$\mathrm{i}(x;y|z) = \mathrm{h}(x|z) - \mathrm{h}(x|y,z)$$
$$\mathrm{i}(x;y|z) = \mathrm{h}(y|z) - \mathrm{h}(y|x,z)$$
$$\mathrm{i}(x;y|z) = \log_2 \frac{p(x|y,z)}{p(x|z)}$$
$$\mathrm{I}(X;Y|Z) = \langle \mathrm{i}(x;y|z) \rangle$$

- *i(x;y|z)* may be positive or negative (as per *i(x;y)*)

Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.3

Conditional Mutual information (CMI) – code

– $I(X;Y|Z)$ is the reduction in uncertainty or surprise about one variable X that we obtain from another variable Y, given the value of another variable Z.

$$\mathrm{i}(x;y|z) = \mathrm{h}(x|z) + \mathrm{h}(y|z) - \mathrm{h}(x,y|z)$$
$$\mathrm{I}(X;Y|Z) = \mathrm{H}(X|Z) + \mathrm{H}(Y|Z) - \mathrm{H}(X,Y|Z)$$

– Exercise: Let's code it! (empirical only)

1. Edit the Matlab function `conditionalmutualinformationempirical(xn,yn,zn)` to return the CMI between X *and* Y *given* Z from empirical samples x_n,y_n,z_n:
   a. Input is samples x_n,y_n,z_n.
   b. Trick: can we use our existing `conditionalentropyempirical()`?
   c. Test: `CMI([0,0,1,1], [0,1,0,1], [0,1,0,1])` =0
   d. Test: `CMI([0,0,1,1], [0,0,1,1], [0,1,1,0])` =1
   e. *Challenge*: compute using $\mathrm{I}(X;Y|Z) = \mathrm{I}(X;Y,Z) - \mathrm{I}(X;Z)$
   f. *Challenge*: write `conditionalmutualinformation (p)` (p is a 3D matrix!)

Conditional and unconditional mutual information

Conditioning on Z in $I(X;Y|Z)$, as compared to $I(X;Y)$ can:

- Have no effect (if all variables are independent)
- Serve to decrease $I(X;Y|Z)$ compared to $I(X;Y)$
  - Y and Z carry redundant information about X.
  - Z explained away some of what could be detected by Y
  - e.g. If X=Y=Z are iid random bits, $I(X;Y|Z)=0$ although $I(X;Y)=1$
- Serve to increase $I(X;Y|Z)$ compared to $I(X;Y)$
  - Y and Z together provide synergistic information about X, which cannot be detected by examining either alone.
  - e.g. If X=Y `xor` Z, iid random bits, $I(X;Y|Z)=1$ although $I(X;Y)=0$.

- $I(X;Y|Z)$ - $I(X;Y)$ being positive implies presence of synergy, or being negative implies presence of redundancy.
- But you can have both redundancy and synergy at once!
- Cannot measure redundancy and synergy with traditional info theory …

Williams and Beer, “Nonnegative decomposition of multivariate information”. arXiv:1004.2515, 2010.
Bossomaier, Barnett, Harré, Lizier, "An Introduction to Transfer Entropy: Information Flow in Complex Systems", Springer, Cham, 2016; section 3.2.3.1

Chain rule for mutual information

- Chain rule for information:
  - $\mathrm{I}(X;Y,Z) = \mathrm{I}(X;Y) + \mathrm{I}(X;Z|Y)$
  - $\mathrm{I}(X;Y,Z) = \mathrm{I}(X;Z) + \mathrm{I}(X;Y|Z)$

$$I(X_1, X_2, \dots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y | X_1 .. X_{i-1})$$

  - This is an information regression!

  - Same applies for $i(x;y)$, $I(X;Y|Z)$ and $i(x;y|z)$.

T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley-Interscience, New York, 1991. Section 2.5

Aside: Mutual information – derivation

– Local mutual information between X and Y:

$$\mathrm{i}(x;y) = \log_2 \frac{p(x|y)}{p(x)}$$

and by implication $I(X;Y) = \langle i(x;y) \rangle$

– Is a unique form that satisfies four axioms:
  – Once-differentiability w.r.t. $p(x)$ and $p(x|y)$
  – Conditional form $i(x;y|z)$ matches $i(x;y)$ but with all PDFs conditioned on z
  – Additivity – $i(x;y,z) = i(x;z) + i(x;y|z)$
  – Separation for independent ensembles:
    - $p(x,y,u,v) = p(x,y)p(u,v) \rightarrow i(x,u;y,v) = i(x;y) + i(u,v)$

Fano, R.M.: Transmission of information: a statistical theory of communications. M.I.T. Press, Cambridge, MA, USA (1961)

What is information: summary

- We've been introduced to the ideas of uncertainty and surprise.
- Understand the meaning of information as uncertainty reduction
- Know how to calculate fundamental measures of information theory, from PDFs and empirically from data.

- *Next lecture*: Move onto using a more advanced toolkit, and dealing with continuous-valued variables using a number of different estimators.

Questions