Data 100, Fall 2021

Discussion #1

Welcome to Data 100!

Linear Algebra Fundamentals

1. Linear algebra is what powers linear regression, logistic regression, and PCA (concepts we will be studying in this course). Moving forward, you will need to understand how matrix-vector operations work. That is the aim of this problem.

   Fernando, Alvin, and Kobe are shopping for fruit at Berkeley Bowl. Berkeley Bowl, true to its name, only sells fruit bowls. A fruit bowl contains some fruit and the price of a fruit bowl is the total price of all of its individual fruit.

   Berkeley Bowl has apples for \$2, bananas for \$1, and cantaloupes for \$4. (expensive!). The price of each of these can be written in a vector:

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

   Berkeley Bowl sells the following fruit bowls:

   1. 2 of each fruit
   2. 5 apples and 8 bananas
   3. 2 bananas and 3 cantaloupes
   4. 10 cantaloupes

   (a) Define a matrix B such that

$$B\vec{v}$$

   evaluates to a length 4 column vector containing the price of each fruit bowl. The first entry of the result should be the cost of fruit bowl 1, the second entry the cost of fruit bowl 2, etc.

(b) Fernando, Alvin, and Kobe make the following purchases:

- Fernando buys 2 fruit bowl 1s and 1 fruit bowl 2.
- Alvin buys 1 of each fruit bowl.
- Kobe buys 10 fruit bowl 4s (he really like cantaloupes).

Define a matrix A such that the matrix expression

$$AB\vec{v}$$

evaluates to a length 3 column vector containing how much each of them spent. The first entry of the result should be the total amount spent by Fernando, the second entry the amount sent by Alvin, etc.

(c) Let's suppose Berkeley Bowl changes their fruit prices, but you don't know what they changed their prices to. Fernando, Alvin, and Kobe buy the same quantity of fruit baskets and the number of fruit in each basket is the same, but now they each spent these amounts:

$$\vec{x} = \begin{bmatrix} 80 \\ 80 \\ 100 \end{bmatrix}$$

In terms of A, B, and $\vec{x}$, determine $\vec{v_2}$ (the new prices of each fruit).

Staff Notes: The problems in this discussion are quite simple, and they all have relatively friendly answers. But as always, getting to the answer isn't really the goal; what you really want to stress in your videos and live sections is the thought process behind why we're doing things.

In this problem, you shouldn't skip right to multiplying out a matrix. You should instead first calculate how much each of the four fruit baskets cost individually, and then try and translate that into a matrix-vector operation. Some students might not have seen matrix-vector multiplication before so spell it out for them (but don't be too slow – most will have).

A nice challenge problem to throw on at the end is, why can't we write $\vec{v_2}$ as $B^{-1}A^{-1}\vec{x}$? (because A and B aren't square matrices, and so inverses have no meaning). Another bonus discussion topic could be around how pseudoinverses, which are defined for non-square matrices, still wouldn't be helpful in this instance. This is due to the fact that a pseudoinverse on matrix B for instance, would produce a possibly inexact approximation since it is a function which outputs vectors in $\mathbb{R}^3$, which is smaller than its input vectors in $\mathbb{R}^4$.

For matrix B, we can see this in action since its pseudoinverse is given by $(B^T B)^{-1} B^T$ (this is also why least squares is inexact for a tall matrix that we cannot invert!). Hence, if $+$ denotes a pseudoinverse (note that the numbers are rounded to the nearest tenth):

$$BB^+ = B(B^T B)^{-1} B^T = \begin{pmatrix} 0.4 & 0.2 & -0.4 & 0.2 \\ 0.2 & 0.9 & 0.1 & -0.1 \\ -0.4 & 0.1 & 0.8 & 0.1 \\ 0.2 & -0.1 & 0.1 & 0.9 \end{pmatrix}$$

Note that this has a connection to SVD as well because this pseudoinverse is equal to $V \Sigma^{-1} U^T$ for tall matrices.