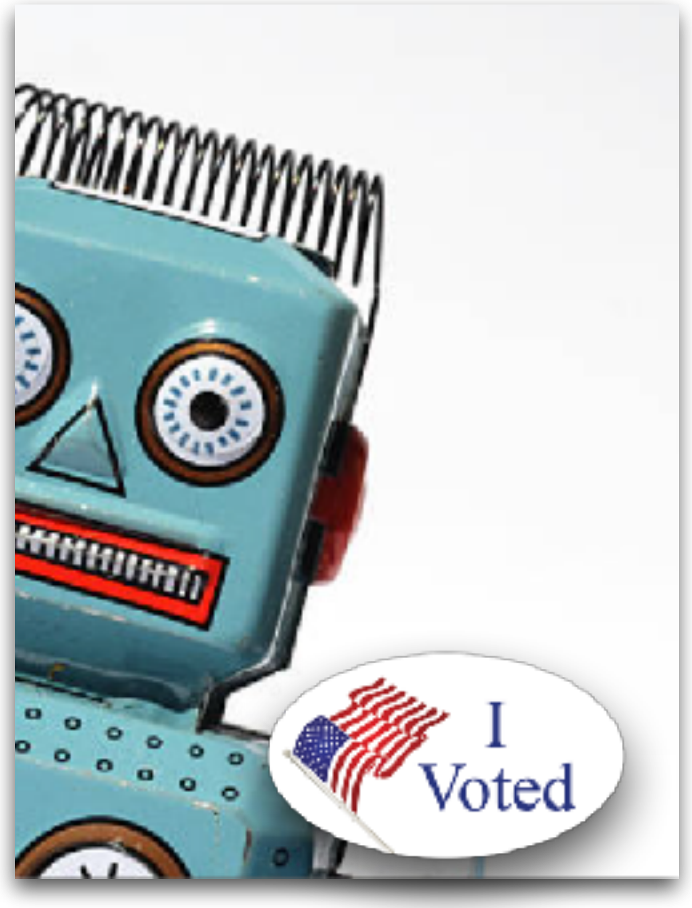
Modeling Democracy

Lecture 5 - **Apportionment and weighted voting**



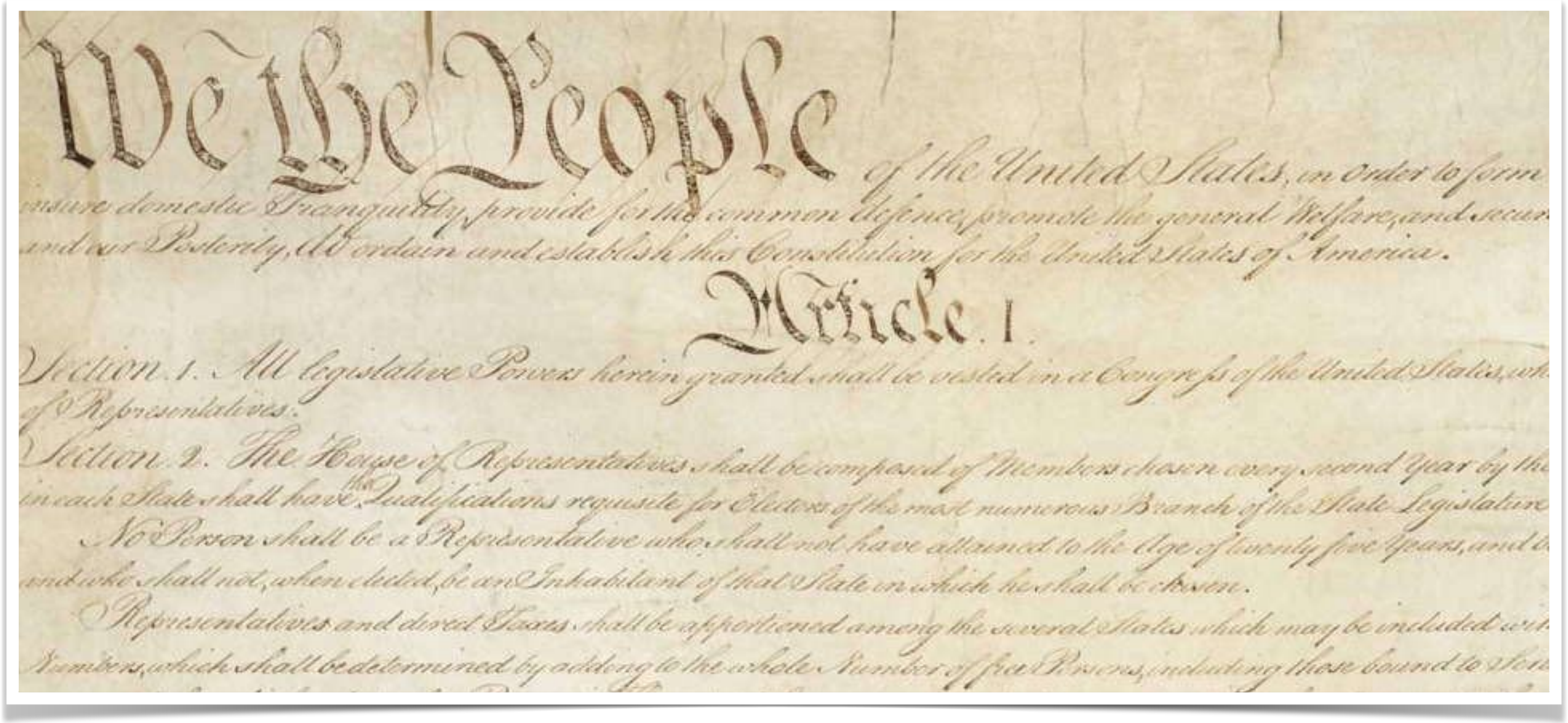
Moon Duchin, Winter 2026 — Wed Jan 21

What is apportionment?

- Giving things out according to a scheme, especially giving out integer allotments according to a quota

- Major example from U.S. Constitution, Article I: “Representatives… shall be apportioned among the several states … according to their respective numbers… The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative.”

- Generally if there are m goods to be allocated, quotas are q_i that sum to m and we want integers $m_i \approx q_i$ that sum to m.



Grocery store problem

- The problem: you want to buy n items at the grocery store, and you're estimating the total cost as you go. But if you round each item to the nearest dollar, it is unlikely that the sum of the roundings is the same as the rounding of the sum.

- (For that to happen with an even number of items, you'd need a specific number to round down and a specific number to round up. At the most likely, that's half and half, which has a probability of $\binom{n}{n/2}/2^n = O(n^{-1/2})$ if the fractional parts are uniformly distributed.

Basic ingredients

- Let's let M_i be the population of the ith state, with $M = \sum M_i$, so that the ith state's quota of seats is $q_i = (M_i/M) \cdot m$, generally not an integer.

- Let's define the lower quota to be $\lfloor q_i \rfloor$ and the upper quota to be $\lceil q_i \rceil$. It would be nice if everything was rounded to one of those two— we can call this the **Quota Rule**. (It's a property of some rounding schemes.)

- Clearly if you round everything down via $m_i = \lfloor q_i \rfloor$, the sum will be too low, if you round up it will be too high, and if you round to the nearest integer you're not terribly likely to hit the sum you want.

Hamilton's method, Jefferson's method

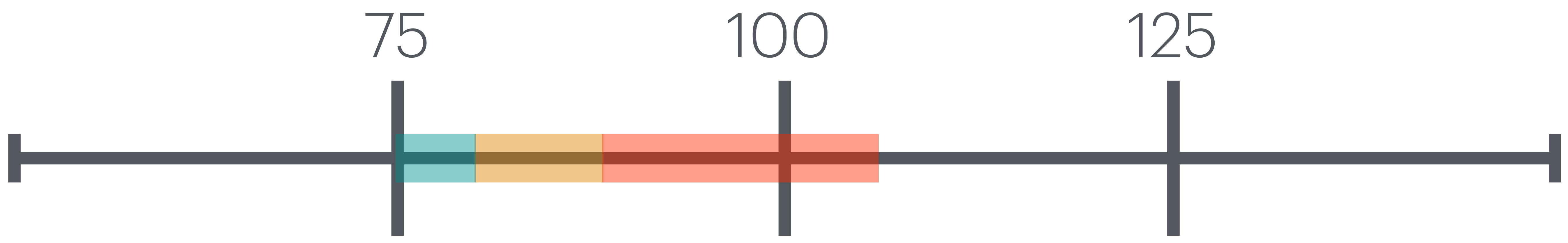
- **Hamilton**: Round everything down, then give out remaining seats in order of fractional part.
- Presented as a bill in March 1792 — gets first presidential veto!
- Instead, **Jefferson**: round down with a *divisor method*
  - Find D such that $\sum \lfloor M_i/D \rfloor = m$ and let $m_i = \lfloor M_i/D \rfloor$.
  - Uncharitable description: "modify the U.S. population till the numbers cooperate."
  - Charitable: $q_i = (M_i/M) \cdot m = \dfrac{M_i}{M/m}$, and M/m is ideal district size. So divisor methods are just modifying the ideal size.
- Jefferson's method used from 1792-1830.

Well-defined and easy to execute

- **Independent of choice of divisor**: Suppose D, D' both work, i.e., $\sum \lfloor M_i/D \rfloor = m$. Then they give the same apportionment, because $D > D' \implies M_i/D < M_i/D'$. So either all allotments are the same under D, D' or else the D allotments add up to less.

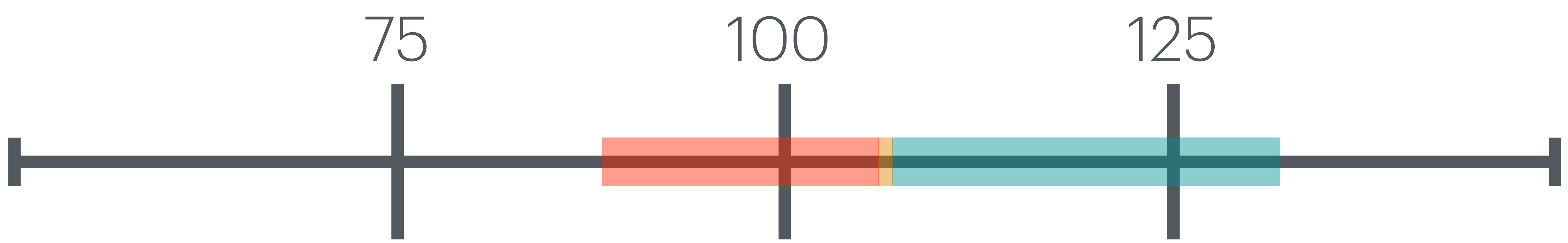
- Example: $M_1 = 150,\ M_2 = 320,\ M_3 = 530$, so that population shares are 15%, 32%, 53%.

- If m=10, then natural divisor D=100 gives out 9 seats. We should *decrease* the divisor.



Adams, Webster, Huntington-Hill

- **Adams**: Divisor method just like Jefferson, but round up instead of down!



- **Webster**: Same but round to nearest integer.

- **Huntington-Hill**: Same but round based on *geometric mean*:
  - if $k < q_i < k + 1$, then rounding cutoffs are $\begin{cases} k, k+1 & \text{Adams} \\ k + \frac{1}{2} & \text{Webster} \\ \sqrt{k(k+1)} & \text{H-H} \end{cases}$

Size bias

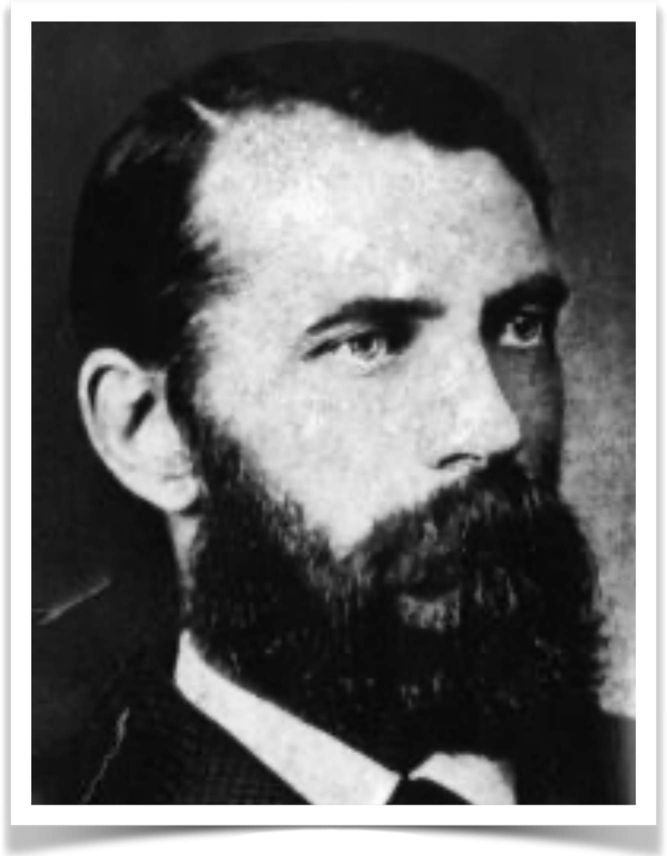
- There's a sense in which Adams is biased in favor of small states and Jefferson in favor of large
- Example: in our scenario with 150, 320, 530, where natural divisor gives (1,3,5) but m=10
  - Jefferson (round down) gets to (1,3,**6**) apportionment
  - Adams (round up) gets to (**2**,3,5) apportionment
- WHY? At baseline, the ideal district size is 100. Jefferson will drive it down while Adams will drive it up.
- At divisor 90, there are 10 fewer people needed per seat. A initially had 50 "extra" people while C has 30 "extra." But the reduced demands free up 10 people in A but 50 people in C. So reducing the divisor lets C catch up and overtake A.
- Likewise, at divisor 110, there are 10 more people needed per seat. This takes 10 away from A's surplus but takes 50 away from C's surplus.

After 1790 Census

- there is a long historical period of battle over **both** the apportionment method and the size of the house
- from 1850–1900 (with the exception of 1860 as Civil War rages) Congress hedges by choosing a size each year at which Webster and Hamilton will give the same answer!

State	M_i	$M_i / 30{,}000$	m_i
Connecticut	236,841	7.895	8
Delaware	55,540	1.851	2
Georgia	70,835	2.361	2
Kentucky	68,705	2.290	2
Maryland	278,514	9.284	9
Massachusetts	475,327	15.844	16
New Hampshire	141,822	4.727	5
New Jersey	179,570	5.986	6
New York	331,589	11.053	11
North Carolina	353,523	11.784	12
Pennsylvania	432,879	14.419	14
Rhode Island	68,446	2.282	2
South Carolina	206,236	6.875	7
Vermont	85,533	2.851	3
Virginia	630,560	21.019	21
Total	3,615,920	120	120

Alabama Paradox



- Hamilton is the law of the land in 1880, but this happens:
- Census chief clerk Charles Seaton has provided Congress with tables to figure out how to apportion; they give the Hamilton apportionment for every House size from m=275 to 350
- From m=299 to m=300, Alabama **lost** a seat (and two others gained one)
- So then and for the next bunch of decades, the House size was chosen so that Hamilton and Webster agreed

Paradoxes abound

- **Alabama paradox**: enlarging the House can make a state lose a seat
    - happened again in 1900 to CO and ME
- **Population paradox**: a slower-growing state (even a shrinking state) can gain a seat while a faster-growing state loses
    - also in 1900, VA grew +1.06%, ME grew +0.7%, but ME got a seat from VA
- **New states paradox**: given a fixed apportionment, it's possible to add a new state, enlarge the House by the allotment its quota would have received, and cause other states to lose
    - OK became a state in 1907, would have deserved 5 seats in the then-current scheme
    - House size bumped by 5, causing OK to get those 5, but NY lost a seat to ME
- **Quota violation**: it's possible for a rule to award a state an allocation that differs by more than 1 from its quota. (So it's past the ceiling or the floor.)

Huntington's Theorem

- Huntington apportionment has the following property:
- $\forall i, j$ s.t. $\dfrac{M_i}{m_i} \leq \dfrac{M_j}{m_j}$, we have $\dfrac{M_i/m_i}{M_j/m_j} > \dfrac{M_j/(m_j+1)}{M_i/(m_i-1)}$.
- In other words, the HH allocation is globally optimal for **proportions** of population to seats, as no swap can improve it.
- E.V. Huntington was a model theorist (logician) at Harvard, and he thought this (proving only one scheme has this property)was a mic drop for Congress…
- …it was not.

But, Impossibility

- Call a rule *neutral* if the m_i depend only on the M_i.
- **Balinski–Young Impossibility Theorem**: No neutral apportionment rule can rule out both population paradoxes and quota violations.

😕

Divisor methods generally

- Call a real function $f : \mathbb{R}_+ \to \mathbb{N}$ a **rounding function** if it stabilizes integers and is weakly increasing.

- Then adjusting the divisor gives a class of apportionment methods as long as input numbers are in general position ("no coincidences") — same proof as before means well-defined.

- Huntington showed that any divisor method that is optimal for proportions is equivalent to Huntington-Hill (geometric mean rounding).

More properties

- Define **population monotonic** to mean that if one state gains and another loses, then the first one grew OR the second one shrank.
- i.e., $m_i < m_i'$ and $m_j > m_j' \implies M_i < M_i'$ or $M_j > M_j'$.
- **Theorem**: All divisor methods are population monotonic.
- Proof: Assume the seat allocations satisfy $m_i < m_i'$ and $m_j > m_j'$ (i gained while j lost). Then $\frac{M_i}{D} < \frac{M_i'}{D'}$ and $\frac{M_j}{D} > \frac{M_j'}{D'}$ (if the rounded value goes up, the unrounded value must go up too, by definition of rounding). That says $M_i' > \dfrac{D'}{D}M_i$ and $M_j' < \dfrac{D'}{D}M_j$. No matter whether D or D′ is larger, one of those gives what we need. $\square$

More properties

- Now define **house monotonic** to mean an increased house can't make someone lose a seat.

- **Theorem**: All divisor methods are house monotonic.

- Proof (easy): For the house to grow overall, someone must gain. But divisor methods are population monotonic, so you can't have someone gain and someone lose while all populations are steady. □

Pop quiz

- Recall: Hamilton rounds down and then gives out by fractional part. Jefferson, Adams, Webster, and H-H are all divisor methods given by rounding down, up, nearest, and geometric.
- Which ones have no **quota violations**? (Every apportionment is floor or ceiling of quota.)
- Answer: only Hamilton! Because B-Y Impossibility says no neutral rule is population monotonic and avoids quota violations.
- Proof of Balinski–Young Impossibility:
    - First note $M_j > M_i \implies m_j \geq m_i$.
    - Then, suppose quota rule holds and derive contradiction from an example, reasoning only from properties.

State	M_i	q_i
1	69,900	6.99
2	5,200	0.52
3	5,000	0.50
4	19,900	1.99
Total	100,000	10

State	M_i'	q_i'
1	68,000	8.02
2	5,500	0.65
3	5,600	0.66
4	5,700	0.67
Total	84,800	10

$$m_1 \leq 7, \quad m_4 \leq 2$$

$$m_2 \geq 1 \text{ or } m_3 \geq 1$$

$$m_2 \geq 1$$

$$m_1' \geq 8$$

$$m_2' = 0$$

Weighted voting

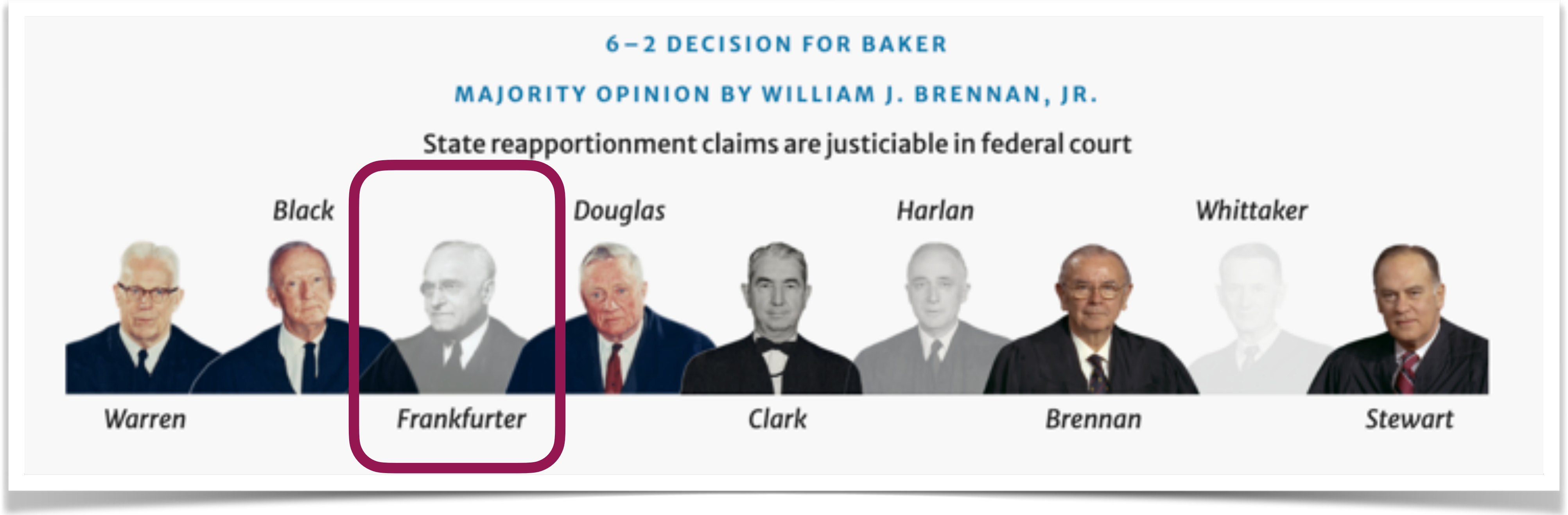
The basic question: What is the **weight** of a vote?

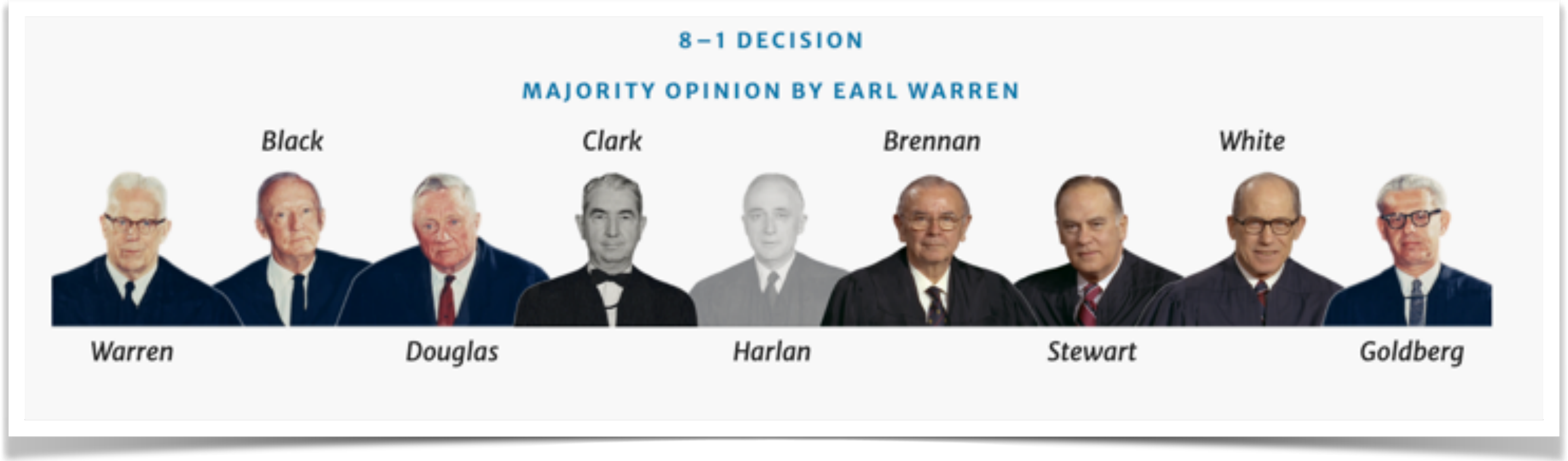
- That is, what is it that we should equalize in "one person, one vote"?

- One standard approach to this is to measure the (*prima facie*) **probability of casting a decisive vote** — just taking into account the voting rule, not the candidates and other particulars.

- How to measure this is contested! But one thing is clear: if districts have equal number of voters, these probabilities should be equal.

- *Baker v. Carr* (1962), *Reynolds v. Sims* (1964) — the Supreme Court accepts this reasoning and initiates its population-balancing requirement



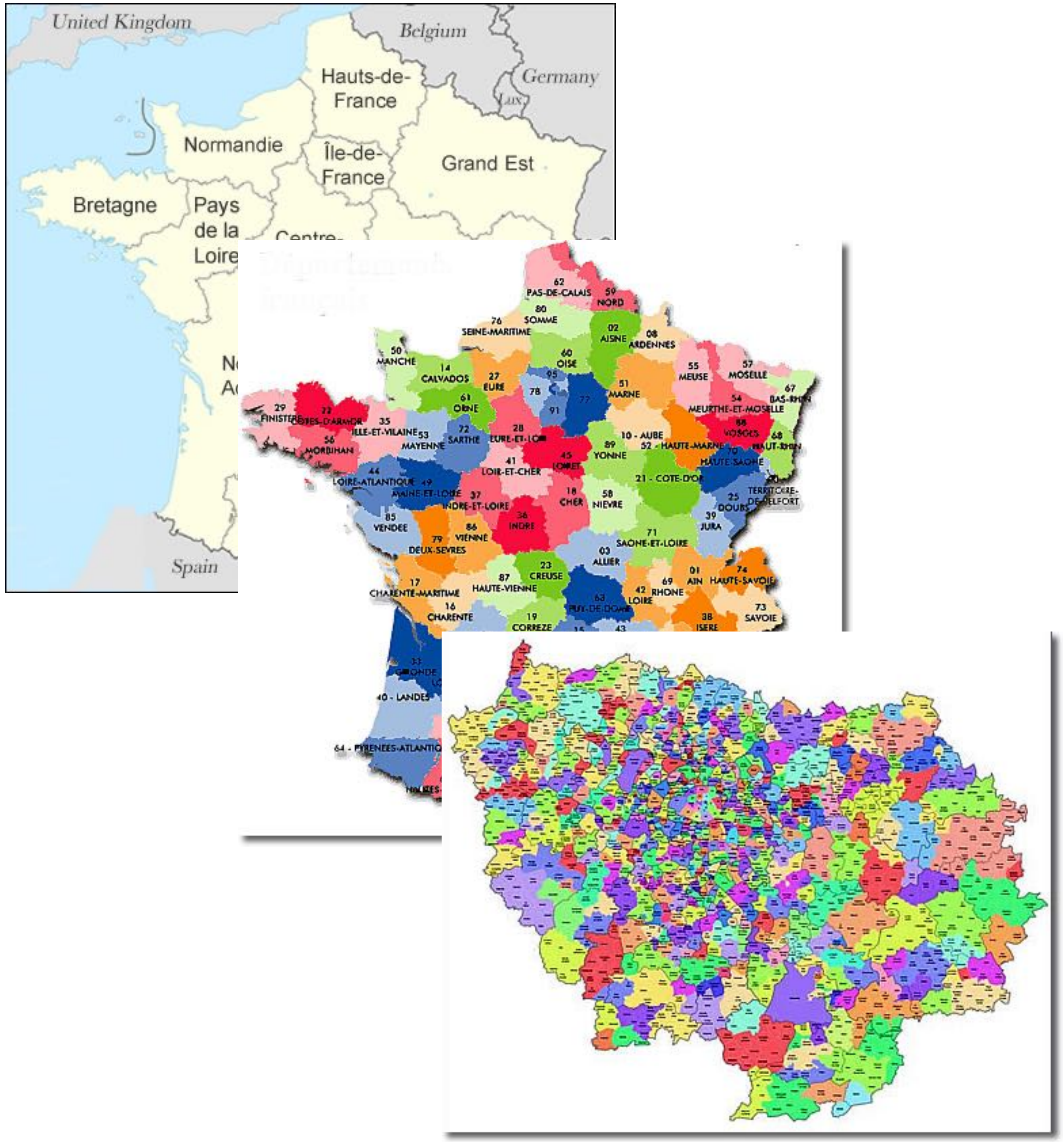


Is this compatible with fixed districts?

- Sometimes there are good reasons to want to leave electoral areas unchanged — like the U.S. states, the départements of France, the states of Brazil, etc.

- Within the U.S., counties are in some cases a powerfully relevant unit of identity and of administration.

- A natural idea is to keep fixed boundaries but **weight the votes** of the representatives.



Example: Canada

(+ note obvious parallel to U.S. electoral college)

House of Commons Seat Allocation by Province 2022–2032

On October 15th 2021, the Chief Electoral Officer (CEO) calculated the House of Commons seats to be allocated to each province using the Representation Formula found in the Constitution and the population estimates provided by Statistics Canada.

On June 27, 2022, the CEO had to redo the calculation in light of the *Act to amend the Constitution Act, 1867* (electoral representation), which amended Rule 2 of subsection 51(1) of the *Constitution Act, 1867* (the Grandfather Clause). The new calculation changed the number of MPs assigned to Quebec from 77 MPs to 78 MPs, which is the same number of MPs that Québec had in the 43rd Parliament.

This seat allocation will only take effect when new Representation Orders come into force. Consult the federal redistribution timeline to find out more.

Allocation of Seats in the House of Commons

Will take effect when the Representation Orders come into force

Province/ Territory	Population Estimate	÷ Electoral Quotient [1]	= Initial Seat Allocation	+ Senatorial Clause	+ Grandfather Clause	+ Representation Rule	= Total Seats
British Columbia	5,214,805		43	-	-	-	**43**
Alberta	4,442,879		37	-	-	-	**37**
Saskatchewan	1,179,844		10	-	4	-	**14**
Manitoba	1,383,765		12	-	2	-	**14**
Ontario	14,826,276		122	-	-	-	**122**
Quebec	8,604,495	121,891	71	-	7	-	**78**
New Brunswick	789,225		7	3	-	-	**10**
Nova Scotia	992,055		9	1	1	-	**11**
Prince Edward Island	164,318		2	2	-	-	**4**
Newfoundland and Labrador	520,553		5	1	1	-	**7**
Yukon	42,986			n/a			**1**
Northwest Territories	45,504			n/a			**1**
Nunavut	39,403			n/a			**1**
Total	**38,246,108**						**343**

A shot over the bow

WEIGHTED VOTING DOESN'T WORK: A MATHEMATICAL ANALYSIS

John F. Banzhaf III *

In response to recent decisions requiring that both houses of state legislatures be apportioned so that equal numbers of citizens have substantially equal representation, weighted voting has been widely suggested as an alternative to actual reapportionment. Weighted voting has already been adopted in at least three state legislative bodies, including the New Jersey Senate; ordered by one federal district court; considered in several other states; and used by one county governing body for a number of years. In New Jersey and New Mexico, state courts rejected, on state grounds, the weighted systems which had been adopted.[1] Under such systems, in lieu of actual redistricting or reap-

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1. N.M. Stat. Ann. §§ 2-7-1 to 2-7-13 (Supp. 1964) established a weighted voting plan for the New Mexico state legislature; this plan was later held to conflict with the state

Representative	Number of Votes
A	5
B	1
C	1
D	1
E	1

Representative	Number of Votes
R	8
S	8
T	8
U	8
V	1