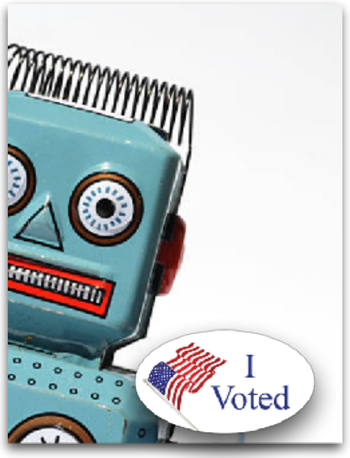
Modeling Democracy

Lecture 2 - **Voting rules and properties**



Moon Duchin, Winter 2026 — Wed Jan 7

Announcements

- **Assignment requirements**: PhD - 5 HW + project or presentation
  MS/Ugrad: 5 HW, where presentation can replace one HW and a project is fully optional
  After the first one, HW will be due Fridays on gradescope.

- **Slides and materials**: no Canvas, but we'll set up a github today. That will also include a link to a friendly intro-level social choice textbook.

- **Office Hours**: Alec will be available immediately after class on both Mon and Wed. Moon's OH TBD.

- **Fridays**: hold 1:30-2:30 (same as class time MW) for presentations. Attending these is "extra credit" for MS/Ugrad.

- **Skills**: overleaf, VoteKit. VoteKit trainings next week! Tue 10-12, Thu 1-3, Fri 1:30-3:30.

Key concepts recap

- **Ballot** — for now, ranking: a (possibly truncated) permutation of m candidates
- **Preference profile** — table of ballots
- **Voting rule** — maps profiles to outcomes (e.g., rankings to winner sets)
- **Condorcet candidate** — one who beats all others H2H (head-to-head)
- **Pairwise comparison graph** (PWCG) / tournament graph — directed complete graph on $\mathscr{C}$ with edges weighted by H2H margins
- **Condorcet cycle** — directed cycle in PWCG
- **Threshold** or **quota** — number/fraction of votes needed to be elected in IRV/STV

Reminder: Exercises from last time

- A Condorcet 3-cycle exists on 3 candidates when some three cyclically permuted columns have been selected n_1, n_2, n_3 times and those numbers satisfy the triangle inequality
- If a preference profile consists of complete rankings, then all margins in the PWCG have the same *parity* (all even or all odd)

IRV/STV recap

- You fix a threshold or quota when electing for k seats — common choice is “Droop quota” of $T = \frac{N}{k+1}$ (possibly rounded) for a profile with N valid ballots.
- An alternative is “Hare quota” N/k — variants include dynamical quota, changing by round
- Round 1: Suppose highest current candidate support is M.
  - If $M > T$ change candidate status to “elected.” Each vote currently supporting them is transferred with weight $\frac{M-T}{M}$ (share of support that was surplus).
  - Else, eliminate candidate with least support and transfer fully.
- Repeat in later rounds until k seats are filled.

Plurality	Positional (Borda)	Sequential
Pairwise (Lhull)	Condo-Borda	Top-Two
IRV	Coombs	STV
Beatpath (Schulze)	Smith	Smithified rules
Dodgson	Kemeny	“Secondality”

- **Plurality** — most first-place votes (FPV) wins
- **Positional (Borda)** — voter gives out points by position of ranking (e.g., 3 for 1st, 2 for 2nd, 1 for 3rd)
- **Sequential** — order fixed in advance, candidates face each other pairwise with one advancing based on head-to-head (H2H) comparison

Plurality	Positional (Borda)	Sequential
Pairwise (Lhull)	**Condo-Borda**	**Top-Two**
IRV	Coombs	STV
Beatpath (Schulze)	Smith	Smithified rules
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- **Pairwise** — each candidate gets 1 point per H2H win, 1/2 per tie
- **Condo-Borda** — Condorcet candidate if they exist, else Borda winner (a “Frankenstein” rule)
- **Top-Two** — two candidates with most FPV advance to instant runoff, decided by H2H

Plurality	Positional (Borda)	Sequential
Pairwise (Lhull)	Condo-Borda	Top-Two
IRV	**Coombs**	**STV**
Beatpath (Schulze)	Smith	Smithified rules
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- **IRV (instant runoff voting)** — successively eliminate those with fewest FPV, transferring their votes, until someone has a majority

- **Coombs**— same, but eliminate those with most last-place votes

- **STV (single transferable vote)** — set a threshold of election (typically $\frac{1}{k+1}$ or so) and successively elect and eliminate until k are elected

Beatpaths and domsets

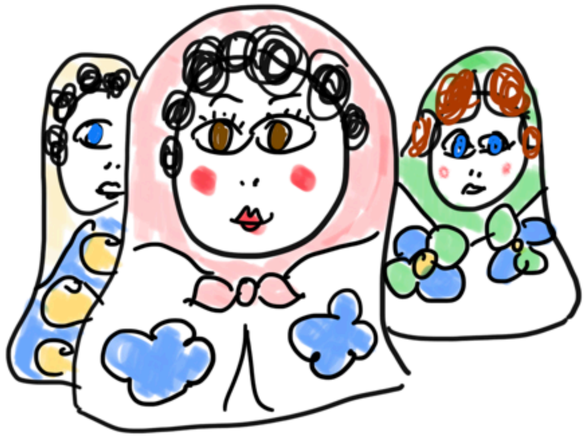
Beatpath: a directed path in the pairwise comparison graph

Strength of a beatpath: its lowest margin

Beatpath **elimination**: A eliminates B (denoted $A \triangleright B$) if A has a beatpath to B (strictly) stronger than any beatpath from B to A

Dominating set ("**domset**"): a nonempty subset of candidates that beats everyone else head-to-head

Fact you will prove: Domsets are **nested**!
So there is a smallest domset formed by intersecting them all.
This is called the **Smith set**, and its members are called "strong" candidates (and others are called "weak").



Plurality	Positional (Borda)	Sequential
Pairwise (Lhull)	Condo-Borda	Top-Two
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Beatpath (Schulze)	**Smith**	**Smithified rules**
Dodgson	Kemeny	“Secondality”

- **Beatpath** — candidate A eliminates B $(A \triangleright B)$ if there is a beatpath from A to B in pairwise comparison graph stronger than any path from B to A. Winner set is anyone not eliminated after all pairs are considered.

- **Smith**— candidates stratified into dominating tiers, where a dominating set has “all arrows pointing out”; Smith set / strong candidates/ winners are members of smallest domset

- **Smithified rules** — Smithified X: first consolidate to Smith set, then run method X

Transitivity

Recall that a relation R is transitive if $X \, R \, Y, \, Y \, R \, Z \implies X \, R \, Z$.

(Examples: *taller than*, *divisible by*. Nonexamples: *sibling of*.)

Head-to-head wins are not transitive — this is the **Condorcet paradox**. That means that if you tried to eliminate a candidate from contention because someone else beat them head-to-head, there would not be a well-defined outcome; it would depend on the order you considered them in.

However, as you will prove, beatpath elimination IS transitive, and this is enough to ensure that a beatpath election has at least one winner and that the winners are independent of the order of pairwise comparison.

Swaps and such

Let's say that two ballots differ by a "**neighbor swap**" if one can be transformed to the other by exchanging candidates in neighboring positions, such as

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \to \begin{pmatrix} A \\ C \\ B \\ D \end{pmatrix}$$

This can be considered an **elementary move** on ballots. (Another reasonable example of an elementary move is deleting a candidate name from the end of your ballot, or adding a name to the bottom of a partial ballot.)

You can get from any ballot to any other ballot this way. For complete ballots, this can be phrased in mathematical language by noting that neighbor swaps are a **generating set** for the symmetric group S_n.

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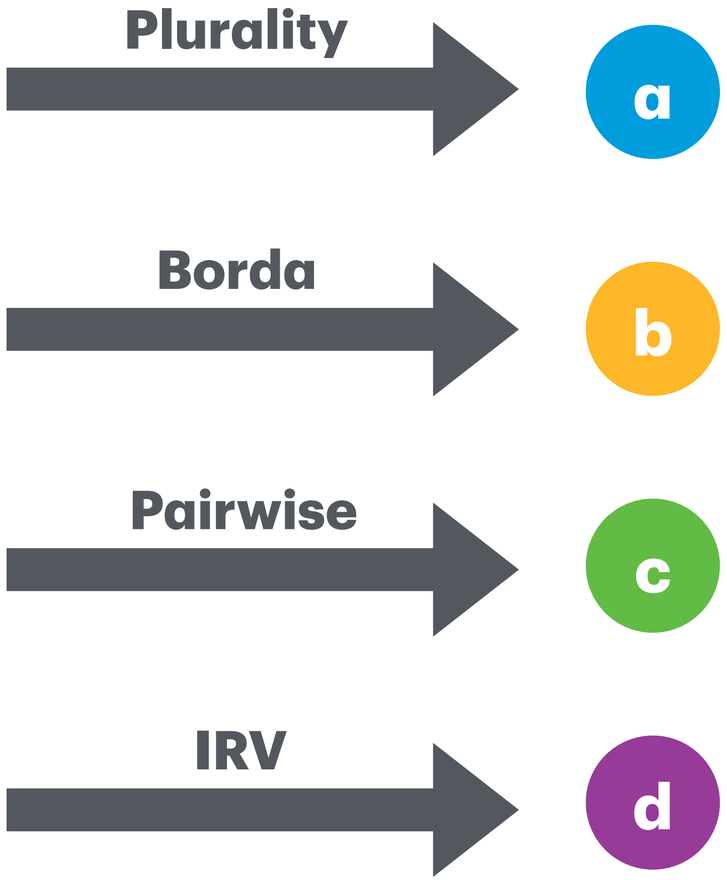
Dodgson	**Kemeny**	**“Secondality”**

- **Dodgson** — winner is candidate who requires the fewest voter ballot swaps to become Condorcet
- **Kemeny**— put a metric on ballots, say by swap distance/bubble sort. Winning ranking is minimizer of sum of distances to cast ballots.
- **Secondality** —most second-place votes wins!

ok so who *should*
win this one:

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a



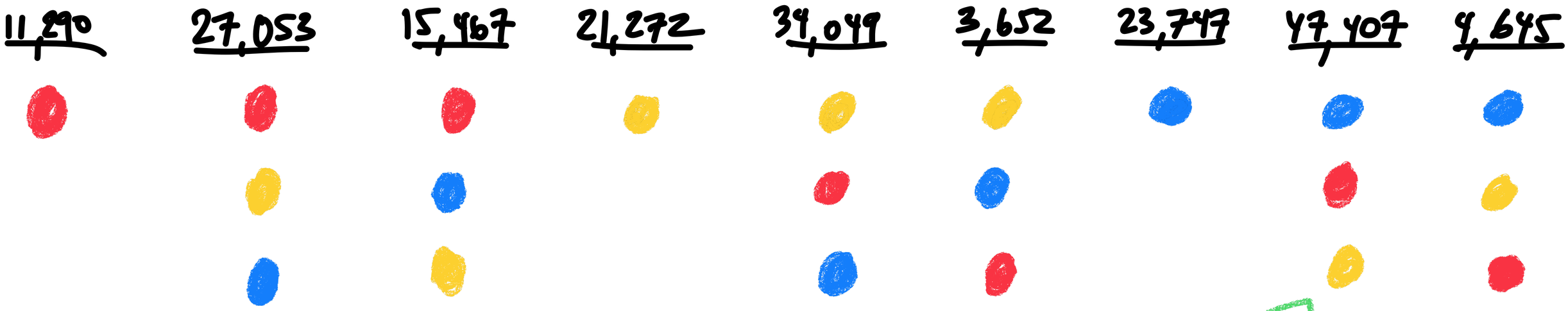
What are we after? Axioms of good voting rules

(Classic) fairness criteria

Majority-fair	**Condorcet-fair**	**Pareto efficient**
Unanimity-fair	Monotonic	Strongly monotonic
Strategy-proof	No (weak) spoilers	Smith-fair
Voter anonymity	Candidate anonymity	Independence of irrelevant alternatives

- **Majority-fair** — in a single-winner contest, a strict majority of FPV guarantees a win
- **Condorcet-fair** — if there is a Condorcet candidate, they win
- **Pareto efficient** — if everyone has same first-place choice, they win (or are ranked first if the output is a ranking)

2022 Special Election by IRV for U.S. Congress from Alaska



- Begich is Condorcet
- Begich is first to be eliminated
- Peltola easily defeats Palin H2H and wins


IOP FELLOWS
Dan Caldwell
Former Advisor to the U.S. Secretary of Defense
Yangyang Cheng
Research Scholar in Law and Fellow at Yale Law School's Paul Tsai China Center
Mary Peltola
Former U.S. Representative from Alaska
David Pressman
Former U.S. Ambassador to Hungary
Alex Wagner
Former Advisor to the U.S. Secretary of Defense
Vince Warren
Executive Director of the Center for Constitutional Rights
Come to one seminar or come to all of them — seminars are open to all students!
SEMINAR SCHEDULE
MONDAYS
TUESDAYS
WEDNESDAYS
THURSDAYS
12:30 – 1:45 PM
DAN CALDWELL
12:30 – 1:45 PM
ALEX WAGNER
3:30 – 4:45 PM
DAVID PRESSMAN
3:30 – 4:45 PM
VINCE WARREN
WEEKS OF NOV 3 & NOV 10
YANGYANG CHENG
WEEK OF OCT 20
MARY PELTOLA
REGISTER FOR SEMINARS
SIGN UP FOR ONE-ON-ONE OFFICE HOURS

More axioms

(Classic) fairness criteria

Majority-fair	Condorcet-fair	Pareto efficient
Unanimity-fair	**Monotonic**	**Strongly monotonic**
Strategy-proof	No (weak) spoilers	Smith-fair
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- **Unanimity property**— for single-winner, if everyone ranks X>Y then $Y \notin \mathscr{W}$; for ranking output, if everyone ranks X>Y then society ranks X>Y
- **Monotonic** — if $X \in \mathscr{W}$ and profile changes only by moves favorable to X, then $X \in \mathscr{W}'$
- **Strongly monotonic** — if $X \in \mathscr{W}$ and profile changes only by moves favorable to or neutral to X, then $X \in \mathscr{W}'$

More axioms

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- **Strategy-proof** — given a profile P with ballot B, there should be no B', P' so that $f(P')$ is preferable according to B
- **Spoilers** — X is ***not*** a spoiler if, when P,P' differ by removal of candidate X from P, one of these holds. If none of these, then X is a spoiler.
  (a) $\mathscr{W} = \mathscr{W}'$, (b) $\mathscr{W} = \{X\}$, (c) $\mathscr{W}' = \mathscr{W} \backslash \{X\}$.
- **Smith-fair** — only the strong can win. ($\mathscr{W} \subseteq \mathscr{S}$)

"weak spoiler" — a spoiler who is outside the Smith set

More axioms

(Classic) fairness criteria

Majority-fair	Condorcet-fair	Pareto efficient
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Strategy-proof	No (weak) spoilers	Smith-fair
Voter anonymity	**Candidate anonymity**	**Independence of irrelevant alternatives**

- **Voter anonymity** — independence of voter names or order
- **Candidate anonymity** — independence of candidate names or order
- **IIA** (for rankings) — whether X>Y in final ranking depends only on whether X>Y on ballots