Abstract Algebra Assignments © BinaryPhi

Name: __________ **Assignment:** Number 3

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Problem 1: Definitions

(a) Assuming H is a non-empty subset of group G while H is also a group with respect to the operation of G, we call H a __________ of G.

(b) H is a subgroup of a group G. If $H = \{e\}$ or $H = G$, H is called a ________________.
Other subgroups are called the _____________________.

(c) Prove that the following statements are equivalent if H is a non-empty subset of G.

1. $H < G$.
2. $a, b \in H \Longrightarrow a \circ b \in H, a^{-1} \in H$.
3. $a, b \in H \Longrightarrow a \circ b^{-1} \in H$.

(d) Assume H is a subgroup of group G, $a \in G$, then:

$$a \circ H = \{a \circ h \mid h \in H\}, H \circ a = \{h \circ a \mid h \in H\}$$

(Or often written as: $aH = \{ah \mid h \in H\}, Ha = \{ha \mid h \in H\}$) are called the __________ and __________ of H with the representative el ement a, respectively.

(e) Assuming H is a subgroup of group G and $aRb \iff a^{-1}b \in H$,
i) prove that the relation R in G is an equivalent relation and
ii) the equivalent class of a, $\overline{a}$, is exactly the left coset of H represented by a: aH;
iii) thus the set of all left cosets of H : $\{aH\}$ is a partition of G.

(f) The quotient set G/R of group G with respect to the equivalent relation $aRb \iff a^{-1}b \in H, H < G$ is called the ______________________________________ or __________________, denoted by $G/H^{\mathbb{L}}$.

(g) The ________ of a subgroup H in a group G is the number of left cosets or right cosets of H in G, which is denoted by $[G : H]$ or $|G : H|$.

(h) Assuming a group G has a subgroup $H < G$, we define H to be a __________________ of G (denoted by $H \lhd G$), if:

$$ghg^{-1} \in H, \forall g \in G, \forall h \in H.$$

(i) Prove the following statements are equivalent assuming G is a group and $H < G$:

1) $H \lhd G$;
2) $gH = Hg, \forall g \in G$;
3) $g_1 H \cdot g_2 H = g_1 g_2 H = \{g_1 h_1 g_2 h_2 \mid h_1, h_2 \in H\}$.

(j) Assuming G is a group and $H < G$, R is a relation defined by $aRb \iff a^{-1}b \in H$, then:

$$R \text{ is a congruence relation in } G \iff H \lhd G.$$

More importantly, the quotient set G/R and the operation with respect to the congruence relation R is, a group, which is also called the ________________ or ________________ of G by H, denoted by G/H.

Problem 2: Prove:

(a) Assuming H is a non-empty and finite subset of group G, we have

$$H < G \Longleftrightarrow H \text{ is closed under the operation of } G$$

(b) If H_1 and H_2 are both subgroup of group G, then $H_1 \cap H_2 < G$.

(c) $[\mathbb{Z} : m \circ \mathbb{Z}] = m$, where $m \in \mathbb{N}$

Problem 3: **Lagrange Theorem:**

For a finite group G, $H < G$, then we have:

$$|G| = [G : H] \cdot |H|,$$

which means the order of the subgroup H is a factor of the order of G.

Problem 4: **Corollary** of Lagrange Theorem:

If G is a finite group and $K < G, H < K$, we have:

$$[G : H] = [G : K] \cdot [K : H].$$

Problem 5: Which of the following are true?

(a) _______ There exists a group in which the cancellation law fails.

(b) _______ Every group has exactly two improper subgroups.

(c) _______ Every group is a subgroup of itself.

(d) _______ A subgroup can be defined as the subset of a group.

(e) _______ Every set of numbers that is a group under addition is also a group under multiplication.

Problem 6: Prove that

if G is an abelian group, written multiplicatively, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G.

Problem 7: Assume H, K are two normal subgroups of group G and $H \cap K = \{1\}$. Prove the following

$$hk = kh, \forall h \in H, \forall k \in K.$$

Problem 8: Assume H is a normal subgroup of group G. Prove that the sufficient prerequisite for G/H to be an abelian group is the following:

$$gkg^{-1}k^{-1} \in H, \forall g, k \in G.$$