Abstract Algebra Assignments © BinaryPhi

Name: __________ **Assignment:** Number 2

Score: __________ **Last Edit:** May 26, 2022 PDT

Problem 1: Definitions

(a) Let "$\circ$" be the binary operation in the non-empty set S, and satisfies the following:

$$(a \circ b) \circ c = a \circ (b \circ c), \quad \forall a, b, c \in S.$$

Then, the algebraic system $\{S; \circ\}$ is called a ____________ (S is a ____________ for short)

(b) If two elements e_1 and e_2 in the semigroup satisfy:

$$e_1 \circ a = a,$$
$$a \circ e_2 = a, \quad \forall a \in S$$

e_1 is called the ______________ of S, and e_2 is called the ______________ of S.

If an element e in the semigroup satisfies:

$$e \circ a = a \circ e = a, \quad \forall a \in S,$$

e is called the _______________ of S.

The semigroup that has _______________ is called a _______________.

(c) Assuming a monoid $\{S; \circ\}$ has the identity element e and an element $a \in S$, if:

$$a_1 \circ a = e,$$
$$a \circ a_2 = e, \quad \forall a_1, a_2 \in S$$

a_1 is called the ______________ of a, and a_2 is called the ______________ of a.

If:

$$a_3 \circ a = a \circ a_3 = e, \quad \forall a_3 \in S,$$

a_3 is called the _______________ of a, and denoted by $a_3 = a^{-1}$.

(d) If every element in monoid $\{S; \circ\}$ is invertible, then S is called a __________.

(e) A group is a set S with an operation "$\circ$" that satisfies the following:

Closure: ______________________________;

Associativity: ______________________________;

Identity: ______________________________;

Invertibility: ______________________________;

(f) Unilateral definition of the previous definition. Prove that a semigroup S is a group if it satisfies the following:

- $\forall a \in S,\ \exists\ b \in S$, so $b \circ a = e$;
- $\forall a \in S,\ \exists\ e \in S$, so $e \circ a = a$;

(g) Interesting Question: Does the previous conclusions still hold if the semigroup has a left inverse and a right identity:

- $\forall a \in S,\ \exists\ a^{-1} \in S$, so $a^{-1} \circ a = e$;
- $\forall a \in S,\ \exists\ e \in S$, so $a \circ e = a$.

(h) Let the operation "$\circ$" in an algebraic system be commutative, the group $\{S;\circ\}$ is called the ____________________ or _________________________.

(i) Prove that the operation "$\circ$" in group $\mathbb{G}$ is left(right) **Cancellative**:

$$\begin{aligned}\forall a,b,c \in \mathbb{G},\ a\circ b = a\circ c &\implies b=c\\ b\circ a = c\circ a &\implies b=c.\end{aligned}$$

(j) The number of elements in group $\mathbb{G}$ is called the __________ of $\mathbb{G}$, denoted by $|\mathbb{G}|$. If $|\mathbb{G}|$ is finite, we call $\mathbb{G}$ a ________________. If $|\mathbb{G}|$ has infinite order, we call $\mathbb{G}$ a ________________.

(k) Assuming the group $\mathbb{G}$ has an operation (multiplication or addtion) and a is an element of $\mathbb{G}$, if $\forall k \in \mathbb{N}$, $a^k \neq 1 (\neq e)$ or $ka \neq 0 (\neq e)$, we call the order of element a is ________________. If $\exists\ k \in N$, $a^k = e$ or $ka = 0$, the order of element a is _________________________________.

Problem 2: Prove:

1) There is only one inverse element of any element a in group $\mathbb{G}$.
2) For a group $\mathbb{G}$, $\forall a, b \in \mathbb{G}$, equations $a \circ x = b$ and $x \circ a = b$ have one and only one solution.
3) If $\forall a, b \in S$ for which S is a semigroup, S is a group if $a \circ x = b,\ x \circ a = b$ both have solutions.

Problem 3: Check if the following options are semigroups, monoids, or groups?

1) In $\mathbb{Z}$, $a \circ b = a - b$;
2) In $\mathbb{Z}$, $a \circ b = a + b + ab$;
3) In $\mathbb{Z}$, $a \circ b = a + b - ab$;

Problem 4: Define operation "$\circ$" in $S = \{x \mid x \in \mathbb{R}, x \neq -1\}$: $a \circ b = a + b + ab$. Prove that S is a group with respect to the operation "$\circ$". Then, solve equation $2 \circ x \circ 3 = 7$.

Problem 5: Prove:

$\mathbb{G}$ is an Abelian Group if the order of every non-identity element is 2.

Problem 6: Assuming M is a monoid, $m \in M$. Define another multiplication rule "$\circ$": $a \circ b = amb$.

Prove that M is a semigroup with respect to "$\circ$".

When is M a monoid with respect to "$\circ$"?

Problem 7: Assuming M is a monoid with an identity element e. It is said that the element a of M is invertible if there exists an element a^{-1} that satisfies $a^{-1}a = aa^{-1} = e$.

Prove the following statements:

1) If $a, b, c \in M$ and $ab = ca = e$, then a is invertible and $a^{-1} = b = c$.
2) If $a \in M$ is invertible, then $b = a^{-1}$ when and only when $aba = a,\ ab^2a = e$.
3) The sufficient prerequisite of G, the subset of M, being a group is that every element in G is invertible and for all $g_1, g_2 \in G$, we have $g_1^{-1}g_2 \in G$.
4) All invertible elements in M is a group.