Abstract Algebra Assignments © BinaryPhi

Name: __________ **Assignment:** Number 4

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Problem 1: Definitions

(a) Assuming $\{G_1; \circ\}$ and $\{G_2; *\}$ are two groups and f is a map from G_1 to G_2, if:

$$f(a \circ b) = f(a) * f(b), \quad \forall a, b \in G_1,$$

the map f is called a ____________________.

If G_1 and G_2 are two same groups,

f is called an ________________.

If a ____________________ f is an injection (one-to-one),

f is called a ________________.

If a ____________________ f is a surjection (onto),

f is called an ________________.

If a ____________________ f is a bijection (one-to-one correspondence, invertible),

f is called an ________________, and G_1 and G_2 are ________________, which is denoted by $G_1 \cong G_2$.

(b) Supplement:

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$$f : A \to B, \forall\, a, b \in A, \text{ such that } f(a) = f(b) \Longrightarrow a = b.$$

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$$f : A \to B, \forall\, b \in B, \exists\, a \in A \text{ s.t. } f(a) = b.$$

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$$f : A \to B, \forall\, b \in B, \text{exists a unique } a \in A \text{ s.t. } f(a) = b.$$

(c) Assuming f is a group homomorphism from group G_1 to group G_2, then the set of all elements from G_1 which map to element e in G_2 is called the _________ of group homomorphism f, which is denoted by _____. Mathematically written as:

$$\_\_\_\_\_ := \{g_1 \in G_1 \mid f(g_1) = e\}.$$

(d) Assuming f is a group homomorphism from group G_1 to group G_2, e_1, e_2 are the identity elements in G_1, G_2 respectively, $\circ, *$ are the operations in G_1, G_2 respectively, prove that $f(e_1) = e_2$ and $\forall a \in G_1, f(a^{-1}) = f(a)^{-1}$.

(e) Assuming f is a group homomorphism from group G_1 to group G_2, $H < G_1$, prove that the image set of H, $f(H)$ is a subgroup of G_2.

(f) Assuming G is a group, $H \lhd G$, ι is a map from G to G/H:

$$\iota(a) = aH,\ \forall a \in G.$$

Then, ι is an epimorphism, and is called the ________________________ from group G to quotient group G/H.

(g) **Group Isomorphism Theorem I | Fundamental Theorem on Group Homomorphisms**

Prove that if f is an epimorphism from group G_1 to group G_2, $G_1/\ker f \cong G_2$.

(h) **Group Isomorphism Theorem II**

Let G be a group, $N \lhd G$, and H is a subgroup of G. Then:

1. HN is a subgroup of G which contains N.
2. $(H \cap N) \lhd H$.
3. $HN/N \cong H/(H \cap N)$.

(i) **Group Isomorphism Theorem III**

Let G be a group, $N \lhd G, N \lhd G, N \subseteq H$. Then:

1. $H/N \lhd G/N$
2. $(G/N)/(H/N) \cong G/H$

(j) **Group Isomorphism Theorem IV | Correspondence Theorem**

Assume f is an epimorphism from group G_1 to G_2, and the kernel of group homomorphism f is $F = \ker f$. We have:

1. The map from a subgroup of G_1 that contains N to a subgroup of G_2 is bijective.
2. The bijection from the subgroup of G_1 that contains N to the subgroup of G_2 is also a map from a normal subgroup onto a normal subgroup.
3. For a normal subgroup $H \lhd G_1$ such that H contains N, $G_1/H \cong G_2/f(H)$.

Problem 2: Prove:

(a) Assuming f is a group homomorphism from group G_1 to G_2, we have $\ker f \lhd G_1$.

(b) Assuming f is a group homomorphism from group G_1 to group G_2, then

$$f \text{ is monomorphism } \iff \ker f = \{e_1\}, \text{ where } e_1 \text{ is the identity of } G_1.$$

Problem 3: Define a binary operation $\circ$ in the integer set $\mathbb{Z}$ such that:

$$a \circ b = a + b - a \times b, \quad \forall a, b \in \mathbb{Z}.$$

Prove that $\{\mathbb{Z}, \circ\}$ is a monoid, and is isomorphic to a monoid of $\mathbb{Z}$ with respect to the operation multiplication "$\times$".

Problem 4: Let G be a group, prove the following statements:

$$m \longrightarrow m^{-1} \text{ is an automorphism of } G \text{ if and only if } G \text{ is an Abelian Group.}$$

Problem 5: Assume G is an abelian group, prove that

$$\forall n \in \mathbb{Z}, m \longrightarrow m^n \text{ is an endomorphism of } G$$

Problem 6: Let $\phi : G \longrightarrow H$ be a group homomorphism.

Prove that $\phi(G)$ is abelian if and only if $\forall a, b \in G, aba^{-1}b^{-1} \in \ker \phi$.

Problem 7: The map $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $\phi(n) = n - 1$ for $n \in \mathbb{Z}$ is bijective. Give the expression of the binary operation "*" on $\mathbb{Z}$ such that ϕ is isomorphic.

$$\{\mathbb{Z}, \times\} \longrightarrow \{\mathbb{Z}, *\}$$

Problem 8: The map $\phi : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined by $\phi(n) = 2n + 1$ for $n \in \mathbb{Q}$ is bijective. Give the expression of the binary operation "*" on $\mathbb{Q}$ such that ϕ is isomorphic.

$$\{\mathbb{Q}, *\} \longrightarrow \{\mathbb{Q}, +\}$$