Solution - Abstract Algebra Assignments © BinaryPhi

Name: __________ **Assignment:** Number 1

Score: __________ **Last Edit:** May 30, 2022 PDT

Problem 1: Definitions

(a) Assuming A and B are two sets, the **Direct Product** (or Cartesian Product) of set A and set B is defined as

$$A \times B = \{\underline{(a, b) \mid a \in A, b \in B}\}.$$

(b) Assuming A, B and C are three non-empty sets, the mapping from $A \times B$ to C is called an **Algebraic Operation**.

(c) Most of the time, we have $A = B = C$(an algebraic operation from A and A to A), which is called the **Binary Operation** in A.

(d) **Associative Property** denotes to a type of binary operation "$\circ$" in set A if

$$\underline{(a \circ b) \circ c = a \circ (b \circ c)}, \forall a, b, c \in A.$$

(e) **Commutative Property** denotes to a type of binary operation "$\circ$" in set A if

$$\underline{a \circ b = b \circ a}, \forall a, b \in A.$$

(f) "$\circ$" is left-distributive over "$+$" if

$$\underline{a \circ (b + c) = a \circ b + a \circ c}, \ \forall a, b, c \in A.$$

"$\circ$" is right-distributive over "$+$" if

$$\underline{(b + c) \circ a = b \circ a + c \circ a}, \ \forall a, b, c \in A.$$

"$\circ$" is **Distributive** over "$+$" if it is both left- and right-distributive.

(g) Assuming A is a non-empty set and R is a subset of $A \times A$, $a, b \in A$, if $(a, b) \in R$, we define that a and b have a relation R, denoted by $\underline{aRb\ (a \sim b)}$. R denotes a **relation** of A.

(h) An **Equivalent Relation** R from set A satisfies the following, $\forall (a, b, c) \in A$:

1. **Reflexive** Property: $\underline{aRa}$
2. **Symmetric** Property: $\underline{aRb \Rightarrow bRa}$
3. **Transitive** Property: $\underline{aRb, bRc \Rightarrow aRc}$

(i) A set of non-empty subsets of A, such that every element of A is included in exactly one subset of A, is defined as a **Partition** of set A.

(j) Assuming R is an equivalent relation in set A, $a \in A$, the set of all elements that have the relation R with a: $\{b \in A \mid bRa\}$, is defined as the **Equivalence Class** of a (also denoted by $\underline{\ \bar{a}\ }$). a is called a **representative** of the class.

(k) Assuming R is an equivalent relation in set A, then the set of all equivalence classes of A with respect to the relation R: $\{\bar{a} \mid a \in A\}$, is called the **Quotient (Set)** of A by R, and is denoted by $\underline{A/R}$.

(l) Assuming R is an equivalent relation in set A, then the map

$$\iota : A \to A/R,\ \iota(a) = \bar{a},\ \forall a \in A,$$

is called the **Cononical Map** from A to A/R.

(m) Assuming a binary operation "$\circ$" is in set A, if an equivalent relation R of A satisfies under this binary operation:

$$\underline{aRb, cRd \Longrightarrow (a \circ c)R(b \circ d), \quad \forall a, b, c, d \in A},$$

R is a **Congruence Relation** with respect to operation "$\circ$", by definition. For the equivalence class of a, $\bar{a}$ is called the **Congruence Class**.

Problem 2: Justify if R of each of the following relation is an equivalence relation:

1) For two $m \times n$ matrix A and B, we have ARB if there exists an invertible $n \times n$ matrix P and an invertible $m \times m$ matrix Q that satisfy $A = PBQ$.

Reflexive Property:

$$A = PAQ$$

Commutative Property:

$$A = PBQ \Longrightarrow B = PAQ$$

Associative Property:

$$B = PAQ, C = PBQ \Longrightarrow C = PAQ$$

2) For two $m \times n$ matrix A and B, we have ARB if there exists an $n \times n$ matrix P and an $m \times m$ matrix Q that satisfy $A = PBQ$.

Incorrect Statement.

3) For two $m \times m$ matrix A and B, we have ARB if there exists an invertible $n \times n$ matrix P that satisfy $A = PBP^{-1}$.

Reflexive Property:

$$A = PAP^{-1}$$

Commutative Property:

$$A = PBP^{-1} \Longrightarrow B = PAP^{-1}$$

Associative Property:

$$B = PAP^{-1}, C = PBP^{-1} \Longrightarrow C = PAP^{-1}$$

Problem 3: Which of the following binary operation "$\sim$" has commutative property? Which of the following has associative property?

1) $a \sim b = a - b, \quad \forall a, b \in \mathbb{Z};$

Commutative Property: None;

Associative Property: None;

2) $a \sim b = a^b, \quad \forall a, b \in \mathbb{N};$

Commutative Property: None;

Associative Property: None;

3) $a \sim b = a^b b^a, \quad \forall a, b \in \mathbb{N};$

Commutative Property: Exist;

Associative Property: None:

$$\begin{aligned} \forall c \in \mathbb{N}, (a \sim b) \sim c &= (a^b b^a)^c c^{(a^b b^a)} \\ a \sim (b \sim c) &= a^{(b^c c^b)} (b^c c^b)^a \end{aligned}$$

4) $a \sim b = a^2 b^2, \quad \forall a, b \in \mathbb{Q};$

Commutative Property: Exist;

Associative Property: Exist;