Abstract Algebra Assignments © BinaryPhi

Name: __________ **Assignment:** Number 1

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Problem 1: Definitions

(a) Assuming A and B are two sets, the **Direct Product** (or ______________) of set A and set B is defined as

$$A \times B = \{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}\}.$$

(b) Assuming A, B and C are three non-empty sets, the mapping from $A \times B$ to C is called a(an) ____________________.

(c) Most of the time, we have $A = B = C$(an algebraic operation from A and A to A), which is called the ____________________ in A.

(d) **Associative Property** denotes to a type of binary operation "$\circ$" in set A if

$$\underline{\qquad\qquad\qquad\qquad\qquad}, \forall a, b, c \in A.$$

(e) **Commutative Property** denotes to a type of binary operation "$\circ$" in set A if

$$\underline{\qquad\qquad\qquad\qquad\qquad}, \forall a, b \in A.$$

(f) "$\circ$" is left-distributive over "$+$" if

$$\underline{\qquad\qquad\qquad\qquad\qquad}, \forall a, b, c \in A.$$

"$\circ$" is right-distributive over "$+$" if

$$\underline{\qquad\qquad\qquad\qquad\qquad}, \forall a, b, c \in A.$$

"$\circ$" is **Distributive** over "$+$" if it is both left- and right-distributive.

(g) Assuming A is a non-empty set and R is a subset of $A \times A$, $a, b \in A$, if $(a, b) \in R$, we define that a and b have a relation R, denoted by ________________. R denotes a **relation** of A.

(h) An **Equivalent Relation** R from set A satisfies the following, $\forall (a, b, c) \in A$:

1. _____________ Property: ___________________
2. _____________ Property: ___________________
3. _____________ Property: ___________________

(i) A set of non-empty subsets of A, such that every element of A is included in exactly one subset of A, is defined as a _____________ of set A.

(j) Assuming R is an equivalent relation in set A, $a \in A$, the set of all elements that have the relation R with a: $\{b \in A \mid bRa\}$, is defined as the ___________________ of a (also denoted by ___). a is called **representative** of the class.

(k) Assuming R is an equivalent relation in set A, then the set of all equivalence classes of A with respect to the relation R: $\{\bar{a} \mid a \in A\}$, is called the ___________________ of A by R, and is denoted by _____________.

(l) Assuming R is an equivalent relation in set A, then the map

$$\iota : A \to A/R, \ \iota(a) = \bar{a}, \ \forall a \in A,$$

is called the ___________________ from A to A/R.

(m) Assuming a binary operation "$\circ$" is in set A, if an equivalent relation R of A satisfies under this binary operation:

____________________________________,

we denotes R to be a **Congruence Relation** with respect of operation "$\circ$". For the equivalence class of a, $\bar{a}$ is called the ___________________.

Problem 2: Justify if R of each of the following relation is an equivalence relation:

1) For two $m \times n$ matrix A and B, we have ARB if there exists an invertible $n \times n$ matrix P and an invertible $m \times m$ matrix Q that satisfy $A = PBQ$.

2) For two $m \times n$ matrix A and B, we have ARB if there exists an $n \times n$ matrix P and an $m \times m$ matrix Q that satisfy $A = PBQ$.

3) For two $m \times m$ matrix A and B, we have ARB if there exists an invertible $n \times n$ matrix P that satisfy $A = PBP^{-1}$.

Problem 3: Which of the following binary operation "$\sim$" has commutative property? Which of the following has associative property?

1) $a \sim b = a - b, \quad \forall a, b \in \mathbb{Z}$;

2) $a \sim b = a^b, \quad \forall a, b \in \mathbb{N}$;

3) $a \sim b = a^b b^a, \quad \forall a, b \in \mathbb{N}$;

4) $a \sim b = a^2 b^2, \quad \forall a, b \in \mathbb{Q}$;

Problem 4: Assuming R is an equivalence relation in set A and "$\circ$" is an operation in A, the operation "$\bar{\circ}$" in the quotient set A/R is:

$$\bar{a} \mathbin{\bar{\circ}} \bar{b} = \overline{a \circ b}, \quad \forall a, b \in A.$$

Prove that the sufficient prerequisite for "$\bar{a} = \bar{c},\ \bar{b} = \bar{d} \Longrightarrow (\overline{a \circ b}) = (\overline{c \circ d})$" is that R is a congruence relation in A with respect to operation "$\circ$".