
Abstract Algebra Assignments © BinaryPhi

Name: _____

Assignment: Number 3

Score: _____

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Problem 1: Definitions

- (a) Assuming H is a non-empty subset of group G while H is also a group with respect to the operation of G , we call H a _____ of G .
- (b) H is a subgroup of a group G . If $H = \{e\}$ or $H = G$, H is called a _____. Other subgroups are called the _____.
- (c) Prove that the following statements are equivalent if H is a non-empty subset of G .
1. $H < G$.
 2. $a, b \in H \implies a \circ b \in H, a^{-1} \in H$.
 3. $a, b \in H \implies a \circ b^{-1} \in H$.

(d) Assume H is a subgroup of group G , $a \in G$, then:

$$a \circ H = \{a \circ h \mid h \in H\}, H \circ a = \{h \circ a \mid h \in H\}$$

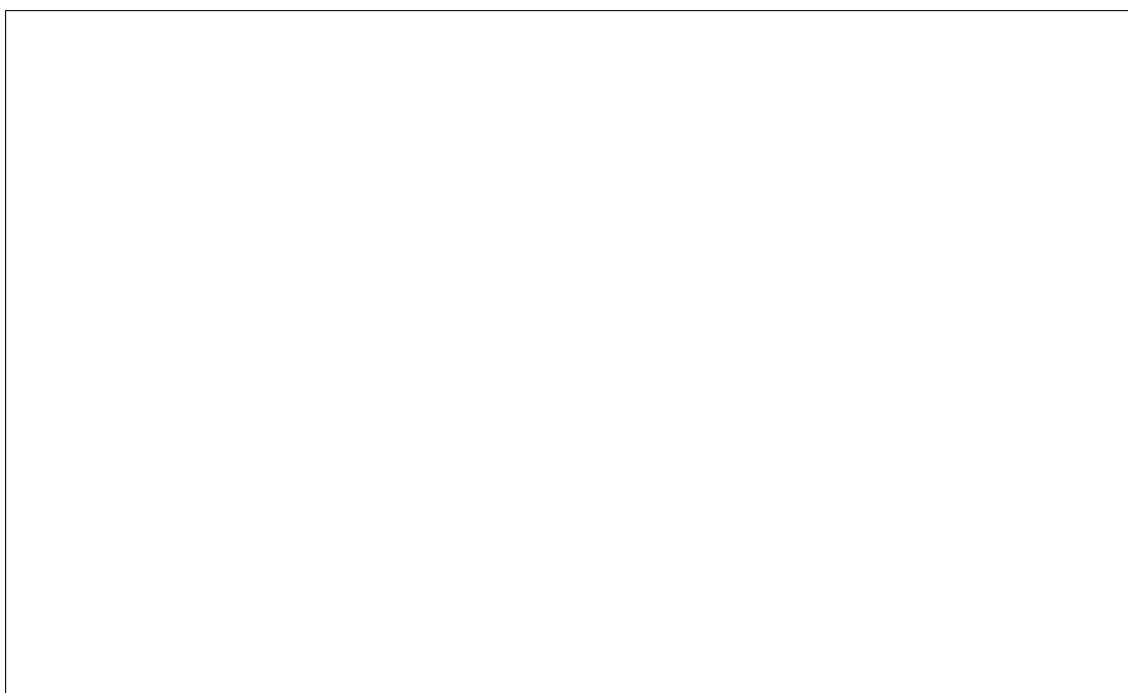
(Or often written as: $aH = \{ah \mid h \in H\}, Ha = \{ha \mid h \in H\}$) are called the _____ and _____ of H with the representative element a , respectively.

(e) Assuming H is a subgroup of group G and $aRb \iff a^{-1}b \in H$,

i) prove that the relation R in G is an equivalent relation and

ii) the equivalent class of a, \bar{a} , is exactly the left coset of H represented by a : aH ;

iii) thus the set of all left cosets of H : $\{aH\}$ is a partition of G .



(f) The quotient set G/R of group G with respect to the equivalent relation $aRb \iff a^{-1}b \in H, H < G$ is called the _____ or _____, denoted by G/H^L .

(g) The _____ of a subgroup H in a group G is the number of left cosets or right cosets of H in G , which is denoted by $[G : H]$ or $|G : H|$.

- (h) Assuming a group G has a subgroup $H < G$, we define H to be a _____ of G (denoted by $H \triangleleft G$), if:

$$ghg^{-1} \in H, \forall g \in G, \forall h \in H.$$

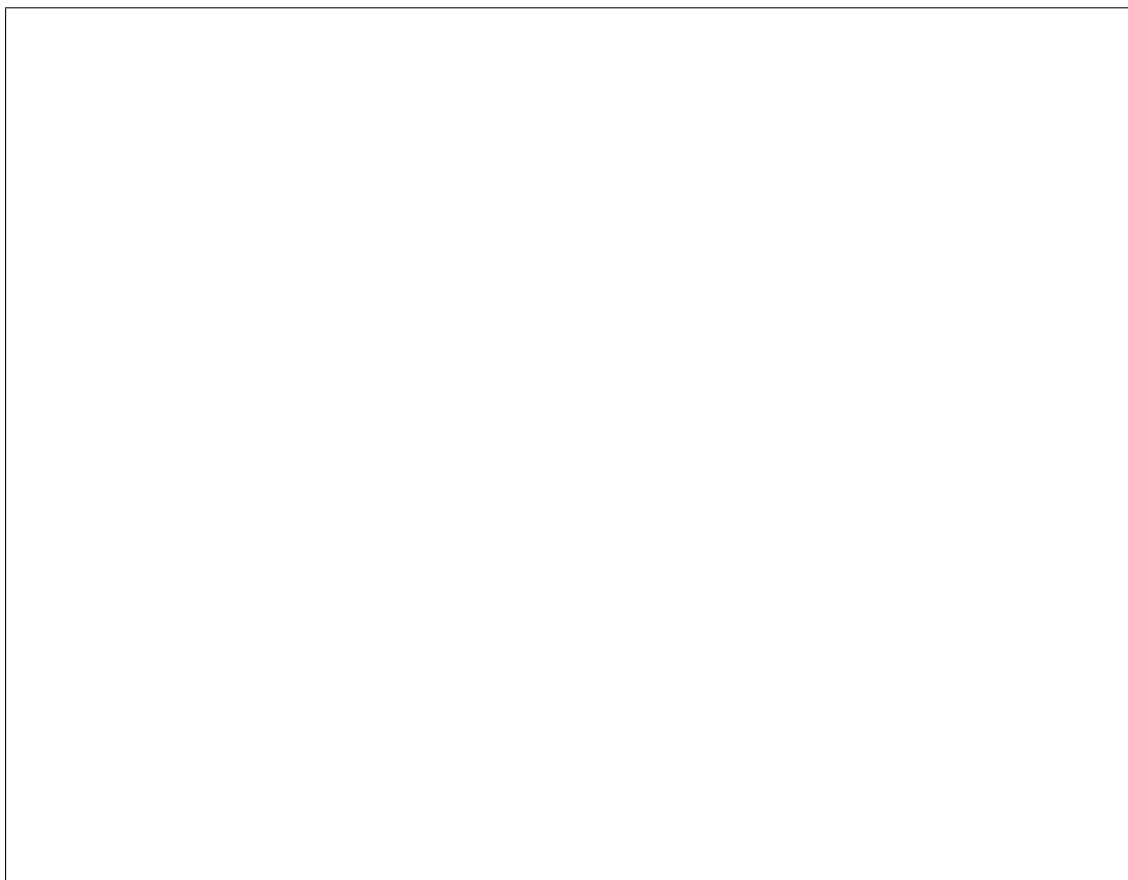
- (i) Prove the following statements are equivalent assuming G is a group and $H < G$:

- 1) $H \triangleleft G$;
- 2) $gH = Hg, \forall g \in G$;
- 3) $g_1H \cdot g_2H = g_1g_2H = \{g_1h_1g_2h_2 \mid h_1, h_2 \in H\}$.



- (j) Assuming G is a group and $H < G$, R is a relation defined by $aRb \iff a^{-1}b \in H$, then:

$$R \text{ is a congruence relation in } G \iff H \triangleleft G.$$



More importantly, the quotient set G/R and the operation with respect to the congruence relation R is, a group, which is also called the _____ or _____ of G by H , denoted by G/H .

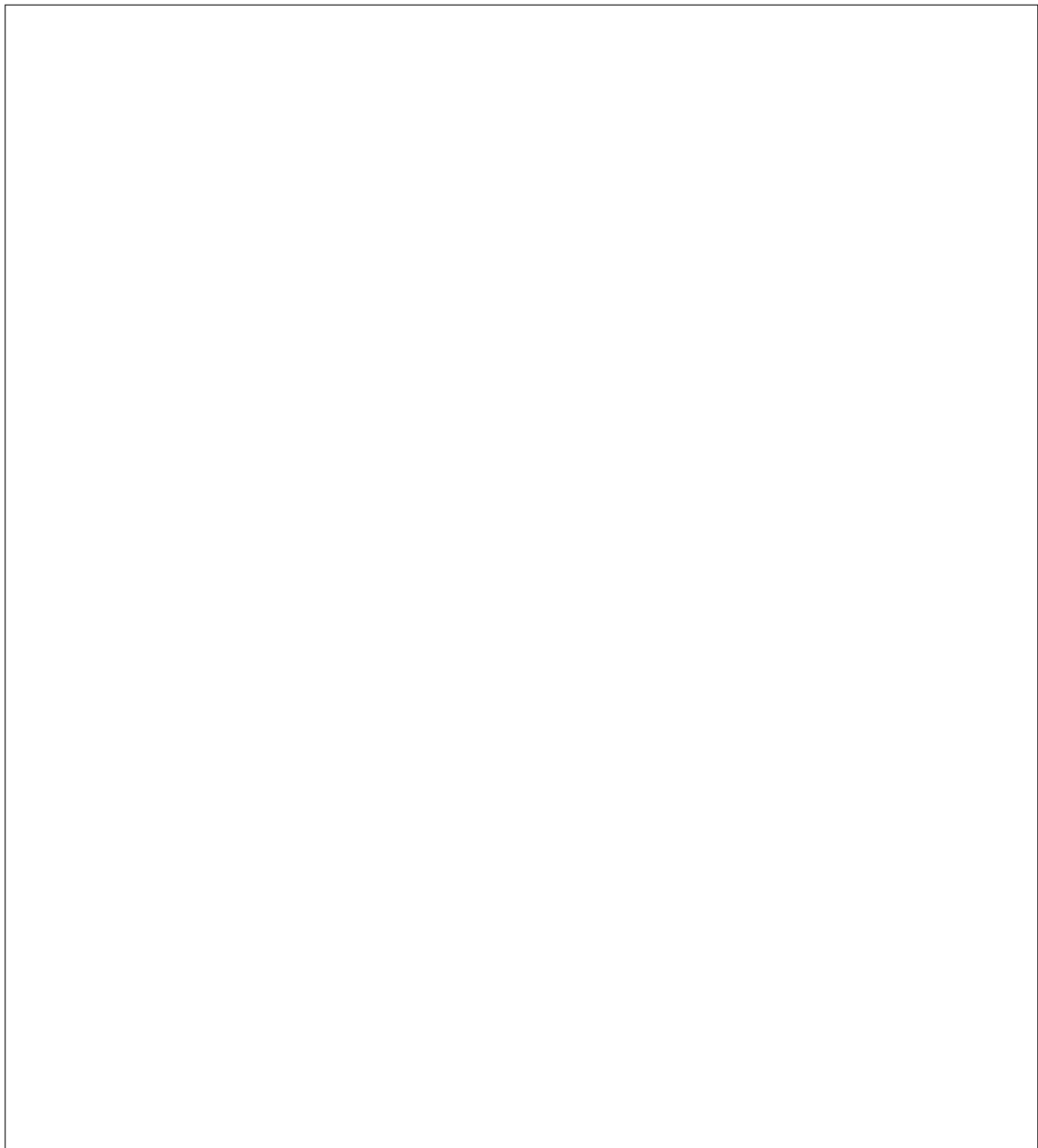
Problem 2: Prove:

(a) Assuming H is a non-empty and finite subset of group G , we have

$$H < G \iff H \text{ is closed under the operation of } G$$

(b) If H_1 and H_2 are both subgroup of group G , then $H_1 \cap H_2 < G$.

(c) $[\mathbb{Z} : m \circ \mathbb{Z}] = m$, where $m \in \mathbb{N}$

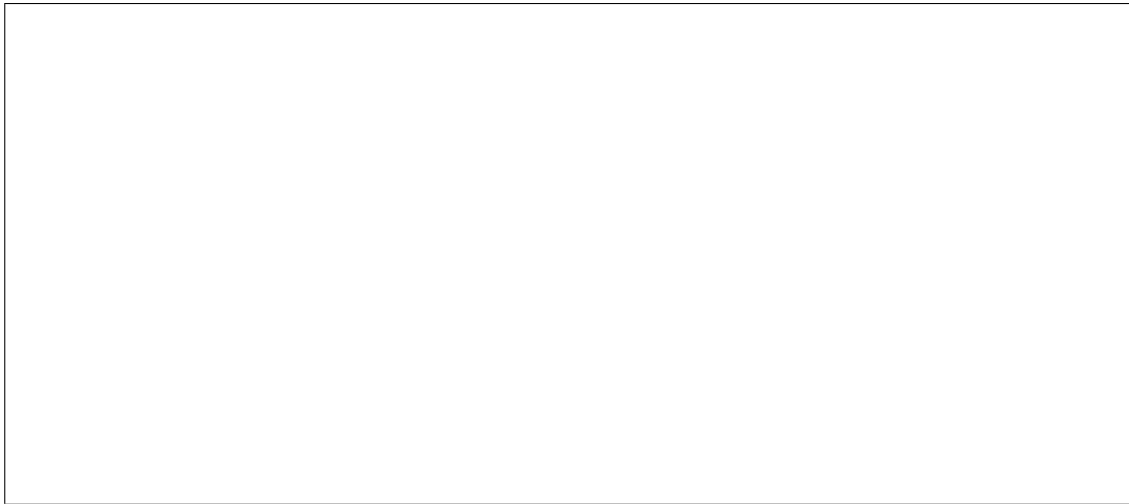


Problem 3: Lagrange Theorem:

For a finite group G , $H < G$, then we have:

$$|G| = [G : H] \cdot |H|,$$

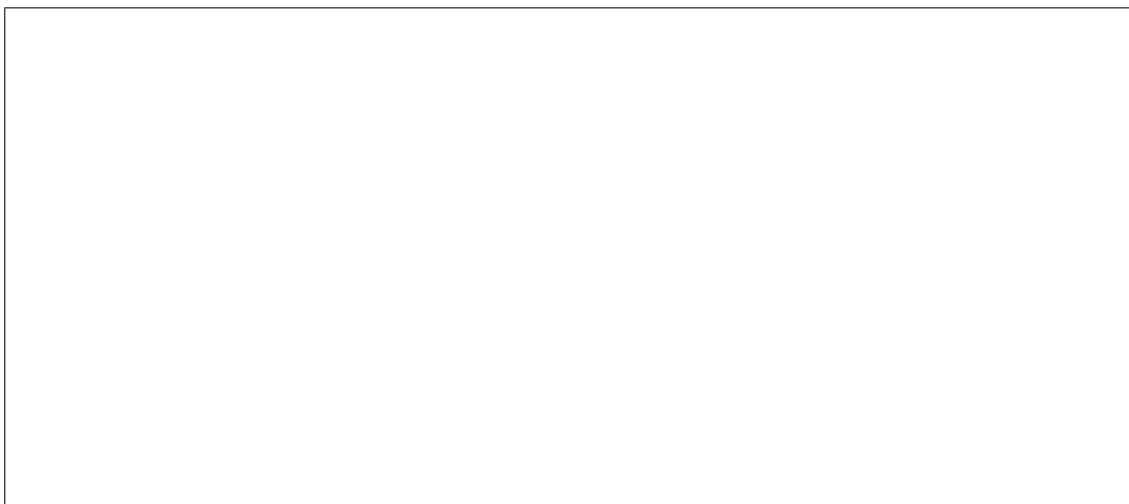
which means the order of the subgroup H is a factor of the order of G .



Problem 4: Corollary of Lagrange Theorem:

If G is a finite group and $K < G, H < K$, we have:

$$[G : H] = [G : K] \cdot [K : H].$$

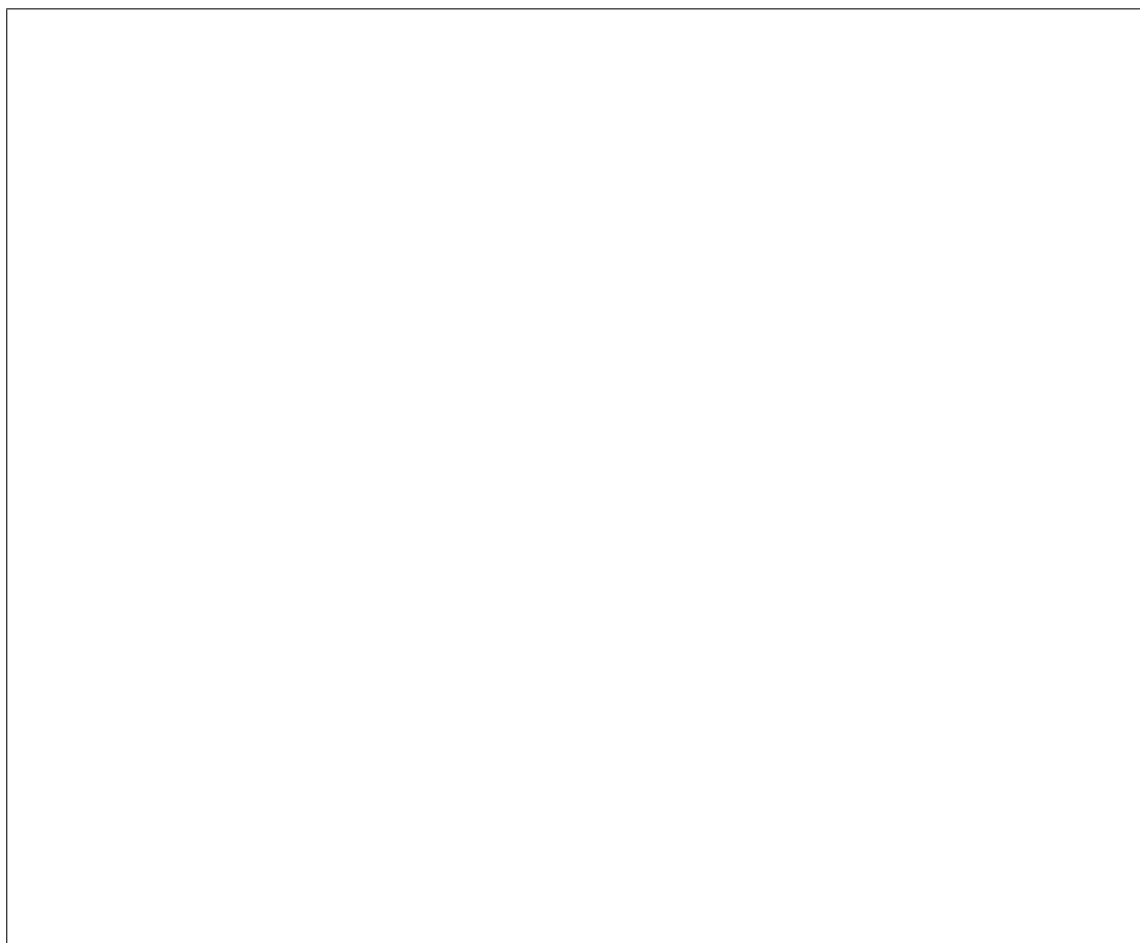


Problem 5: Which of the following are true?

- (a) _____ There exists a group in which the cancellation law fails.
- (b) _____ Every group has exactly two improper subgroups.
- (c) _____ Every group is a subgroup of itself.
- (d) _____ A subgroup can be defined as the subset of a group.
- (e) _____ Every set of numbers that is a group under addition is also a group under multiplication.

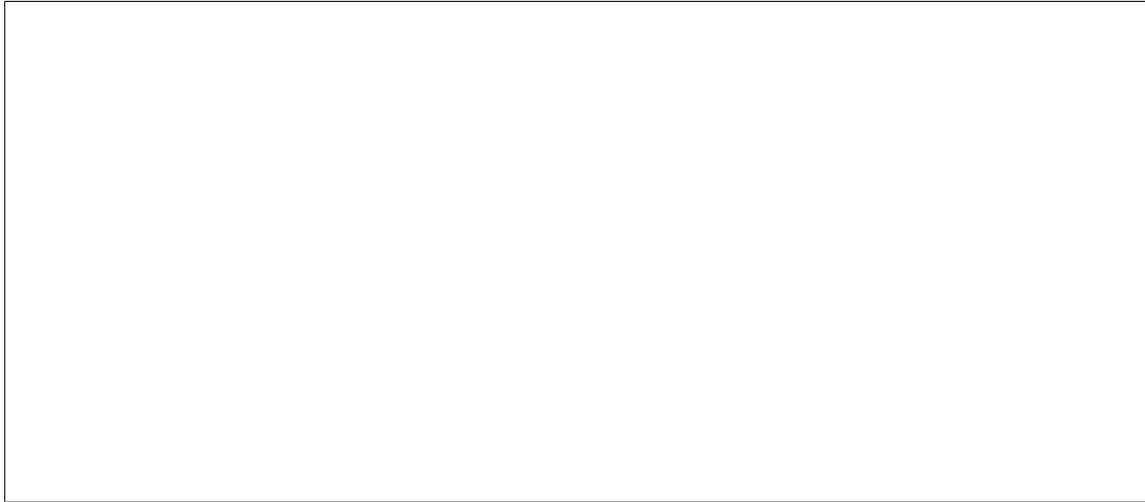
Problem 6: Prove that

if G is an abelian group, written multiplicatively, with identity element e , then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G .



Problem 7: Assume H, K are two normal subgroups of group G and $H \cap K = \{1\}$. Prove the following

$$hk = kh, \forall h \in H, \forall k \in K.$$



Problem 8: Assume H is a normal subgroup of group G . Prove that the sufficient prerequisite for G/H to be an abelian group is the following:

$$gkg^{-1}k^{-1} \in H, \forall g, k \in G.$$

