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## Abstract Algebra Assignments © BinaryPhi

Name: \_\_\_\_\_

Assignment: Number 2

Score: \_\_\_\_\_

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### Problem 1: Definitions

- (a) Let " $\circ$ " be the binary operation in the non-empty set  $S$ , and satisfies the following:

$$(a \circ b) \circ c = a \circ (b \circ c), \quad \forall a, b, c \in S.$$

Then, the algebraic system  $\{S; \circ\}$  is called a \_\_\_\_\_ ( $S$  is a \_\_\_\_\_ for short)

- (b) If two elements  $e_1$  and  $e_2$  in the semigroup satisfy:

$$\begin{aligned} e_1 \circ a &= a, \\ a \circ e_2 &= a, \quad \forall a \in S \end{aligned}$$

$e_1$  is called the \_\_\_\_\_ of  $S$ , and  $e_2$  is called the \_\_\_\_\_ of  $S$ .

If an element  $e$  in the semigroup satisfies:

$$e \circ a = a \circ e = a, \quad \forall a \in S,$$

$e$  is called the \_\_\_\_\_ of  $S$ .

The semigroup that has \_\_\_\_\_ is called a \_\_\_\_\_.

- (c) Assuming a monoid  $\{S; \circ\}$  has the identity element  $e$  and an element  $a \in S$ , if:

$$\begin{aligned} a_1 \circ a &= e, \\ a \circ a_2 &= e, \quad \forall a_1, a_2 \in S \end{aligned}$$

$a_1$  is called the \_\_\_\_\_ of  $a$ , and  $a_2$  is called the \_\_\_\_\_ of  $a$ .

If:

$$a_3 \circ a = a \circ a_3 = e, \quad \forall a_3 \in S,$$

$a_3$  is called the \_\_\_\_\_ of  $a$ , and denoted by  $a_3 = a^{-1}$ .

- (d) If every element in monoid  $\{S; \circ\}$  is invertible, then  $S$  is called a \_\_\_\_\_.

(e) A group is a set  $S$  with an operation " $\circ$ " that satisfies the following:

- Closure:** \_\_\_\_\_;  
**Associativity:** \_\_\_\_\_;  
**Identity:** \_\_\_\_\_;  
**Invertibility:** \_\_\_\_\_;

(f) Unilateral definition of the previous definition. Prove that a semigroup  $S$  is a group if it satisfies the following:

- $\forall a \in S, \exists b \in S, \text{ so } b \circ a = e;$
- $\forall a \in S, \exists e \in S, \text{ so } e \circ a = a;$

(g) Interesting Question: Does the previous conclusions still hold if the semigroup has a left inverse and a right identity:

- $\forall a \in S, \exists a^{-1} \in S, \text{ so } a^{-1} \circ a = e;$
- $\forall a \in S, \exists e \in S, \text{ so } a \circ e = a.$

(h) Let the operation "o" in an algebraic system be commutative, the group  $\{S; \circ\}$  is called the \_\_\_\_\_ or \_\_\_\_\_.

(i) Prove that the operation "o" in group  $\mathbb{G}$  is left(right) **Cancellative**:

$$\begin{aligned}\forall a, b, c \in \mathbb{G}, \quad a \circ b = a \circ c &\implies b = c \\ b \circ a = c \circ a &\implies b = c.\end{aligned}$$

(j) The number of elements in group  $\mathbb{G}$  is called the \_\_\_\_\_ of  $\mathbb{G}$ , denoted by  $|\mathbb{G}|$ .  
If  $|\mathbb{G}|$  is finite, we call  $\mathbb{G}$  a \_\_\_\_\_. If  $|\mathbb{G}|$  has infinite order, we call  $\mathbb{G}$  a \_\_\_\_\_.

(k) Assuming the group  $\mathbb{G}$  has an operation (multiplication or addition) and  $a$  is an element of  $\mathbb{G}$ , if  $\forall k \in \mathbb{N}, a^k \neq 1 (\neq e)$  or  $ka \neq 0 (\neq e)$ , we call the order of element  $a$  is \_\_\_\_\_. If  $\exists k \in \mathbb{N}, a^k = e$  or  $ka = 0$ , the order of element  $a$  is \_\_\_\_\_.

**Problem 2: Prove:**

- 1) There is only one inverse element of any element  $a$  in group  $\mathbb{G}$ .
- 2) For a group  $\mathbb{G}$ ,  $\forall a, b \in \mathbb{G}$ , equations  $a \circ x = b$  and  $x \circ a = b$  have one and only one solution.
- 3) If  $\forall a, b \in S$  for which  $S$  is a semigroup,  $S$  is a group if  $a \circ x = b$ ,  $x \circ a = b$  both have solutions.

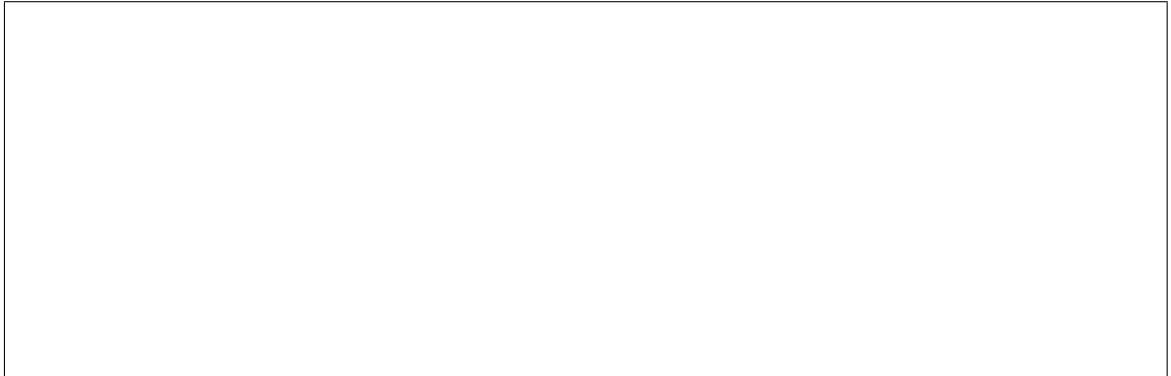
**Problem 3:** Check if the following options are semigroups, monoids, or groups?

- 1) In  $\mathbb{Z}$ ,  $a \circ b = a - b$ ;
- 2) In  $\mathbb{Z}$ ,  $a \circ b = a + b + ab$ ;
- 3) In  $\mathbb{Z}$ ,  $a \circ b = a + b - ab$ ;

**Problem 4:** Define operation " $\circ$ " in  $S = \{x \mid x \in \mathbb{R}, x \neq -1\}$ :  $a \circ b = a + b + ab$ . Prove that  $S$  is a group with respect to the operation " $\circ$ ". Then, solve equation  $2 \circ x \circ 3 = 7$ .

**Problem 5:** Prove:

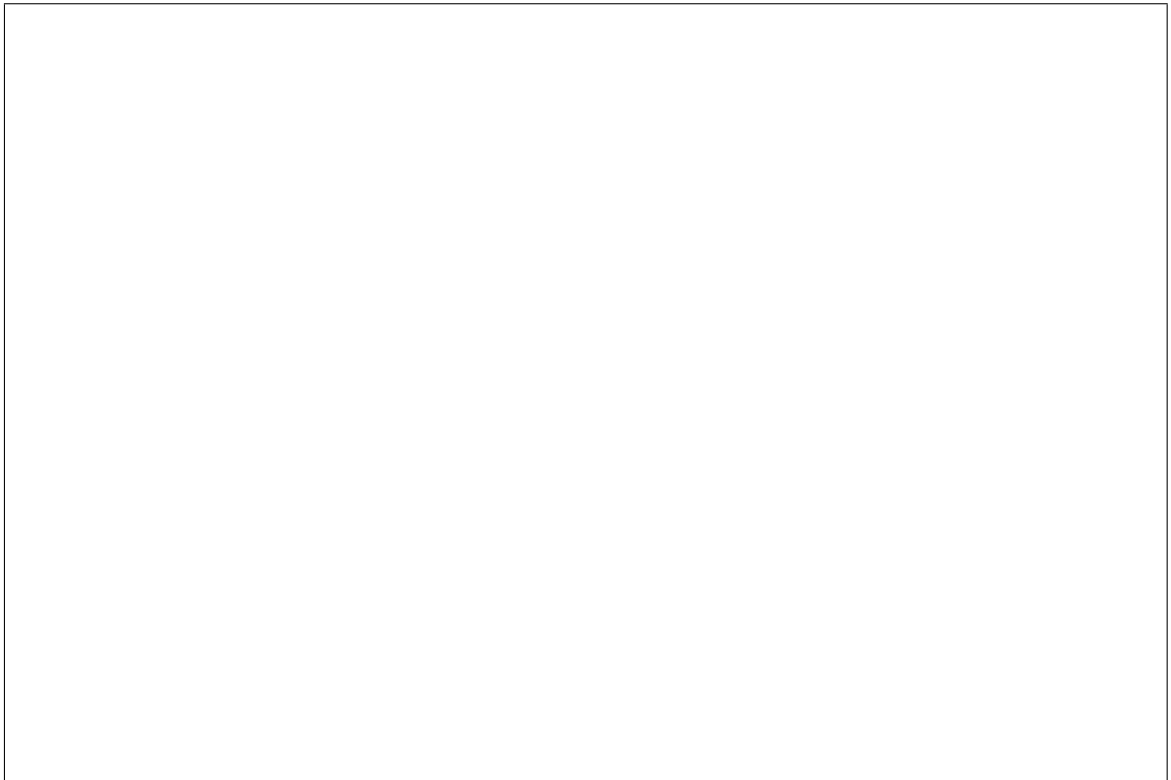
$\mathbb{G}$  is an Abelian Group if the order of every non-identity element is 2.



**Problem 6:** Assuming  $M$  is a monoid,  $m \in M$ . Define another multiplication rule " $\circ$ ":  
 $a \circ b = amb$ .

Prove that  $M$  is a semigroup with respect to " $\circ$ ".

When is  $M$  a monoid with respect to " $\circ$ "?



**Problem 7:** Assuming  $M$  is a monoid with an identity element  $e$ . It is said that the element  $a$  of  $M$  is invertible if there exists an element  $a^{-1}$  that satisfies  $a^{-1}a = aa^{-1} = e$ .

Prove the following statements:

- 1) If  $a, b, c \in M$  and  $ab = ca = e$ , then  $a$  is invertible and  $a^{-1} = b = c$ .
- 2) If  $a \in M$  is invertible, then  $b = a^{-1}$  when and only when  $aba = a$ ,  $ab^2a = e$ .
- 3) The sufficient prerequisite of  $G$ , the subset of  $M$ , being a group is that every element in  $G$  is invertible and for all  $g_1, g_2 \in G$ , we have  $g_1^{-1}g_2 \in G$ .
- 4) All invertible elements in  $M$  is a group.

