Moment of Inertia © BinaryPhi

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Definitions:

Moment of Inertia, often denoted by I (SI Unit: kg·m^2), is a quantitative measurement of the inertia for an object to rotate about a specific axis. It can be calculated by the summation of moment of inertia of every infinitesimal part of the rigid body ($\Longrightarrow$ in other words, a **mass point**). The moment of inertia of a mass point is:

$$I = m\,r^2, \tag{1}$$

where m is the mass of the mass point and r is the distance between this mass point and the rotational axis. By using summation, the moment of inertia of a system can be written as:

$$I = \sum_i^{\infty} m_i\,r_i^2. \tag{2}$$

By using integral, the moment of inertia can be written as:

$$I = \int r^2 dM. \tag{3}$$

Depending on the circumstances, we can transform (3) to the following:

$$I = \int \rho r^2 dV; \tag{4}$$

$$I = \int \sigma r^2 dS; \tag{5}$$

$$I = \int \lambda r^2 dL. \tag{6}$$

In (4) (5) (6), the convention is ρ denotes the Volume Density, σ denotes the Area Density, and λ denotes the Linear Density. In addtion,

$$dL = Rd\theta$$

is also useful in some cases.

Infinite Thin Rod : **Center** (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{12}ml^2. \tag{7}$$

proof

$$\begin{aligned}
I &= \int r^2 dM = \int_0^m r^2 dM \\
&= \int r^2(\lambda dL) = \int_{-l/2}^{l/2} \frac{m}{l} L^2 dL \\
&= \frac{m}{l}\int_{-l/2}^{l/2} L^2 dL = \frac{m}{l} \cdot \left[\frac{1}{3}L^3\right]_{-l/2}^{l/2} \\
&= \frac{m}{l}\left[\frac{1}{3}\left(\frac{l}{2}\right)^3 - \frac{1}{3}\left(-\frac{l}{2}\right)^3\right] \\
&\boxed{= \frac{1}{12}ml^2.}
\end{aligned}$$

Infinite Thin Rod : **One End** (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{3}ml^2. \tag{8}$$

proof

$$\begin{aligned}
I &= \int r^2 dM = \int_0^m r^2 dM \\
&= \int r^2(\lambda dL) = \int_0^l \frac{m}{l} L^2 dL \\
&= \frac{m}{l}\int_0^l L^2 dL \\
&= \frac{m}{l} \cdot \left[\frac{1}{3}L^3\right]_0^l = \frac{m}{l}\left(\frac{1}{3}l^3 - 0\right) \\
&\boxed{= \frac{1}{3}ml^2.}
\end{aligned}$$

Parallel Axis Theorem

The moment of inertia of a rigid body about a rotating axis is:

$$I = I_{cm} + md^2. \tag{9}$$

I_{cm} is the moment of inertia when rotating about the axis that passes through the center of mass. d is the distance between an arbitrary rotational axis and the axis that passes through the center of mass of the object.

proof

For the sake of simplicity, let's imagine the rigid body rotating about the z-axis, which passes through the center of mass of the body. You could use tensor to do a more generalized proof (I will do a tensor version of this whole topic if I have time).

$$\begin{aligned}
I_{cm} &= \int r^2 dM \\
&= \int (x^2 + y^2 + z^2) dM = \int (x^2 + y^2) dM \\
I &= \int \left((x - d_1)^2 + (y - d_2)^2\right) dM \\
&= \int \left(x^2 + y^2 - 2d_1 x - 2d_2 y + d_1^2 + d_2^2\right) dM \\
&= \int (x^2 + y^2) dM + (d_1^2 + d_2^2) \int dM \\
&\qquad - 2d_1 \int x dM - 2d_2 \int y dM \\
\because \quad & \int x dM = \int y dM = 0 \\
\therefore I &= \int (x^2 + y^2) dM + d^2 \cdot \int dM \\
& \boxed{= I_{cm} + md^2}
\end{aligned}$$

Circular Thin Loop : **Orthogonal Center** (Mass: **m**; Radius: **R**)

$$I_z = mR^2. \tag{10}$$

proof

$$\begin{aligned} I_z &= \int r^2 dM = \int R^2 dM \\ &= R^2 \int dM = R^2 \cdot m \\ &\boxed{= mR^2} \end{aligned}$$

Circular Thin Loop : **Diameter** (Mass: **m**; Radius: **R**)

$$I_x = I_y = \frac{1}{2} mR^2. \tag{11}$$

proof

$$\begin{aligned} I_x = I_y &= \int r^2 dM \\ &= \int r^2 (\lambda dL) = \int_0^{2\pi} r^2 \left(\frac{m}{2\pi R} R d\theta \right) \\ &= \int_0^{2\pi} (R\cos\theta)^2 \left(\frac{m}{2\pi} \right) d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \cos^2\theta d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta \\ &= R^2 \frac{m}{2\pi} \left(\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} \right) \\ &= R^2 \frac{m}{2\pi} (\pi - 0 + 0 - 0) \\ &\boxed{= \frac{1}{2} mR^2.} \end{aligned}$$