Moment of Inertia © BinaryPhi

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Definitions:

Moment of Inertia, often denoted by I (SI Unit: $\mathrm{kg{\cdot}m^2}$), is a quantitative measurement of the inertia for an object to rotate about a specific axis. It can be calculated by the summation of moment of inertia of every infinitesimal part of the rigid body ($\Longrightarrow$ in other words, a **mass point**). The moment of inertia of a mass point is:

$$I = m\,r^2, \tag{1}$$

where m is the mass of the mass point and r is the distance between this mass point and the rotational axis. By using summation, the moment of inertia of a system can be written as:

$$I = \sum_{i}^{\infty} m_i\,r_i^2. \tag{2}$$

By using integral, the moment of inertia can be written as:

$$I = \int r^2 dM. \tag{3}$$

Depending on the circumstances, we can transform (3) to the following:

$$I = \iiint \rho r^2 dV; \tag{4}$$

$$I = \iint \sigma r^2 dS; \tag{5}$$

$$I = \int \lambda r^2 dL. \tag{6}$$

In (4) (5) (6), the convention is ρ denotes the Volume Density, σ denotes the Area Density, and λ denotes the Linear Density. In addtion,

$$dL = Rd\theta; \quad dA = rd\theta dr$$

are also useful in some cases.

Eq.1: Infinite Thin Rod : **Center** (Mass: **m**; Lenght: **l**)

Eq.2: Infinite Thin Rod : **One End** (Mass: **m**; Lenght: **l**)

Theorem.1: Parallel Axis Theorem

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Eq.7: Circular Thin Disk : **Diameter** (Mass: **m**; Radius: **R**)

Eq.8: Solid Cylinder : **Transverse Axis & Center** (Mass: **m**; Radius: **R**; Height: **H**)

Eq.9: Hollow Cylinder (Thick Tube) : **Orthogonal Center**

Eq.10: Hollow Cylinder (Thick Tube) : **Transverse Axis & Center**

Eq.11: Thin Spherical Shell (Mass: **m**; Radius: **R**)

Eq.12: Solid Sphere (Mass: **m**; Radius: **R**)

Eq.1: Infinite Thin Rod : Center (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{12} m l^2. \tag{7}$$

proof

$$\begin{aligned}
I &= \int r^2 dM = \int_0^m r^2 dM \\
&= \int r^2 (\lambda dL) = \int_{-l/2}^{l/2} \frac{m}{l} L^2 dL \\
&= \frac{m}{l} \int_{-l/2}^{l/2} L^2 dL = \frac{m}{l} \cdot \left[\frac{1}{3} L^3\right]_{-l/2}^{l/2} \\
&= \frac{m}{l} \left[\frac{1}{3}\left(\frac{l}{2}\right)^3 - \frac{1}{3}\left(-\frac{l}{2}\right)^3\right] \\
&\boxed{= \frac{1}{12} m l^2.}
\end{aligned}$$

Eq.2: Infinite Thin Rod : One End (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{3} m l^2. \tag{8}$$

proof

$$\begin{aligned}
I &= \int r^2 dM = \int_0^m r^2 dM \\
&= \int r^2 (\lambda dL) = \int_0^l \frac{m}{l} L^2 dL \\
&= \frac{m}{l} \int_0^l L^2 dL \\
&= \frac{m}{l} \cdot \left[\frac{1}{3} L^3\right]_0^l = \frac{m}{l} \left(\frac{1}{3} l^3 - 0\right) \\
&\boxed{= \frac{1}{3} m l^2.}
\end{aligned}$$

Theorem.1: Parallel Axis Theorem

The moment of inertia of a rigid body about a rotating axis is:

$$I = I_{cm} + md^2. \tag{9}$$

I_{cm} is the moment of inertia when rotating about the axis that passes through the center of mass. d is the distance between an arbitrary rotational axis and the axis that passes through the center of mass of the object.

proof

For the sake of simplicity, let's imagine the rigid body rotating about the z-axis, which passes through the center of mass of the body. You could use tensor to do a more generalized proof (I will do a tensor version of this whole topic if I have time).

$$\begin{aligned}
I_{cm} &= \int r^2 dM \\
&= \int (x^2 + y^2 + z^2) dM = \int (x^2 + y^2) dM \\
I &= \int \left((x - d_1)^2 + (y - d_2)^2\right) dM \\
&= \int \left(x^2 + y^2 - 2d_1 x - 2d_2 y + d_1^2 + d_2^2\right) dM \\
&= \int (x^2 + y^2) dM + (d_1^2 + d_2^2) \int dM \\
&\qquad - 2d_1 \int x dM - 2d_2 \int y dM \\
\because \quad & \int x dM = \int y dM = 0 \\
\therefore I &= \int (x^2 + y^2) dM + d^2 \cdot \int dM \\
&\boxed{= I_{cm} + md^2}
\end{aligned}$$

Eq.3: Circular Thin Loop : Orthogonal Center (Mass: **m**; Radius: **R**)

$$I_z = mR^2. \tag{10}$$

proof

$$\begin{aligned} I_z &= \int r^2 dM = \int R^2 dM \\ &= R^2 \int dM = R^2 \cdot m \\ &\boxed{= mR^2} \end{aligned}$$

Eq.4: Circular Thin Loop : Diameter (Mass: **m**; Radius: **R**)

$$I_x = I_y = \frac{1}{2}mR^2. \tag{11}$$

proof

$$\begin{aligned} I_x = I_y &= \int r^2 dM \\ &= \int r^2(\lambda dL) = \int_0^{2\pi} r^2 \left(\frac{m}{2\pi R} R d\theta\right) \\ &= \int_0^{2\pi} (R\cos\theta)^2 \left(\frac{m}{2\pi}\right) d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \cos^2\theta d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta \\ &= R^2 \frac{m}{2\pi} \left(\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \right) \\ &= R^2 \frac{m}{2\pi} (\pi - 0 + 0 - 0) \\ &\boxed{= \frac{1}{2}mR^2.} \end{aligned}$$

Theorem.2: Perpendicular Axis Theorem

For a flat, thin, and uniform object, the moment of inertia, I_z, about a rotating axis, say z-axis, that is orthogonal to the center of mass, is equal to the sum of the moments of inertia of the object about two other rotating axes orthogonal to each other, I_x, I_y:

$$I_z = I_x + I_y. \tag{12}$$

I_z is the moment of inertia when rotating about the z-axis, I_x, I_y are the moments of inertia when rotating about the x-axis and y-axis respectively. Note that the object is flat among the xy surface. We have:

$$I_z = 2I_x = 2I_y; \quad I_x = I_y \tag{13}$$

when the rigid body has rotational symmetry in xy surface.

proof

$$\begin{aligned} I_z &= \int r^2 dM \\ &= \int (x^2 + y^2) dM \\ &= \int x^2 dM + \int y^2 dM \\ &= I_y + I_x. \\ \text{Note that: } \int & x^2 dM = I_y, \\ \text{and } \int & y^2 dM = I_x. \end{aligned}$$

Since the circular loop is a flat, thin, and uniform rigid body, we have:

$$(10) : I_z = mR^2$$

$$(11) : I_x = I_y = \frac{1}{2} mR^2$$

which satisfy

$$I_z = I_x + I_y.$$

Eq.5: Circular Thin Disk : Orthogonal Center (Mass: **m**; Radius: **R**)

$$I_z = \frac{1}{2} m R^2. \tag{14}$$

proof

$$\begin{aligned}
I_z &= \int r^2 dM = \iint r^2(\sigma dA) \\
&= \int_0^R \int_{-\pi}^{\pi} r^2 \sigma r d\theta dr = \int_0^R \sigma r^3 \left(\int_{-\pi}^{\pi} d\theta \right) dr \\
&= 2\pi\sigma \int_0^R r^3 dr \\
&= 2\pi\sigma \left[\frac{1}{4} r^4 \right]_0^R = 2\pi \frac{m}{\pi R^2} \left(\frac{1}{4} R^4 - 0 \right) \\
&\boxed{= \frac{1}{2} m R^2}
\end{aligned}$$

Eq.6: Solid Cylinder : Orthogonal Center (Mass: **m**; Radius: **R**; Hight: **H**)

$$I_z = \frac{1}{2} m R^2. \tag{15}$$

proof

$$\begin{aligned}
I_z &= \int r^2 dM = \iiint r^2(\rho dV) \\
&= \iint r^2 \rho dA \int_0^H dh \\
&= H\rho \int_0^R r^3 \cdot \left(\int_{-\pi}^{\pi} d\theta \right) \cdot dr \\
&= H \frac{m}{\pi R^2 H} 2\pi \int_0^R r^3 dr \\
&= \frac{2m}{R^2} \left(\frac{1}{4} R^4 \right) \\
&\boxed{= \frac{1}{2} m R^2}
\end{aligned}$$

Eq.7: Circular Thin Disk : Diameter (Mass: **m**; Radius: **R**)

$$I_x = I_y = \frac{1}{4}mR^2. \tag{16}$$

proof

$$\begin{aligned}
I_x = I_y &= \int r^2 dM \\
dM &= \sigma dA \\
&= \sigma r d\theta dr \\
I_x = I_y &= \int_0^R \int_0^{2\pi} (r\cos\theta)^2 (\sigma r d\theta dr) \\
&= \int_0^R \sigma r^3 \left(\int_0^{2\pi} \cos^2\theta d\theta \right) dr \\
&= \int_0^R \sigma r^3 \left(\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta \right) dr \\
&= \int_0^R \sigma r^3 \left(\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \right) dr \\
&= \int_0^R \sigma r^3 \pi dr \\
&= \frac{m}{\pi R^2} \pi \int_0^R r^3 dr \\
&= \frac{m}{R^2} \left[\frac{1}{4} r^4 \right]_0^R \\
&= \frac{m}{R^2} \frac{1}{4} R^4 \\
&\boxed{= \frac{1}{4} mR^2}
\end{aligned}$$

Eq.8: Solid Cylinder : **Transverse Axis & Center** (Mass: **m**; Radius: **R**; Height: **H**)

$$I_x = I_y = \frac{1}{4}mR^2 + \frac{1}{12}mH^2. \tag{17}$$

proof

$$\begin{aligned}
I_x = I_y = \int r^2 dM &:= I \\
dI = dI_x = dI_y &= r^2 dM \\
\because I &= I_{cm} + md^2 \\
&= I_{cm} + mx^2 \\
\therefore dI &= d(I_{cm} + mx^2) \\
&= dI_{cm} + x^2 \cdot dM \\
&= \frac{1}{4}R^2 \cdot dM + x^2 \cdot dM \\
I &= \int \frac{1}{4}R^2 \cdot dM + \int x^2 \cdot dM \\
&= \int_{-H/2}^{H/2} \frac{1}{4}R^2 \left(\frac{m}{H}dx\right) + \int_{-H/2}^{H/2} x^2 \left(\frac{m}{H}dx\right) \\
&= \frac{1}{4}R^2 \frac{m}{H} \int_{-H/2}^{H/2} dx + \frac{m}{H} \int_{-H/2}^{H/2} x^2 dx \\
&= \frac{1}{4}R^2 \frac{m}{H} \Big[x\Big]_{-H/2}^{H/2} + \frac{m}{H} \left[\frac{1}{3}x^3\right]_{-H/2}^{H/2} \\
&= \frac{1}{4}mR^2 + \frac{m}{H}\left(\frac{1}{3}\left(\frac{H}{2}\right)^3 - \frac{1}{3}\left(-\frac{H}{2}\right)^3\right) \\
&= \frac{1}{4}mR^2 + \frac{m}{H}\left(\frac{1}{12}H^3\right) \\
&= \frac{1}{4}mR^2 + \frac{1}{12}mH^2.
\end{aligned}$$

$$\boxed{\therefore I_x = I_y = \frac{1}{4}mR^2 + \frac{1}{12}mH^2.}$$

Eq.9: Hollow Cylinder (Thick Tube) : Orthogonal Center

(Mass: **m**; Inner Radius: $\mathbf{R_1}$; Outer Radius: $\mathbf{R_2}$; Height: **H**)

$$I_z = \frac{1}{2}m(R_1^2 + R_2^2) \tag{18}$$

proof

$$\begin{aligned}
I_z &= \int r^2 dM \\
&= \iiint r^2 (\rho dV) \\
&= \iint r^2 \rho dA \int_0^H dh \\
&= \iint r^2 H\rho (r d\theta dr) \\
&= \int_{R_1}^{R_2} \int_{-\pi}^{\pi} r^3 H\rho d\theta dr \\
&= \int_{R_1}^{R_2} r^3 H\rho \left(\int_{-\pi}^{\pi} d\theta \right) dr \\
&= \int_{R_1}^{R_2} r^3 H\rho 2\pi dr \\
&= 2\pi H \frac{m}{H(\pi R_2^2 - \pi R_1^2)} \int_{R_1}^{R_2} r^3 dr \\
&= \frac{2m}{R_2^2 - R_1^2} \left[\frac{1}{4} r^4 \right]_{R_1}^{R_2} \\
&= \frac{2m}{R_2^2 - R_1^2} \left(\frac{1}{4} R_2^4 - \frac{1}{4} R_1^4 \right) \\
&= \frac{m}{2\left(R_2^2 - R_1^2\right)} \left(R_2^2 + R_1^2\right)\left(R_2^2 - R_1^2\right) \\
&\boxed{= \frac{1}{2}m(R_1^2 + R_2^2)}
\end{aligned}$$

Eq.10: Hollow Cylinder (Thick Tube) : Transverse Axis & Center

(Mass: **m**; Inner Radius: $\mathbf{R_1}$; Outer Radius: $\mathbf{R_2}$; Height: **H**)

$$I_x = I_y = \frac{1}{4}m\left(R_1^2 + R_2^2\right) + \frac{1}{12}mH^2. \tag{19}$$

proof

$$\begin{aligned}
I_x = I_y &= \int r^2 dM := I \\
\because I &= I_{cm} + mx^2 \\
\therefore dI &= d(I_{cm} + mx^2) \\
&= dI_{cm} + x^2 \cdot dM \\
&= d\left(\frac{1}{2}\left(\frac{1}{2}m(R_1^2 + R_2^2)\right)\right) + x^2 \cdot dM \\
&= \frac{1}{4}(R_1^2 + R_2^2) \cdot dM + x^2 \cdot dM \\
I &= \int \frac{1}{4}(R_1^2 + R_2^2) \cdot dM + \int x^2 \cdot dM \\
&= \int_{-H/2}^{H/2} \frac{1}{4}(R_1^2 + R_2^2)\left(\frac{m}{H}dx\right) + \int_{-H/2}^{H/2} x^2\left(\frac{m}{H}dx\right) \\
&= \frac{1}{4}(R_1^2 + R_2^2)\frac{m}{H}\int_{-H/2}^{H/2} dx + \frac{m}{H}\int_{-H/2}^{H/2} x^2 dx \\
&= \frac{1}{4}(R_1^2 + R_2^2)\frac{m}{H}\Big[x\Big]_{-H/2}^{H/2} + \frac{m}{H}\left[\frac{1}{3}x^3\right]_{-H/2}^{H/2} \\
&= \frac{1}{4}m(R_1^2 + R_2^2) + \frac{m}{H}\left(\frac{1}{3}\left(\frac{H}{2}\right)^3 - \frac{1}{3}\left(-\frac{H}{2}\right)^3\right) \\
&= \frac{1}{4}m(R_1^2 + R_2^2) + \frac{m}{H}\left(\frac{1}{12}H^3\right) \\
&= \frac{1}{4}m\left(R_1^2 + R_2^2\right) + \frac{1}{12}mH^2
\end{aligned}$$

$$\boxed{\therefore I_x = I_y = \frac{1}{4}m\left(R_1^2 + R_2^2\right) + \frac{1}{12}mH^2.}$$

Eq.11: Thin Spherical Shell (Mass: **m**; Radius: **R**)

$$I = \frac{2}{3}mR^2. \tag{20}$$

proof

$$\begin{aligned}
I &= \int r^2 dM \\
&= \iint r^2\sigma \cdot dA \\
&= \iint r^2\sigma r d\theta dr \\
&= \iint r^2\sigma r d\theta R d\phi \\
&= \int_0^{2\pi} r^3\sigma R \int_0^{2\pi} d\theta d\phi \\
&= 2\pi\sigma R \int_0^{2\pi} R^3 \sin^3\phi d\phi \\
&= 2\pi\sigma R^4 \int_0^{2\pi} \sin\phi \sin^2\phi d\phi \\
&= 2\pi\sigma R^4 \int_0^{2\pi} \sin\phi(1-\cos^2\phi) d\phi \\
&= 2\pi\sigma R^4 \left(\int_0^{2\pi} \sin\phi d\phi - \int_0^{2\pi} \sin\phi\cos^2\phi d\phi\right) \\
&= 2\pi\sigma R^4 \left(\int_0^{2\pi} \sin\phi d\phi + \int_1^{-1} \cos^2\phi d(\cos\phi)\right) \\
&= 2\pi\sigma R^4 \left(\int_0^{2\pi} \sin\phi d\phi + \int_1^{-1} u^2 du\right) \\
&= 2\pi\frac{m}{4\pi R^2}R^4\left(2 + \frac{-2}{3}\right) \\
&= \frac{m}{2}R^2 \cdot \frac{4}{3} \\
&\boxed{= \frac{2}{3}mR^2.}
\end{aligned}$$

Eq.12: Solid Sphere (Mass: **m**; Radius: **R**)

$$I = \frac{2}{5}mR^2. \tag{21}$$

proof

(Spherical Shell)

$$\begin{aligned}
I &= \int dI = \int \left(\frac{2}{3}r^2 dM\right) \\
&= \int \frac{2}{3}r^2 \rho 4\pi r^2 dr = \int_0^R \frac{2}{3}4\pi\rho r^4 dr \\
&= \frac{2}{3}4\pi\rho \int_0^R r^4 dr \\
① &= \frac{8\pi}{3}\frac{m}{\frac{4}{3}\pi R^3}\left[\frac{1}{5}r^5\right]_0^R = \frac{2m}{R^3}\frac{1}{5}R^5 \\
&\boxed{= \frac{2}{5}mR^2.} \\
② &= \frac{2}{3}4\pi\rho\left[\frac{1}{5}r^5\right]_0^R = 2\left(\rho\frac{4}{3}\pi R^3\right)\frac{1}{5}R^2 \\
&\boxed{= \frac{2}{5}mR^2.}
\end{aligned}$$

proof

(Thin Disk)

$$\begin{aligned}
I &= \int dI = \int \left(\frac{1}{2}r^2 dM\right) \\
&= \int \frac{1}{2}r^2 \rho\pi r^2 dh = \int \frac{1}{2}\pi\rho r^4 dh \\
&= \int_{-R}^R \frac{1}{2}\pi\rho(R^2 - h^2)^2 dh \\
&= \frac{1}{2}\pi\rho \int_{-R}^R (R^4 - R^2h^2 + h^4)dh = \frac{1}{2}\pi\rho\left(\frac{16}{15}R^5\right) \\
&\boxed{= \frac{2}{5}mR^2.}
\end{aligned}$$

proof

(Parallel Axis Theorem)

$$
\begin{aligned}
dI &= dI_{cm} + x^2 dM = \frac{1}{4} r^2 dM + x^2 dM \\
dM &= \rho dV = \rho\pi r^2 dx = \rho\pi(R^2 - x^2)dx \\
I &= \int dI = \int \left(\frac{1}{4}r^2 + x^2\right) dM \\
&= \int_{-R}^{R} \left(\frac{1}{4}(R^2 - x^2) + x^2\right) \rho\pi(R^2 - x^2)dx \\
&= \rho\pi \int_{-R}^{R} \left(\frac{1}{4}R^4 + \frac{1}{2}R^2x^2 - \frac{3}{4}x^4\right) dx = \frac{8}{15}\rho\pi R^5 \\
&\boxed{= \frac{2}{5}mR^2.}
\end{aligned}
$$

proof

(Interesting Method)

$$
\begin{aligned}
I &:= I_x = I_y = I_z \\
&= \frac{1}{3}(I_x + I_y + I_z) \\
&= \frac{1}{3} \iiint \rho(y^2 + z^2)dV + \frac{1}{3} \iiint \rho(x^2 + z^2)dV \\
&\qquad + \frac{1}{3} \iiint \rho(x^2 + y^2)dV \\
&= \frac{1}{3} \iiint 2\rho(x^2 + y^2 + z^2)dV \\
&= \frac{2}{3}\rho \iiint r^2 r^2 \text{sin}\theta dr d\theta d\phi \Longleftarrow \text{Can be easier} \\
&= \frac{2}{3}\rho \int_0^R r^4 \int_0^\pi \text{sin}\theta \int_0^{2\pi} d\phi d\theta dr \\
&\boxed{= \frac{2}{5}mR^2.}
\end{aligned}
$$