

## Definitions:

**Moment of Inertia**, often denoted by  $I$  (SI Unit:  $\text{kg}\cdot\text{m}^2$ ), is a quantitative measurement of the inertia for an object to rotate about a specific axis. It can be calculated by the summation of moment of inertia of every infinitesimal part of the rigid body ( $\implies$  in other words, a **mass point**). The moment of inertia of a mass point is:

$$I = m r^2, \quad (1)$$

where  $m$  is the mass of the mass point and  $r$  is the distance between this mass point and the rotational axis. By using summation, the moment of inertia of a system can be written as:

$$I = \sum_i^{\infty} m_i r_i^2. \quad (2)$$

By using integral, the moment of inertia can be written as:

$$I = \int r^2 dM. \quad (3)$$

Depending on the circumstances, we can transform (3) to the following:

$$I = \iiint \rho r^2 dV; \quad (4)$$

$$I = \iint \sigma r^2 dS; \quad (5)$$

$$I = \int \lambda r^2 dL. \quad (6)$$

In (4) (5) (6), the convention is  $\rho$  denotes the Volume Density,  $\sigma$  denotes the Area Density, and  $\lambda$  denotes the Linear Density. In addition,

$$dL = R d\theta; \quad dA = r d\theta dr$$

are also useful in some cases.

**Eq.1: Infinite Thin Rod : Center** (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{12}ml^2. \quad (7)$$

*proof*

$$\begin{aligned} I &= \int r^2 dM = \int_0^m r^2 dM \\ &= \int r^2 (\lambda dL) = \int_{-l/2}^{l/2} \frac{m}{l} L^2 dL \\ &= \frac{m}{l} \int_{-l/2}^{l/2} L^2 dL = \frac{m}{l} \cdot \left[ \frac{1}{3} L^3 \right]_{-l/2}^{l/2} \\ &= \frac{m}{l} \left[ \frac{1}{3} \left( \frac{l}{2} \right)^3 - \frac{1}{3} \left( -\frac{l}{2} \right)^3 \right] \\ &= \frac{1}{12} ml^2. \end{aligned}$$

**Eq.2: Infinite Thin Rod : One End** (Mass: **m**; Lenght: **l**)

$$I = \frac{1}{3}ml^2. \quad (8)$$

*proof*

$$\begin{aligned} I &= \int r^2 dM = \int_0^m r^2 dM \\ &= \int r^2 (\lambda dL) = \int_0^l \frac{m}{l} L^2 dL \\ &= \frac{m}{l} \int_0^l L^2 dL \\ &= \frac{m}{l} \cdot \left[ \frac{1}{3} L^3 \right]_0^l = \frac{m}{l} \left( \frac{1}{3} l^3 - 0 \right) \\ &= \frac{1}{3} ml^2. \end{aligned}$$

### Theorem.1: Parallel Axis Theorem

The moment of inertia of a rigid body about a rotating axis is:

$$I = I_{cm} + md^2. \quad (9)$$

$I_{cm}$  is the moment of inertia when rotating about the axis that passes through the center of mass.  $d$  is the distance between an arbitrary rotational axis and the axis that passes through the center of mass of the object.

#### *proof*

For the sake of simplicity, let's imagine the rigid body rotating about the  $z$ -axis, which passes through the center of mass of the body. You could use tensor to do a more generalized proof (I will do a tensor version of this whole topic if I have time).

$$\begin{aligned} I_{cm} &= \int r^2 dM \\ &= \int (x^2 + y^2 + z^2) dM = \int (x^2 + y^2) dM \\ I &= \int ((x - d_1)^2 + (y - d_2)^2) dM \\ &= \int (x^2 + y^2 - 2d_1x - 2d_2y + d_1^2 + d_2^2) dM \\ &= \int (x^2 + y^2) dM + (d_1^2 + d_2^2) \int dM \\ &\quad - 2d_1 \int x dM - 2d_2 \int y dM \\ \because \quad \int x dM &= \int y dM = 0 \\ \therefore I &= \int (x^2 + y^2) dM + d^2 \cdot \int dM \\ &= I_{cm} + md^2 \end{aligned}$$

**Eq.3: Circular Thin Loop : Orthogonal Center** (Mass: **m**; Radius: **R**)

$$I_z = mR^2. \quad (10)$$

*proof*

$$\begin{aligned} I_z &= \int r^2 dM = \int R^2 dM \\ &= R^2 \int dM = R^2 \cdot m \\ &= mR^2 \end{aligned}$$

**Eq.4: Circular Thin Loop : Diameter** (Mass: **m**; Radius: **R**)

$$I_x = I_y = \frac{1}{2}mR^2. \quad (11)$$

*proof*

$$\begin{aligned} I_x = I_y &= \int r^2 dM \\ &= \int r^2 (\lambda dL) = \int_0^{2\pi} r^2 \left( \frac{m}{2\pi R} R d\theta \right) \\ &= \int_0^{2\pi} (R \cos \theta)^2 \left( \frac{m}{2\pi} \right) d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \cos^2 \theta d\theta \\ &= R^2 \frac{m}{2\pi} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta \\ &= R^2 \frac{m}{2\pi} \left( \left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \right) \\ &= R^2 \frac{m}{2\pi} (\pi - 0 + 0 - 0) \\ &= \frac{1}{2} m R^2. \end{aligned}$$

## Theorem.2: Perpendicular Axis Theorem

For a flat, thin, and uniform object, the moment of inertia,  $I_z$ , about a rotating axis, say  $z$ -axis, that is orthogonal to the center of mass, is equal to the sum of the moments of inertia of the object about two other rotating axes orthogonal to each other,  $I_x, I_y$ :

$$I_z = I_x + I_y. \quad (12)$$

$I_z$  is the moment of inertia when rotating about the  $z$ -axis,  $I_x, I_y$  are the moments of inertia when rotating about the  $x$ -axis and  $y$ -axis respectively. Note that the object is flat among the  $xy$  surface. We have:

$$I_z = 2I_x = 2I_y; \quad I_x = I_y \quad (13)$$

when the rigid body has rotational symmetry in  $xy$  surface.

**proof**

$$\begin{aligned} I_z &= \int r^2 dM \\ &= \int (x^2 + y^2) dM \\ &= \int x^2 dM + \int y^2 dM \\ &= I_y + I_x. \end{aligned}$$

$$\text{Note that: } \int x^2 dM = I_y,$$

$$\text{and } \int y^2 dM = I_x.$$

Since the circular loop is a flat, thin, and uniform rigid body, we have:

$$(10) : I_z = mR^2$$

$$(11) : I_x = I_y = \frac{1}{2}mR^2$$

which satisfy

$$I_z = I_x + I_y.$$

**Eq.5: Circular Thin Disk : Orthogonal Center** (Mass: **m**; Radius: **R**)

$$I_z = \frac{1}{2}mR^2. \quad (14)$$

**proof**

$$\begin{aligned} I_z &= \int r^2 dM = \iint r^2 (\sigma dA) \\ &= \int_0^R \int_{-\pi}^{\pi} r^2 \sigma r d\theta dr = \int_0^R \sigma r^3 \left( \int_{-\pi}^{\pi} d\theta \right) dr \\ &= 2\pi\sigma \int_0^R r^3 dr \\ &= 2\pi\sigma \left[ \frac{1}{4}r^4 \right]_0^R = 2\pi \frac{m}{\pi R^2} \left( \frac{1}{4}R^4 - 0 \right) \\ &= \frac{1}{2}mR^2 \end{aligned}$$

**Eq.6: Solid Cylinder : Orthogonal Center** (Mass: **m**; Radius: **R**; Height: **H**)

$$I_z = \frac{1}{2}mR^2. \quad (15)$$

**proof**

$$\begin{aligned} I_z &= \int r^2 dM = \iiint r^2 (\rho dV) \\ &= \iint r^2 \rho dA \int_0^H dh \\ &= H\rho \int_0^R r^3 \cdot \left( \int_{-\pi}^{\pi} d\theta \right) \cdot dr \\ &= H \frac{m}{\pi R^2 H} 2\pi \int_0^R r^3 dr \\ &= \frac{2m}{R^2} \left( \frac{1}{4}R^4 \right) \\ &= \frac{1}{2}mR^2 \end{aligned}$$

**Eq.7: Circular Thin Disk : Diameter (Mass:  $m$ ; Radius:  $R$ )**

$$I_x = I_y = \frac{1}{4}mR^2. \quad (16)$$

*proof*

$$\begin{aligned} I_x = I_y &= \int r^2 dM \\ dM &= \sigma dA \\ &= \sigma r d\theta dr \\ I_x = I_y &= \int_0^R \int_0^{2\pi} (r \cos \theta)^2 (\sigma r d\theta dr) \\ &= \int_0^R \sigma r^3 \left( \int_0^{2\pi} \cos^2 \theta d\theta \right) dr \\ &= \int_0^R \sigma r^3 \left( \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta \right) dr \\ &= \int_0^R \sigma r^3 \left( \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \right) dr \\ &= \int_0^R \sigma r^3 \pi dr \\ &= \frac{m}{\pi R^2} \pi \int_0^R r^3 dr \\ &= \frac{m}{R^2} \left[ \frac{1}{4} r^4 \right]_0^R \\ &= \frac{m}{R^2} \frac{1}{4} R^4 \\ &= \frac{1}{4} m R^2 \end{aligned}$$

**Eq.8: Solid Cylinder : Transverse Axis & Center** (Mass: **m**; Radius: **R**; Height: **H**)

$$I_x = I_y = \frac{1}{4}mR^2 + \frac{1}{12}mH^2. \quad (17)$$

*proof*

$$\begin{aligned} I_x = I_y &= \int r^2 dM := I \\ dI &= dI_x = dI_y = r^2 dM \\ \therefore I &= I_{cm} + md^2 \\ &= I_{cm} + mx^2 \\ \therefore dI &= d(I_{cm} + mx^2) \\ &= dI_{cm} + x^2 \cdot dM \\ &= \frac{1}{4}R^2 \cdot dM + x^2 \cdot dM \\ I &= \int \frac{1}{4}R^2 \cdot dM + \int x^2 \cdot dM \\ &= \int_{-H/2}^{H/2} \frac{1}{4}R^2 \left( \frac{m}{H} dx \right) + \int_{-H/2}^{H/2} x^2 \left( \frac{m}{H} dx \right) \\ &= \frac{1}{4}R^2 \frac{m}{H} \int_{-H/2}^{H/2} dx + \frac{m}{H} \int_{-H/2}^{H/2} x^2 dx \\ &= \frac{1}{4}R^2 \frac{m}{H} \left[ x \right]_{-H/2}^{H/2} + \frac{m}{H} \left[ \frac{1}{3}x^3 \right]_{-H/2}^{H/2} \\ &= \frac{1}{4}mR^2 + \frac{m}{H} \left( \frac{1}{3} \left( \frac{H}{2} \right)^3 - \frac{1}{3} \left( -\frac{H}{2} \right)^3 \right) \\ &= \frac{1}{4}mR^2 + \frac{m}{H} \left( \frac{1}{12}H^3 \right) \\ &= \frac{1}{4}mR^2 + \frac{1}{12}mH^2. \end{aligned}$$

$$\therefore I_x = I_y = \frac{1}{4}mR^2 + \frac{1}{12}mH^2.$$



**Eq.9: Hollow Cylinder (Thick Tube) : Orthogonal Center**(Mass: **m**; Inner Radius: **R<sub>1</sub>**; Outer Radius: **R<sub>2</sub>**; Height: **H**)

$$I_z = \frac{1}{2}m(R_1^2 + R_2^2) \quad (18)$$

**proof**

$$\begin{aligned}
I_z &= \int r^2 dM \\
&= \iiint r^2 (\rho dV) \\
&= \iint r^2 \rho dA \int_0^H dh \\
&= \iint r^2 H \rho (r d\theta dr) \\
&= \int_{R_1}^{R_2} \int_{-\pi}^{\pi} r^3 H \rho d\theta dr \\
&= \int_{R_1}^{R_2} r^3 H \rho \left( \int_{-\pi}^{\pi} d\theta \right) dr \\
&= \int_{R_1}^{R_2} r^3 H \rho 2\pi dr \\
&= 2\pi H \frac{m}{H(\pi R_2^2 - \pi R_1^2)} \int_{R_1}^{R_2} r^3 dr \\
&= \frac{2m}{R_2^2 - R_1^2} \left[ \frac{1}{4} r^4 \right]_{R_1}^{R_2} \\
&= \frac{2m}{R_2^2 - R_1^2} \left( \frac{1}{4} R_2^4 - \frac{1}{4} R_1^4 \right) \\
&= \frac{m}{2(R_2^2 - R_1^2)} (R_2^2 + R_1^2) (R_2^2 - R_1^2) \\
&= \frac{1}{2}m(R_1^2 + R_2^2)
\end{aligned}$$

**Eq.10: Hollow Cylinder (Thick Tube) : Transverse Axis & Center**(Mass: **m**; Inner Radius: **R<sub>1</sub>**; Outer Radius: **R<sub>2</sub>**; Height: **H**)

$$I_x = I_y = \frac{1}{4}m(R_1^2 + R_2^2) + \frac{1}{12}mH^2. \quad (19)$$

**proof**

$$\begin{aligned}
I_x = I_y &= \int r^2 dM := I \\
\therefore I &= I_{cm} + mx^2 \\
\therefore dI &= d(I_{cm} + mx^2) \\
&= dI_{cm} + x^2 \cdot dM \\
&= d\left(\frac{1}{2}\left(\frac{1}{2}m(R_1^2 + R_2^2)\right)\right) + x^2 \cdot dM \\
&= \frac{1}{4}(R_1^2 + R_2^2) \cdot dM + x^2 \cdot dM \\
I &= \int \frac{1}{4}(R_1^2 + R_2^2) \cdot dM + \int x^2 \cdot dM \\
&= \int_{-H/2}^{H/2} \frac{1}{4}(R_1^2 + R_2^2) \left(\frac{m}{H}dx\right) + \int_{-H/2}^{H/2} x^2 \left(\frac{m}{H}dx\right) \\
&= \frac{1}{4}(R_1^2 + R_2^2) \frac{m}{H} \int_{-H/2}^{H/2} dx + \frac{m}{H} \int_{-H/2}^{H/2} x^2 dx \\
&= \frac{1}{4}(R_1^2 + R_2^2) \frac{m}{H} \left[x\right]_{-H/2}^{H/2} + \frac{m}{H} \left[\frac{1}{3}x^3\right]_{-H/2}^{H/2} \\
&= \frac{1}{4}m(R_1^2 + R_2^2) + \frac{m}{H} \left(\frac{1}{3}\left(\frac{H}{2}\right)^3 - \frac{1}{3}\left(-\frac{H}{2}\right)^3\right) \\
&= \frac{1}{4}m(R_1^2 + R_2^2) + \frac{m}{H} \left(\frac{1}{12}H^3\right) \\
&= \frac{1}{4}m(R_1^2 + R_2^2) + \frac{1}{12}mH^2
\end{aligned}$$

$$\therefore I_x = I_y = \frac{1}{4}m(R_1^2 + R_2^2) + \frac{1}{12}mH^2.$$

**Eq.11: Thin Spherical Shell (Mass:  $m$ ; Radius:  $R$ )**

$$I = \frac{2}{3}mR^2. \quad (20)$$

*proof*

$$\begin{aligned} I &= \int r^2 dM \\ &= \iint r^2 \sigma \cdot dA \\ &= \iiint r^2 \sigma r d\theta dr \\ &= \iint r^2 \sigma r d\theta R d\phi \\ &= \int_0^{2\pi} r^3 \sigma R \int_0^{2\pi} d\theta d\phi \\ &= 2\pi \sigma R \int_0^{2\pi} R^3 \sin^3 \phi d\phi \\ &= 2\pi \sigma R^4 \int_0^{2\pi} \sin \phi \sin^2 \phi d\phi \\ &= 2\pi \sigma R^4 \int_0^{2\pi} \sin \phi (1 - \cos^2 \phi) d\phi \\ &= 2\pi \sigma R^4 \left( \int_0^{2\pi} \sin \phi d\phi - \int_0^{2\pi} \sin \phi \cos^2 \phi d\phi \right) \\ &= 2\pi \sigma R^4 \left( \int_0^{2\pi} \sin \phi d\phi + \int_1^{-1} \cos^2 \phi d(\cos \phi) \right) \\ &= 2\pi \sigma R^4 \left( \int_0^{2\pi} \sin \phi d\phi + \int_1^{-1} u^2 du \right) \\ &= 2\pi \frac{m}{4\pi R^2} R^4 \left( 2 + \frac{-2}{3} \right) \\ &= \frac{m}{2} R^2 \cdot \frac{4}{3} \\ &= \frac{2}{3} m R^2. \end{aligned}$$

**Eq.12: Solid Sphere** (Mass: **m**; Radius: **R**)

$$I = \frac{2}{5}mR^2. \quad (21)$$

*proof*

$$\begin{aligned} I &= \int r^2 dM \\ &= \iiint r^2 \rho dV \\ &= \end{aligned}$$

**Eq.13: Spherical Thick Shell** (Mass:  $m$ ; Inner Radius:  $R_1$ ; Outer Radius:  $R_2$ )

$$I = \frac{2}{5} \quad (22)$$

*proof*

$$I = \int r^2 dM$$