Factorization Machines

DSAIL @ KAIST

김원중

E-mail: wjkim@kaist.ac.kr

Contents

Existing methods

- Support Vector Machine(SVM)

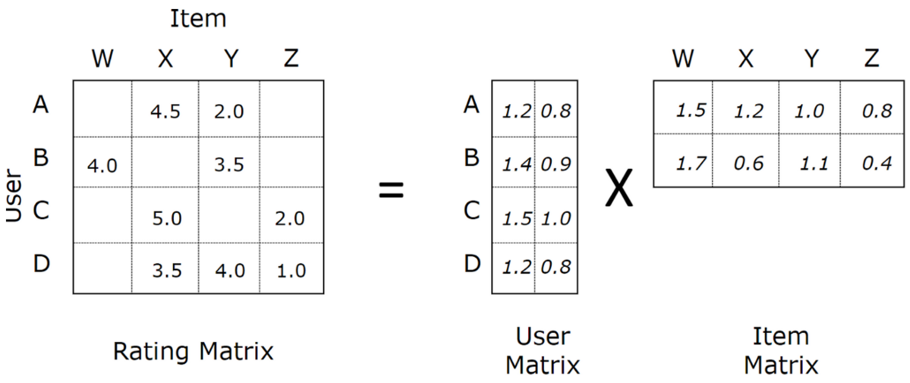
  1) Work on any real valued feature vector

  2) Cannot derive hyperplane in nonlinear kernel spaces under **sparse data**

- Factorization models

  1) Not applicable to **standard prediction data**(a real valued feature vector)

  2) Require effort in modelling since **models derived individually** for a specific task



https://supkoon.tistory.com/31

Factorization Machine (FM)

- Advantage of Factorization Machine(FM)

  1) Working on sparse data
  2) Linear complexity
  3) General Predictor

Input representation

data S = {(Alice, Titanic, 2010-1, 5), (Bob, Star Wars, 2010-2, 3) … }

Feature vector **x**

	User A	User B	User C	User …	Movie TI	Movie NH	Movie SW	Movie ST	Movie …	Other Movies rated TI	Other Movies rated NH	Other Movies rated SW	Other Movies rated ST	Other Movies rated …	Time	Last Movie rated TI	Last Movie rated NH	Last Movie rated SW	Last Movie rated ST	Last Movie rated …	Target y	
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	1	0	0	...	0	0	1	0	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

Input representation

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Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
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$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
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Input representation

data S = {(Alice, Titanic, $\boxed{\text{2010-1}}$ 5), (Bob, Star Wars, 2010-2, 3) … }

Feature vector **x** | Target y

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	…	TI	NH	SW	ST	…	TI	NH	SW	ST	…	Time	TI	NH	SW	ST	…		
$\mathbf{x}^{(1)}$	1	0	0	…	1	0	0	0	…	0.3	0.3	0.3	0	…	$\boxed{13}$	0	0	0	0	…	5	$y^{(1)}$
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Input representation

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Feature vector **x** | Target y

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
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Factorized Machine model

- Model equation

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

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- Model equation

$$\hat{y}(\mathbf{x}) := \boxed{w_0} + \sum_{i=1}^{n} \boxed{w_i} x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \boxed{\mathbf{v}_i, \mathbf{v}_j} \rangle \, x_i \, x_j$$

where the parameters that have to be estimated are: $w_0 \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n, \quad \mathbf{V} \in \mathbb{R}^{n \times k}$

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where the parameters that have to be estimated are: $w_0 \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n, \quad \mathbf{V} \in \mathbb{R}^{n \times k}$

dot product of two vectors: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle := \sum_{f=1}^{k} v_{i,f} \cdot v_{j,f}$

Factorized Machine model

- Model equation

$$\hat{y}(\mathbf{x}) := \boxed{w_0} + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

w_0: global bias

Factorized Machine model

- Model equation

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} \boxed{w_i} x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle x_i x_j$$

w_0: global bias

w_i: strength of the i-th variable

Factorized Machine model

- Model equation

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \boxed{\mathbf{v}_i, \mathbf{v}_j} \rangle x_i x_j$$

w_0: global bias

w_i: strength of the i-th variable

$w_{i,j} \coloneqq \langle \mathbf{v}_i, \mathbf{v}_j \rangle$: interaction between the i -th and j -th variable

$\mathbf{v}_i$: i -th variable with k factors (factor vector)

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
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$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

$$\begin{aligned}\mathbf{x}^{(1)} &= [x_1, \ldots, x_n] \\ &= [1,0,0, \ldots, 1,0,0,0, \ldots, 0.3, 0.3, 0.3, 0, \ldots] \\ &\Rightarrow [\mathbf{v}_1, \ldots, \mathbf{v}_n]\end{aligned}$$

$$\text{where } \mathbf{v}_i = \begin{bmatrix} v_1 \\ v_2 \\ \ldots \\ v_k \end{bmatrix}$$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

Feature vector **x** | Target y

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...		TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
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$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

Interaction between Alice(A) and Star Trek(ST)?

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i\, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j$$

Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	$\boxed{1}$	0	0	...	1	0	0	$\boxed{0}$	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	$\boxed{1}$	0	0	...	0	1	0	$\boxed{0}$	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	$\boxed{1}$	0	0	...	0	0	1	$\boxed{0}$	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

Direct estimate → no interaction ($w_{A,ST} = 0$)

Use **factorized** interaction parameters $w_{A,ST} := \langle \mathbf{v}_\mathrm{A}, \mathbf{v}_\mathrm{ST} \rangle$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
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$\mathbf{x}^{(4)}$	0	$\boxed{1}$	0	...	0	0	$\boxed{1}$	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	$\boxed{4}$	$y^{(3)}$
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$$\langle \mathbf{v}_{\mathrm{B}}, \mathbf{v}_{\mathrm{SW}} \rangle \approx \langle \mathbf{v}_{\mathrm{C}}, \mathbf{v}_{\mathrm{SW}} \rangle$$

$$\rightarrow \mathbf{v}_{\mathrm{B}} \approx \mathbf{v}_{\mathrm{C}}$$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

Feature vector **x** / Target y

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$\langle \mathbf{v}_A, \mathbf{v}_{TI} \rangle \not\approx \langle \mathbf{v}_C, \mathbf{v}_{TI} \rangle$

$\rightarrow \mathbf{v}_A \not\approx \mathbf{v}_C$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i\, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j$$

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$\langle \mathbf{v}_A, \mathbf{v}_{SW} \rangle \approx \langle \mathbf{v}_C, \mathbf{v}_{SW} \rangle$

$\rightarrow \mathbf{v}_A \approx \mathbf{v}_C$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

Feature vector **x**

	User A	User B	User C	User ...	Movie TI	Movie NH	Movie SW	Movie ST	Movie ...	Other Movies rated TI	Other Movies rated NH	Other Movies rated SW	Other Movies rated ST	Other Movies rated ...	Time	Last Movie rated TI	Last Movie rated NH	Last Movie rated SW	Last Movie rated ST	Last Movie rated ...	Target y	
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	1	0	0	...	0	0	1	0	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

$\langle \mathbf{v}_B, \mathbf{v}_{SW} \rangle \approx \langle \mathbf{v}_B, \mathbf{v}_{ST} \rangle$

→ $\mathbf{v}_{SW} \approx \mathbf{v}_{ST}$

Parameter estimation under sparsity

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i\, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j$$

Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	$\boxed{1}$	0	0	...	0	0	$\boxed{1}$	0	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

In total, $\langle \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{ST}} \rangle$ will be similar to $\langle \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{SW}} \rangle$

Linear complexity

$O(k\,n^2)$

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i\, x_i + \boxed{\sum_{i=1}^{n}\sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j}$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle\, x_i\, x_j - \frac{1}{2}\sum_{i=1}^{n} \langle \mathbf{v}_i, \mathbf{v}_i \rangle\, x_i\, x_i$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle := \sum_{f=1}^{k} v_{i,f} \cdot v_{j,f}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{f=1}^{k} v_{i,f}\, v_{j,f}\, x_i\, x_j - \sum_{i=1}^{n}\sum_{f=1}^{k} v_{i,f}\, v_{i,f}\, x_i\, x_i\right)$$

$$= \frac{1}{2}\sum_{f=1}^{k}\left(\left(\sum_{i=1}^{n} v_{i,f}\, x_i\right)\left(\sum_{j=1}^{n} v_{j,f}\, x_j\right) - \sum_{i=1}^{n} v_{i,f}^2\, x_i^2\right)$$

$O(k\,n)$

$$= \frac{1}{2}\sum_{f=1}^{k}\left(\left(\sum_{i=1}^{n} v_{i,f}\, x_i\right)^2 - \sum_{i=1}^{n} v_{i,f}^2\, x_i^2\right)$$

FM as various predictors

- Regression: $\hat{y}(x)$ can be the predictor and optimization criterion
- Binary classification: the sign of $\hat{y}(x)$ is used
- Ranking: $\mathbf{x}$ are ordered by the score of $\hat{y}(x)$

Regularization terms are usually added (ex. $L2$)

Learning FM

- Gradients

$$\hat{y}(\mathbf{x}) := w_0 + \sum_{i=1}^{n} w_i \, x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i \, x_j$$

$$\frac{\partial}{\partial\theta}\hat{y}(\mathbf{x}) = \begin{cases} 1, & \text{if } \theta \text{ is } w_0 \\ x_i, & \text{if } \theta \text{ is } w_i \\ x_i \underbrace{\boxed{\sum_{j=1}^{n} v_{j,f} x_j}}_{\text{Independent of } i} - v_{i,f} x_i^2, & \text{if } \theta \text{ is } v_{i,f} \end{cases} \qquad \cdots \; O(1)$$

Generalization

- d-way FM

$$\hat{y}(x) := w_0 + \sum_{i=1}^{n} w_i\, x_i \, + \sum_{l=2}^{d} \sum_{i_1=1}^{n} \ldots \sum_{i_l=i_{l-1}+1}^{n} \left(\prod_{j=1}^{l} x_{i_j} \right) \left(\sum_{f=1}^{k_l} \prod_{j=1}^{l} v_{i_j,f}^{(l)} \right)$$

FM vs. SVM

- SVM with the linear kernel

$$\hat{y}(\mathbf{x}) = w_0 + \sum_{i=1}^{n} w_i\, x_i, \quad w_0 \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n$$

- SVM with the polynomial kernel

$$\hat{y}(\mathbf{x}) = w_0 + \sqrt{2}\sum_{i=1}^{n} w_i\, x_i + \sum_{i=1}^{n} w_{i,i}^{(2)} x_i^2 + \sqrt{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} w_{i,j}^{(2)}\, x_i\, x_j$$

Feature vector **x**

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	1	0	0	...	0	0	1	0	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

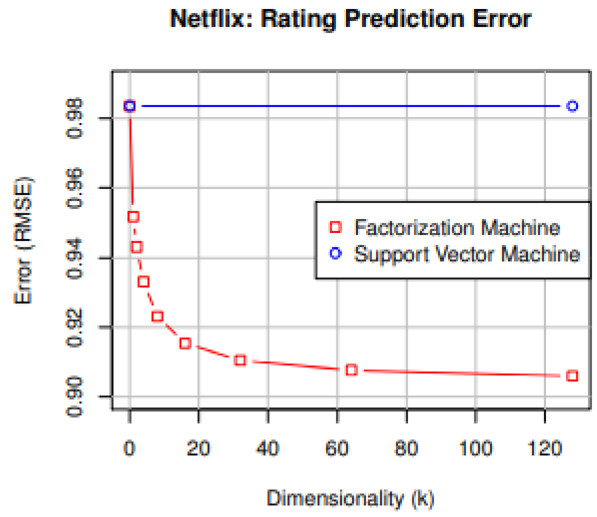
FM vs. SVM

- SVM with the linear kernel

$$\hat{y}(\mathbf{x}) = w_0 + w_u + w_i$$

- SVM with the polynomial kernel

$$\hat{y}(\mathbf{x}) = w_0 + \sqrt{2}(w_u + w_i) + w_{u,u}^{(2)} + w_{i,i}^{(2)} + \boxed{\sqrt{2}w_{u,i}^{(2)}}$$



Feature vector **x** / Target y

	User				Movie					Other Movies rated					Time	Last Movie rated					Target y	
	A	B	C	...	TI	NH	SW	ST	...	TI	NH	SW	ST	...	Time	TI	NH	SW	ST	...		
$\mathbf{x}^{(1)}$	1	0	0	...	1	0	0	0	...	0.3	0.3	0.3	0	...	13	0	0	0	0	...	5	$y^{(1)}$
$\mathbf{x}^{(2)}$	1	0	0	...	0	1	0	0	...	0.3	0.3	0.3	0	...	14	1	0	0	0	...	3	$y^{(2)}$
$\mathbf{x}^{(3)}$	1	0	0	...	0	0	1	0	...	0.3	0.3	0.3	0	...	16	0	1	0	0	...	1	$y^{(2)}$
$\mathbf{x}^{(4)}$	0	1	0	...	0	0	1	0	...	0	0	0.5	0.5	...	5	0	0	0	0	...	4	$y^{(3)}$
$\mathbf{x}^{(5)}$	0	1	0	...	0	0	0	1	...	0	0	0.5	0.5	...	8	0	0	1	0	...	5	$y^{(4)}$
$\mathbf{x}^{(6)}$	0	0	1	...	1	0	0	0	...	0.5	0	0.5	0	...	9	0	0	0	0	...	1	$y^{(5)}$
$\mathbf{x}^{(7)}$	0	0	1	...	0	0	1	0	...	0.5	0	0.5	0	...	12	1	0	0	0	...	5	$y^{(6)}$

FM vs. Other factorization models

- Matrix and Tensor Factorization

$$n := |U \cup I|, \quad x_j := \delta\,(j = i \vee j = u)$$

$$\hat{y}(\mathbf{x}) = w_0 + w_u + w_i + \langle \mathbf{v}_u, \mathbf{v}_i \rangle$$

- SVD++

$$n := |U \cup I \cup L|, \quad x_j := \begin{cases} 1, & \text{if } j = i \vee j = u \\ \frac{1}{\sqrt{|N_u|}}, & \text{if } j \in N_u \\ 0, & \text{else} \end{cases}$$

$$\hat{y}(\mathbf{x}) = \overbrace{w_0 + w_u + w_i + \langle \mathbf{v}_u, \mathbf{v}_i \rangle + \frac{1}{\sqrt{|N_u|}} \sum_{l \in N_u} \langle \mathbf{v}_i, \mathbf{v}_l \rangle}^{\text{SVD++}} + \frac{1}{\sqrt{|N_u|}} \sum_{l \in N_u} \left(w_l + \langle \mathbf{v}_u, \mathbf{v}_l \rangle + \frac{1}{\sqrt{|N_u|}} \sum_{l' \in N_u, l' > l} \langle \mathbf{v}_l, \mathbf{v}_l' \rangle \right)$$

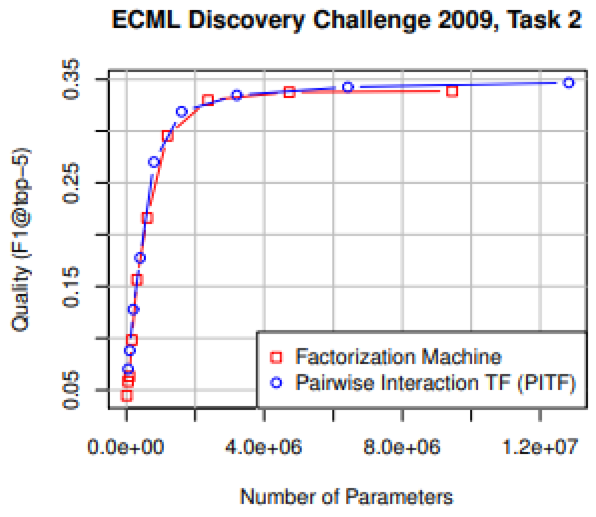
FM vs. Other factorization models

- PITF(Pairwise interaction tensor factorization)

$$n := |U \cup I \cup T|, \quad x_j := \delta\left(j = i \vee j = u \vee j = t\right)$$

$$\hat{y}(\mathbf{x}) = w_0 + w_u + w_i + w_t + \langle \mathbf{v}_u, \mathbf{v}_i \rangle + \langle \mathbf{v}_u, \mathbf{v}_t \rangle + \langle \mathbf{v}_i, \mathbf{v}_t \rangle$$

$$\hat{y}(\mathbf{x}) := w_t + \langle \mathbf{v}_u, \mathbf{v}_t \rangle + \langle \mathbf{v}_i, \mathbf{v}_t \rangle$$



- FPMC(Factorized personalized markov chains)

$$n := |U \cup I \cup L|, \quad x_j := \begin{cases} 1, & \text{if } j = i \vee j = u \\ \frac{1}{|B^u_{t-1}|}, & \text{if } j \in B^u_{t-1} \\ 0, & \text{else} \end{cases}$$

$$\hat{y}(\mathbf{x}) = w_0 + w_u + w_i + \langle \mathbf{v}_u, \mathbf{v}_i \rangle + \frac{1}{|B^u_{t-1}|} \sum_{l \in B^u_{t-1}} \langle \mathbf{v}_i, \mathbf{v}_l \rangle + \frac{1}{|B^u_{t-1}|} \sum_{l \in B^u_{t-1}} \left(w_l + \langle \mathbf{v}_u, \mathbf{v}_l \rangle + \frac{1}{|B^u_{t-1}|} \sum_{l' \in B^u_{t-1}, l' > l} \langle \mathbf{v}_l, \mathbf{v}'_l \rangle \right)$$

$$\hat{y}(\mathbf{x}) = w_i + \langle \mathbf{v}_u, \mathbf{v}_i \rangle + \frac{1}{|B^u_{t-1}|} \sum_{l \in B^u_{t-1}} \langle \mathbf{v}_i, \mathbf{v}_l \rangle$$

Summary

- Able to estimate parameters under sparsity
- Linear complexity
- General predictor that handle any real valued vector
- Identical or very similar to many other models

Implementation

- Google Colab, Tensorflow
- Dataset: Wisconsin Diagnostic Breast Cancer (WDBC)

  30 columns, binary data, scaled with MinMax Scaler
- k (dimension of factor vector) = 10
- Learning rate = 0.01
- Accuracy on test data = 0.958 (epoch: 100)

End of Documents