

APPLICATIONS



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Network Metrics

Applications of Data Science - Class 9

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Centrality

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What makes a node central to a network?

| A central node is... | Centrality |
|--|-----------------------------|
| Connected to many nodes | Degree |
| Connected to other important nodes | Eigenvector, Katz, PageRank |
| Close to many nodes | Closeness |
| "Mediator", without it the network might "break" | Betweenness |

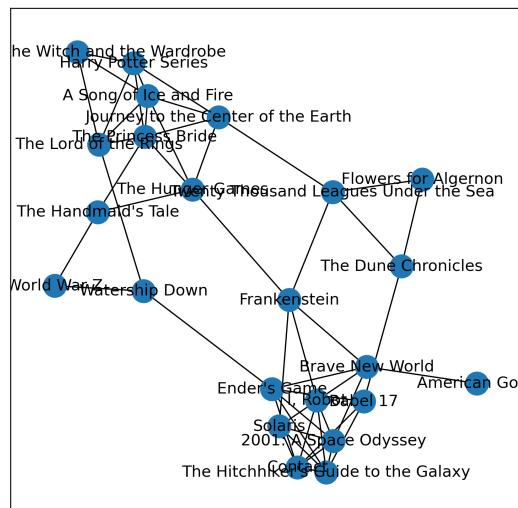
```

import pandas as pd
import numpy as np
import networkx as nx
import matplotlib.pyplot as plt

scifi_edgelist = pd.read_csv('../data/sci_fi_final_edgelist.csv')
G = nx.from_pandas_edgelist(scifi_edgelist, 'book', 'book2', ['correlation'])

nx.draw_networkx(G)
plt.show()

```



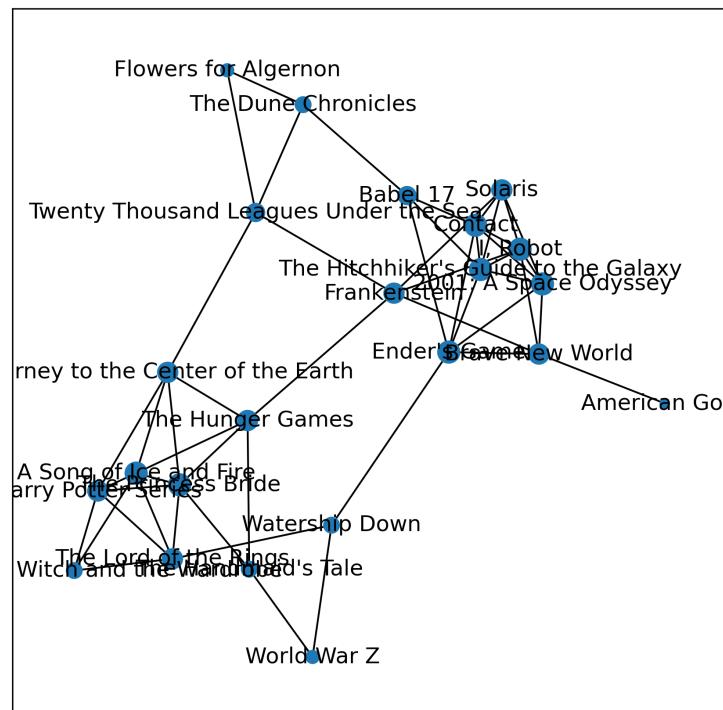
Degree Centrality

- $x_i = k_i = \sum_{j=1}^n A_{ij}$
- NetworkX also normalizes by dividing by $n - 1$
- Or in directed networks: in-degree and out-degree centralities
- Makes sense in social networks, co-citation networks

```
cent_deg = nx.degree_centrality(G)  
pretty_print(get_top_n_dict(cent_deg))
```

```
## I, Robot: 0.27  
## The Princess Bride: 0.27  
## A Song of Ice and Fire: 0.27  
## 2001: A Space Odyssey: 0.27  
## Ender's Game: 0.27
```

```
nx.draw_networkx(G, nodelist = cent_deg.keys(),
    node_size = [c * 500 for c in cent_deg.values()])
plt.show()
```



Eigenvector Centrality

It's not *how many* you're connected to, it's *who* you're connected to...

Let's make your centrality proportional to the sum of centralities of your neighbors:

$$x_i = \delta \sum_{j \in \text{neigh}(i)} x_j = \delta \sum_{j=1}^n A_{ij} x_j$$

The vector of centralities can be written as:

$$\mathbf{x} = \delta \mathbf{A} \mathbf{x}$$

Marking $\lambda = 1/\delta$ we get:

$$\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$$

Which means our vector of centralities is an eigenvector of the adjacency matrix.

Now there are n eigenvectors to this $n \times n$ matrix

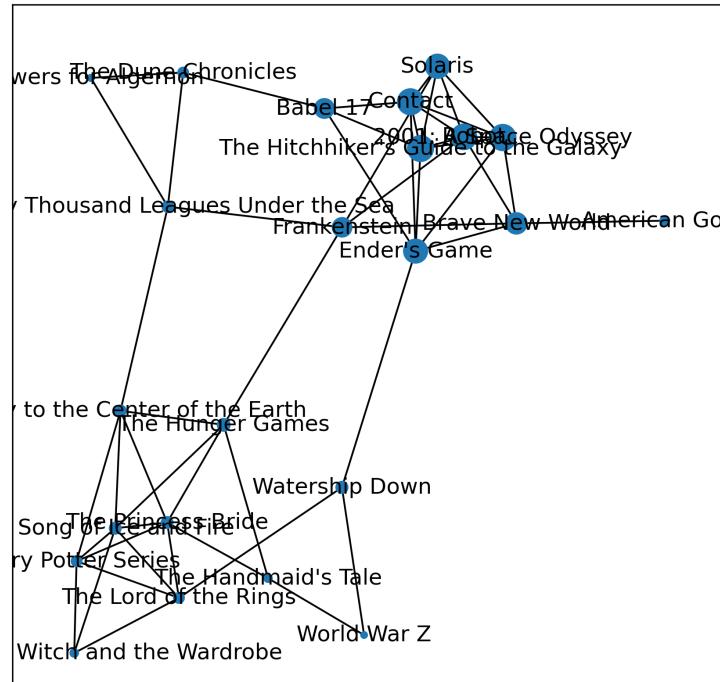
But according to the *Perron-Frobenius* theorem, since A 's elements are all non-negative, there exists only one eigenvector with all elements non-negative, the leading eigenvector, corresponding to the largest eigenvalue λ

```
cent_eig = nx.eigenvector_centrality(G)
pretty_print(get_top_n_dict(cent_eig))

# Compare this to:
# A = nx.to_numpy_array(G)
# eigenvalues, eigenvectors = np.linalg.eigh(A)
# eigenvectors[:, -1]

## 2001: A Space Odyssey: 0.39
## The Hitchhiker's Guide to the Galaxy: 0.38
## Contact: 0.38
## I, Robot: 0.37
## Solaris: 0.33
```

```
nx.draw_networkx(G, nodelist = cent_eig.keys(),
    node_size = [c * 500 for c in cent_eig.values()])
plt.show()
```



Issues with Eigenvector Centrality

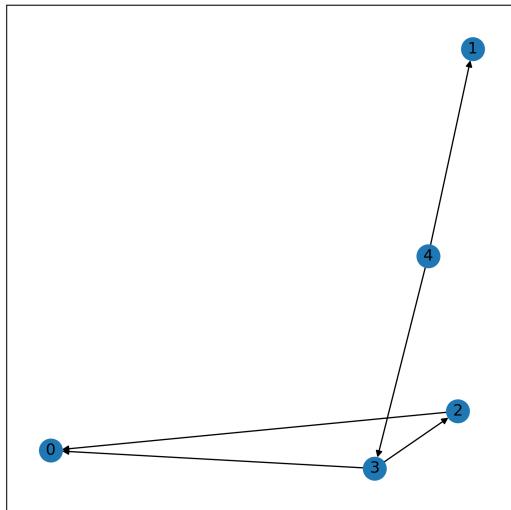
- What to do with directed networks?

Common solution: you are more "central" if many central nodes point *to* you --> definition remains the same only we stress that the eigenvector taken is the *right* leading eigenvector.

- What about DAGs?

```
A = np.array([
[0, 0, 1, 1, 0],
[0, 0, 0, 0, 1],
[0, 0, 0, 1, 0],
[0, 0, 0, 0, 1],
[0, 0, 0, 0, 0]
])
Dag = nx.from_numpy_array(A.transpose(), create_using=nx.DiGraph)
```

```
nx.draw_networkx(Dag)
plt.show()
```



```
nx.eigenvector_centrality(Dag)
```

```
## Error in py_call_impl(callable, dots$args, dots$keywords):
networkx.exception.PowerIterationFailedConvergence:
(PowerIterationFailedConvergence(...), 'power iteration failed to converge
in 100 iterations')
```

Katz Centrality

Let us give each node a "baseline" centrality β :

$$x_i = \alpha \sum_{j=1}^n A_{ij} x_j + \beta$$

It can be shown that:

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1} \cdot \mathbf{1}$$

And therefore β itself isn't really important.

It is custom to have $\beta = 1$ and α vary between 0 (constant β centrality) and $1/\lambda$ where λ is the maximum eigenvalue of A , in which case you get centrality very similar to the eigenvector centrality, plus a little addition to avoid degeneration.

So this now works:

```
nx.katz_centrality(Dag)
```

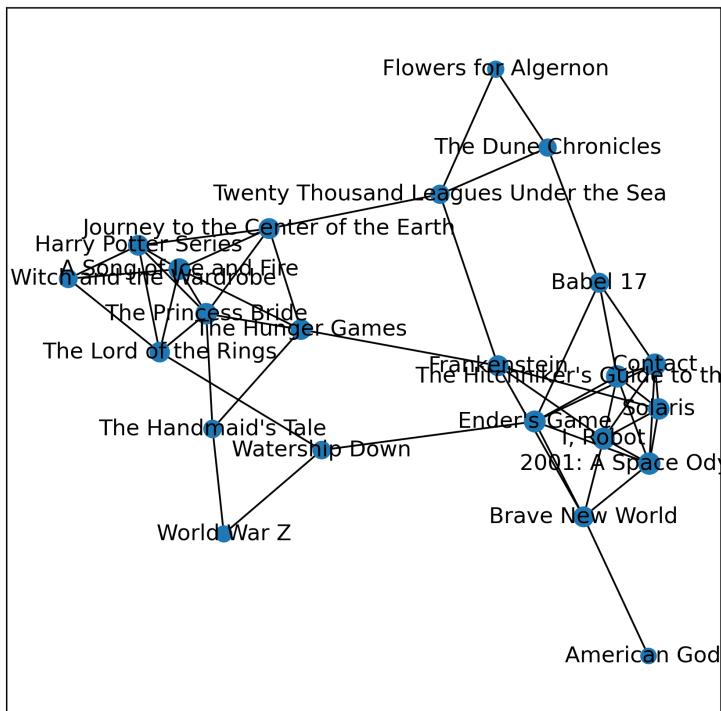
```
## {0: 0.4926375274857429, 1: 0.44381759232949813, 2: 0.44785229771431173, :}
```

And in the Sci-Fi books case:

```
cent_katz = nx.katz_centrality(G)
pretty_print(get_top_n_dict(cent_katz))
```

```
## 2001: A Space Odyssey: 0.25
## The Hitchhiker's Guide to the Galaxy: 0.25
## Contact: 0.25
## I, Robot: 0.25
## Ender's Game: 0.24
```

```
nx.draw_networkx(G, nodelist = cent_katz.keys(),
    node_size = [c * 500 for c in cent_katz.values()])
plt.show()
```



PageRank

One issue with Katz centrality is that it gives equal weights to all of a node's edges. This becomes a problem in large networks like the WWW, in which a directory website could have links to potentially millions and billions of sites. PageRank improves on Katz centrality by dividing the contribution of each node by its (out) degree:

$$x_i = \alpha \sum_{j=1}^n A_{ij} \frac{x_j}{\max(k_j^{\text{out}}, 1)} + \beta$$

It can be shown that:

$$\mathbf{x} = \beta(\mathbf{I} - \alpha \mathbf{A} \mathbf{D}^{-1}) \cdot \mathbf{1}$$

Where \mathbf{D} is a diagonal matrix with $D_{ii} = \max(k_i^{\text{out}}, 1)$

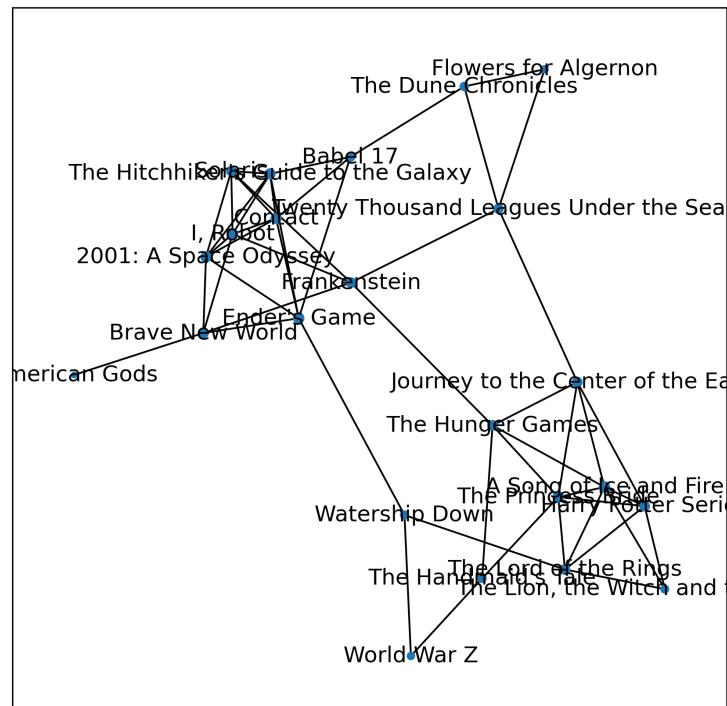
Again β itself isn't really important and is usually set to 1.

For undirected networks it can be shown that α should vary between 0 and 1. For directed networks there is no such limit, often a value close to 1 is used, Google and NetworkX use 0.85 by default.

```
cent_pagerank = nx.pagerank(G)
pretty_print(get_top_n_dict(cent_pagerank))
```

```
## The Princess Bride: 0.06
## A Song of Ice and Fire: 0.06
## Ender's Game: 0.06
## I, Robot: 0.05
## 2001: A Space Odyssey: 0.05
```

```
nx.draw_networkx(G, nodelist = cent_pagerank.keys(),
    node_size = [c * 500 for c in cent_pagerank.values()])
plt.show()
```



Closeness

Closeness is the *inverse mean shortest distance* from node i to every other node:

$$x_i = \frac{n}{\sum_j d(i,j)}$$

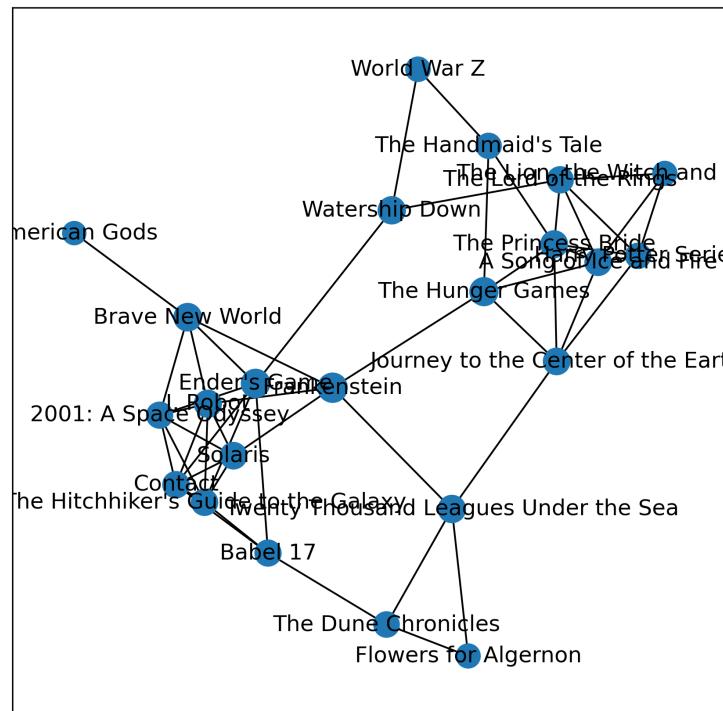


What might be an issue with Closeness centrality?

```
cent_closeness = nx.closeness_centrality(G)
pretty_print(get_top_n_dict(cent_closeness))

## Frankenstein: 0.49
## Ender's Game: 0.46
## The Hunger Games: 0.45
## Brave New World: 0.43
## Twenty Thousand Leagues Under the Sea: 0.42
```

```
nx.draw_networkx(G, nodelist = cent_closeness.keys(),
    node_size = [c * 500 for c in cent_closeness.values()])
plt.show()
```



Betweenness

How "in between" is a node? If we keep pushing messages from one random node to another, how many messages will go through node i ?

$$x_i = \sum_{u,v} \frac{\tau(u,v|i)}{\tau(u,v)}$$

Where $\tau(u, v)$ is the number of shortest paths from node u to node v and $\tau(u, v|i)$ is the number of those shortest paths which pass through node i .

When $\tau(u, v) = 0$ the contribution is "0/0" and is defined as zero.

💡 What is assumed regarding the motivation for Betweenness centrality above?

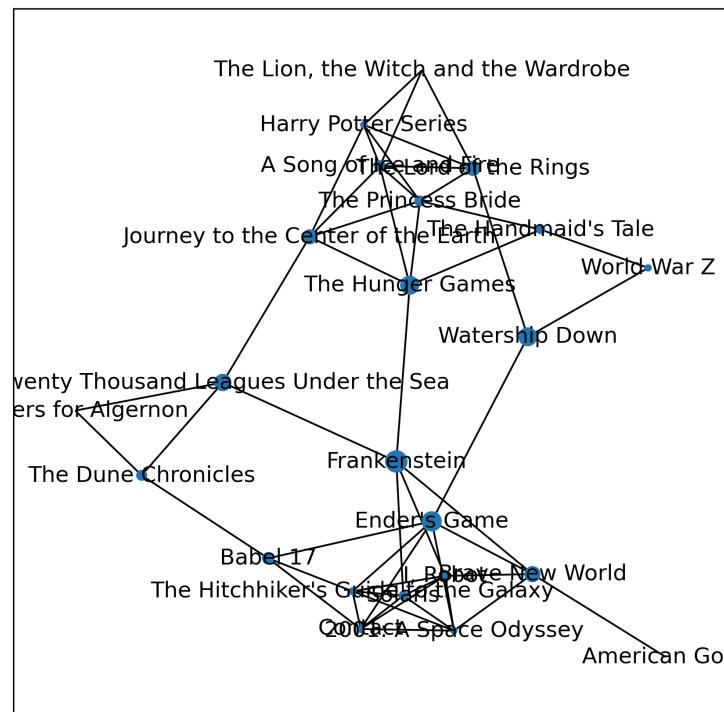
How would you "normalize" Betweenness to range from 0 to 1?

How would you imagine a node with high Betweenness (a *broker*) "looks like"?

```
cent_betweenness = nx.betweenness_centrality(G)
pretty_print(get_top_n_dict(cent_betweenness))
```

```
## Frankenstein: 0.25
## Ender's Game: 0.20
## The Hunger Games: 0.18
## Watership Down: 0.17
## Twenty Thousand Leagues Under the Sea: 0.14
```

```
nx.draw_networkx(G, nodelist = cent_betweenness.keys(),
    node_size = [c * 500 for c in cent_betweenness.values()])
plt.show()
```



Group Centrality

In recent years there is also interest in measuring the centrality of a *group* of nodes. We can naturally extend some of the definitions we have seen:

- Group degree centrality: the fraction of non-group members connected to group members
- Group closeness centrality: inverse mean of the *minimum* shortest distance (minimum across all nodes in the group) to all nodes in the non-group
- Group betweenness centrality: sum of the fraction of number of shortest paths between two nodes in the non-group, passing through any node in the group, from the number of all shortest paths between the two nodes

See NetworkX [docs](#) for further details

Detour: Cliques

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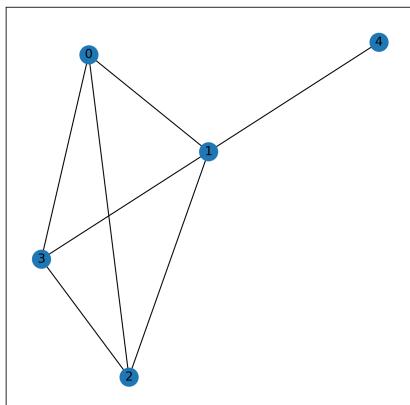


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You know what a clique is...

A clique is a group of nodes in an undirected network so that each node is connected to each other node.

```
Clique = nx.Graph()  
Clique.add_edges_from([(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2,  
nx.draw_networkx(Clique)  
plt.show()
```



End of Detour

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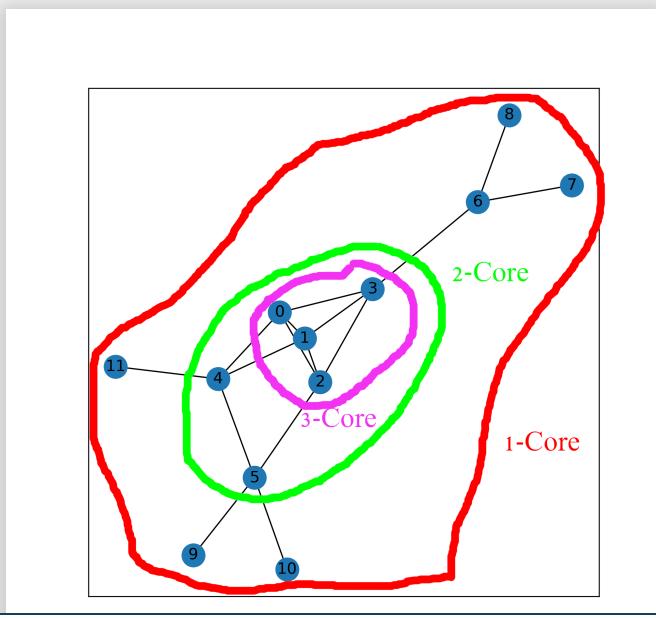


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Cores

k -Cores are a type of relaxation of Cliques which is also useful in describing centrality of a group of nodes.

k -Core is a group of nodes where each node is connected to at least k of the other nodes.



For a given connected component the k -Cores are nested within each other: the 2-cores are a subset of the 1-cores (since if a node is connected to at least 2 nodes it must be connected to at least 1 node), the 3-cores are subsets of the 2-cores, etc.

We get an onion or mountain-like structure which might be useful in describing centrality.

```
k_cores = nx.core_number(G)
pretty_print(get_top_n_dict(k_cores))

## I, Robot: 4.00
## Solaris: 4.00
## 2001: A Space Odyssey: 4.00
## The Hitchhiker's Guide to the Galaxy: 4.00
## Contact: 4.00
```

 How would you find all k-Cores?

Cohesion

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Density

We've seen density before: the fraction of existing edges from potential edges:

$$\rho = \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)}$$

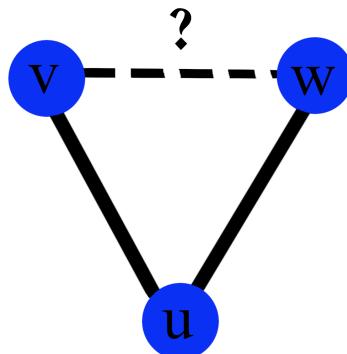
Which means the density for large networks is roughly the ratio of average degree to network size:

$$\rho = \frac{c}{n-1} \propto \frac{c}{n}$$

```
nx.density(G)
```

```
## 0.2015810276679842
```

Transitivity



If Virginia is friends with Ursula, and Ursula is friends with Winona, will Virginia be friends with Winona? Is the *vuw connected triple* or *triad a closed triad*?

The graph *transitivity* is the proportion of closed triads or triangles out of all triads:

$$\text{Transitivity}(G) = \frac{\#\text{closed triads}}{\#\text{triads}} = \frac{3 \times \#\text{triangles}}{\#\text{triads}} = \frac{6 \times \#\text{triangles}}{\#\text{paths of length 2}}$$



Where did the 3 and 6 factors come from?

```
nx.transitivity(G)
```

```
## 0.5223880597014925
```

Is this high? One way is to compare a graph transitivity to its density. If nodes connected to other nodes by random we would expect transitivity of about 20%. This is much higher, implying that the network is far from "random" and there is some logic to its edges.



How would a network with transitivity 0 or 1 look like?

Local Clustering Coefficient

The local equivalent of transitivity to a single node is the proportion of closed triads the node is part of, out of all pairs of neighbors:

$$LCC(i) = \frac{\text{\#closed triads involving node } i}{\text{\#pairs of neighbors of } i} = \frac{2 \times \text{\#closed triads involving node } i}{k_i(k_i - 1)}$$

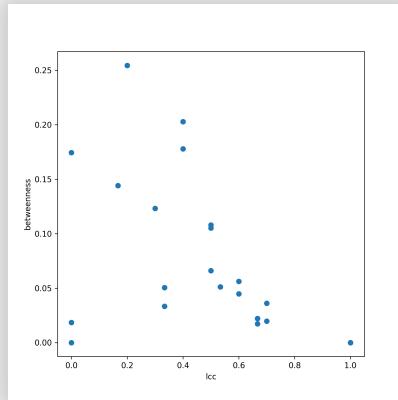


How would a node with clustering coefficient 0 or 1 look like?

```
clust_coefs = nx.clustering(G)
pretty_print(get_top_n_dict(clust_coefs))
```

```
## Flowers for Algernon: 1.00
## The Lion, the Witch and the Wardrobe: 1.00
## Solaris: 0.70
## Harry Potter Series: 0.70
## 2001: A Space Odyssey: 0.67
```

LCC vs. Betweenness Centrality



In many realistic networks we see a negative correlation between LCC and betweenness centrality. Since for a given node the LCC is much easier to compute it is often a good enough approximation of (inverse) betweenness.



Can you explain this pattern?

Similarity

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Structural Equivalence

- Which nodes in a given network are most similar to one another?
- Which node is most similar to a given node i ?

Structural Equivalence is only one type of similarity and it is concerned with n_{ij} : how many neighbors two nodes i and j share.

💡 Why not just use *that* as a measure of similarity?

⚠️ The Sci-Fi books is already a network based on pairwise similarities. Structural Equivalence does not deal with this type of similarity, calculated over a vector of features.

Working with the Adjacency Matrix

Cosine Distance and Pearson Correlation are two obvious choices:

$$S(i, j) = \cos(\theta) = \frac{\sum_k A_{ik}A_{kj}}{\sqrt{\sum_k A_{ik}^2}\sqrt{\sum_k A_{jk}^2}} = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

$$S(i, j) = r(i, j) = \frac{\sum_k (A_{ik} - \bar{A}_i)(A_{jk} - \bar{A}_j)}{\sqrt{\sum_k (A_{ik} - \bar{A}_i)^2}\sqrt{\sum_k (A_{jk} - \bar{A}_j)^2}}$$

⚠️ Not implemented in NetworkX, you could extract the adjacency matrix yourself and use Scikit-Learn for cosine distance and SciPy for pearson correlation.

Jaccard Coefficient

$$S(i, j) = J(i, j) = \frac{|neigh(i) \cap neigh(j)|}{|neigh(i) \cup neigh(j)|} = \frac{n_{ij}}{k_i + k_j - n_{ij}}$$

```
sim_jaccard = nx.jaccard_coefficient(G)

sim_jaccard_top5 = { (u, v): j for u, v, j in sorted(sim_jaccard, }
pretty_print(sim_jaccard_top5)
```

```
## ('The Princess Bride', 'The Lion, the Witch and the Wardrobe'): 0.50
## ('I, Robot', "Ender's Game"): 0.50
## ('Harry Potter Series', 'The Hunger Games'): 0.43
## ('Journey to the Center of the Earth', 'The Lord of the Rings'): 0.43
## ('Solaris', 'Brave New World'): 0.43
```

Homophily

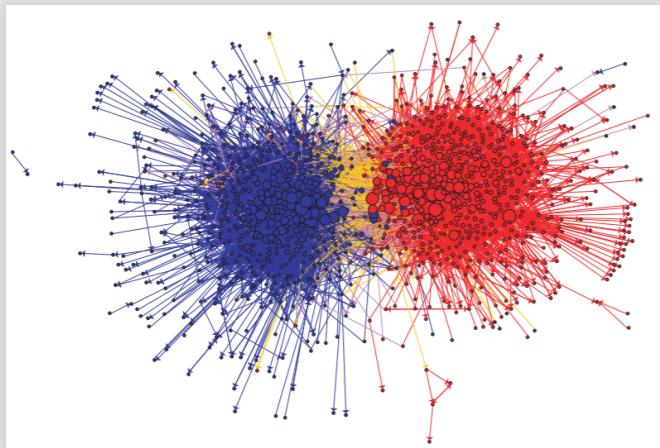
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Assortative Mixing

This has just arrived: people tend to connect with other people similar to them! a.k.a *assortative mixing* (Sorry to burst your bubble)



[The Political Blogosphere and the 2004 U.S. Election: Divided They Blog / Lada Adamic](#)



What would be an easy example of a network displaying *disassortative mixing*?

Categorical Attribute Assortative Mixing

- Is the network of Israeli musical cooperations assortative for musical style?
- Is the Sci-Fi books network assortative for author gender?
- Is the choice of spouse assortative for race?

Behold the joint probability distribution of attribute race for a real-life network:

| | | women | | | | a_i |
|-----|----------|-------|----------|-------|-------|-------|
| | | black | hispanic | white | other | |
| men | black | 0.258 | 0.016 | 0.035 | 0.013 | 0.323 |
| | hispanic | 0.012 | 0.157 | 0.058 | 0.019 | 0.247 |
| | white | 0.013 | 0.023 | 0.306 | 0.035 | 0.377 |
| | other | 0.005 | 0.007 | 0.024 | 0.016 | 0.053 |
| | | b_i | 0.289 | 0.204 | 0.423 | 0.084 |

The percentages in the cells refer to the proportion of edges for which the 1st node (a man) is of race i (e.g. black) and the 2nd node (a woman) is of race j (e.g. white)

We wish to measure to what extent all edges are found on the diagonal of the above matrix (perfect assortativity).

Let M be the joint probability distribution matrix of the specified categorical attribute, so that $\sum_{ij} M_{ij} = 1$. And let a_i and b_j be the marginal probabilities of 1st and 2nd nodes (row and column) being of type i and j respectively, such that:

$$\sum_j M_{ij} = a_i; \quad \sum_i M_{ij} = b_j$$

If the nodes formed edges randomly, the expected probability of 1st and 2nd nodes (row and column) being both of type i would be: $a_i b_i$. So the "distance" between the observed and expected over M is: $\sum_i (M_{ii} - a_i b_i)$ and we divide this by the maximum value this distance can reach under perfect assortativity and get:

$$Assortativity_1(attribute) = \frac{\sum_i M_{ii} - \sum_i a_i b_i}{1 - \sum_i a_i b_i} = \frac{\text{Tr}(M) - ||M^2||}{1 - ||M^2||}$$

Where $||B||$ means the sum of all elements in matrix B .

This *assortative coefficient* is positive for attributes which demonstrate assortativity (up to a maximum of 1), 0 for attributes with no apparent assortativity and negative for attributes which demonstrate disassortativity.

💡 Why not use a χ^2 test for independence?

How would you perform hypothesis testing on such a statistic?

```
scifi_books = pd.read_csv('../data/sci_fi_books.csv')
scifi_author_gender = scifi_books[scifi_books['book'].isin(G.nodes)]
nx.set_node_attributes(G, scifi_author_gender, 'author_gender')
print(nx.attribute_mixing_matrix(G, 'author_gender'))
```

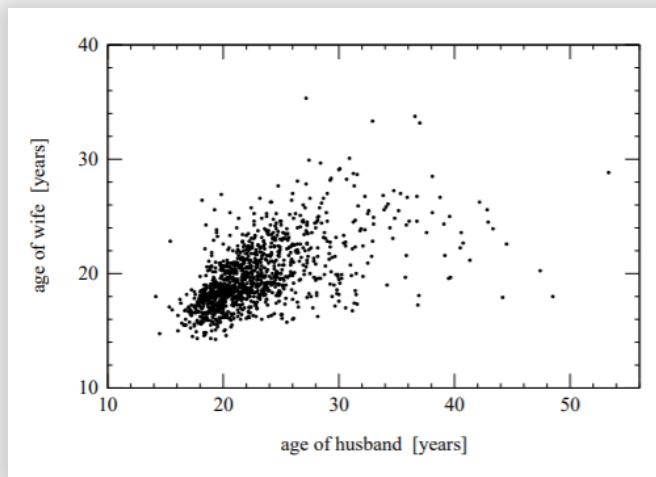
```
## [[0.68627451  0.1372549 ]
##   [0.1372549   0.03921569]]
```

```
nx.attribute_assortativity_coefficient(G, 'author_gender')
```

```
## 0.0555555555555517
```

Ordered/Numeric Attribute Assortative Mixing

- Is the network of Israeli musical cooperations assortative for artists popularity?
- Is the Sci-Fi books network assortative for year of release?
- Is the choice of spouse assortative for age?



For this specific case we *could* use a simple Pearson correlation coefficient.

In general though we might be dealing with a discrete numeric attribute with not many levels (e.g. year, grade) and the *weighted* correlation coefficient is used.

Let M be the same joint probability distribution matrix for a discrete numeric attribute (if not discrete we can discretize it or use integration). And let a_{x_i} and b_{y_j} be the marginal probabilities of 1st and 2nd nodes (row and column) being of type x_i and y_j respectively.

As before: $\sum_{ij} M_{ij} = 1$; $\sum_j M_{ij} = a_{x_i}$; $\sum_i M_{ij} = b_{y_j}$

The weighted mean of attribute (variable) x :

$$\hat{E}(x; a) = \frac{\sum_i x_i a_{x_i}}{\sum_i a_{x_i}} = \sum_i x_i a_{x_i} = \sum_{ij} x_i M_{ij}$$

$$\begin{aligned}
Assortativity_2(attribute) &= r(x, y; M) = \frac{\hat{c}\hat{o}v(x, y; M)}{\hat{\sigma}(x)\hat{\sigma}(y)} = \\
&= \frac{\hat{E}(xy; E) - \hat{E}(x; a)\hat{E}(y; b)}{\hat{\sigma}(x)\hat{\sigma}(y)} = \frac{\sum_{ij} M_{ij}x_iy_j - \sum_i x_i a_{xi} \cdot \sum_j y_j b_{yj}}{\sqrt{\hat{c}\hat{o}v(x, x; M)\hat{c}\hat{o}v(y, y; M)}} \\
&= \frac{\sum_{ij} M_{ij}x_iy_j - \sum_{ij} a_{xi}b_{yj}x_iy_j}{\sqrt{\hat{c}\hat{o}v(x, x; M)\hat{c}\hat{o}v(y, y; M)}} = \frac{\sum_{ij} x_iy_j(M_{ij} - a_{xi}b_{yj})}{\sqrt{\hat{c}\hat{o}v(x, x; M)\hat{c}\hat{o}v(y, y; M)}}
\end{aligned}$$

This is a proper correlation coefficient, which ranges from -1 (perfect disassortativity) to 1 (perfect assortativity).

```

scifi_quarter = scifi_books[scifi_books['book'].isin(G.nodes)].set
nx.set_node_attributes(G, scifi_quarter, 'quarter_century')

# print(nx.numeric_mixing_matrix(G, 'quarter_century'))
nx.numeric_assortativity_coefficient(G, 'quarter_century')

```

0.11555476906742237