Complex Analysis

Md. Aquil Khan

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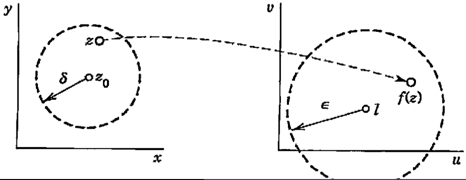

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A complex function is said to be continuous in a region R if it is continuous at every point in R.

Theorem

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$u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0)

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- f is not continuous at $z = 0$.

Theorems on real continuous functions can be extended to complex continuous functions

- Sum, difference and product of continuous functions are also continuous