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54	56
55	58
56	59
57	60
58	61
59	62
60	63
61	64
62	65
63	66
64	67
65	69
66	70
67	71
68	72
69	73
70	74
71	75
72	76
73	77
74	78

1 Multiples of 3 and 5

[Problem]

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

↗ Multiples of 3 and 5

10 未満の自然数のうち、3 もしくは 5 の倍数になっているものは 3,5,6,9 の 4 つがあり、これらの合計は 23 になる。

同じようにして、1000 未満の 3 か 5 の倍数になっている数字の合計を求めよ。

↗ 3 と 5 の倍数

2 Even Fibonacci numbers

[Problem]

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

↗ Even Fibonacci numbers

フィボナッチ数列の項は前の 2 つの項の和である。最初の 2 項を 1, 2 とすれば、最初の 10 項は以下の通りである。

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

数列の項の値が 400 万以下の、偶数値の項の総和を求めよ。

↗ 偶数のフィボナッチ数

3 Largest prime factor

[Problem]

The prime factors of 13195 are 5, 7, 13 and 29.

What is the largest prime factor of the number 600851475143 ?

↗ Largest prime factor

13195 の素因数は 5, 7, 13, 29 である.

600851475143 の素因数のうち最大のものを求めよ.

↗ 最大の素因数

4 Largest palindrome product

[Problem]

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is $9009 = 91 \times 99$.

Find the largest palindrome made from the product of two 3-digit numbers.

↗ Largest palindrome product

左右どちらから読んでも同じ値になる数を回文数という。2桁の数の積で表される回文数のうち、最大のものは $9009 = 91 \times 99$ である。

では、3桁の数の積で表される回文数の最大値を求めよ。

↗ 最大の回文積

5 Smallest multiple

[Problem]

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?

↗ Smallest multiple

2520 は 1 から 10 の数字の全ての整数で割り切れる数字であり，そのような数字の中では最小の値である．

では，1 から 20 までの整数全てで割り切れる数字の中で最小の正の数はいくらになるか．

↗ 最小の倍数

6 Sum square difference

[Problem]

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + \cdots + 10^2 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + \cdots + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$.

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

↗ Smallest multiple

最初の 10 個の自然数について，その二乗の和は，

$$1^2 + 2^2 + \cdots + 10^2 = 385$$

最初の 10 個の自然数について，その和の二乗は，

$$(1 + 2 + \cdots + 10)^2 = 55^2 = 3025$$

これらの数の差は $3025 - 385 = 2640$ となる．同様にして，最初の 100 個の自然数について二乗の和と和の二乗の差を求めよ．

↗ 最小の倍数

7 10001st prime

[Problem]

By listing the first six prime numbers: 2, 3, 5, 7, 11 and 13, we can see that the 6th prime is 13.

What is the 10001st prime number?

↗ 10001st prime

素数を小さい方から6つ並べると2, 3, 5, 7, 11, 13であり, 6番目の素数は13である.

10,001番目の素数を求めよ.

↗ 10001番目の素数

8 Largest product in a series

[Problem]

The four adjacent digits in the 1000-digit number that have the greatest product are $9 \times 9 \times 8 \times 9 = 5832$.

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
```

Find the thirteen adjacent digits in the 1000-digit number that have the greatest product. What is the value of this product?

🔗 Largest product in a series

次の 1000 桁の数字のうち，隣接する 4 つの数字の総乗の中で，最大となる値は， $9 \times 9 \times 8 \times 9 = 5832$ である．

(1000 桁の数字は上述)

この 1000 桁の数字から 13 個の連続する数字を取り出して，それらの総乗を計算する．では，それら総乗のうち，最大となる値はいくらか．

🔗 数字列中の最大の積

9 Special Pythagorean triplet

[Problem]

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2$$

For example, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$. Find the product abc .

☞ Special Pythagorean triplet

ピタゴラス数（ピタゴラスの定理を満たす自然数）とは $a < b < c$ で以下の式を満たす数の組である.

$$a^2 + b^2 = c^2$$

例えば, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$ である.

$a + b + c = 1000$ となるピタゴラスの三つ組が一つだけ存在する. これらの積 abc を計算しなさい.

☞ 特別なピタゴラス数

10 Summation of primes

[Problem]

The sum of the primes below 10 is $2 + 3 + 5 + 7 = 17$.

Find the sum of all the primes below two million.

↗ Summation of primes

10 以下の素数の和は $2 + 3 + 5 + 7 = 17$ である.

200 万以下の全ての素数の和を求めよ.

↗ 素数の和

11 Largest product in a grid

[Problem]

In the 20×20 grid below, four numbers along a diagonal line have been marked in red. The product of these numbers is $26 \times 63 \times 78 \times 14 = 1788696$.

```

08 02 22 97 38 15 00 40 00 75 04 05 07 78 52 12 50 77 91 08
49 49 99 40 17 81 18 57 60 87 17 40 98 43 69 48 04 56 62 00
81 49 31 73 55 79 14 29 93 71 40 67 53 88 30 03 49 13 36 65
52 70 95 23 04 60 11 42 69 24 68 56 01 32 56 71 37 02 36 91
22 31 16 71 51 67 63 89 41 92 36 54 22 40 40 28 66 33 13 80
24 47 32 60 99 03 45 02 44 75 33 53 78 36 84 20 35 17 12 50
32 98 81 28 64 23 67 10 26 38 40 67 59 54 70 66 18 38 64 70
67 26 20 68 02 62 12 20 95 63 94 39 63 08 40 91 66 49 94 21
24 55 58 05 66 73 99 26 97 17 78 78 96 83 14 88 34 89 63 72
21 36 23 09 75 00 76 44 20 45 35 14 00 61 33 97 34 31 33 95
78 17 53 28 22 75 31 67 15 94 03 80 04 62 16 14 09 53 56 92
16 39 05 42 96 35 31 47 55 58 88 24 00 17 54 24 36 29 85 57
86 56 00 48 35 71 89 07 05 44 44 37 44 60 21 58 51 54 17 58
19 80 81 68 05 94 47 69 28 73 92 13 86 52 17 77 04 89 55 40
04 52 08 83 97 35 99 16 07 97 57 32 16 26 26 79 33 27 98 66
88 36 68 87 57 62 20 72 03 46 33 67 46 55 12 32 63 93 53 69
04 42 16 73 38 25 39 11 24 94 72 18 08 46 29 32 40 62 76 36
20 69 36 41 72 30 23 88 34 62 99 69 82 67 59 85 74 04 36 16
20 73 35 29 78 31 90 01 74 31 49 71 48 86 81 16 23 57 05 54
01 70 54 71 83 51 54 69 16 92 33 48 61 43 52 01 89 19 67 48

```

What is the greatest product of four adjacent numbers in the same direction (up, down, left, right, or diagonally) in the 20×20 grid?

☞ Largest product in a grid

上の 20×20 の格子のうち、斜めに並んだ 4 つの数字が赤くマークされている。

(20×20 の格子は上述)

200 万以下の全ての素数の和を求めよ。それらの数字の積は $26 \times 63 \times 78 \times 14 = 1788696$ となる。

上の 20×20 の格子のうち、上下左右斜めのいずれかの方向で連続する 4 つの数字の積のうち最大のものはいくつか?

☞ 格子内の最大の積

12 Highly divisible triangular number

[Problem]

The sequence of triangle numbers is generated by adding the natural numbers. So the 7th triangle number would be $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. The first ten terms would be:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Let us list the factors of the first seven triangle numbers:

1 : 1

3 : 1, 3

6 : 1, 2, 3, 6

10 : 1, 2, 5, 10

15 : 1, 3, 5, 15

21 : 1, 3, 7, 21

28 : 1, 2, 4, 7, 14, 28

We can see that 28 is the first triangle number to have over five divisors.

What is the value of the first triangle number to have over five hundred divisors?

↗ Highly divisible triangular number

三角数の数列は自然数の和で表わされ、7 番目の三角数は $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ である。三角数の最初の 10 項は：

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

となる。

最初の 7 項について、その約数を列挙すると、以下のとおり。

(上述)

これから、7 番目の三角数である 28 は、5 個より多く約数をもつ最初の三角数であることが分かる。

では、500 個より多く約数をもつ最初の三角数はいくつか。

↗ 高度整除三角数

13 Large sum

[Problem]

Work out the first ten digits of the sum of the following one-hundred 50-digit numbers.

37107287533902102798797998220837590246510135740250
74324986199524741059474233309513058123726617309629
23067588207539346171171980310421047513778063246676
28112879812849979408065481931592621691275889832738
47451445736001306439091167216856844588711603153276
62176457141856560629502157223196586755079324193331
92575867718337217661963751590579239728245598838407
80181199384826282014278194139940567587151170094390
86515506006295864861532075273371959191420517255829
54370070576826684624621495650076471787294438377604
36123272525000296071075082563815656710885258350721
17423706905851860660448207621209813287860733969412
51934325451728388641918047049293215058642563049483
15732444386908125794514089057706229429197107928209
18336384825330154686196124348767681297534375946515
78182833757993103614740356856449095527097864797581
48403098129077791799088218795327364475675590848030
59959406895756536782107074926966537676326235447210
41052684708299085211399427365734116182760315001271
35829035317434717326932123578154982629742552737307
88902802571733229619176668713819931811048770190271
36270218540497705585629946580636237993140746255962
9143028819710328859780666976089293863828502533403
23053081172816430487623791969842487255036638784583
63783299490636259666498587618221225225512486764533
95548255300263520781532296796249481641953868218774
37774242535411291684276865538926205024910326572967
29798860272258331913126375147341994889534765745501
38298203783031473527721580348144513491373226651381
40957953066405232632538044100059654939159879593635
41698116222072977186158236678424689157993532961922
23189706772547915061505504953922979530901129967519
11306739708304724483816533873502340845647058077308
97623331044818386269515456334926366572897563400500
55121603546981200581762165212827652751691296897789
75506164965184775180738168837861091527357929701337
32924185707147349566916674687634660915035914677504
73267460800591547471830798392868535206946944540724
97142617910342598647204516893989422179826088076852
10848802521674670883215120185883543223812876952786
6218407357239974223406235393808339651327408011116
60661826293682836764744779239180335110989069790714
66024396409905389607120198219976047599490197230297
16730939319872750275468906903707539413042652315011
78639167021187492431995700641917969777599028300699
40789923115535562561142322423255033685442488917353
41503128880339536053299340368006977710650566631954
82616570773948327592232845941706525094512325230608
77158542502016545090413245809786882778948721859617
20849603980134001723930671666823555245252804609722

46376937677490009712648124896970078050417018260538
91942213363574161572522430563301811072406154908250
89261670696623633820136378418383684178734361726757
44274228917432520321923589422876796487670272189318
70386486105843025439939619828917593665686757934951
64906352462741904929101432445813822663347944758178
58203565325359399008402633568948830189458628227828
35398664372827112653829987240784473053190104293586
71693888707715466499115593487603532921714970056938
53282654108756828443191190634694037855217779295145
45876576172410976447339110607218265236877223636045
81142660418086830619328460811191061556940512689692
62467221648435076201727918039944693004732956340691
55037687525678773091862540744969844508330393682126
80386287592878490201521685554828717201219257766954
16726320100436897842553539920931837441497806860984
87086987551392711854517078544161852424320693150332
69793950679652694742597709739166693763042633987085
65378607361501080857009149939512557028198746004375
94953759765105305946966067683156574377167401875275
25267680276078003013678680992525463401061623866526
24074486908231174977792365466257246923322810917141
344130655780162127815921815005561868836468420090470
11487696932154902810424020138335124462181441773470
67720186971698544312419572409913959008952310058822
76085327132285723110424803456124867697064507995236
23701913275725675285653248258265463092207058596522
18495701454879288984856827726077713721403798879715
34829543829199918180278916522431027392251122869539
29746152185502371307642255121183693803580388584903
62467957194401269043877107275048102390895523597457
8618808822587531452958409925120382900940770775672
82959174767140363198008187129011875491310547126581
4284628018351707052783183942588214552122751250327
3223819573432933994643750190783694576588335239986
62177842752192623401942399639168044983993173312731
99518671430235219628894890102423325116913619626622
7684182252467441716151403642798227334805556214818
8778364618279934631376775430780936333018982642090
71329612474782464538636993009049310363619763878039
66627891981488087797941876876144230030984490851411
85786944089552990653640447425576083659976645795096
6491398268003297315603712004137790378556608589252
94809377245048795150954100921645863754710598436791
15368713711936614952811305876380278410754449733078
44889911501440648020369068063960672322193204149535
8123488067321046739058568557934581403627822703280
2291880205877319719839450180888072429661980811197
72107838435069186155435662884062257473692284509516
53503534226472524250874054075591789781264330331960

↗ Large sum

以下の 50 桁の数字 100 個の合計の上から 10 桁を求めなさい。

(50 桁の数字 100 個は上述)

↗ 大きな数の足し算

15

14 Longest Collatz sequence

[Problem]

The following iterative sequence is defined for the set of positive integers:

$$n \rightarrow n / 2 \quad (n \text{ is even})$$

$$n \rightarrow 3n + 1 \quad (n \text{ is odd})$$

Using the rule above and starting with 13, we generate the following sequence:

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

NOTE: Once the chain starts the terms are allowed to go above one million.

↗ Longest Collatz sequence

正の整数に以下の式で繰り返し生成する数列を定義する.

$$n \rightarrow n / 2 \quad (n \text{ は偶数})$$

$$n \rightarrow 3n + 1 \quad (n \text{ は奇数})$$

13 から始めるとこの数列は以下ようになる.

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

13 から 1 まで 10 個の項になる. この数列はどのような数字からはじめても最終的には 1 になると考えられているが, まだそのことは証明されていない (コラッツ問題)

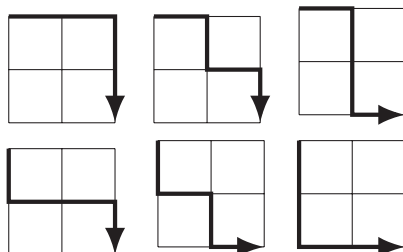
さて, 100 万未満の数字の中でどの数字からはじめれば最長の数列を生成するか.

↗ 最長のコラッツ数列

15 Lattice paths

[Problem]

Starting in the top left corner of a 2×2 grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner.



How many such routes are there through a 20×20 grid?

↗ Lattice paths

2×2 のマス目の左上からスタートした場合、引き返しなしで右下に行くルートは 6 つある。
では、 20×20 のマス目ではいくつのルートがあるか。

↗ 格子経路

16 Power digit sum

[Problem]

$2^{15} = 32768$ and the sum of its digits is $3 + 2 + 7 + 6 + 8 = 26$.

What is the sum of the digits of the number 2^{1000} ?

↗ Power digit sum

$2^{15} = 32768$ であり、各位の数字の和は $3 + 2 + 7 + 6 + 8 = 26$ となる。

同様に、 2^{1000} の各位の数字の和を求めよ。

↗ 各位の数字の和

17 Number letter counts

[Problem]

If the numbers 1 to 5 are written out in words: one, two, three, four, five, then there are $3 + 3 + 5 + 4 + 4 = 19$ letters used in total.

If all the numbers from 1 to 1000 (one thousand) inclusive were written out in words, how many letters would be used?

NOTE: Do not count spaces or hyphens. For example, 342 (three hundred and forty-two) contains 23 letters and 115 (one hundred and fifteen) contains 20 letters. The use of “and” when writing out numbers is in compliance with British usage.

☞ Number letter counts

1 から 5 までの数字を英単語で書けば one, two, three, four, five であり, 全部で $3+3+5+4+4 = 19$ の文字が使われている. では 1 から 1000 (one thousand) までの数字をすべて英単語で書けば, 全部で何文字になるか.

注: 空白文字やハイフンを数えないこと. 例えば, 342 (three hundred and forty-two) は 23 文字, 115 (one hundred and fifteen) は 20 文字と数える. なお, “and” を使用するのは英国の慣習.

☞ 数字の文字数

18 Maximum path sum I

[Problem]

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
      3
     7 4
    2 4 6
   8 5 9 3
```

That is, $3 + 7 + 4 + 9 = 23$.

Find the maximum total from top to bottom of the triangle below:

```
      75
     95 64
    17 47 82
   18 35 87 10
  20 04 82 47 65
 19 01 23 75 03 34
 88 02 77 73 07 63 67
 99 65 04 28 06 16 70 92
 41 41 26 56 83 40 80 70 33
 41 48 72 33 47 32 37 16 94 29
 53 71 44 65 25 43 91 52 97 51 14
 70 11 33 28 77 73 17 78 39 68 17 57
 91 71 52 38 17 14 91 43 58 50 27 29 48
 63 66 04 68 89 53 67 30 73 16 69 87 40 31
 04 62 98 27 23 09 70 98 73 93 38 53 60 04 23
```

NOTE: As there are only 16384 routes, it is possible to solve this problem by trying every route. However, Problem 67, is the same challenge with a triangle containing one-hundred rows; it cannot be solved by brute force, and requires a clever method! ;o)

19 Counting Sundays

[Problem]

You are given the following information, but you may prefer to do some research for yourself.

- 1 Jan 1900 was a Monday.
- Thirty days has September, April, June and November.
- All the rest have thirty-one, Saving February alone, Which has twenty-eight, rain or shine. And on leap years, twenty-nine.
- A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400.

How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?

20 Factorial digit sum

[Problem]

$n!$ means $n \times (n - 1) \times \cdots \times 3 \times 2 \times 1$

For example, $10! = 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 3628800$, and the sum of the digits in the number $10!$ is $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$.

Find the sum of the digits in the number $100!$

21 Amicable numbers

[Problem]

Let $d(n)$ be defined as the sum of proper divisors of n (numbers less than n which divide evenly into n).

If $d(a) = b$ and $d(b) = a$, where $a \neq b$, then a and b are an amicable pair and each of a and b are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore $d(220) = 284$. The proper divisors of 284 are 1, 2, 4, 71 and 142; so $d(284) = 220$.

Evaluate the sum of all the amicable numbers under 10000.

22 Names scores

[Problem]

Using names.txt (right click and 'Save Link/Target As...'), a 46K text file containing over five-thousand first names, begin by sorting it into alphabetical order. Then working out the alphabetical value for each name, multiply this value by its alphabetical position in the list to obtain a name score.

For example, when the list is sorted into alphabetical order, COLIN, which is worth $3 + 15 + 12 + 9 + 14 = 53$, is the 938th name in the list. So, COLIN would obtain a score of $938 \times 53 = 49714$.

What is the total of all the name scores in the file?

23 Non-abundant sums

[Problem]

A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be $1 + 2 + 4 + 7 + 14 = 28$, which means that 28 is a perfect number.

A number n is called deficient if the sum of its proper divisors is less than n and it is called abundant if this sum exceeds n .

As 12 is the smallest abundant number, $1 + 2 + 3 + 4 + 6 = 16$, the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.

Find the sum of all the positive integers which cannot be written as the sum of two abundant numbers.

24 Lexicographic permutations

[Problem]

A permutation is an ordered arrangement of objects. For example, 3124 is one possible permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are:

012 021 102 120 201 210

What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9?

25 1000-digit Fibonacci number

[Problem]

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1 \text{ and } F_2 = 1.$$

Hence the first 12 terms will be:

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 2$$

$$F_4 = 3$$

$$F_5 = 5$$

$$F_6 = 8$$

$$F_7 = 13$$

$$F_8 = 21$$

$$F_9 = 34$$

$$F_{10} = 55$$

$$F_{11} = 89$$

$$F_{12} = 144$$

The 12th term, F_{12} , is the first term to contain three digits.

What is the index of the first term in the Fibonacci sequence to contain 1000 digits?

26 Reciprocal cycles

[Problem]

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$$1 / 2 = 0.5$$

$$1 / 3 = 0.(3)$$

$$1 / 4 = 0.25$$

$$1 / 5 = 0.2$$

$$1 / 6 = 0.1(6)$$

$$1 / 7 = 0.(142857)$$

$$1 / 8 = 0.125$$

$$1 / 9 = 0.(1)$$

$$1 / 10 = 0.1$$

Where $0.1(6)$ means $0.166666\ldots$, and has a 1-digit recurring cycle. It can be seen that $1 / 7$ has a 6-digit recurring cycle.

Find the value of $d < 1000$ for which $1 / d$ contains the longest recurring cycle in its decimal fraction part.

27 Quadratic primes

[Problem]

Euler discovered the remarkable quadratic formula:

$$n^2 + n + 41$$

It turns out that the formula will produce 40 primes for the consecutive integer values $0 \leq n \leq 39$. However, when $n = 40$, $40^2 + 40 + 41 = 40(40 + 1) + 41$ is divisible by 41, and certainly when $n = 41$, $41^2 + 41 + 41$ is clearly divisible by 41.

The incredible formula $n^2 - 79n + 1601$ was discovered, which produces 80 primes for the consecutive values $0 \leq n \leq 79$. The product of the coefficients, -79 and 1601 , is -126479 .

Considering quadratics of the form:

$$n^2 + an + b, \text{ where } |a| < 1000 \text{ and } |b| < 1000$$

where $|n|$ is the modulus/absolute value of n

e.g. $|11| = 11$ and $|-4| = 4$

Find the product of the coefficients, a and b , for the quadratic expression that produces the maximum number of primes for consecutive values of n , starting with $n = 0$.

28 Number spiral diagonals

[Problem]

Starting with the number 1 and moving to the right in a clockwise direction a 5 by 5 spiral is formed as follows:

21	22	23	24	25
20	7	8	9	10
19	6	1	2	11
18	5	4	3	12
17	16	15	14	13

It can be verified that the sum of the numbers on the diagonals is 101.

What is the sum of the numbers on the diagonals in a 1001 by 1001 spiral formed in the same way?

29 Distinct powers

[Problem]

Consider all integer combinations of ab for $2 \leq a \leq 5$ and $2 \leq b \leq 5$:

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$3^2 = 9$$

$$3^3 = 27$$

$$3^4 = 81$$

$$3^5 = 243$$

$$4^2 = 16$$

$$4^3 = 64$$

$$4^4 = 256$$

$$4^5 = 1024$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

$$5^5 = 3125$$

If they are then placed in numerical order, with any repeats removed, we get the following sequence of 15 distinct terms:

4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125

How many distinct terms are in the sequence generated by ab for $2 \leq a \leq 100$ and $2 \leq b \leq 100$?

30 Digit fifth powers

[Problem]

Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits:

$$1634 = 1^4 + 6^4 + 3^4 + 4^4$$

$$8208 = 8^4 + 2^4 + 0^4 + 8^4$$

$$9474 = 9^4 + 4^4 + 7^4 + 4^4$$

As $1 = 1^4$ is not a sum it is not included.

The sum of these numbers is $1634 + 8208 + 9474 = 19316$.

Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.

31 Coin sums

[Problem]

In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation:

1p, 2p, 5p, 10p, 20p, 50p, £1(100p) and £2(200p).

It is possible to make £2 in the following way:

$$1 \times £1 + 1 \times 50p + 2 \times 20p + 1 \times 5p + 1 \times 2p + 3 \times 1p$$

How many different ways can £2 be made using any number of coins?

32 Pandigital products

[Problem]

We shall say that an n -digit number is pandigital if it makes use of all the digits 1 to n exactly once; for example, the 5-digit number, 15234, is 1 through 5 pandigital.

The product 7254 is unusual, as the identity, $39 \times 186 = 7254$, containing multiplicand, multiplier, and product is 1 through 9 pandigital.

Find the sum of all products whose multiplicand/multiplier/product identity can be written as a 1 through 9 pandigital.

HINT: Some products can be obtained in more than one way so be sure to only include it once in your sum.

33 Digit cancelling fractions

[Problem]

The fraction $49 / 98$ is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that $49 / 98 = 4 / 8$, which is correct, is obtained by cancelling the 9s.

We shall consider fractions like, $30 / 50 = 3 / 5$, to be trivial examples.

There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator.

If the product of these four fractions is given in its lowest common terms, find the value of the denominator.

34 Digit factorials

[Problem]

145 is a curious number, as $1! + 4! + 5! = 1 + 24 + 120 = 145$.

Find the sum of all numbers which are equal to the sum of the factorial of their digits.

Note: as $1! = 1$ and $2! = 2$ are not sums they are not included.

35 Circular primes

[Problem]

The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime.

There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular primes are there below one million?

36 Double-base palindromes

[Problem]

The decimal number, $585 = 1001001001_2$ (binary), is palindromic in both bases.

Find the sum of all numbers, less than one million, which are palindromic in base 10 and base 2.

(Please note that the palindromic number, in either base, may not include leading zeros.)

37 Truncatable primes

[Problem]

The number 3797 has an interesting property. Being prime itself, it is possible to continuously remove digits from left to right, and remain prime at each stage: 3797, 797, 97, and 7. Similarly we can work from right to left: 3797, 379, 37, and 3.

Find the sum of the only eleven primes that are both truncatable from left to right and right to left.

NOTE: 2, 3, 5, and 7 are not considered to be truncatable primes.

38 Pandigital multiples

[Problem]

Take the number 192 and multiply it by each of 1, 2, and 3:

$$192 \times 1 = 192$$

$$192 \times 2 = 384$$

$$192 \times 3 = 576$$

By concatenating each product we get the 1 to 9 pandigital, 192384576. We will call 192384576 the concatenated product of 192 and (1,2,3)

The same can be achieved by starting with 9 and multiplying by 1, 2, 3, 4, and 5, giving the pandigital, 918273645, which is the concatenated product of 9 and (1,2,3,4,5).

What is the largest 1 to 9 pandigital 9-digit number that can be formed as the concatenated product of an integer with (1,2, ..., n) where $n > 1$?

39 Integer right triangles

[Problem]

If p is the perimeter of a right angle triangle with integral length sides, $\{a, b, c\}$, there are exactly three solutions for $p = 120$.

$$\{20, 48, 52\}, \{24, 45, 51\}, \{30, 40, 50\}$$

For which value of $p \leq 1000$, is the number of solutions maximised?

40 Champernowne's constant

[Problem]

An irrational decimal fraction is created by concatenating the positive integers:

0.123456789101112131415161718192021...

It can be seen that the 12th digit of the fractional part is 1.

If d_n represents the n th digit of the fractional part, find the value of the following expression.

$$d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10000} \times d_{100000} \times d_{1000000}$$

41 Pandigital prime

[Problem]

We shall say that an n -digit number is pandigital if it makes use of all the digits 1 to n exactly once.

For example, 2143 is a 4-digit pandigital and is also prime.

What is the largest n -digit pandigital prime that exists?

42 Coded triangle numbers

[Problem]

The n th term of the sequence of triangle numbers is given by, $t_n = (1/2)n(n + 1)$; so the first ten triangle numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

By converting each letter in a word to a number corresponding to its alphabetical position and adding these values we form a word value. For example, the word value for SKY is $19 + 11 + 25 = 55 = t_{10}$. If the word value is a triangle number then we shall call the word a triangle word.

Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, how many are triangle words?

43 Sub-string divisibility

[Problem]

The number, 1406357289, is a 0 to 9 pandigital number because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property.

Let d_1 be the 1st digit, d_2 be the 2nd digit, and so on. In this way, we note the following:

- $d_2d_3d_4 = 406$ is divisible by 2
- $d_3d_4d_5 = 063$ is divisible by 3
- $d_4d_5d_6 = 635$ is divisible by 5
- $d_5d_6d_7 = 357$ is divisible by 7
- $d_6d_7d_8 = 572$ is divisible by 11
- $d_7d_8d_9 = 728$ is divisible by 13
- $d_8d_9d_{10} = 289$ is divisible by 17

Find the sum of all 0 to 9 pandigital numbers with this property.

44 Pentagon numbers

[Problem]

Pentagonal numbers are generated by the formula, $P_n = n(3n - 1)/2$. The first ten pentagonal numbers are:

$$1, 5, 12, 22, 35, 51, 70, 92, 117, 145, \dots$$

It can be seen that $P_4 + P_7 = 22 + 70 = 92 = P_8$. However, their difference, $70 - 22 = 48$, is not pentagonal.

Find the pair of pentagonal numbers, P_j and P_k , for which their sum and difference are pentagonal and $D = |P_k - P_j|$ is minimised; what is the value of D ?

45 Triangular, pentagonal, and hexagonal

[Problem]

Triangle, pentagonal, and hexagonal numbers are generated by the following formulae:

$$\text{Triangle} \quad T_n = n(n + 1) / 2 \quad 1, 3, 6, 10, 15, \dots$$

$$\text{Pentagonal} \quad P_n = n(3n - 1) / 2 \quad 1, 5, 12, 22, 35, \dots$$

$$\text{Hexagonal} \quad H_n = n(2n - 1) \quad 1, 6, 15, 28, 45, \dots$$

It can be verified that $T_{285} = P_{165} = H_{143} = 40755$.

Find the next triangle number that is also pentagonal and hexagonal.

46 Goldbach's other conjecture

[Problem]

It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square.

$$9 = 7 + 2 \times 1^2$$

$$15 = 7 + 2 \times 2^2$$

$$21 = 3 + 2 \times 3^2$$

$$25 = 7 + 2 \times 3^2$$

$$27 = 19 + 2 \times 2^2$$

$$33 = 31 + 2 \times 1^2$$

It turns out that the conjecture was false.

What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?

47 Distinct primes factors

[Problem]

The first two consecutive numbers to have two distinct prime factors are:

$$14 = 2 \times 7$$

$$15 = 3 \times 5$$

The first three consecutive numbers to have three distinct prime factors are:

$$644 = 2^2 \times 7 \times 23$$

$$645 = 3 \times 5 \times 43$$

$$646 = 2 \times 17 \times 19$$

Find the first four consecutive integers to have four distinct prime factors each. What is the first of these numbers?

48 Self powers

[Problem]

The series, $1^1 + 2^2 + 3^3 + \dots + 10^{10} = 10405071317$.

Find the last ten digits of the series, $1^1 + 2^2 + 3^3 + \dots + 1000^{1000}$.

49 Prime permutations

[Problem]

The arithmetic sequence, 1487, 4817, 8147, in which each of the terms increases by 3330, is unusual in two ways: (i) each of the three terms are prime, and, (ii) each of the 4-digit numbers are permutations of one another. There are no arithmetic sequences made up of three 1-, 2-, or 3-digit primes, exhibiting this property, but there is one other 4-digit increasing sequence.

What 12-digit number do you form by concatenating the three terms in this sequence?

50 Consecutive prime sum

[Problem]

The prime 41, can be written as the sum of six consecutive primes:

$$41 = 2 + 3 + 5 + 7 + 11 + 13$$

This is the longest sum of consecutive primes that adds to a prime below one-hundred.

The longest sum of consecutive primes below one-thousand that adds to a prime, contains 21 terms, and is equal to 953.

Which prime, below one-million, can be written as the sum of the most consecutive primes?

↗ Consecutive prime sum

51 Prime digit replacements

[Problem]

By replacing the 1st digit of the 2-digit number $*3$, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3rd and 4th digits of $56**3$ with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

☞ Prime digit replacements

52 Permuted multiples

[Problem]

It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order.

Find the smallest positive integer, x , such that $2x$, $3x$, $4x$, $5x$, and $6x$, contain the same digits.

↗ Permuted multiples

53 Combinatoric selections

[Problem]

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation, ${}^5C_3 = 10$.

In general,

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ where } r \leq n, n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1, 0! = 1$$

It is not until $n = 23$, that a value exceeds one-million: ${}^{23}C_{10} = 1144066$.

How many, not necessarily distinct, values of nC_r , for $1 \leq n \leq 100$, are greater than one-million?

↗ Combinatoric selections

54 Poker hands

[Problem]

In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way:

- **High Card:** Highest value card.
- **One Pair:** Two cards of the same value.
- **Two Pairs:** Two different pairs.
- **Three of a Kind:** Three cards of the same value.
- **Straight:** All cards are consecutive values.
- **Flush:** All cards of the same suit.
- **Full House:** Three of a kind and a pair.
- **Four of a Kind:** Four cards of the same value.
- **Straight Flush:** All cards are consecutive values of same suit.
- **Royal Flush:** Ten, Jack, Queen, King, Ace, in same suit.

The cards are valued in the order: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair of eights beats a pair of fives (see example 1 below). But if two ranks tie, for example, both players have a pair of queens, then highest cards in each hand are compared (see example 4 below); if the highest cards tie then the next highest cards are compared, and so on.

Consider the following five hands dealt to two players:

Hand	Player 1	Player 2	Winner
1	5H 5C 6S 7S KD Pair of Fives	2C 3S 8S 8D TD Pair of Eights	Player 2
2	5D 8C 9S JS AC Highest card Ace	2C 5C 7D 8S QH Highest card Queen	Player 1
3	2D 9C AS AH AC Three Aces	3D 6D 7D TD QD Flush with Diamonds	Player 2
4	4D 6S 9H QH QC Pair of Queens Highest card Nine	3D 6D 7H QD QS Pair of Queens	player 1
5	2H 2D 4C 4D 4S Full House With Three Fours	3C 3D 3S 9S 9D Full House with Three Threes	player 1

The file, poker.txt, contains one-thousand random hands dealt to two players. Each line of the file contains ten cards (separated by a single space): the first five are Player 1's cards and the last five are Player 2's cards. You can assume that all hands are valid (no invalid characters or repeated cards), each player's hand is in no specific order, and in each hand there is a clear winner.

How many hands does Player 1 win?

🔗 [Poker hands](#)

55 Lychrel numbers

[Problem]

If we take 47, reverse and add, $47 + 74 = 121$, which is palindromic.

Not all numbers produce palindromes so quickly. For example,

$$349 + 943 = 1292$$

$$1292 + 2921 = 4213$$

$$4213 + 3124 = 7337$$

That is, 349 took three iterations to arrive at a palindrome.

Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a palindrome. In fact, 10677 is the first number to be shown to require over fifty iterations before producing a palindrome: 4668731596684224866951378664 (53 iterations, 28-digits).

Surprisingly, there are palindromic numbers that are themselves Lychrel numbers; the first example is 4994.

How many Lychrel numbers are there below ten-thousand?

NOTE: Wording was modified slightly on 24 April 2007 to emphasise the theoretical nature of Lychrel numbers.

↗ Lychrel numbers

56 Powerful digit sum

[Problem]

A googol (10^{100}) is a massive number: one followed by one-hundred zeros; 100^{100} is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form, a^b , where $a, b < 100$, what is the maximum digital sum?

☞ Powerful digit sum

57 Square root convergents

[Problem]

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$$\sqrt{2} = 1 + 1 / (2 + 1 / (2 + 1 / (2 + \dots))) = 1.414213\dots$$

By expanding this for the first four iterations, we get:

$$1 + 1 / 2 = 3 / 2 = 1.5$$

$$1 + 1 / (2 + 1 / 2) = 7 / 5 = 1.4$$

$$1 + 1 / (2 + 1 / (2 + 1 / 2)) = 17 / 12 = 1.41666\dots$$

$$1 + 1 / (2 + 1 / (2 + 1 / (2 + 1 / 2))) = 41 / 29 = 1.41379\dots$$

The next three expansions are $99/70$, $239/169$, and $577/408$, but the eighth expansion, $1393/985$, is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than denominator?

↗ Square root convergents

58 Spiral primes

[Problem]

Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed.

37	36	35	34	33	32	31
38	17	16	15	14	13	30
39	18	5	4	3	12	29
40	19	6	1	2	11	28
41	20	7	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of $8 / 13 \approx 62\%$

If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

↗ Spiral primes

59 XOR decryption

[Problem]

Please note that the file associated with this problem, p059_cipher.txt, was changed on Tuesday 5 February 2019. If you downloaded the file before this date, please download the new version. This notice will remain for seven days.

Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (*) = 42, and lowercase k = 107.

A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example, $65 \text{ XOR } 42 = 107$, then $107 \text{ XOR } 42 = 65$.

For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message.

Unfortunately, this method is impractical for most users, so the modified method is to use a password as a key. If the password is shorter than the message, which is likely, the key is repeated cyclically throughout the message. The balance for this method is using a sufficiently long password key for security, but short enough to be memorable.

Your task has been made easy, as the encryption key consists of three lower case characters. Using p059_cipher.txt (right click and 'Save Link/Target As...'), a file containing the encrypted ASCII codes, and the knowledge that the plain text must contain common English words, decrypt the message and find the sum of the ASCII values in the original text.

🔗 XOR decryption

60 Prime pair sets

[Problem]

The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property.

Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.

[↗ Prime pair sets](#)

61 Cyclical figurate numbers

[Problem]

Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae:

Triangle	$P_{3,n} = n(n + 1) / 2$	1, 3, 6, 10, 15, ...
Square	$P_{4,n} = n^2$	1, 4, 9, 16, 25, ...
Pentagonal	$P_{5,n} = n(3n - 1) / 2$	1, 5, 12, 22, 35, ...
Hexagonal	$P_{6,n} = n(2n - 1)$	1, 6, 15, 28, 45, ...
Heptagonal	$P_{7,n} = n(5n - 3) / 2$	1, 7, 18, 34, 55, ...
Octagonal	$P_{8,n} = n(3n - 2)$	1, 8, 21, 40, 65, ...

The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties.

1. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first).
2. Each polygonal type: triangle ($P_{3,127} = 8128$), square ($P_{4,91} = 8281$), and pentagonal ($P_{5,44} = 2882$), is represented by a different number in the set.
3. This is the only set of 4-digit numbers with this property.

Find the sum of the only ordered set of six cyclic 4-digit numbers for which each polygonal type: triangle, square, pentagonal, hexagonal, heptagonal, and octagonal, is represented by a different number in the set.

↗ Cyclical figurate numbers

62 Cubic permutations

[Problem]

The cube, 41063625 (345^3), can be permuted to produce two other cubes: 56623104 (384^3) and 66430125 (405^3). In fact, 41063625 is the smallest cube which has exactly three permutations of its digits which are also cube.

Find the smallest cube for which exactly five permutations of its digits are cube.

↗ Cubic permutations

63 Powerful digit counts

[Problem]

The 5-digit number, $16807 = 7^5$, is also a fifth power. Similarly, the 9-digit number, $134217728 = 8^9$, is a ninth power.

How many n -digit positive integers exist which are also an n th power?

↗ Powerful digit counts

64 Odd period square roots

[Problem]

All square roots are periodic when written as continued fractions and can be written in the form:

$$\sqrt{N} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

For example, let us consider $\sqrt{23}$,

$$\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \frac{1}{\frac{1}{\sqrt{23} - 4}} = 4 + \frac{1}{1 + \frac{\sqrt{23} - 3}{7}}$$

If we continue we would get the following expansion:

$$\sqrt{23} = 4 + \frac{4}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{8 + \dots}}}}$$

The process can be summarised as follows:

$a_0 = 4,$	$\frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7}$
$a_1 = 1,$	$\frac{7}{\sqrt{23} - 3} = \frac{7(\sqrt{23} + 3)}{14} = 3 + \frac{\sqrt{23} - 3}{2}$
$a_2 = 3,$	$\frac{2}{\sqrt{23} - 3} = \frac{2(\sqrt{23} + 3)}{14} = 1 + \frac{\sqrt{23} - 4}{7}$
$a_3 = 1,$	$\frac{7}{\sqrt{23} - 4} = \frac{7(\sqrt{23} + 4)}{7} = 8 + \sqrt{23} - 4$
$a_4 = 8,$	$\frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7}$
$a_5 = 1,$	$\frac{7}{\sqrt{23} - 3} = \frac{7(\sqrt{23} + 3)}{14} = 3 + \frac{\sqrt{23} - 3}{7}$
$a_6 = 1,$	$\frac{7}{\sqrt{23} - 3} = \frac{2(\sqrt{23} + 3)}{14} = 1 + \frac{\sqrt{23} - 4}{7}$
$a_7 = 1,$	$\frac{7}{\sqrt{23} - 4} = \frac{7(\sqrt{23} + 4)}{7} = 8 + \sqrt{23} - 4$

It can be seen that the sequence is repeating. For conciseness, we use the notation $\sqrt{23} = [4; (1, 3, 1, 8)]$, to indicate that the block (1, 3, 1, 8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

$\sqrt{2} = [1; (2)],$	period = 1
$\sqrt{3} = [1; (1, 2)],$	period = 2
$\sqrt{5} = [2; (4)],$	period = 1
$\sqrt{6} = [2; (2, 4)],$	period = 2
$\sqrt{7} = [2; (1, 1, 1, 4)],$	period = 4
$\sqrt{8} = [2; (1, 4)],$	period = 2
$\sqrt{10} = [3; (6)],$	period = 1
$\sqrt{11} = [3; (3, 6)],$	period = 2
$\sqrt{12} = [3; (2, 6)],$	period = 2
$\sqrt{13} = [3; (1, 1, 1, 1, 6)],$	period = 5

Exactly four continued fractions, for $N \leq 13$, have an odd period.

How many continued fractions for $N \leq 10000$ have an odd period?

↗ Odd period square roots

65 Convergents of e

[Problem]

The square root of 2 can be written as an infinite continued fraction.

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

The infinite continued fraction can be written, $\sqrt{2} = [1; (2)]$, (2) indicates that 2 repeats ad infinitum. In a similar way, $\sqrt{23} = [4; (1, 3, 1, 8)]$.

It turns out that the sequence of partial values of continued fractions for square roots provide the best rational approximations. Let us consider the convergents for $\sqrt{2}$.

$$\begin{aligned} 1 + 1 &= 2 \\ 1 + \frac{1}{2 + \frac{1}{2}} &= \frac{7}{5} \\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} &= \frac{17}{12} \\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} &= \frac{41}{29} \end{aligned}$$

Hence the sequence of the first ten convergents for $\sqrt{2}$ are:

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}, \dots$$

What is most surprising is that the important mathematical constant,

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots, 1, 2k, 1,]$$

The first ten terms in the sequence of convergents for e are:

$$2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465}, \frac{1457}{536}, \dots$$

The sum of digits in the numerator of the 10th convergent is $1 + 4 + 5 + 7 = 17$.

Find the sum of digits in the numerator of the 100th convergent of the continued fraction for e .

☞ Convergents of e

66 Diophantine equation

[Problem]

Consider quadratic Diophantine equations of the form:

$$x^2 - Dy^2 = 1$$

For example, when $D = 13$, the minimal solution in x is $649^2 - 13 \times 180^2 = 1$.

It can be assumed that there are no solutions in positive integers when D is square.

By finding minimal solutions in x for $D = \{2, 3, 5, 6, 7\}$, we obtain the following:

$$3^2 - 2 \times 2^2 = 1$$

$$2^2 - 3 \times 1^2 = 1$$

$$9^2 - 5 \times 4^2 = 1$$

$$5^2 - 6 \times 2^2 = 1$$

$$8^2 - 7 \times 3^2 = 1$$

Hence, by considering minimal solutions in x for $D \leq 7$, the largest x is obtained when $D = 5$.

Find the value of $D \leq 1000$ in minimal solutions of x for which the largest value of x is obtained.

☞ Diophantine equation

67 Maximum path sum II

[Problem]

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
      3
     7 4
    2 4 6
   8 5 9 3
```

That is, $3 + 7 + 4 + 9 = 23$.

Find the maximum total from top to bottom in triangle.txt (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows.

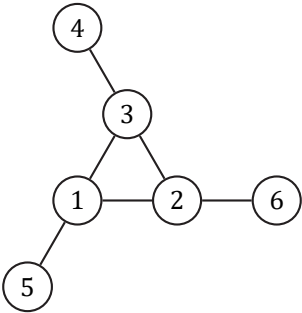
NOTE: This is a much more difficult version of Problem 18. It is not possible to try every route to solve this problem, as there are 2^{99} altogether! If you could check one trillion (10^{12}) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

↗ Maximum path sum II

68 Magic 5-gon ring

[Problem]

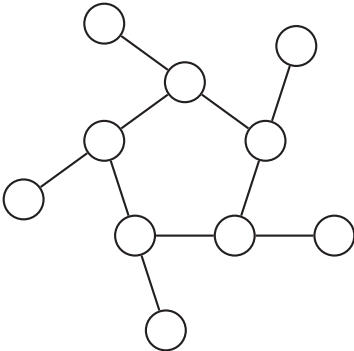
Consider the following “magic” 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine.



Working **clockwise**, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3.

Total	Solution Set
9	4,2,3; 5,3,1; 6,1,2
9	4,3,2; 6,2,1; 5,1,3
10	2,3,5; 4,5,1; 6,1,3
10	2,5,3; 6,3,1; 4,1,5
11	1,4,6; 3,6,2; 5,2,4
11	1,6,4; 5,4,2; 3,2,6
12	1,5,6; 2,6,4; 3,4,5
12	1,6,5; 3,5,4; 2,4,6

By concatenating each group it is possible to form 9-digit strings; the maximum string for a 3-gon ring is 432621513. Using the numbers 1 to 10, and depending on arrangements, it is possible to form 16- and 17-digit strings. What is the maximum 16-digit string for a “magic” 5-gon ring?



☑ Magic 5-gon ring

69 Totient maximum

[Problem]

Euler's Totient function, $\varphi(n)$ [sometimes called the phi function], is used to determine the number of numbers less than n which are relatively prime to n . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, $\varphi(9) = 6$.

n	Relatively Prime	$\varphi(n)$	$n / \varphi(n)$
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that $n=6$ produces a maximum $n / \varphi(n)$ for $n \leq 10$.

Find the value of $n \leq 1000000$ for which $n / \varphi(n)$ is a maximum.

↗ Totient maximum

70 Totient permutation

[Problem]

Euler's Totient function, $\varphi(n)$ [sometimes called the phi function], is used to determine the number of positive numbers less than or equal to n which are relatively prime to n . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, $\varphi(9) = 6$. The number 1 is considered to be relatively prime to every positive number, so $\varphi(1) = 1$.

Interestingly, $\varphi(87109) = 79180$, and it can be seen that 87109 is a permutation of 79180.

Find the value of n , $1 < n < 107$, for which $\varphi(n)$ is a permutation of n and the ratio $n / \varphi(n)$ produces a minimum.

☞ Totient permutation

71 Ordered fractions

[Problem]

Consider the fraction, n / d , where n and d are positive integers. If $n < d$ and $\text{HCF}(n, d) = 1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}, \frac{7}{8}$$

It can be seen that $2 / 5$ is the fraction immediately to the left of $3 / 7$.

By listing the set of reduced proper fractions for $d \leq 1000000$ in ascending order of size, find the numerator of the fraction immediately to the left of $3 / 7$.

🔗 Ordered fractions

72 Counting fractions

[Problem]

Consider the fraction, n / d , where n and d are positive integers. If $n < d$ and $\text{HCF}(n, d) = 1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}, \frac{7}{8}$$

It can be seen that there are 21 elements in this set.

How many elements would be contained in the set of reduced proper fractions for $d \leq 1000000$?

↗ Counting fractions

73 Counting fractions in a range

[Problem]

Consider the fraction, n / d , where n and d are positive integers. If $n < d$ and $\text{HCF}(n, d) = 1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}, \frac{7}{8}$$

It can be seen that there are 3 fractions between $1 / 3$ and $1 / 2$.

How many fractions lie between $1 / 3$ and $1 / 2$ in the sorted set of reduced proper fractions for $d \leq 12000$?

↗ Counting fractions in a range

74 Digit factorial chains

[Problem]

The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145:

$$1! + 4! + 5! = 1 + 24 + 120 = 145$$

Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist:

$$169 \rightarrow 363601 \rightarrow 1454 \rightarrow 169$$

$$871 \rightarrow 45361 \rightarrow 871$$

$$872 \rightarrow 45362 \rightarrow 872$$

It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example,

$$69 \rightarrow 363600 \rightarrow 1454 \rightarrow 169 \rightarrow 363601(\rightarrow 1454)$$

$$78 \rightarrow 45360 \rightarrow 871 \rightarrow 45361(\rightarrow 871)$$

$$540 \rightarrow 145(\rightarrow 145)$$

Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms.

How many chains, with a starting number below one million, contain exactly sixty non-repeating terms?

↗ Digit factorial chains