Artemev Dmitrii
group 5

Question 1

$$\begin{cases} loss(\hat\beta) = (y-\hat y)^T(y-\hat y) + \lambda \hat\beta^T\hat\beta \\ \hat y = \hat\beta_1 x + \hat\beta_2 x \end{cases}$$

$$loss(\hat\beta) = \underbrace{(y-\hat\beta_1 x - \hat\beta_2 x)^T(y-\hat\beta_1 x - \hat\beta_2 x)} + \lambda \hat\beta^T\hat\beta \to \min_{\hat\beta_1, \hat\beta_2}$$

1) $(\vdots)\cdot(\dots) = y^2 - \hat\beta_1 xy - \hat\beta_2 xy - \hat\beta_1 xy + \hat\beta_1^2 x^2 + \hat\beta_1\hat\beta_2 x^2 - \hat\beta_2 xy + \hat\beta_1\hat\beta_2 x^2 + \hat\beta_2^2 x^2 = y^2 - 2xy(\hat\beta_1+\hat\beta_2) + 2\hat\beta_1\hat\beta_2 x^2 + \hat\beta_1^2 x^2 + \hat\beta_2^2 x^2$

2) $(\because)\cdot(\cdot\cdot) = \lambda(\hat\beta_1^2 + 2\hat\beta_1\hat\beta_2 + \hat\beta_2^2) = \lambda(\hat\beta_1+\hat\beta_2)^2$

F.O.C:

$$+\begin{cases} (loss(\hat\beta))'_{\hat\beta_1} = -2xy + 2\hat\beta_2 x^2 + 2\hat\beta_1 x^2 + 2\lambda(\hat\beta_1+\hat\beta_2) = 0 \\ (loss(\hat\beta))'_{\hat\beta_2} = -2xy + 2\hat\beta_1 x^2 + 2\hat\beta_2 x^2 + 2\lambda(\hat\beta_1+\hat\beta_2) = 0 \end{cases}$$

$$4\hat\beta_1 x^2 + 4\hat\beta_2 x^2 + 4\lambda(\hat\beta_1+\hat\beta_2) = 0 \quad |:4$$

$$x^2(\hat\beta_1+\hat\beta_2) + \lambda(\hat\beta_1+\hat\beta_2) = 0$$

$$(\hat\beta_1+\hat\beta_2)(x^2+\lambda) = 0$$

$$\Downarrow$$

$$\left.\begin{cases} \hat\beta_1^* = -\hat\beta_2^* \\ \hat\beta_2^* = -\hat\beta_1^* \end{cases}\right\} \text{under fixed } \lambda$$

b) Optimal $\hat\beta_2, \hat\beta_1$ dont change

c) $\lim(\hat\beta_1 + \beta_2) = 0$

Question 2

$y = X\beta + u$

$E(u/X) = 0$

$Var(u|X) = \sigma^2 I_n$

a) $\hat y = X\hat\beta_{OLS} = \underbrace{X(X^TX)^{-1}X^T}_{\text{projection matrix}} y$

$\hat\beta_{OLS} = (X^TX)^{-1}X^Ty$

$E(\hat\beta|X) = E(\hat\beta) = (X^TX)^{-1}X^T \underbrace{E(y)}_{X\beta} = \underbrace{(X^TX)^{-1}X^TX}_{I}\hat\beta = \hat\beta_{OLS} \to$ unbiased estimator

b) $V\hat{a}r(\hat\beta|X) = ((X^TX)^{-1}X^T)\underbrace{Var(y)}_{\sigma^2}((X^TX)^{-1}X^T)^T = (\underbrace{(X^TX)^{-1}X^T)(X}_{I}(X^TX)^{-1})\sigma^2 = \sigma^2(X^TX)^{-1}$

Question 3

$$X=\begin{pmatrix}2&1\\-1&2\\1&1\end{pmatrix}$$

$$X^TX=\begin{pmatrix}2&-1&1\\1&2&1\end{pmatrix}\cdot\begin{pmatrix}2&1\\-1&2\\1&1\end{pmatrix}=\begin{pmatrix}6&1\\1&6\end{pmatrix}$$

$$\begin{vmatrix}6-\lambda&1\\1&6-\lambda\end{vmatrix}=(6-\lambda)^2-1=0\to \begin{matrix}\lambda_1=5\\ \lambda_2=7\end{matrix}$$

1) $\lambda=5$:

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}x_1+x_2=0\\x_1+x_2=0\end{cases}\Rightarrow x_1=-x_2\to V_1=\begin{pmatrix}1\\-1\end{pmatrix}$$

2) $\lambda=7$:

$$\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}-x_1+x_2=0\\x_1-x_2=0\end{cases}\Rightarrow x_1=x_2\to V_2=\begin{pmatrix}1\\1\end{pmatrix}$$

$$C=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$$C^{-1}:\left(\begin{array}{cc|cc}1&1&1&0\\-1&1&0&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&1&1&0\\0&2&1&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&1&1&0\\0&1&0{,}5&0{,}5\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&0{,}5&-0{,}5\\0&1&0{,}5&0{,}5\end{array}\right)\to C^{-1}=\begin{pmatrix}0{,}5&-0{,}5\\0{,}5&0{,}5\end{pmatrix}$$

$$\underset{A}{\begin{pmatrix}6&1\\1&6\end{pmatrix}}=\underset{C}{\begin{pmatrix}1&1\\-1&1\end{pmatrix}}\cdot\underset{D}{\begin{pmatrix}5&0\\0&7\end{pmatrix}}\cdot\underset{C^{-1}}{\begin{pmatrix}0{,}5&0\\0{,}5&0\end{pmatrix}}$$

b) $$XX^T=\begin{pmatrix}2&1\\-1&2\\1&1\end{pmatrix}\begin{pmatrix}2&-1&1\\1&2&1\end{pmatrix}=\begin{pmatrix}5&0&3\\0&5&1\\3&1&2\end{pmatrix}$$

$$\begin{pmatrix}5-\lambda&0&3\\0&5-\lambda&1\\3&1&2-\lambda\end{pmatrix}=(5-\lambda)^2(2-\lambda)-9(5-\lambda)-(5-\lambda)=0$$

$$(5-\lambda)((5-\lambda)(2-\lambda)-9-1)=0$$

$\lambda_1=5$ $\quad \lambda^2-7\lambda=0$

$\lambda(\lambda-7)=0$

$\lambda_2=0 \quad \lambda_3=7$

1) $\lambda=5$:

$$\begin{pmatrix}0&0&3\\0&0&1\\3&1&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}3x_3=0\\1x_3=0\\3x_1+x_2-3x_3=0\end{cases}\Rightarrow \begin{matrix}x_3=0\\x_1=-\frac{x_2}{3}\end{matrix}\to V_1=\begin{pmatrix}-1\\3\\0\end{pmatrix}/\sqrt{10}$$

2) $\lambda=0$:

$$\begin{pmatrix} 5 & 0 & 3 \\ 0 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 5x_1+3x_3=0 \\ 5x_2+x_3=0 \\ 3x_1+x_2+2x_3=0 \end{cases} \to \begin{cases} x_1=-0{,}6x_3 \\ x_2=-0{,}2x_3 \\ -1{,}8x_3-0{,}2x_3+2x_3=0 \end{cases} \to \begin{cases} x_3=10 \\ x_2=-2 \\ x_1=-6 \end{cases} \to V_2=\begin{pmatrix} -6 \\ -2 \\ 10 \end{pmatrix}\Big/\sqrt{140}$$

3) $\lambda=7$:

$$\begin{pmatrix} -2 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_1+3x_3=0 \\ -2x_2+x_3=0 \\ 3x_1+x_2-5x_3=0 \end{cases} \to \begin{cases} x_1=1{,}5x_3 \\ x_2=0{,}5x_3 \\ 4{,}5x_3+0{,}5x_3-5x_3=0 \end{cases} \to \begin{cases} x_3=2 \\ x_2=1 \\ x_1=3 \end{cases} \to V_3=\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}\Big/\sqrt{14}$$

$$V=\left(V_1,V_2,V_3\right), \quad U^T=\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}^T$$

$$X=\begin{pmatrix} \frac{-1}{\sqrt{10}} & -\frac{6}{\sqrt{140}} & \frac{3}{\sqrt{14}} \\ \frac{3}{\sqrt{10}} & -\frac{2}{\sqrt{140}} & \frac{1}{\sqrt{14}} \\ 0 & \frac{10}{\sqrt{140}} & \frac{2}{\sqrt{14}} \end{pmatrix}\cdot\begin{pmatrix} \sqrt5 & 0 \\ 0 & \sqrt7 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$$

c) $$\sqrt7\cdot\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}\cdot\begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}=\begin{pmatrix} 1{,}5 & 0{,}5 & 1 \\ 1{,}5 & 0{,}5 & 1 \end{pmatrix}, rk=1$$

Question 4

a) $$\begin{cases} \|Xw\|^2 \to \max \\ \text{s.t. } \|w\|^2=1 \end{cases}$$

$$L=\|Xw\|^2-\lambda(1-\|w\|^2)\to\max_{x,w,\lambda}$$

F.O.C:

$$d\langle Xw;Xw\rangle=2\langle Xw;Xdw\rangle$$

$$d\|Xw\|^2=2\langle X^TXw;dw\rangle=2X^TXw$$

$$d(-\lambda\|w\|^2)=-2\lambda\langle w;dw\rangle=-2\lambda w$$

$$L'=2X^TXw-2\lambda w=0$$

$$2X^TXw=2\lambda w$$

$$X^TXw=\lambda w$$

$$X^TX=\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}=\lambda\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

eigenvalues: $\lambda_1=5, \lambda_2=7$

1) $\lambda = 5$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

$$\begin{cases} w_1 + w_2 = 0 \\ w_1 + w_2 = 0 \end{cases} \to w_1 = -w_2, \ V_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \left.\begin{matrix} \Delta_1 > 1 \\ \Delta_2 = 0 \end{matrix}\right\} \Rightarrow$$ positive semidefinite, not max

2) $\lambda = 7$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -w_1 + w_2 \\ w_1 - w_2 \end{cases} \to w_1 = w_2, \ V_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \left.\begin{matrix} \Delta_1 = -1 \\ \Delta_2 = 0 \end{matrix}\right\} \Rightarrow$$ negative semidefinite, can be max

$$\begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$