Krasnobaev Artem Group № 5

Home assignment № 3

Problem № 1

a) $\hat{y} = \hat{\beta}_1 x + \hat{\beta}_2 x = x(\hat{\beta}_1 + \hat{\beta}_2)$ $\quad SSE_{L_2} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + \lambda \sum_{j=1}^{p} \beta_j^2$

$$loss(\hat{\beta}) = (y - \hat{y})^T (y - \hat{y}) + \lambda \hat{\beta}^T \hat{\beta} = (y - x(\hat{\beta}_1 + \hat{\beta}_2))^T (y - x(\hat{\beta}_1 + \hat{\beta}_2)) +$$

$$+ \lambda(\hat{\beta}_1^2 + \hat{\beta}_2^2) = y^T y - 2xy^T(\hat{\beta}_1 + \hat{\beta}_2) + xx^T(\hat{\beta}_1 + \hat{\beta}_2)^2 + \lambda(\hat{\beta}_1^2 + \hat{\beta}_2^2) \to \min_{\hat{\beta}_1, \hat{\beta}_2}$$

F.O.C.:

$$\begin{cases} (loss(\hat{\beta}))'_{\hat{\beta}_1} = -2xy^T + 2xx^T\hat{\beta}_1 + 2xx^T\hat{\beta}_2 + 2\lambda\hat{\beta}_1 = 0 \\ (loss(\hat{\beta}))'_{\hat{\beta}_2} = -2xy^T + 2xx^T\hat{\beta}_1 + 2xx^T\hat{\beta}_2 + 2\lambda\hat{\beta}_2 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \hat{\beta}_1 = \dfrac{2xy^T - 2xx^T\hat{\beta}_2}{2xx^T + 2\lambda} \\ \hat{\beta}_2 = \dfrac{2xy^T - 2xx^T\hat{\beta}_1}{2xx^T + 2\lambda} \end{cases}$$

by the symmetry and the information given we can say that $\hat{\beta}_1 = \hat{\beta}_2$ $\Rightarrow$

$$\Rightarrow \hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta} \Rightarrow \hat{\beta} = \frac{2xy^T - 2xx^T\hat{\beta}}{2xx^T + 2\lambda} \Rightarrow \hat{\beta}(2xx^T + 2\lambda) = 2xy^T -$$

$$- 2xx^T\hat{\beta} \Rightarrow \hat{\beta}(4xx^T + 2\lambda) = 2xy^T \Rightarrow \hat{\beta} = \frac{xy^T}{2xx^T + \lambda}$$

b) if $\lambda \to \infty$ then $\hat{\beta} = \dfrac{xy^T}{2xx^T + \lambda} \to 0$

c) $\hat{\beta}_1 + \hat{\beta}_2 = \hat{\beta} + \hat{\beta} = 2\hat{\beta} = \dfrac{2xy^T}{2xx^T + \lambda}$ $\quad$ if $\lambda \to 0$ then $2\hat{\beta} = \dfrac{xy^T}{xx^T}$

Problem № 2

a) $\hat{\beta}_{OLS} = \dfrac{x^T y}{x^T x}$

$$E(\hat{\beta} \mid x) = E\left(\frac{x^T y}{x^T x} \mid x\right) = E\left(\frac{x^T(x\beta + u)}{x^T x} \mid x\right) = \frac{x^T(x\beta + E(u|x))}{x^T x} =$$

$$= \frac{x^T x \beta}{x^T x} = \beta$$

$$E(\hat{\beta}) = E(E(\hat{\beta} \mid x)) = E(\beta) = \beta$$

b) $Var(\hat{\beta}|x) = Var\left(\frac{x^T(x\beta+u)}{x^T x}\Big| x\right) = \frac{Var(u|x)}{(x^T x)^2} = \frac{\sigma^2 w}{(x^T x)^2}$

d) $Cov(y, \hat{\beta}|x) = Cov\left(x\beta+u, \frac{x^T y}{x^T x}\Big| x\right) =$

$= Cov\left(x\beta+u, \frac{x^T\cdot(x\beta+u)}{x^T\cdot x}\Big| x\right) = \frac{x^T}{x^T x}\cdot Cov(u, u|x) =$

since x is a const

$= \frac{x^T}{x^T x} Var(u|x) = \frac{x^T \sigma^2 w}{x^T x}$

Problem № 3

$x = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$

a) $x^T = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$x^T x = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix} = A$

diagonalization: $A - \lambda\cdot I = \begin{pmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{pmatrix}$

$\det = (6-\lambda)(6-\lambda) - 1 = 36 - 6\lambda - 6\lambda + \lambda^2 - 1 = \lambda^2 - 12\lambda + 35 = 0 \Rightarrow$

$\Rightarrow D = 144 - 140 = 4 \quad \lambda_1 = \frac{12+2}{2} = 7 \quad \lambda_2 = \frac{12-2}{2} = 5$