Omarov Timur | DS for Economics | HW – 3

№ 2

(a) since β is non-random then $\mathbb{E}(\hat{\beta}) = \beta$

$y = X\beta + u$

$u = y - X\beta$

First, let's derive β:

- β should be such that it gives us minimal value of SSR

$SSR = u^T u = (y - X\beta)^T (y - X\beta) = (y^T - X^T \beta^T)(y - X\beta) =$

$= y^T y - \underline{y^T X \beta} - \underline{\underline{\beta^T X^T y}} + \beta^T X^T X \beta$

$y^T X \beta$ – scalar $\Rightarrow$ scalar = scalar transposed $\Rightarrow$

$$\underbrace{\underset{1\times n}{(n\times 1)^T} \; \underbrace{n\times K \;\; K\times 1}_{n\times 1}}_{1\times 1}$$

$\Rightarrow \underline{y^T X \beta} = (y^T X \beta)^T = \underline{\underline{\beta^T X^T y}}$

$$SSR = y^T y - \beta^T X^T y - \beta^T X^T y + \beta^T X^T X \beta = y^T y - 2\beta^T X^T y + \beta^T X^T X \beta \rightarrow \min_{\beta}$$

$$\frac{\partial SSR}{\partial \beta} = 0 - 2 \cdot 1 \cdot X^T y + 2\beta \cdot X^T X = -2X^T y + 2\beta X^T X = 0$$

$$\beta \cdot X^T X = X^T y$$

$$\boxed{\beta = \frac{X^T y}{X^T X}} \quad \text{OR} \quad (X^T X)^{-1} X^T y$$

$$\hat{\beta} = (X^T X)^{-1} X^T y = (X^T X)^{-1} X^T (X\beta + u) = \cancel{(X^T X)^{-1}(X^T X)}\, \beta + (X^T X)^{-1} X^T u =$$

$$= \beta + X^{-1} \cancel{(X^T)^{-1}} \cancel{X^T} u = \beta + X^{-1} u$$

$$\mathbb{E}(\hat{\beta} \mid X) = \beta + X^{-1} \cdot \mathbb{E}(u \mid X) = \beta + X^{-1} \cdot 0 = \beta + 0 = \beta$$

(b) $\hat{\beta} = \beta + x^{-1} u$

$$Var(\hat{\beta} \mid X) = Var(\beta + x^{-1} u \mid X) = 0 + Var(x^{-1} u \mid X) =$$

$$= (x^{-1})^2 \cdot Var(u \mid X) = X^{-2} \cdot \sigma^2 W$$

(c) Standard confidence interval assumes h~~etero~~omoscedasticity i.e. $Var(u \mid X) = \sigma^2 I$. However, we've got $Var(u \mid X) = \sigma^2 W$, i.e. heteroscedasticity $\Rightarrow$ the standard confidence interval will be invalid in this case.

(d) $\text{Cov}(y, \hat{\beta} \mid X) = \text{Cov}(y, \underbrace{(X^T X)^{-1} X^T} y \mid X) = (X^T X)^{-1} X^T \cdot \text{Var}(y \mid X) =$

$= (X^T X)^{-1} X^T \cdot \text{Var}(X\beta + u \mid X) = (X^T X)^{-1} X^T \cdot \text{Var}(u \mid X) = (X^T X)^{-1} X^T \sigma^2 W$