№ 4

(d) Constrained Optimization Problem:

$$\begin{cases} \|Xw\|^2 \to \max \\ \text{s.t.} \\ \|w\|^2 = 1 \end{cases}$$

$$\|w\|^2 = w^T w = 1$$

$$\|Xw\|^2 = (Xw)^T (Xw) = w^T X^T X w$$

$$\boxed{L = w^T X^T X w - \lambda (w^T w - 1) \to \max_{\lambda \geq 0}}$$

(e) F.O.C.:

$$\begin{cases} \frac{\partial L}{\partial \lambda} = -(w^T w - 1) = 1 - w^T w = 0 \Rightarrow \boxed{w^T w = 1} \\ \frac{\partial L}{\partial w} = \left(w^T (X^T X) w\right)'_w - \lambda \cdot (w^T w)'_w = 2(X^T X) w - \lambda \cdot 2w = 0 \end{cases}$$

$$\Rightarrow \boxed{X^T X w = \lambda w}$$

(f) $w = k_1 \hat{v}_1 + k_2 \hat{v}_2 = (\hat{v}_1 \ \hat{v}_2) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = V \cdot K$

$\|Xw\|^2 \to \max$

$$\|Xw\|^2 = w^T X^T X w = (VK)^T X^T X (VK) = K^T V^T X^T X V K$$

$$X^T X = (UDV^T)^T (UDV^T) = V D^T U^T U D V^T$$

U - orthogonal $\Rightarrow$ $U^T U = I$

$$\Rightarrow X^T X = V D^T (U^T U) D V^T = V D^T I D V^T = V D^T D V^T$$

$$\|Xw\|^2 = K^T V^T (X^T X) V K = K^T \underbrace{V^T (V}_{I} D^T D \underbrace{V^T) V}_{I} K = K^T D^T D K =$$

$$= K_1^2 \sigma_1^2 + K_2^2 \sigma_2^2$$

$\sigma_1 > \sigma_2$

$K_1 + K_2 = 1$

$K_1, K_2 \in [0; 1]$

$\Rightarrow \|Xw\|^2$ is maximized at $K_1 = 1$ $\Rightarrow$

$$\Rightarrow \|Xw\|^2_{max} = 1^2 \sigma_1^2 + 0^2 \sigma_2^2 = \sigma_1^2$$

$\Rightarrow$ optimal $w = K_1 \hat{V}_1 +$ $K_2 \hat{V}_2$ ~~corresponding to σ_1~~ $= 1 \cdot \hat{V}_1 + 0 \cdot \hat{V}_2 = \hat{V}_1$ OR $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$