Omarov Timur | DS for Economics | HW – 3

№3

(a) $X^TX = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$

ordinary Diagonalisation

$X^TX = PDP^{-1}$, $P = (V_1\ V_2)$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$|X^TX - \lambda I| = \begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} = (6-\lambda)^2 - 1 = 36 + \lambda^2 - 12\lambda - 1 =$

$= \lambda^2 - 12\lambda + 35 = (\lambda-5)(\lambda-7) \Rightarrow \left[\begin{array}{l} \lambda_1 = 5 \\ \lambda_2 = 7 \end{array}\right. \Rightarrow D = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$

$X^TX - \lambda_1 I = \begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X^TX - \lambda_2 I = \begin{pmatrix} 6-7 & 1 \\ 1 & 6-7 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$P^{-1} = \dfrac{1}{(1)(1) - (-1)(1)} \begin{pmatrix} 1 & -(1) \\ -(-1) & 1 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$$X^T X = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(b) SVD of X: $\boxed{X = UDV^T}$

U - matrix of left singular vectors of X OR matrix of normalized eigenvectors of XX^T

V - matrix of right singular vectors of X OR matrix of normalized eigenvectors of X^TX

D (or else Σ) - Diagonal matrix of singular values of X^TX OR XX^T ~~on the Diagonal~~ in Descending order

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$$D = \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}}_{3\times 2 \text{ just like } X}, \quad \left.\begin{array}{l} \sigma_1 = \sigma_{max} = \sqrt{\lambda^{X^TX}_{max}} = \sqrt{7} \\ \sigma_2 = \sigma_{min} = \sqrt{\lambda^{X^TX}_{min}} = \sqrt{5} \end{array}\right\} \Rightarrow D = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \hat{v}_1 & \hat{v}_2 \end{pmatrix}$$

$$\hat{v}_1 (\sigma_1) = \frac{1}{\sqrt{(1)^2 + (1)^2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\hat{v}_2 (\sigma_2) = \frac{1}{\sqrt{(1)^2 + (-1)^2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \Rightarrow V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$U = \begin{pmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \end{pmatrix}$$

$$A\hat{v}_1 = \sigma_1 \cdot \hat{u}_1 \Rightarrow \hat{u}_1 = \frac{1}{\sigma_1} A\hat{v}_1 = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}$$

$$A\hat{v}_2 = \sigma_2 \cdot \hat{u}_2 \Rightarrow \hat{u}_2 = \frac{1}{\sigma_2} \cdot A\hat{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \\ 0 \end{pmatrix}$$

$$rk(X) = 2 \Rightarrow \hat{u}_3 \text{ lies in } N(X) \Rightarrow \begin{cases} \langle \hat{u}_3, \hat{u}_1 \rangle = 0 \\ \langle \hat{u}_3, \hat{u}_2 \rangle = 0 \end{cases}$$

$$\hat{u}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\langle \hat{u}_3, \hat{u}_1 \rangle = \frac{3}{\cancel{\sqrt{14}}} x + \frac{1}{\cancel{\sqrt{14}}} y + \frac{2}{\cancel{\sqrt{14}}} z = 0$$

$$\boxed{3x + y + 2z = 0}$$

$$\langle \hat{u}_3, \hat{u}_2 \rangle = \frac{1}{\sqrt{10}} x - \frac{3}{\sqrt{10}} y + 0 \cdot z = 0$$

$$\boxed{x - 3y = 0} \Rightarrow x = 3y$$

$$\Rightarrow \quad 9y + y + 2z = 0$$

$$10y = -2z$$

$$y = -\frac{1}{5} z$$

Let $\underline{z = -5} \Rightarrow \underline{y = 1}$

$$\Rightarrow \underline{x = 3} \Rightarrow u_3 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$$

$$\hat{u}_3 = \frac{1}{\sqrt{3^2 + 1^2 + (-5)^2}} u_3 = \begin{pmatrix} 3/\sqrt{35} \\ 1/\sqrt{35} \\ -5/\sqrt{35} \end{pmatrix}$$

So,

$$U = \begin{pmatrix} 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \\ 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \end{pmatrix}$$

SVD:

$$X = UDV^T = \boxed{\begin{pmatrix} 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \\ 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}$$

$$\underbrace{\underbrace{3\times 3 \quad\quad 3\times 2}_{3\times 2} \quad\quad 2\times 2}_{3\times 2 \text{ (like } X)}$$

(c) rank 1 $\Rightarrow$ $\tilde{X} = \sigma_{max} \cdot u_{\sigma_{max}} \cdot v^T_{\sigma_{max}}$

$\sigma_{max} = \sqrt{7}$

$$u_{\sigma_{max}} = \hat{u}(\sigma = \sqrt{7}) = \hat{u}_1 = \begin{pmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}$$

$$v_{\sigma_{max}} = \hat{v}(\sigma = \sqrt{7}) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{X} = \sqrt{7} \cdot \underbrace{\begin{pmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{1\times 2} = \boxed{\begin{pmatrix} 3/2 & 3/2 \\ 1/2 & 1/2 \\ 1 & 1 \end{pmatrix}}$$

3×2 (like X)