1. We have two absolutely identical preliminary standardized regressors x and x. The dependent variable y is centered.

   In the ridge regression one minimizes the loss function

   $$\text{loss}(\hat{\beta}) = (y - \hat{y})^T(y - \hat{y}) + \lambda\hat{\beta}^T\hat{\beta}, \quad \hat{y} = \hat{\beta}_1 x + \hat{\beta}_2 x.$$

   (a) Find the optimal $\hat{\beta}_1$ and $\hat{\beta}_2$ for fixed λ.

   (b) What happens to the estimates when $\lambda \to \infty$?

   (c) What happens to the sum $\hat{\beta}_1 + \hat{\beta}_2$ when $\lambda \to 0$?

2. Consider the model $y = X\beta + u$ where β is non-random, $\mathbb{E}(u \mid X) = 0$, the matrix X of size $n \times k$ has rank $X = k$, but $\mathbb{V}\text{ar}(u \mid X) = \sigma^2 W$ with $W \neq I$. Let $\hat{\beta}$ be the standard OLS estimator of β.

   (a) Find $\mathbb{E}(\hat{\beta} \mid X)$, $\mathbb{E}(\hat{\beta})$.

   (b) Find $\mathbb{V}\text{ar}(\hat{\beta} \mid X)$.

   (c) How do you think, will the standard confidence interval for β be valid in this case?

   (d) Find $\mathbb{C}\text{ov}(y, \hat{\beta} \mid X)$.

3. Consider the matrix

   $$X = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}.$$

   (a) Find the matrix $X^T X$ and diagonalize it.

   (b) Find the SVD of X.

   (c) Find the best approximation to X with rank equal to 1.

   Remark: in the principal component analysis the variables in the matrix X should be standardized. If you can't do this by bare hands, feel free to use python, but provide code!

4. The columns of X are standardized. You know the SVD of the matrix $X = UDV^T$. The diagonal elements of D are positive and ordered from highest to lowest, $d_{11} > d_{22} > \cdots > 0$.

   Let's maximize $\|Xw\|^2$ by choosing an optimal vector w subject to $\|w\|^2 = 1$.

   (d) Write the Lagrangian function for this problem.

   (e) Find the first order conditions. Differential is your friend!

   (f) Find the optimal w in terms of columns of V.

   Hint: one may interpret the FOC in terms of eigenvalues and eigenvectors!