Problem 1

$$loss(\hat{B}_1, \hat{B}_2) = (y-\hat{y})^T(y-\hat{y}) + \lambda B^T B$$

$$\hat{y} = \hat{B}_1 x_1 + \hat{B}_2 x_2$$

a)

$$loss(B_1, B_2) = (y - \hat{B}_1 x - \hat{B}_2 x)(y - \hat{B}_1 x - \hat{B}_2 x)^T + \lambda(B_1^2 + B_2^2)$$

$$loss(B_1, B_2) = y^T y - 2(B_1 + B_2) y^T x + (B_1 + B_2)^2 x^T x + \lambda(B_1^2 + B_2^2)$$

Minimization by FOC

1) $$\frac{\partial\, loss(B_1, B_2)}{\partial B_1} = 0$$

$$0 - 2y^T x + 2(B_1 + B_2) x^T x + 2\lambda B_1 + 0 = 0$$

$\cancel{-2y^T x + C x^T x + \lambda \hat{B}_1 = 0}$ $$(B_1 + B_2) x^T x + \lambda \hat{B}_1 = y^T x$$

$$\lambda \hat{B}_1 = y^T x - (B_1 + B_2) x^T x$$

$$\hat{B}_1 = \cancel{\frac{y^T x - (\hat{B}_1 + \hat{B}_2) x^T x}{\lambda}} \quad \frac{y_x^T x - (\hat{B}_1 + \hat{B}_2) x^T x}{\lambda}$$ minimized by $\hat{B}_1$

The results of minimization for $\hat{B}_1$ and $\hat{B}_2$ are symmetric. Therefore, $\hat{B}_1 = \hat{B}_2$ can be used

$$\hat{\beta}_1 = \frac{y^T x - 2\beta_1 x^T x}{1}$$

$$\hat{\beta}_1 = \frac{y^T x}{1 + 2x^T x}$$

$$\hat{\beta}_2 = \frac{y^T x}{1 + 2x^T x}$$ due to symmetry

x regressor is standartized. It means that $Var(x) = 1$ and $E(x) = 0$. That means:

$$x^T x = \Sigma x_i^2 = n$$

$$\hat{\beta}_1 = \hat{\beta}_2 = \frac{y^T x}{1 + 2n}$$

Answer: $\hat{\beta}_1 = \hat{\beta}_2 = \frac{y^T x}{1 + 2n}$

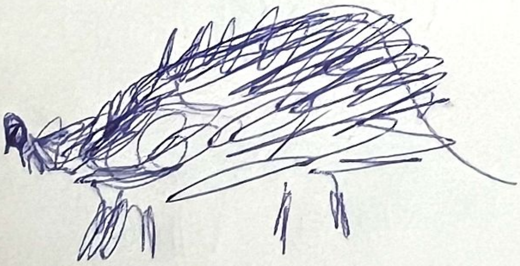
b) $\hat{\beta}_1 = \hat{\beta}_2 = \dfrac{y^T x}{\lambda + 2n}$

$$\lim_{\lambda \to \infty} \beta_1 = \lim_{\lambda \to \infty} \beta_2 = \lim_{\lambda \to \infty} \frac{y^T x}{\lambda + 2n} = 0$$

Ans: $\hat{\beta}_1 = \hat{\beta}_2 = 0$

c) $\hat{\beta}_1 = \hat{\beta}_2 = \dfrac{y^T x}{\lambda + 2n}$

$$\lim_{\lambda \to 0} \beta_1 = \lim_{\lambda \to 0} \beta_2 = \lim_{\lambda \to 0} \frac{y^T x}{\lambda + 2n} = \frac{y^T x}{0 + 2n} = \frac{y^T x}{2n}$$



Ans: $\hat{\beta}_1 = \hat{\beta}_2 = \dfrac{y^T x}{2n}$

Problem 2

$$\hat{B} = (X^T X)^{-1} X^T y = (X^T X)^{-1} X^T (XB + u)$$

a) $E(\hat{B} | X) = E((X^T X)^{-1} X^T (XB + u)) = E((X^T X)^{-1} (X^T X) B) +$

$(X^T X)^{-1} X^T E(u/X) = B + (X^T X)^{-1} X^T \cdot 0 = B$

Ans: $E(\hat{B}|X) = B$

$E(\hat{B}) = E[E(\hat{B}|X)] = E(B) = B$

previous proof   Ans: $E(\hat{B}) = B$

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b) $Var(\hat{B}|X) = (X^T X)^{-1} X^T \; Var(u|X) \, X (X^T X)^{-1} =$

$(X^T X)^{-1} X^T \sigma^2 W X (X^T X)^{-1} = \sigma^2 W (X^T X)^{-1}$

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c) No. The confidence interval assumes homoskedasticity ($W = I$). However, it is given that $W \neq I$

Therefore, $\sigma^2 (X^T X)^{-1}$ formula for variance is incorrect.

Problem 3

$$X = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$$

a) $X^T = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$$X^T X = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$C_{11} = 2 \cdot 2 + (-1)(-1) + (1)(1) = 6$

$C_{12} = 2 \cdot 1 + (-1) \cdot 2 + 1 \cdot 1 = 1$

$C_{21} = 1 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1 = 1$

$C_{22} = 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 6$

$$\boxed{\text{Answer: } X^T X = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}}$$

Diagnozalization of $\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$

1) $\det\begin{pmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{pmatrix} = 0$

$(6-\lambda)(6-\lambda) - 1 \cdot 1 = 0$

$(6-\lambda)^2 = 1$

$\lambda_1 = 7 \quad \lambda_2 = 5$ eigenvalues

2) for eigenvalue $\lambda = 7$

$$(X^T X - 7I)V = 0$$

$$\left(\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right) V = 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -V_1 + V_2 &= 0 \\ V_1 - V_2 &= 0 \end{aligned}\right\} \rightarrow V_1 = V_2$$

$$\boxed{V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$ eigenvector one

for eigenvalue $\lambda = 5$

$$(X^T X - 5I)V = 0$$

$$\left(\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right) V = 0$$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix}=0$$

$$\begin{aligned} V_1+V_2&=0\\ V_1+V_2&=0 \rightarrow \end{aligned}$$

$$V_1=-V_2$$

$$\boxed{V=\begin{pmatrix}1\\ -1\end{pmatrix}}$$ eigenvector two

Diagnalization:

$$X^TX = PDP^{-1}$$

1) $P=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

2) $D=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}5 & 0\\ 0 & 7\end{pmatrix}$

3) $P^{-1}=\,?$

$$P^{-1}=\begin{pmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{pmatrix}$$

Diagnolized form: of X^TX

$$\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 7\end{pmatrix}\begin{pmatrix}0,5 & -0,5\\ 0,5 & 0,5\end{pmatrix}$$

b)

$$X = V \Sigma V^T$$

1) singular values of Σ

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{5}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{7}$$

$$\Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix}$$

2)

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\|v_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\|v_2\| = \sqrt{1^2 + (1)^2} = \sqrt{2}$$

normalized eigenvectors:

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

3)

$$U = XV\Sigma^{-1}$$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sqrt{7} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix}$$

$$XV = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{14}} & \frac{3}{\sqrt{10}} \\ \frac{2}{\sqrt{7}} & 0 \end{bmatrix}$$

$$X = U\Sigma V^T = \begin{bmatrix} \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{14}} & \frac{-3}{\sqrt{10}} \\ \frac{2}{\sqrt{7}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

SVD decomposition

SVD decomposition:

$$X = U \Sigma V^T = \begin{bmatrix} \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{14}} & \frac{3}{\sqrt{10}} \\ \frac{2}{\sqrt{17}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$