1 Late submission

This file has been submitted withing two days of the original deadline. According to the honey-pot policy, having used two honey-pots for the first practical home assignment, I have two honey-pots remaining, and this late submission must be counted towards my score.

2 Problem 2

$y = X\beta + u,\ \mathbb{E}(u\,|\,X) = 0$

a) $y = \mathbb{E}(y\,|\,X) = \mathbb{E}(X\hat{\beta} + u\,|\,X) = X\mathbb{E}(\hat{\beta}\,|\,X)$

$(X^TX)^{-1}X^Ty = (X^TX)^{-1}X^TX\mathbb{E}(\hat{\beta}\,|\,X) \Rightarrow \mathbb{E}(\hat{\beta}\,|\,X) = (X^TX)^{-1}X^Ty$

$y = \mathbb{E}(y) = \mathbb{E}(X\hat{\beta} + u) = X\mathbb{E}(\hat{\beta}) + X\mathbb{E}(u)$ (assuming u and β are independent)

3 Problem 3

a) $X = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$; $X^T = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$; $X^TX = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$

Let's diagonalize the newfound matrix.$X^TX - \lambda I = 0 \Leftrightarrow (6-\lambda)^2 - 1 = 0 \Leftrightarrow \lambda_1 = 5,\ \lambda_2 = 7$

$Ker(X^TX - \lambda_1) = Ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = Lin\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Ker(X^TX - \lambda_2) = Ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = Lin\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The transition matrix $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$P^T = P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

The matrix diagonalized is:

$$X^TX = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

b) Find $V, U, \Sigma:\ \ X = U\Sigma V^T$

We have found the eigenvectors of X^TX to be $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Let's find the eigenvectors of XX^T; $XX^T = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

The eigenvectors are $\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$, $\frac{1}{\sqrt{35}}\begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}$

This all yield us a form $U\Sigma V^T$, where:

$$\Sigma = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix};\quad V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix};\quad U = \begin{pmatrix} \frac{3}{\sqrt{14}} & \frac{-1}{\sqrt{10}} & \frac{-3}{\sqrt{35}} \\ \frac{1}{\sqrt{14}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & 0 & \frac{5}{\sqrt{35}} \end{pmatrix}$$

Checking with a calculator, the product $U\Sigma V^T$ does in fact yield X. I am beyond content.

Remark: the order of the eigenvalues has been changed because I wanted to display the diagonal matrix with eigenvalues in descending order.

c) The best low-rank approximation will be obtained if the smallest entries the diagonal matrix of the SVD form are removed. For a rank-1 approximation, we will remove the lower of the two eigenvalues, 5. We will obtain $U\Sigma' V^T$, where $\Sigma' = \begin{pmatrix} 7 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

The result of the low-rank approximation is $\begin{pmatrix} 1.5 & 1.5 \\ 0.5 & 0.5 \\ 1 & 1 \end{pmatrix}$.