

# DSLs of Mathematics: limit of functions

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# Math book quote: The limit of a function

We say that  $f(x)$  **approaches the limit**  $L$  as  $x$  **approaches**  $a$ , and we write

$$\lim_{x \rightarrow a} f(x) = L,$$

*if the following condition is satisfied:*

*for every number  $\varepsilon > 0$  there exists a number  $\delta > 0$ , possibly depending on  $\varepsilon$ , such that if  $0 < |x - a| < \delta$ , then  $x$  belongs to the domain of  $f$  and*

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

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*if*

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

*such that if*

$$0 < |x - a| < \delta,$$

*then*

$$x \in \text{Dom } f \wedge |f(x) - L| < \varepsilon$$

First attempt at translation:

$$\lim_{x \rightarrow a} f(x) = L = \forall \epsilon > 0. \exists \delta > 0. P_{\epsilon, \delta}$$

$$\text{where } P_{\epsilon, \delta} = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f(x) - L| < \epsilon)$$

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where  $P \epsilon \delta = (0 < |x - a| < \delta) \Rightarrow$   
 $(x \in \text{Dom } f \wedge |f x - L| < \epsilon)$

Scope check:

- $a, f, L$  bound in the def. of  $\lim$ .

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- Anything missing?

Finally (after adding a binding for  $x$ ):

$$\lim_{x \rightarrow a} f(x) = L \iff \forall \epsilon > 0. \exists \delta > 0. P \in \delta$$

$$\text{where } P \in \delta = \forall x. Q \in \delta \ x$$

$$Q \in \delta \ x = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f(x) - L| < \epsilon)$$



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Lesson learned: be careful with scope and binding (of  $x$  in this case).

## Variants of *lim* and some properties

- Typing: let  $X \subseteq \mathbb{R}$ ;  $Y \subseteq \mathbb{R}$ ;  $a : X$ ;  $L : Y$ ; and  $f : X \rightarrow Y$   
Version “limProp”     *lim*     :  $X \rightarrow (X \rightarrow Y) \rightarrow Y \rightarrow Prop$   
Version “limMaybe”   *lim*     :  $X \rightarrow (X \rightarrow Y) \rightarrow Maybe\ Y$   
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• lim can be used as a partial function lim, or *lim*:

$$\forall a, f, L_1, L_2. (\text{lim } a \ f \ L_1 \wedge \text{lim } a \ f \ L_2) \Rightarrow L_1 == L_2$$

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$$lim\ a\ (f \oplus g) = lim\ a\ f + lim\ a\ g$$

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$$f \oplus g = \lambda x \rightarrow f\ x + g\ x$$

## Example 2: derivative

The **derivative** of a function  $f$  is another function  $f'$  defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

at all points  $x$  for which the limit exists (i.e., is a finite real number). If  $f'(x)$  exists, we say that  $f$  is **differentiable** at  $x$ .

We can write

$$D f x = \lim_{h \rightarrow 0} g \quad \text{where} \quad g h = \frac{f(x+h) - f x}{h}; \quad g :: H \rightarrow Y; \text{type } H = \mathbb{R}^+$$

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$$D f x = \lim 0 (\varphi x) \quad \text{where} \quad \varphi x h = \frac{f(x+h) - f x}{h}; \quad \varphi :: X \rightarrow (H \rightarrow Y)$$



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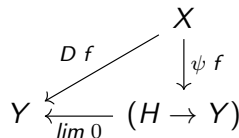
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$$D f = \lim 0 \circ \psi \ f \text{ where } \psi \ f \ x \ h = \frac{f(x+h) - f \ x}{h};$$



Examples:

$$D : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R})$$

$$sq\ x = x^2$$

$$double\ x = 2 * x$$

$$c_2\ x = 2$$

$$sq' == D\ sq == D\ (\lambda x \rightarrow x^2) == D\ (^2) == (2*) == double$$

$$sq'' == D\ sq' == D\ double == c_2 == const\ 2$$

Note: we cannot *implement*  $D$  (of this type) in Haskell.

Given only  $f : \mathbb{R} \rightarrow \mathbb{R}$  as a “black box” we cannot compute the actual derivative  $f' : \mathbb{R} \rightarrow \mathbb{R}$ .

We need the “source code” of  $f$  to apply rules from calculus.