D as a linear transform

Patrik Jansson

$V = \mathbb{R} \to \mathbb{R}$

LinTran(D) ?

$D : V \to V$

$H_0(D, 0_F, 0_F)$
$\& \; H_2(D, (+_F), (+_F))$
$\& \; \forall c.\, H_1(D, \mathit{scale}_F\, c, \mathit{scale}_F\, c)$

Actually, the subset of differentiable functions: $V = C^1(\mathbb{R})$

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$V = \mathbb{R} \to \mathbb{R}$

$\mathit{LinTran}(D)$!

$D : V \to V$

$$H_0(D, 0_F, 0_F) \equiv D\,(\mathit{const}\,0) = \mathit{const}\,0 \qquad \text{OK}$$

$$\& \; H_2(D, (+_F), (+_F)) \equiv \forall f, g.\; D(f +_F g) = D f +_F D g \qquad \text{OK}$$

$$\& \; \forall c.\; H_1(D, \mathit{scale}_F\,c, \mathit{scale}_F\,c)$$

$$\equiv \forall c. \forall f.\; D\,(\mathit{scale}_F\,c\,f) = \mathit{scale}_F\,c\,(D f) \qquad \text{OK}$$

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$D : V \to V$

Examples: $\exp : \mathbb{R} \to \mathbb{R}$

$$D \exp = \exp$$

$s : \mathbb{R}$

$g_s : \mathbb{R} \to \mathbb{R}$

$$g_s\ t = \underbrace{\exp(-s \cdot t)}$$

$$D\ g_s\ t = \underbrace{\exp(-s \cdot t)} \cdot (-s)$$

$$= -s \cdot g_s\ t$$

$$D\ g_s = \backslash t \to -s \cdot g_s\ t$$

$$= scale_F\ (-s)\ g_s$$

$scale_F : \mathbb{R} \to V \to V$

$scale_F\ c\ f = \backslash t \to c \cdot f\ t$

D as a linear transform

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$D : V \to V$

Examples: $v_1 = \exp : \mathbb{R} \to \mathbb{R}$ $\quad D \exp = \exp$

let $g_s(t) = \exp(-s \cdot t)$

then $g_s : V$ for every $s : \mathbb{R}$

$D\, g_s = \backslash t \to -s \cdot \exp(-s \cdot t) = \text{scale } (-s)\ g_s$

D as a linear transform

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$$D : V \to V$$

Examples: $v_1 = \exp : \mathbb{R} \to \mathbb{R}$

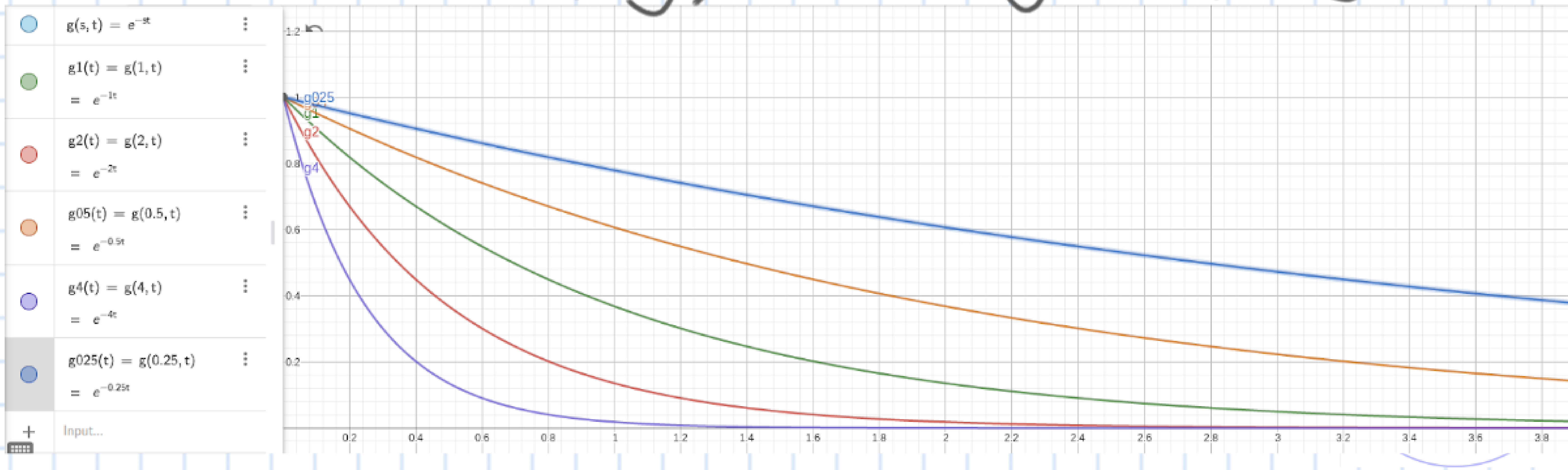
$$g_s(t) = \exp(-s \cdot t)$$

$$\begin{aligned} g_s(0) &= \exp(-s \cdot 0) \\ &= \exp 0 \\ &= 1 \end{aligned}$$

$$D \exp = \exp$$

$$D g_s = \text{scale}\,(-s)\; g_s$$

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$



$\int$ as a linear transform

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$D : V \to V$

$D(f \cdot g) = Df \cdot g + f \cdot Dg$

$I : V \to V$

$$I\,f\,x = \int_0^x f = \int_0^x f(t)\,dt$$

$$I\,(D f)\,x = \int_0^x f'(t)\,dt = \left[f(t) \right]_0^x = f\,x - f\,0$$

$\int$ as a linear transform

Patrik Jansson

$D : V \to V$ $\qquad D(f \cdot g) = Df \cdot g + f \cdot Dg$

$I : V \to V$

$$I\,f\,x = \int_0^x f = \int_0^x f(t)\,dt$$

$$I(Df)\;x = f\,x - f\,0$$

$$I(D(f \cdot g))\;x = (f \cdot g)\;x - (f \cdot g)\;0$$
$$= f\,x \cdot g\,x - f\,0 \cdot g\,0$$
$$= f(x) \cdot g(x) - f(0) \cdot g(0)$$

$\int$ as a linear transform

Patrik Jansson

$$D : V \to V \qquad D(f\cdot g) = Df\cdot g + f\cdot Dg$$

$$I : V \to V$$

$$I\, f\, x = \int_0^x f = \int_0^x f(t)\, dt$$

$$I(Df)\; x = f x - f 0$$

$$I(D(f\cdot g))\, x = (f\cdot g)\, x - (f\cdot g)\, 0 = I(Df\cdot g)\, x + I(f\cdot Dg)\, x$$

Towards Laplace

Patrik Jansson

$$(f \cdot g)\, x - (f \cdot g)\, 0 = I\,(Df \cdot g)\, x + I\,(f \cdot Dg)\, x$$

let $g = g_s = \backslash t \to e^{-st}$; $D g_s = \text{scale}\,(-s)\, g_s$

Assume $(f \cdot g_s)\, x \to 0$ as $x \to \infty$

$$0 - f\,0 \cdot g_s\,0 = I\,(Df \cdot g_s)\,\infty + I\,(\overbrace{f \cdot \text{scale}\,(-s)\, g_s}^{\text{scale}\,(-s)\,(f \cdot g_s)})\,\infty$$

$$\Leftrightarrow \quad -f\,0 = I\,(Df \cdot g_s)\,\infty - s \cdot I\,(f \cdot g_s)\,\infty$$

Towards Laplace

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$$(f \cdot g)\, x - (f \cdot g)\, 0 = I\,(Df \cdot g)\, x + I\,(f \cdot Dg)\, x$$

let $g = g_s = \backslash t \to e^{-st}$; $D\, g_s = \text{scale}\,(-s)\, g_s$

Assume $(f \cdot g_s)\, x \to 0$ as $x \to \infty$

$$0 - f\,0 \cdot g_s\,0 = I\,(Df \cdot g_s)\,\infty + I\,(f \cdot \text{scale}\,(-s)\, g_s)\,\infty$$

$$-f\,0 = I\,(Df \cdot g_s)\,\infty - s \cdot I\,(f \cdot g_s)\,\infty$$

$$-f\,0 = L\,(Df)\,s - s \cdot L\,f\,s$$

$$L\,(Df)\,s = -f\,0 + s \cdot L\,f\,s$$

$$L\,f\,s = I\,(f \cdot g_s)\,\infty$$

$$= \int_0^\infty f \cdot g_s$$

Towards Laplace

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$$(f \cdot g)\, x - (f \cdot g)\, 0 = I\,(Df \cdot g)\, x + I\,(f \cdot Dg)\, x$$

let $g = g_s = \backslash t \to e^{-st}$; $D g_s = \text{scale}\,(-s)\, g_s$

Assume $(f \cdot g_s)\, x \to 0$ as $x \to \infty$

$$0 - f\,0 \cdot g_s\,0 = I\,(Df \cdot g_s)\,\infty + I\,(f \cdot \text{scale}\,(-s)\, g_s)\,\infty$$

$$-f\,0 = I\,(Df \cdot g_s)\,\infty - s \cdot I\,(f \cdot g_s)\,\infty$$

$$-f\,0 = L\,(Df)\,s - s \cdot L\,f\,s$$

$$L\,(Df)\,s = -f\,0 + s \cdot L\,f\,s$$

let $L\,f\,s = \overbrace{I\,(f \cdot g_s)\,\infty}^{\mathbb{C}}$

$\mathbb{R} \to \mathbb{R}$ $\quad \mathbb{C}$

$L : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{C} \to \mathbb{C})$

Laplace transform

Patrik Jansson

$L\,f\,s = I\,(f \cdot g_s)\,\infty$

$$= \int_0^\infty f(t) \cdot e^{-s \cdot t}\,dt$$

$g_s\,t = e^{-st}$

Assume $f\,x \cdot g_s\,x \to 0$ as $x \to \infty$

$V \cong \mathbb{R} \to \mathbb{R}$

$W \cong \mathbb{C} \to \mathbb{C}$

$L\,(D\,f)\,s = -f\,0 + s \cdot L\,f\,s$

$L : V \to W$

$\mathrm{LinTrans}\,(L, V, W)$ (exercise!)