

Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \rightarrow W$$

"linear transform"

$$\text{LinTran}(h, V, W) = \begin{aligned} & \text{fl}_0(h, \text{zero}_V, \text{zero}_W) \\ & \wedge \text{fl}_2(h, (+_V), (+_W)) \\ & \wedge \forall s. H(h, \text{scale}_V s, \text{scale}_W s) \end{aligned}$$

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$$\text{LinTran}(h, V, W) = \text{fl}_0(h, \text{zero}_V, \text{zero}_W) \\ \wedge \text{fl}_2(h, (+_V), (+_W)) \\ \wedge \forall s. H(h, \text{scale}_V s, \text{scale}_W s)$$

Examples:

- Hs. $\text{LinTran}(\text{scale}_V s, V, V)$
- Hc. $\text{linTran}(\text{applyc}, a \rightarrow s, s)$
"projections"

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$$h : V \rightarrow W$$

$$\text{LinTran}(h, V, W) = \text{Hl}_0(h, \text{zero}_V, \text{zero}_W)$$

$$\wedge \text{Hl}_2(h, (+_V), (+_W))$$

$$v : V = G \rightarrow S \quad \wedge \forall s. H(h, \text{scale}_V s, \text{scale}_W s)$$

$$\begin{aligned} hv &= h\left(\sum_i^V \text{scale}_V v_i e_i\right) = \sum_i^W h(\text{scale}_V v_i e_i) \\ &= \sum_i^W \text{scale}_W v_i (he_i) \end{aligned}$$

$$w = hv : W = G' \rightarrow S$$

$$M_h = \begin{pmatrix} | & | & | \\ he_0 & he_1 & he_2 \dots \\ | & | & | \end{pmatrix}$$

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$$h : V \rightarrow W$$

$$\text{LinTran}(h, V, W) = \text{Hl}_0(h, \text{zero}_V, \text{zero}_W)$$

$$\wedge \text{Hl}_2(h, (+_V), (+_W))$$

$$\wedge \forall s. H(h, \text{scale}_V s, \text{scale}_W s)$$

$$h v = h \left(\sum_i \text{scale}_V v_i e_i \right) = \sum_i \text{scale}_W h(v_i e_i) =$$

$$= \sum_i \text{scale}_W v_i (h e_i)$$

$$w = h v : W = G' \rightarrow S$$

$$M_h = \begin{pmatrix} | & | & | \\ h e_0 & h e_1 & h e_2 \dots \\ | & | & | \end{pmatrix}$$

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$$h: V \rightarrow W$$

$$h v = h \left(\sum_i^V \text{scale}_v v_i e_i \right) = \sum_i^W h(\text{scale}_v v_i e_i) = \\ = \sum_i^W \text{scale}_w v_i (h e_i)$$

Syntax = Matrix s

Semantics = $V \rightarrow W$

$$\text{eval}_{MV} m v = \sum_i^W \text{scale}_w (v_i) (m_i)$$

$$M_h = \begin{pmatrix} | & | & | \\ he_0 & he_1 & he_2 & \dots \\ | & | & | \end{pmatrix}$$

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Example: $\text{der} : P_3 \rightarrow P_2$ "polynomial derivative"

$P_n = \{\text{polynomials of degree } \leq n\} = \{0..n\} \rightarrow \mathbb{R}$ (coeff.)

$\text{eval}_P : P_n \rightarrow \mathbb{R} \rightarrow \mathbb{R}$

$\text{eval}_P a = \sum_i \text{scale } a_i \cdot p_i$

$p_i(x) = x^i$

$\text{der } p_i = \text{scale } i \cdot p_{i-1}$

$$\text{DER} = \left(\begin{array}{c|c|c|c}
x^0 & x^1 & x^2 & x^3 \\
\hline
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{array} \right)$$

x^0 x^1 x^2 x^3

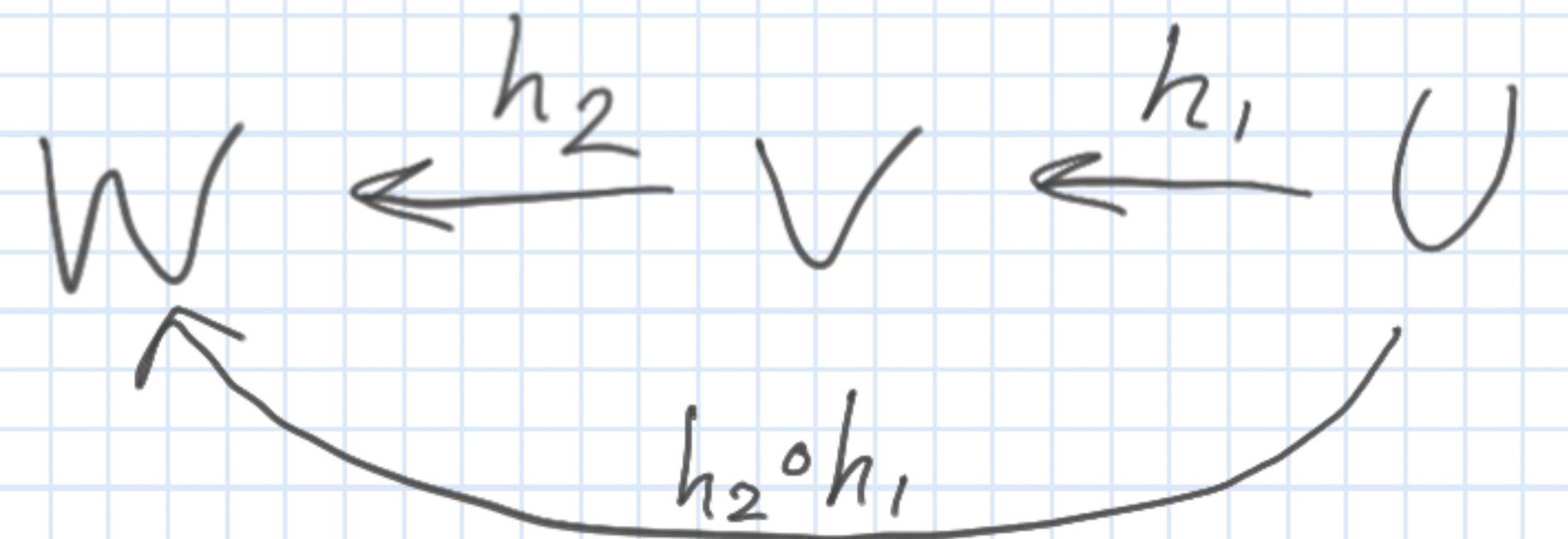
$\text{eval}_{MV} : M \rightarrow V \rightarrow V$

$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$

\downarrow

$\text{LinTran}(h_2 \circ h_1, U, W)$

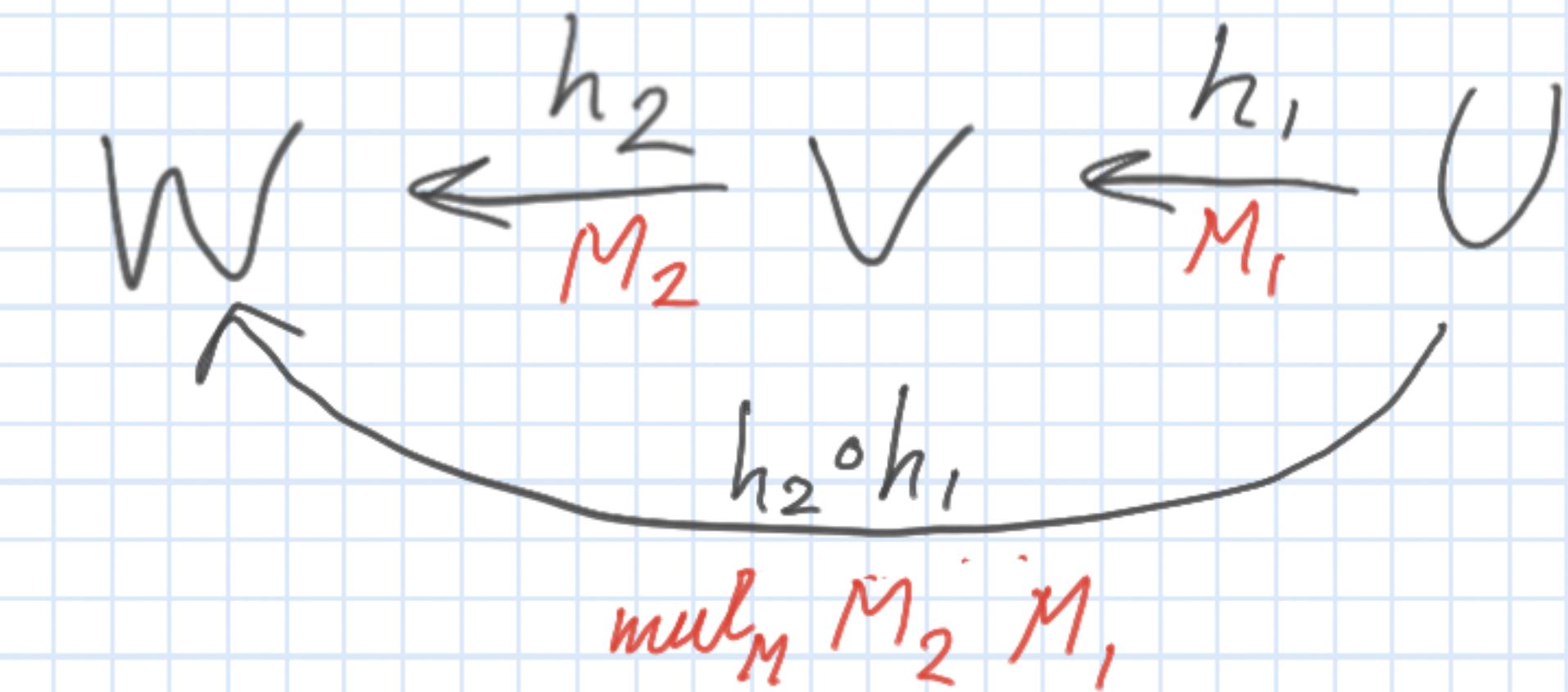


$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$

\downarrow

$\text{LinTran}(h_2 \circ h_1, U, W)$



$$\begin{aligned} \text{eval}_{MV} M_1 &= h_1 \\ \text{eval}_{MV} M_2 &= h_2 \end{aligned} \quad \Rightarrow$$

$$\begin{aligned} h_2 \circ h_1 &= \text{eval}_{MV} M_2 \circ \text{eval}_{MV} M_1 \\ &= \text{eval}_{MV} (\text{mul}_M M_2 M_1) \end{aligned}$$

$H_2(\text{eval}_{MV}, \underline{\text{mul}}_{M_1}(\circ))$

Specification, but what
is the implementation?

type $Vsa = a \rightarrow s$

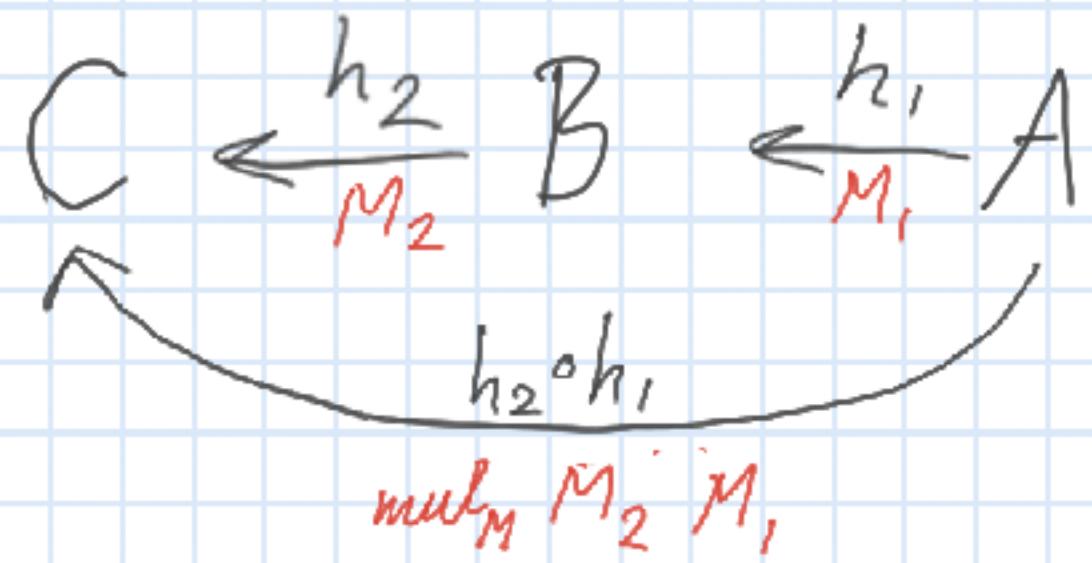
type $Msab = b \rightarrow a \rightarrow s = b \rightarrow Vs a$

transpose : $Msab \rightarrow Msba$

transpose $m = \lambda j i \rightarrow m^{ij} = flip m$

getCol : $Msab \rightarrow a \rightarrow Vs b$

getCol = flip = transpose



$$\begin{pmatrix} & & & \\ & | & | & | \\ & he_0 & he_1 & he_2 \\ & | & | & | \end{pmatrix}$$

type $Vsa = a \rightarrow s$

type $Msab = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

transpose : $Msab \rightarrow Msba$

transpose $m = \lambda j \ i \rightarrow m^{ij} = \text{flip } m$

getCol : $Msab \rightarrow a \rightarrow Vs b$

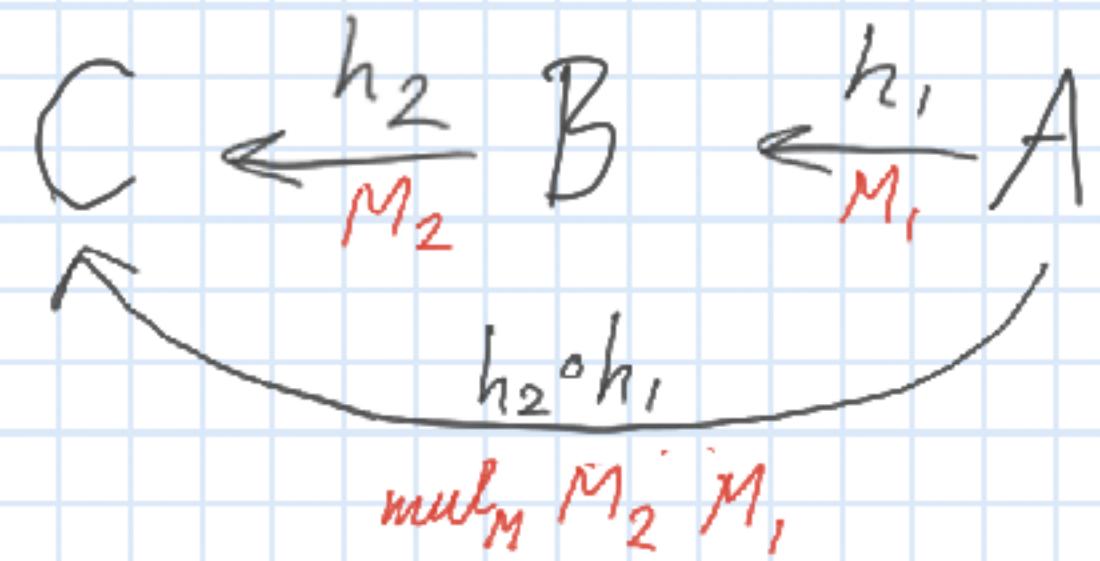
getCol = flip = transpose

Example: $a = \{X, Y\}$, $b = \{0, 1, 2\}$, $c = \{1\}$

$$M_1 = \begin{pmatrix} 1 & | & 1 \\ h_1 e_X & h_1 e_Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

$h_2 = \text{apply } 1$ (projection)



type $Vsa = a \rightarrow s$

type $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vs a$

transpose : $M_{sab} \rightarrow M_{sba}$

transpose $m = \lambda j i \rightarrow m^{ij} = \text{flip } m$

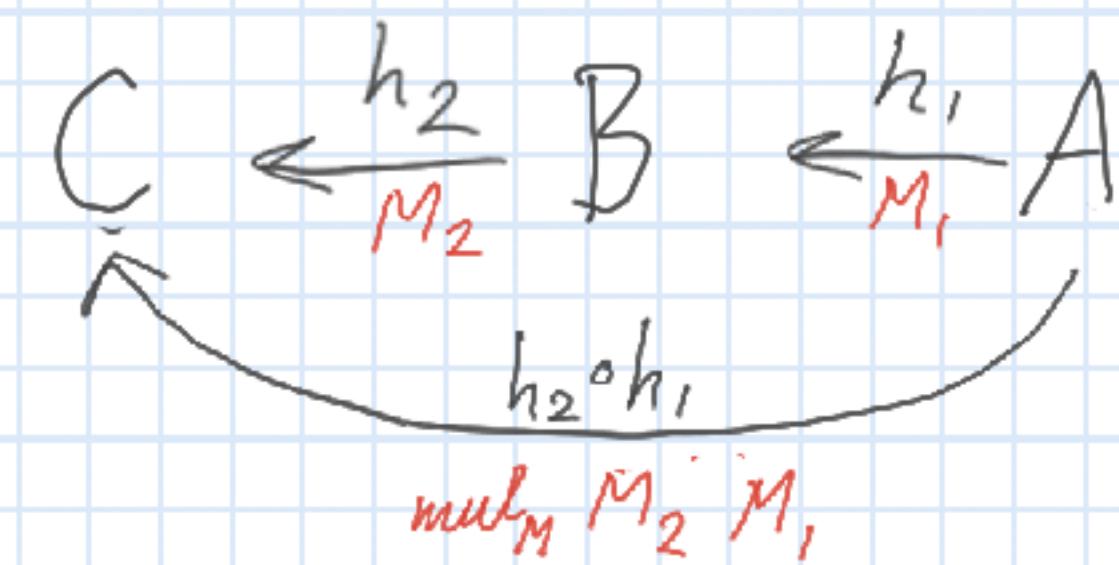
getCol : $M_{sab} \rightarrow a \rightarrow Vs b$

getCol = flip = transpose

Spec. of eval_{MV}: getCol $m^i = \text{eval}_{MV} m (e^i)$

getCol $m = \text{eval}_{MV} m \circ e$

$\text{flip}(\text{flip } m) = m$ = $\text{flip}(\text{eval}_{MV} m \circ e) =$
 h $\text{flip}(h \circ e)$



type $Vsa = a \rightarrow s$

type $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip : $M_{sab} \rightarrow M_{sba}$

Spec. of eval_{MV} : $m = \text{flip}(\text{eval}_{MV} m \circ e)$

$\text{getCol}(\text{mul}_M m_2 m_1)_i = h_2(h_1 e_i)$

$\text{eval}_{MV} m_2$

$\text{getCol } m_1, i$

A diagram illustrating the components of the matrix multiplication. At the top right, there is a triangular arrangement of three sets: A, B, and C. Set A is at the bottom left, set B is at the top center, and set C is at the top right. There are two horizontal arrows pointing from left to right: one labeled h_1 above and M_1 below, and another labeled h_2 above and M_2 below. A curved arrow labeled $h_2 \circ h_1$ above and $\text{mul}_M M_2 M_1$ below connects the bottom of set A to the top of set C.

type $Vsa = a \rightarrow s$

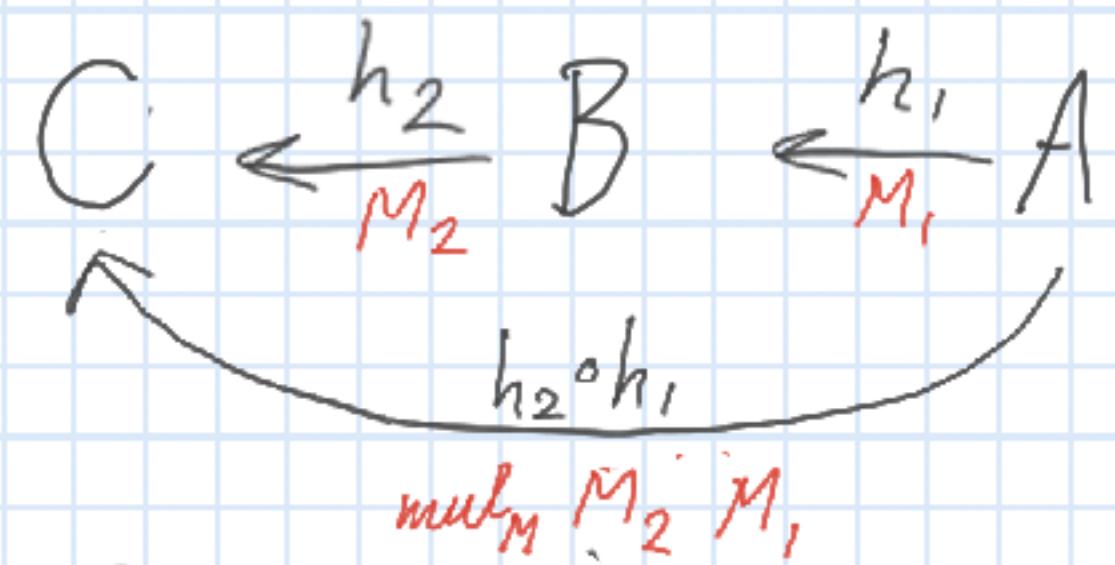
type $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip : $M_{sab} \rightarrow M_{sba}$

Spec. of eval_{MV} : $m = \text{flip}(\text{eval}_{MV} m \circ e)$

$\text{getCol}(\text{mul}_M m_2 m_1)_i = h_2(h_1 e_i) =$

$= \text{eval}_{MV} m_2 (\text{getCol } m_1 i^\circ)$



type $Vsa = a \rightarrow s$

type $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip : $M_{sab} \rightarrow M_{sba}$

Spec. of eval_{MV}: $m = \text{flip}(\text{eval}_{MV} m \circ e)$

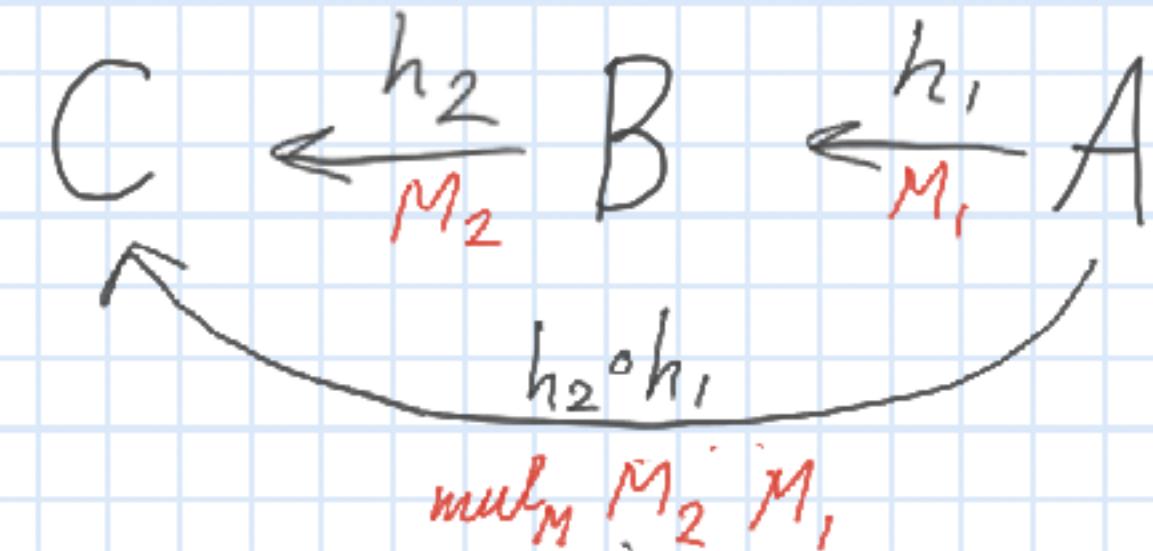
getCol($\text{mul}_M m_2 m_1$)_i = $h_2(h_1 e_i) =$

= $\text{eval}_{MV} m_2 (\text{getCol } m_1, i)$

getCol($\text{mul}_M m_2 m_1$) = $\text{eval}_{MV} m_2 \circ \text{getCol } m_1$

$\boxed{\text{mul}_M m_2 m_1 = \text{flip}(\text{eval}_{MV} m_2 \circ \text{flip } m_1)}$

where $\text{eval}_{MV} m v = \sum_i \text{scale } v_i m_i$



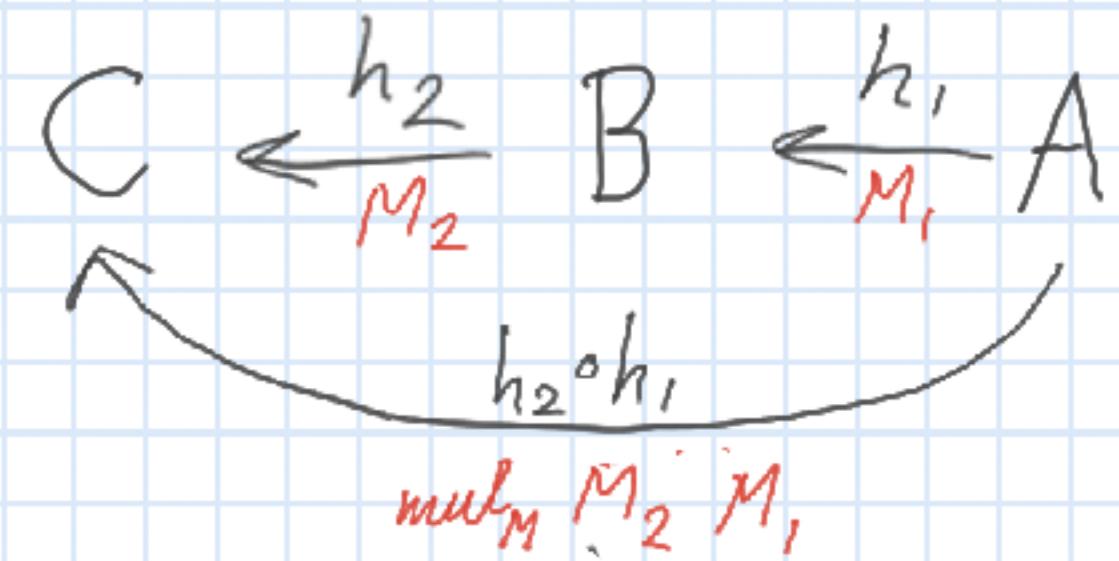
Det. of
 mul_M

type $Vsa = a \rightarrow s$

type $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

$\text{flip} : M_{sab} \rightarrow M_{sba}$

$\text{flip} = \text{transpose} = \text{getCol}$



$$\boxed{\text{mul}_M m_2 m_1 = \text{flip} (\text{eval}_{MV} m_2 \circ \text{flip} m_1)} \quad \begin{array}{l} A = Vsa \\ B = Vsb \\ C = Vsc \end{array}$$

where $\text{eval}_{MV} m v = \sum_i \text{scale} v_i m_i$

$\text{mul}_M : M_{sab} \rightarrow M_{sbc} \rightarrow M_{sac}$

$\text{eval}_{MV} = \text{mul}_{MV} : M_{sab} \rightarrow Vsa \rightarrow Vsb$