

Partial derivatives

tions. Our example here is by Mac Lane [1986, page 169], where we read

- 1 [...] a function $z = f(x, y)$ for all points (x, y) in some open set U
- 2 of the cartesian (x, y) -plane. [...] If one holds y fixed, the quantity z
- 3 remains just a function of x ; its derivative, when it exists, is called
- 4 the *partial derivative* with respect to x . Thus at a point (x, y) in U this
- 5 derivative for $h \neq 0$ is

$$U \subseteq \mathbb{R} \times \mathbb{R}$$

$$\partial z / \partial x = f'_x(x, y) = \lim_{h \rightarrow 0} (f(x + h, y) - f(x, y)) / h \quad \mathbb{R}$$

Types: $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ $f: U \rightarrow \mathbb{R}$ $\mathbb{R} \rightarrow \mathbb{R}$

$x, y: \mathbb{R}$

$z: \mathbb{R} \rightarrow \mathbb{R}$

$\frac{\partial z}{\partial x}: \mathbb{R} \rightarrow \mathbb{R}$

$f_x: U \rightarrow \mathbb{R}$

[...] a function $z = f(x, y)$ for all points (x, y) in some ~~open set~~^{subset} ~~U~~^{lesson} of the cartesian (x, y) -plane. [...] If one holds y fixed, the quantity z remains just a function of x ; its derivative, when it exists, is called the *partial derivative* with respect to x . Thus at a point (x, y) in U this derivative for $h \neq 0$ is

$$\frac{\partial z}{\partial x} = f'_x(x, y) = \lim_{h \rightarrow 0} (f(x + h, y) - f(x, y)) / h$$

$$D_1 f(x, y) = \lim_{h \rightarrow 0} (\psi_1 f(x, y)) \quad D_1 : (R^2 \rightarrow R) \rightarrow (R^2 \rightarrow R)$$
$$\psi_1 : (U \rightarrow R) \rightarrow U \rightarrow (R^+ \rightarrow R)$$

$$\psi_1 f(x, y) \quad h = \frac{1}{h} (f(x+h, y) - f(x, y)) \quad D_2$$
$$\psi_2 f(x, y) \quad h = \frac{1}{h} (f(x, y+h) - f(x, y))$$

The partial derivative with respect to x . Thus at a point (x, y) in U this derivative for $h \neq 0$ is

$$z = f(x, y)$$
$$\frac{\partial z}{\partial x} : \mathbb{R} = f'_x(x, y) = \lim_{h \rightarrow 0} (f(x + h, y) - f(x, y)) / h$$

$$D_1, D_2 : (\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$$

Here $D_1 = \frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial y}$ where $f(x, y) = \dots$

Be careful with scoping
& the difference between
expressions & functions

$$\frac{\partial f}{\partial x} : \mathbb{R}^2 \rightarrow \mathbb{R}$$

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DSL $\rightarrow \delta\sigma\lambda$
DSLs of Math

DSLsofMath: Typing Mathematics (Week 3)

the Lagrangian Equations

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Starting point: a math quote (the Lagrangian)

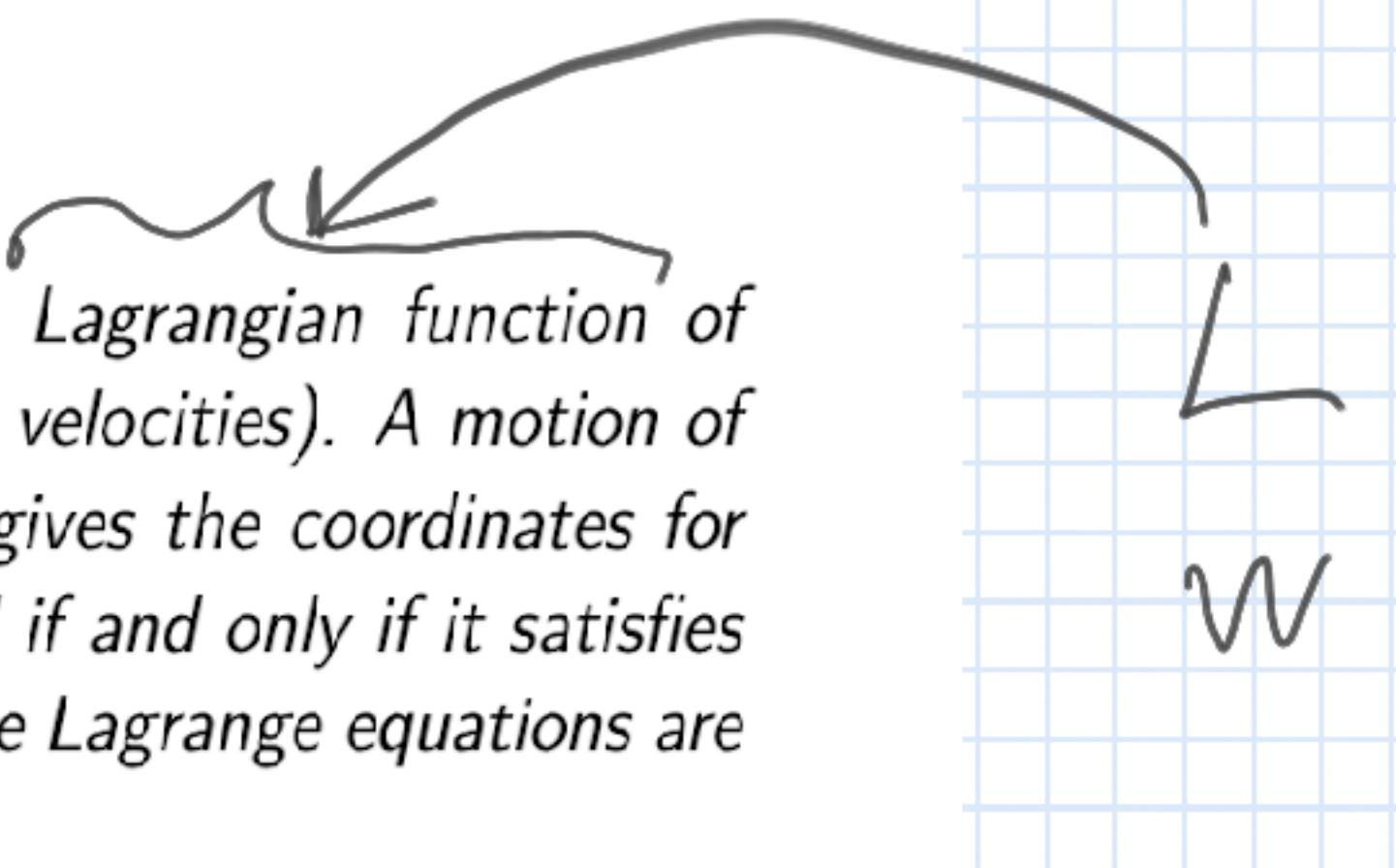


From [Sussman and Wisdom, 2013]:

A mechanical system is described by a Lagrangian function of the system state (time, coordinates, and velocities). A motion of the system is described by a path that gives the coordinates for each moment of time. A path is allowed if and only if it satisfies the Lagrange equations. Traditionally, the Lagrange equations are written

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

What could this expression possibly mean?



- *valid path*

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The use of notation for “partial derivative”, $\partial L / \partial q$, suggests that L is a function of at least a pair of arguments:

$$L : \mathbb{R}^i \rightarrow \mathbb{R}, i \geq 2$$

This is consistent with the description: “Lagrangian function of the system state (time, coordinates, and velocities)”. So, if we let “coordinates” be just one coordinate, we can take $i = 3$:

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}$$

The “system state” here is a triple, of type $S = T \times Q \times V$, and we can call the three components $t : T$ for time, $q : Q$ for coordinate, and $v : V$ for velocity. ($T = Q = V = \mathbb{R}$.)

Lagrangian, cont.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Looking again at $\partial L / \partial q$, q is the name of a variable, one of the 3 args to L . In the context, which we do not have, we would expect to find somewhere the definition of the Lagrangian as

$$L : T \times Q \times V \rightarrow \mathbb{R}$$
$$L(t, q, v) = \dots$$

- therefore, $\partial L / \partial q$ should also be a function of the same triple:

$$(\partial L / \partial q) : T \times Q \times V \rightarrow \mathbb{R}$$

It follows that the equation expresses a relation between *functions*, therefore the 0 on the right-hand side is *not* the real number 0, but rather the constant function *const 0*:

$$\text{const 0} : T \times Q \times V \rightarrow \mathbb{R}$$
$$\text{const 0}(t, q, v) = 0$$



Lagrangian, cont.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$\underbrace{\mathbb{R} \rightarrow \mathbb{R}}_{\mathbb{R} \rightarrow \mathbb{R}}$ $\mathbb{R}^3 \rightarrow \mathbb{R}$

- We now have a problem: d / dt can only be applied to functions of **one** real argument t , and the result is a function of one real argument:

$$(d / dt)(\partial L / \partial \dot{q}) : T \rightarrow \mathbb{R}$$

Since we subtract from this the function $\partial L / \partial q$, it follows that this, too, must be of type $T \rightarrow \mathbb{R}$. But we already typed it as $T \times Q \times V \rightarrow \mathbb{R}$, contradiction!

- The expression $\partial L / \partial \dot{q}$ appears to also be malformed. We would expect a variable name where we find \dot{q} , but \dot{q} is the same as dq/dt , a function.

$$D : (\mathbb{R} \rightarrow \mathbb{R}) \downarrow (\mathbb{R} \rightarrow \mathbb{R})$$

$$q' = D q = \dot{q}$$

var-
namen

$$\frac{\partial L}{\partial \left(\frac{dq}{dt} \right)} \vdash V$$

Lagrangian, cont.

$$D = \frac{d}{dt}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The only immediate candidate for an application of d/dt is “a path that gives the coordinates for each moment of time”. Thus, the path is a function of time, let us say

$w : T \rightarrow Q$ -- with T for time and Q for coords ($q : Q$)

We can now guess that the use of the plural form “equations” might have something to do with the use of “coordinates”. In an n -dim. space, a position is given by n coordinates. A path would then be

$w : T \rightarrow Q$ -- with $Q = \mathbb{R}^n$

which is equivalent to n functions of type $T \rightarrow \mathbb{R}$, each computing one coordinate as a function of time. We would then have an equation for each of them. We will use $n = 1$ for the rest of this example.

Lagrangian, cont.

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LagrangeEqs.pdf

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Now that we have a path, the coordinates at any time are given by the path. And as the time derivative of a coordinate is a velocity, we can actually compute the trajectory of the full system state $T \times Q \times V$ starting from just the path.

$$q : T \rightarrow Q$$

$$q = w \quad \text{-- or, equivalently, } q(t) = w(t)$$

$$\sqrt{=} \quad \dot{q} : T \rightarrow V$$

$$\sqrt{=} \quad \dot{q} = D w \quad \text{-- or, equivalently, } \dot{q}(t) = dw(t) / dt$$

We combine these in the “combinator” *expand*, given by

$$\textit{expand} : (T \rightarrow Q) \rightarrow (T \rightarrow T \times Q \times V)$$

$$\textit{expand} w t = (t, w t, D w t)$$

Lagrangian, cont.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- With *expand w* in our toolbox we can fix the typing problem.

$$(\partial L / \partial q) \circ (\text{expand } w) : T \rightarrow \mathbb{R}$$

- We now move to using D for d / dt , D_2 for $\partial / \partial q$, and D_3 for $\partial / \partial \dot{q}$. In combination with *expand w* we find these type correct combinations for the two terms in the equation:

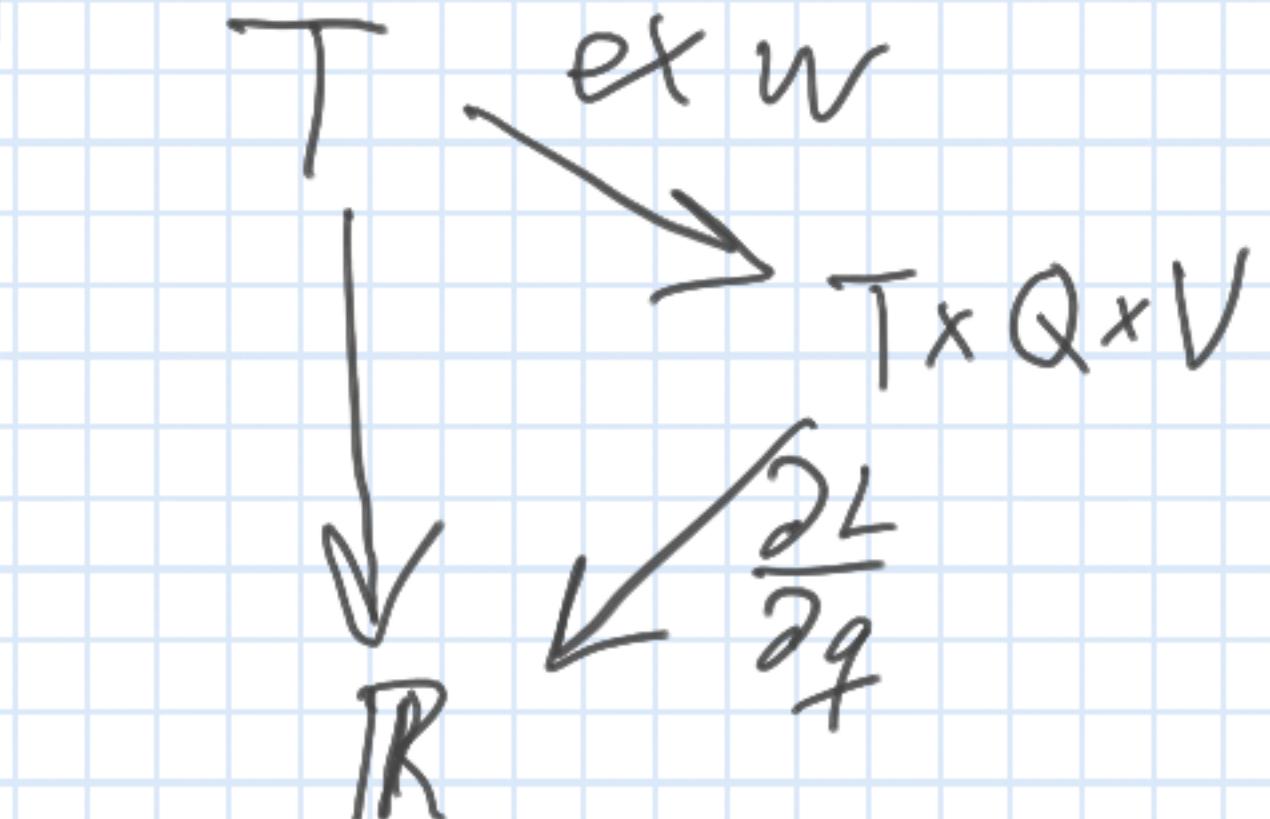
$$D((D_3 L) \circ (\text{expand } w)) : T \rightarrow \mathbb{R}$$
$$(D_2 L) \circ (\text{expand } w) : T \rightarrow \mathbb{R}$$

The equation becomes

$$D((D_3 L) \circ (\text{expand } w)) - (D_2 L) \circ (\text{expand } w) = \text{const } 0$$

or, after simplification:

$$D(D_3 L \circ \text{expand } w) = D_2 L \circ \text{expand } w$$



$$\frac{\partial}{\partial \dot{q}} = \frac{\partial}{\partial v} = D_3$$

Case 3: Lagrangian, summary

“A path is allowed if and only if it satisfies the Lagrange equations” means that this equation is a predicate on paths:

$$\text{Lagrange}(L, w) = D(D_3 L \circ \text{expand } w) == D_2 L \circ \text{expand } w$$

Thus: If we can describe a mechanical system in terms of “a Lagrangian” ($L : S \rightarrow \mathbb{R}$), then we can use the predicate to check if a particular candidate path $w : T \rightarrow \mathbb{R}$ qualifies as a “motion of the system” or not. The unknown of the equation is the path w , and the equation is an example of a partial differential equation (a PDE).

Example of Lagrange eq.s

$$L : T \times Q \times V \rightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$D_3 L : T \times Q \times V \rightarrow \mathbb{R}$$

$$D_3 L(t, x, v) = m \cdot v$$

$$D_2 L(t, x, v) = -m \cdot g$$

$$\text{Lagrange}(L, w) = D(D_3 L \circ \text{expand } w) :: D_2 L \circ \text{expand } w$$

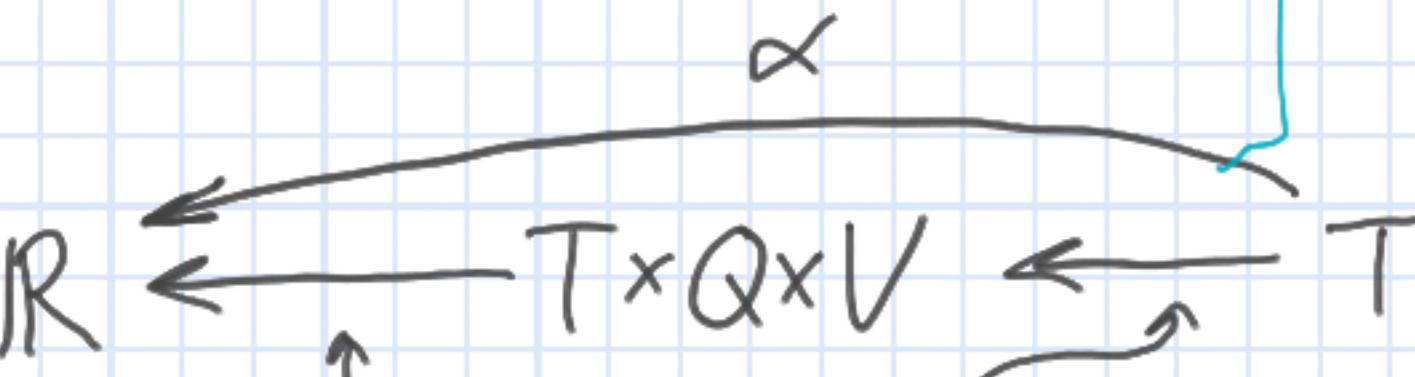
Example of Lagrange eq.s

$$L : T \times Q \times V \rightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$\text{ex} : (T \rightarrow Q) \rightarrow (T \rightarrow T \times Q \times V)$$

$$\text{ex } w \ t = (t, w_t, D_w t)$$



$$\text{Lagrange } (L, w) = D \underbrace{(D_3 L \circ \text{expand } w)}_{\alpha} :: D_2 L \circ \text{expand } w \underbrace{\text{expand } w}_{\beta}$$

Example of Lagrange eq.s

$$L : T \times Q \times V \rightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$D_3 L(-, -, v) = m \cdot v$$

$$D_2 L(-, -, -) = -m \cdot g$$

$$\alpha t = D_3 L(\text{ex w } t) = m \cdot 2 \cdot B \cdot t$$

$$D\alpha t = 2 \cdot m \cdot B$$

$$D\alpha = \underbrace{\text{const}}_{\text{const}}$$

$$\text{Lagrange}(L, w) =$$

$$D(D_3 L \circ \text{expand } w) = D_2 L \circ \text{expand } w$$

α

β

$$w : T \rightarrow Q$$

$$w t = A + B \cdot t^2$$

$$\text{ex } w t = (t, A + B \cdot t^2, 2 \cdot B \cdot t)$$

$$\beta t = \underbrace{-m \cdot g}_{\text{const}}$$

$$\beta = \text{const}$$

$$2 \cdot B = -g$$

$$\text{const}(2 \cdot m \cdot B) == \text{const}(-m \cdot g)$$

Example of Lagrange eq.s

$$L : T \times Q \times V \rightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$D_3 L(-, -, v) = m \cdot v$$

$$D_2 L(-, -, -) = -m \cdot g$$

$$\alpha t = D_3 L(\text{ex w } t) = 2 \cdot B \cdot m \cdot t$$

$$D\alpha t = 2 \cdot B \cdot m$$

$$\text{Lagrange}(L, w) =$$

$$D(D_3 L \circ \text{expand } w) = D_2 L \circ \text{expand } w$$

$$w : T \rightarrow Q$$

$$w(t) = A + B \cdot t^2$$

$$\text{ex } w(t) = (t, A + B \cdot t^2, 2 \cdot B \cdot t)$$

$$\beta t = -m \cdot g$$

$$m=0 \text{ or } B=-g/2$$

$$\text{const}(2 \cdot B \cdot m) = \text{const}(-m \cdot g)$$

α

β

Example of Lagrange eq.s

$$L : T \times Q \times V \rightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$\text{Lagrange } (L, w) = D(D_3 L \circ \text{expand } w) = D_2 L \circ \text{expand } w$$

$$\Rightarrow m=0$$

with A, B free

$$\text{or } w \cdot t = A - \frac{g \cdot t^2}{2}$$

with A free

(free fall with acc. $-g$ from height A)

$$w : T \rightarrow Q$$

$$w \cdot t = A + B \cdot t^2$$

