# DSLs of Mathematics: limit of functions

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# Math book quote: The limit of a function

*We say that* $f(x)$ ***approaches the limit*** $L$ *as* $x$ ***approaches*** $a$*, and we write*

$$\lim_{x \to a} f(x) = L,$$

*if the following condition is satisfied:*
*for every number* $\varepsilon > 0$ *there exists a number* $\delta > 0$*, possibly depending on* $\varepsilon$*, such that if* $0 < |x - a| < \delta$*, then* $x$ *belongs to the domain of* $f$ *and*

$$|f(x) - L| < \varepsilon$$

\- Adams & Essex, Calculus - A Complete Course

# Limit of a function – continued: fragments translated to logic

$$\lim_{x \to a} f(x) = L,$$

*if*

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

*such that if*

$$0 < |x - a| < \delta,$$

*then*

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

# Limit of a function – continued

First attempt at translation:

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\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow \\
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Scope check:

- $a$, $f$, $L$ bound in the def. of *lim*.

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- Anything missing?

# Limit of a function – continued

Finally (after adding a binding for $x$):

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Lesson learned: be careful with scope and binding (of $x$ in this case).

# Variants of *lim* and some properties

- Typing: let $X \subseteq \mathbb{R}$; $Y \subseteq \mathbb{R}$; $a : X$; $L : Y$; and $f : X \to Y$

  Version "limProp" $\quad \mathit{lim} \quad : \quad X \to (X \to Y) \to Y \to \mathit{Prop}$

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- $\underline{lim}$ can be used as a partial function $\underline{lim}$, or $lim$:

$$\forall\ a, f, L_1, L_2.\ \ (\underline{lim}\ a\ f\ \ L_1 \wedge \underline{lim}\ a\ f\ \ L_2) \Rightarrow L_1 == L_2$$
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$$f \oplus g = \lambda x \to f\ x + g\ x$$

# Example 2: derivative

*The **derivative** of a function $f$ is another function $f'$ defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points $x$ for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that $f$ is **differentiable** at $x$.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}; \qquad g :: \ H \to Y; \text{type } H = \mathbb{R}^{\neq 0}$$

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$$D\ f \quad = lim\ 0 \circ \psi\ f\ \text{ where } \psi\ f\ \ x\ \ h = \frac{f\ (x+h) - f\ x}{h};$$

$$\begin{tikzcd} & X \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\ Y & (H \to Y) \arrow[l, "lim\ 0"] \end{tikzcd}$$

# Derivatives, cont.

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&sq\ x \qquad = x\hat{\ }2 \\
&double\ x = 2 * x \\
&c_2\ x \qquad = 2 \\
&sq' \ \texttt{==}\ D\ sq \ \texttt{==}\ D\ (\lambda x \to x\hat{\ }2)\ \texttt{==}\ D\ (\hat{\ }2)\ \texttt{==}\ (2*)\ \texttt{==}\ double \\
&sq'' \ \texttt{==}\ D\ sq' \ \texttt{==}\ D\ double\ \texttt{==}\ c_2\ \texttt{==}\ const\ 2
\end{aligned}
$$

Note: we cannot *implement* $D$ (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of $f$ to apply rules from calculus.