Domain-Specific Languages of Mathematics
Course codes: DAT326 / DIT983 / DIT982

Patrik Jansson

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Contact	Patrik Jansson, 072 985 2033.
Results	Announced within 15 workdays
Exam check	Book a slot by email to patrikj AT chalmers.se
Aids	One textbook of your choice (Domain-Specific Languages of Mathematics, or Beta - Mathematics Handbook, or Rudin, or Adams and Essex, or ...). No printouts, no lecture notes, no notebooks, etc.
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain-Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure:** a DSL for Quasigroups

   Consider the following mathematical definition (adapted from Quasigroup, Wikipedia):

   > A quasigroup $(Q, (**), (\backslash\backslash), (//))$ is a set Q equipped with three binary operators (called multiplication, left- and right-division) satisfying the following identities for all x and y in Q:
   >
   > $y = (y ** x) // x$
   > $y = (y // x) ** x$
   > $y = x \backslash\backslash (x ** y)$
   > $y = x ** (x \backslash\backslash y)$
   >
   > In other words: multiplication and division in either order, one after the other, on the same side by the same element, have no net effect.

   (a) Define a type class *Quasi* that corresponds to the Quasigroup structure.

   (b) Define a datatype $Q\ v$ for the language of quasigroup expressions (with variables of type v) and define a *Quasi* instance for it. (These are expressions formed from applying the quasigroup operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Quasigroup* class. Make sure the laws are satisfied.

   (d) Give a type signature for, and define, a general evaluator for $Q\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Quasi* instances defined in (1c). Take three quasigroup expressions of type $Q\ String$, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**

   Consider the following coupled differential equations:

$$
\begin{aligned}
f' - f &= g, & f(0) &= 1 \\
g' - 4g &= exp - 2f, & g(0) &= 3
\end{aligned}
$$

   (a) [10p] Solve the equations assuming that f and g can be expressed by power series *fs* and *gs*, that is, use *integ* and the differential equations to express the relation between *fs*, *fs'*, *gs*, and *gs'*. What are the first three coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equations using the Laplace transform. You should need this formula and the rules for linearity + derivative:

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

   Show that your solutions do indeed satisfy the four requirements.

3. [25p] **Proofs: Unique limits**

   Consider the statement: "The limit of a convergent sequence is unique."

   Let $X \subseteq \mathbb{R}$, and $\mathit{Seq}\ X = \mathbb{N} \to X$. Then the statement can be formalised as T where

   $T : \mathit{Prop}$
   $T = \forall\ a : \mathit{Seq}\ X.\ \ U\ a$

   $U : \mathit{Seq}\ X \to \mathit{Prop}$ -- "a has a unique limit"
   $U\ a = \forall\ L_1 : \mathbb{R}.\ \forall\ L_2 : \mathbb{R}.\ \ ((Q\ a\ L_1) \wedge (L_1 \not\equiv L_2)) \Rightarrow \neg\,(Q\ a\ L_2)$

   $Q : \mathit{Seq}\ X \to X \to \mathit{Prop}$ -- "a converges to L"
   $Q\ a\ L = \forall\ \epsilon > 0.\ \ P\ a\ L\ \epsilon$

   $P : \mathit{Seq}\ X \to X \to \mathbb{R}_{>0} \to \mathit{Prop}$ -- "some tail of a is near L"
   $P\ a\ L\ \epsilon = \exists\ N : \mathbb{N}.\ \ \forall\ n : \mathbb{N}.\ \ (n \geqslant N) \Rightarrow (|a\ n - L| < \epsilon)$

   i.e., if a sequence converges to a limit (L_1), then it doesn't converge to anything else (L_2).

   (a) [10p] Let $nQ\ a\ L_2 = \neg(Q\ a\ L_2)$. Simplify this to eliminate the negation. (By pushing the negation $\neg$ inwards until it meets an ordering which it can negate). Explain the steps in your equational reasoning.

   (b) [5p] Give nQ a functional interpretation – that is, explain what form a value *prf* of type $nQ\ a\ L_2$ would have.

   (c) [10p] Sketch a proof of T using the functional interpretation – that is, provide pseudo code for $t : T$.

4. [25p] **Typing maths: differentials**

   Consider the following (slightly edited) quote from [Adams, p105]:

   > The Newton quotient $[f(x+h)-f(x)]/h$, whose limit we take to find the derivative dy/dx, can be written in the form $\Delta y/\Delta x$, where [...].
   > The Newton quotient $\Delta y/\Delta x$ is actually the quotient of two quantities, Δy and Δx. It is not at all clear, however, that the derivative dy/dx, the limit of $\Delta y/\Delta x$ as Δx approaches zero, can be regarded as a quotient. If y is a continuous function of x, then Δy approaches zero when Δx approaches zero, so dy/dx appears to be the meaningless quantity $0/0$. Nevertheless, it is sometimes useful to be able to refer to quantities dy and dx in such a way that quotient is the derivative dy/dx. We can justify this by regarding dx as a new *independent* variable (called **the differential of** x) and defining a new *dependent* variable dy (**the differential of** y) as a function of x and dx by
   >
   > $$dy = \frac{dy}{dx}dx = f'(x)dx.$$

   (a) [10p] Give the types of x, y, f, dy/dx, dx, dy, f'. Explain your reasoning.

   (b) [5p] What is dy if $y = f(x^2)$?

   (c) [10p] What would be the corresponding notion of differential dz for a two-variable function $z = g(x_1, x_2)$? It should be a function of $2 * 2$ independent variables.