

DSLs of Mathematics: limit of functions

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Math book quote: The limit of a function

We say that $f(x)$ **approaches the limit** L as x **approaches** a , and we write

$$\lim_{x \rightarrow a} f(x) = L,$$

if the following condition is satisfied:

for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε , such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

Limit of a function – continued

$$\lim_{x \rightarrow a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \text{Dom } f \wedge |f(x) - L| < \varepsilon$$

Limit of a function – continued

First attempt at translation:

$$\lim_{x \rightarrow a} f(x) = L = \forall \epsilon > 0. \exists \delta > 0. P_{\epsilon \delta}$$

$$\text{where } P_{\epsilon \delta} = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f(x) - L| < \epsilon)$$

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Scope check:

- a, f, L bound in the def. of \lim .

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- Anything missing?

Finally (after adding a binding for x):

$$\lim_{x \rightarrow a} f(x) = L = \forall \epsilon > 0. \exists \delta > 0. P \in \delta$$

$$\text{where } P \in \delta = \forall x. Q \in \delta \wedge x$$

$$Q \in \delta \wedge x = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f(x) - L| < \epsilon)$$

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Lesson learned: be careful with scope and binding (of x in this case).

Variants of *lim* and some properties

- Typing: let $X \subseteq \mathbb{R}$; $Y \subseteq \mathbb{R}$; $a : X$; $L : Y$; and $f : X \rightarrow Y$
Version “limProp” *lim* : $X \rightarrow (X \rightarrow Y) \rightarrow Y \rightarrow Prop$
Version “limMaybe” *lim* : $X \rightarrow (X \rightarrow Y) \rightarrow Maybe\ Y$
Version “limSloppy” *lim* : $X \rightarrow (X \rightarrow Y) \rightarrow Y$

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- lim can be used as a partial function lim, or *lim*:
$$\forall a, f, L_1, L_2. (\text{lim } a\ f\ L_1 \wedge \text{lim } a\ f\ L_2) \Rightarrow L_1 == L_2$$
$$\forall a, f, L. (\text{lim } a\ f\ L) \Rightarrow (\text{lim } a\ f = Just\ L) \wedge (lim\ a\ f = L)$$

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- The function $lim\ a : (X \rightarrow Y) \rightarrow Y$ is linear:

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$$f \oplus g = \lambda x \rightarrow f x + g x$$

Example 2: derivative

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number).

We can write

$$D f x = \lim 0 g \quad \text{where} \quad g h = \frac{f (x+h) - f x}{h}; \quad g :: H \rightarrow Y; \text{type } H = \mathbb{R}_{\neq 0}$$

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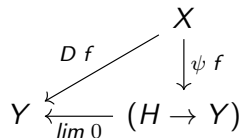
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$$D f = \lim 0 \circ \psi \ f \quad \text{where} \quad \psi \ f \ x \ h = \frac{f(x+h) - f \ x}{h};$$



Examples:

$$D : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R})$$

$$sq\ x = x^2$$

$$double\ x = 2 * x$$

$$c_2\ x = 2$$

$$sq' == D\ sq == D\ (\lambda x \rightarrow x^2) == D\ (^2) == (2*) == double$$

$$sq'' == D\ sq' == D\ double == c_2 == const\ 2$$

Note: we cannot *implement* D (of this type) in Haskell.

Given only $f : \mathbb{R} \rightarrow \mathbb{R}$ as a “black box” we cannot compute the actual derivative $f' : \mathbb{R} \rightarrow \mathbb{R}$.

We need the “source code” of f to apply rules from calculus.

Next: We implement a DSL (embedded in Haskell) to solve this and then *we implement* a “symbolic derivative” function.