DSLs of Mathematics: limit of functions

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Math book quote: The limit of a function

*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

Limit of a function – continued

$$\lim_{x \to a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

Limit of a function – continued

First attempt at translation:

$$
\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow \\
&\qquad\qquad\qquad\qquad (x \in \mathit{Dom}\ f \wedge |f\ x - L| < \epsilon)
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Scope check:

- a, f, L bound in the def. of *lim*.

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Scope check:

- a, f, L bound in the def. of *lim*.
- Forall binds ϵ, exists binds δ (and then again in $P\ \epsilon\ \delta$).
- Anything missing?

Limit of a function – continued

Finally (after adding a binding for x):

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Lesson learned: be careful with scope and binding (of x in this case).

Variants of *lim* and some properties

- Typing: let $X \subseteq \mathbb{R}$; $Y \subseteq \mathbb{R}$; $a : X$; $L : Y$; and $f : X \to Y$

Version "limProp"	$\underline{lim}$	:	$X \to (X \to Y) \to Y \to Prop$
Version "limMaybe"	$\underline{lim}$	:	$X \to (X \to Y) \to Maybe\ Y$
Version "limSloppy"	lim	:	$X \to (X \to Y) \to Y$

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- $\underline{lim}$ can be used as a partial function $\underline{lim}$, or lim:

$$\forall\ a, f, L_1, L_2.\ \ (\underline{lim}\ a\ f\ \ L_1 \wedge \underline{lim}\ a\ f\ \ L_2) \Rightarrow L_1 == L_2$$

$$\forall\ a, f, L.\ \ (\underline{lim}\ a\ f\ \ L) \Rightarrow (\underline{lim}\ a\ f = Just\ L) \wedge (lim\ a\ f = L)$$

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$$lim\ a\ (f \oplus g) = lim\ a\ f + lim\ a\ g$$
$$lim\ a\ (c \triangleleft f)\ \ = c * (lim\ a\ f)$$

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\mathit{lim}\ a\ (f \oplus g) & = \mathit{lim}\ a\ f + \mathit{lim}\ a\ g \\
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- What is $(\oplus) : (X \to Y) \to (X \to Y) \to (X \to Y)$?

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- What is $(\oplus) : (X \to Y) \to (X \to Y) \to (X \to Y)$?

$$f \oplus g = \lambda x \to f\ x + g\ x$$

Example 2: derivative

*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}; \qquad g :: \quad H \to Y; \text{type } H = \mathbb{R}^{+}$$

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$$D\ f\ x = lim\ 0\ (\varphi\ x)\ \text{where} \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}; \quad \varphi :: X \to (H \to Y)$$

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$$D\ f \quad = \mathit{lim}\ 0 \circ \psi\ f \text{ where } \psi\ f\ \ x\ \ h = \tfrac{f\ (x+h) - f\ x}{h};$$

$$\begin{tikzcd} & X \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\ Y & (H \to Y) \arrow[l, "\mathit{lim}\ 0"] \end{tikzcd}$$

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&sq\ x \qquad = x\hat{}2 \\
&double\ x = 2 * x \\
&c_2\ x \qquad = 2 \\
&sq' \ == D\ sq\ == D\ (\lambda x \to x\hat{}2) == D\ (\hat{}2) == (2*) == double \\
&sq'' == D\ sq' == D\ double == c_2 == const\ 2
\end{aligned}
$$

Note: we cannot *implement* D (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of f to apply rules from calculus.