

Power series and ODEs

Patrik Jansson

Ordinary Differential Equation (ODE)

Examples:

$$f(x) + f'(x) = 1, \quad f(0) = 0$$

$$e(x) - e'(x) = 0, \quad e(0) = 1$$

$$s(x) + s''(x) = 0, \quad s(0) = 0, s'(0) = 1$$

Power series and ODEs:

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$,

Then $f'(x) = D\left(\sum_{i=0}^{\infty} a_i \cdot x^i\right) x = \sum_{i=0}^{\infty} i \cdot a_i \cdot x^{i-1}$

$$f(0) = a_0 = 0$$

$$\text{LHS} = (\underbrace{a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + \dots}_{+}) + \\ (\underbrace{0 \cdot a_1 + 2 \cdot a_2 \cdot x + 3 \cdot a_3 \cdot x^2 + \dots}_{+})$$

$$\text{RHS} = 1 + 0 \cdot x + \dots$$

$a_0 = 0$	$a_1 + 2a_2 = 0$	$a_3 = \dots$
$a_1 = 1$	$a_2 = -1/2$	

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Then $f'(x) = \sum_{i=1}^{\infty} i \cdot a_i \cdot x^{i-1}$
 $f(0) = a_0 = 0$

$$f(x) \approx 0 + x \cdot \left(-\frac{1}{2}x + \frac{x^2}{6} - \frac{x^3}{24} \right)$$

$$\text{LHS} = (a_0 + 1 \cdot a_1) + (a_1 + 2 \cdot a_2) \cdot x + (a_2 + 3 \cdot a_3) \cdot x^2 + \dots$$

$$\text{RHS} = 1 + 0 \cdot x + 0 \cdot x^2 + \dots$$

$$a_1 = 1 \wedge a_2 = -\frac{1}{2} \wedge a_3 = \frac{1}{2 \cdot 3} \wedge a_4 = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \wedge \dots$$

Power series and ODEs:

Transform

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Example: $f(x) + f'(x) = 1, f(0) = 0$

Idea: solve using power series. $f = \text{eval as}$

$$\left\{ \begin{array}{l} as + as' = 1 : 0 : 0 : \dots \\ as = \text{integ}(f(0)) \end{array} \right.$$

$f' = \text{eval as}'$

$$as' = f(0) : \text{zipWith}(1) as'[1..]$$

$$as = 0 : 1 : -\frac{1}{2} : \frac{1}{2 \cdot 3} : -\frac{1}{2 \cdot 3 \cdot 4} : \frac{1}{5!} : -\frac{1}{6!} : \dots$$

$$as' = 1 : -1 : \frac{1}{2} : -\frac{1}{6} : \dots$$

$$rhs = 1 : 0 : 0 : 0 : \dots$$

Power series and ODEs:

Transform

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. $f = \text{eval as}$

$$\left\{ \begin{array}{l} as + as' = 1:0:0:\dots \\ as = \text{integ}(f0) \end{array} \right.$$

$f' = \text{eval as}'$

$$as' = [f0]: \text{zipWith}(\lambda) as'[1..]$$

$$as = 0$$

$$as' =$$

$$rhs = 1$$

Power series and ODEs:

Transform

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. $f = \text{eval as}$

$$\left\{ \begin{array}{l} as + as' = 1 : 0 : 0 : \dots \\ as = \text{integ}(f(0)) \end{array} \right.$$

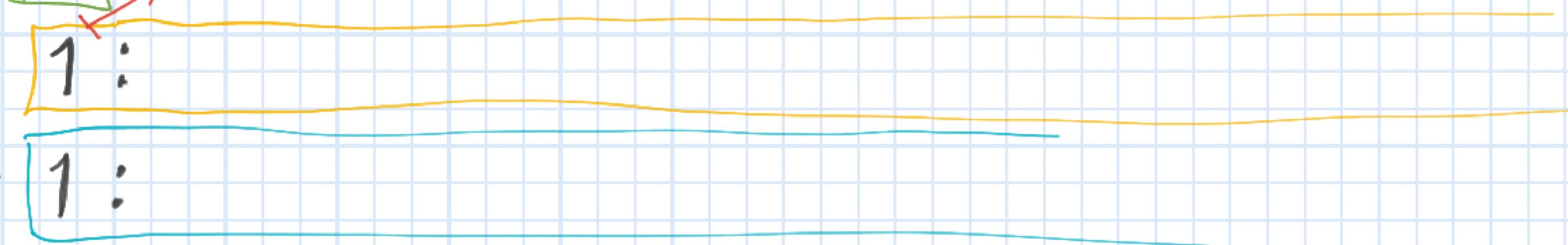
$f' = \text{eval as}'$

$as' = f(0) : \text{zipWith}(\lambda) as'[1..]$

$$as = 0 :$$

$$as' = 1 :$$

$$rhs = 1 :$$



Power series and ODEs:

Transform

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Example:

$$f(x) + f'(x) = 1$$

$$f(0) = 0$$

Idea: solve using power series.

$f = \text{eval as}$

$f' = \text{eval as}'$

$$\left\{ \begin{array}{l} as + as' = 1 : 0 : 0 : \dots \\ as = \text{integ}(f(0)) \end{array} \right.$$

$as' = f(0) : \text{zipWith}(\lambda) as'[1..]$

$$as = 0 : 1/1 : -\frac{1}{2}$$

$$as' = 1 : -1 :$$

$$rhs = 1 : 0 : 0$$

Power series and ODEs:

Transform

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. $f = \text{eval as}$

$$\left\{ \begin{array}{l} as + as' = 1:0:0:\dots \\ as = \text{integ}(f0) \quad as' = f0 \end{array} \right.$$

$f' = \text{eval as}'$

$f' = \text{eval as}'$: $\text{zipWith} (1) as'[1..]$

$$as = 0:1/1:-\frac{1}{2}:\frac{1}{6}:-\frac{1}{24}:$$

$$as' = 1:-1:1/2:-1/6:$$

$$rhs = 1:0:0:0:$$

Power series and ODEs:

Transform

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$as = \text{integ}(f0)$ $as' = f0 : \text{zipWith}(\lambda) as'[1..]$

Background:

$$I : R \rightarrow (R \rightarrow R) \rightarrow (R \rightarrow R)$$

$$I^c g^x = c + \int_0^x g = c + \int_0^x g(t) dt$$

$$\text{Law 3: } D(I^c g) = g$$

$$I(f0)(Df) = f$$

Semantics

$f = \text{eval } as$

$f' = \text{eval } as'$

$$\text{deriv(integ } c \text{ as)} = as$$

$$\boxed{\text{integ (head as)} as' = as}$$

Syntax

Power series and ODEs:

Transform

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Transform example: $f' + 2 \cdot f = r$, $f(0) = c_0$
let $f = \text{eval as}$; $f' = \text{eval as}'$; $r = \text{eval rhs}$

in $\begin{cases} \underline{as' = \text{rhs} - \text{scale } 2 \text{ as}} \\ \underline{as = \text{integ } c_0 \text{ as}'} \end{cases}$

Then solve the system starting from c_0

$$\begin{matrix} as &= & c_0 & : & : & : \\ as' &= & : & \downarrow & \nearrow & \downarrow & \nearrow \end{matrix}$$

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Examples: $e(x) - e'(x) = 0, \quad e(0) = 1$

$$\Rightarrow \begin{cases} as' = as \\ as = \int 1 \, as' \end{cases}$$

$$\Rightarrow as = \int 1 \, as$$

$$as = 1 : 1 : \frac{1}{2} : \frac{1}{2 \cdot 3} : \frac{1}{2 \cdot 3 \cdot 4} :$$

$$f(x) = \text{eval as } x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$= \exp(x)$$

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Examples: $e(x) - e'(x) = 0, \quad e(0) = 1$

$$\Rightarrow \begin{cases} a_s' = a_s \\ a_s = \int 1 \, a_s' \end{cases} \Rightarrow a_s = \int 1 \, a_s'$$

$$a_s = 1 : \frac{1}{1} : \frac{1}{2} : \frac{1}{6} : \frac{1}{24} : \dots : \underbrace{\frac{1}{n!}}_{\text{:}}$$

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Examples: $s(x) + s''(x) = 0, s(0) = 0, s'(0) = 1$

$$\left\{ \begin{array}{l} as'' = -as \\ as' = \text{integ } 1 as'' \\ as = \text{integ } 0 as' \end{array} \right.$$

$$\left\{ \begin{array}{l} as'' = -0 : \\ as' = 1 : \frac{0}{1} \\ as = 0 : \frac{1}{1} : \frac{0}{2} \end{array} \right.$$

DSL $\rightarrow \delta\sigma\lambda$
DSL_{Math}

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Examples: $s(x) + s''(x) = 0, s(0) = 0, s'(0) = 1$

$$\left\{ \begin{array}{l} as'' = -as \\ as' = \underline{\text{integ } 1 \ as''} \\ as = \underline{\text{integ } 0 \ as'} \end{array} \right.$$

$$\left\{ \begin{array}{l} as'' = 0 : -1 : \\ as' = 1 : 0 : \\ as = 0 : 1 : 0 \end{array} \right. \quad \begin{array}{l} (11) \\ (12) \\ (13) \\ (14) \end{array}$$

DSL $\rightarrow \delta\sigma^\lambda$
DSL_{Math}

Power series and ODEs

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Examples: $s(x) + s''(x) = 0, s(0) = 0, s'(0) = 1$

$$\left\{ \begin{array}{l} as'' = -as \\ as' = \text{integ } 1 \text{ as''} \\ as = \text{integ } 0 \text{ as'} \end{array} \right.$$

$$s(x) = \text{eval as } x$$

$$\approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\left\{ \begin{array}{l} as'' = 0 : -1 : 0 : \frac{1}{6} : 0 \\ as' = 1 : 0 : -\frac{1}{2} : 0 : \frac{1}{24} : \\ as = 0 : 1 : 0 : -\frac{1}{6} : 0 : \frac{1}{120} : \end{array} \right.$$

(1) (2) (3) (4) (5)

DSL \rightarrow $\delta\sigma^\lambda$
DSL_{Math}

Taylor & MacLaurin

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$$f(x) \approx f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \dots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \dots$$

"Taylor series at 0" or "MacLaurin series"

$$f \xrightarrow{\text{DAll}} [f, f', f'', \dots, f^{(n)}, \dots] : \text{DS } (\mathbb{R} \rightarrow \mathbb{R})$$

$\downarrow \text{map (apply 0)}$

$$[f(0), f'(0), f''(0), \dots, f^{(n)}(0), \dots] : \text{DS R}$$

$\downarrow \backslash \text{ds} \rightarrow \text{zipWith } (/) \text{ ds factorials}$

$$[f(0), f'(0), \frac{f''(0)}{2}, \dots, f^{(n)}(0)/n!, \dots] : \text{PS R}$$

DSL $\rightarrow \lambda$
DSLs of Math

Taylor & MacLaurin

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$$f(x) \approx f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \dots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \dots$$

"Taylor series at 0" or "MacLaurin series"

$$f \xrightarrow{\text{DAll}} [f, f', f'', \dots, f^{(n)}, \dots] : DS(\mathbb{R} \rightarrow \mathbb{R})$$

$$\text{DAll} : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow DS(\mathbb{R} \rightarrow \mathbb{R})$$

$$DS_a = [a]$$

$$\begin{matrix} \text{eval} \uparrow & & \uparrow \text{map eval} \\ & & \end{matrix}$$

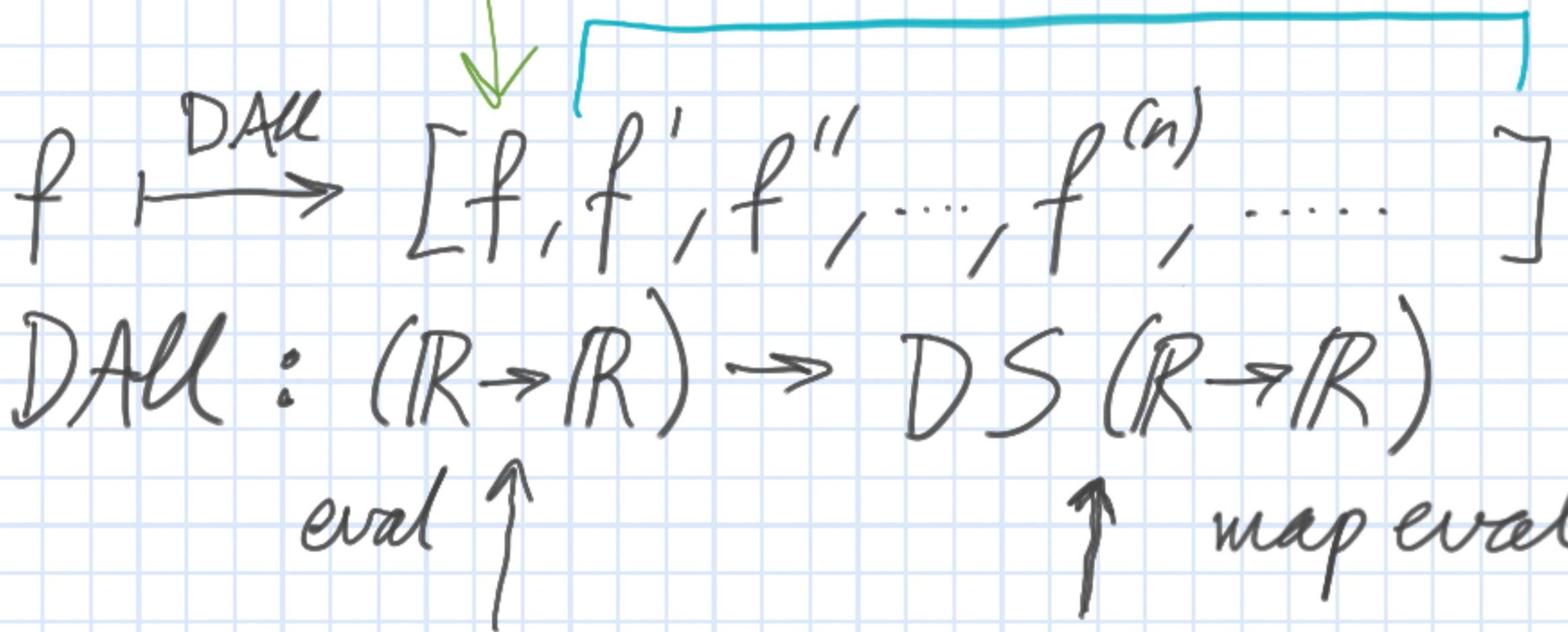
$$\text{derAll} : \text{FunExp} \rightarrow DS \text{FunExp}$$

Taylor & MacLaurin

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Note: $DAll(Df) = tail(DAll f)$
 $f = head(DAll f)$

$H1(DerAll, D, tail)$



$DAll : (R \rightarrow R) \rightarrow DS(R \rightarrow R)$

$derAll : FunExp \rightarrow DS \bar{FunExp}$

Taylor & MacLaurin

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Note: $DAll(Df) = tail(DAll f)$
 $f = head(DAll f)$

Examples: $DAll id = id : \underline{\lambda \rightarrow 1 : zero} :: DS(R \rightarrow R)$

$DAll sin = \sin : \cos : -\sin : -\cos : \dots$

$DAll exp = \exp : \exp : \exp : \dots$

Taylor & MacLaurin

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Note: $DAll(Df) = tail(DAll f)$
 $f = head(DAll f)$

Examples: $DAll id = id : _ \rightarrow 1 : zero$
 $\quad \quad \quad 0 : 1 : 0 : 0 : \dots$

$DAll sin = sin : cos : -sin : -cos :$
 $\quad \quad \quad 0 : 1 : 0 : -1 : \dots$

Potens-
serie

$$\frac{1}{1-x} \approx 1:1:1:\dots$$

$DAll exp = exp : exp : exp : \dots$
 $\quad \quad \quad 1 : 1 : 1 : 1 : \dots$