Power series and ODEs

Patrik Jansson

Ordinary Differential Equation (ODE)

Examples:

$$f(x) + f'(x) = 1 \quad , \quad f(0) = 0$$

$$e(x) - e'(x) = 0 \quad , \quad e(0) = 1$$

$$s(x) + s''(x) = 0 \quad , \quad s(0) = 0 , s'(0) = 1$$

Power series and ODEs:

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Example : $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Then $f'(x) =$

$f(0) =$

$$LHS = (a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots) + (\quad)$$

$$RHS = 1 + 0 \cdot x + \cdots$$

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Example: $f(x) + f'(x) = 1 \quad , f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Then $f'(x) = \sum_{i=1}^{\infty} i \cdot a_i \cdot x^{i-1}$ $\qquad f(x) \approx 0 + x \cdot \left(1 - \frac{1}{2}x + \frac{x^2}{6} - \frac{x^3}{24}\right)$

$f(0) = a_0 = 0$

$LHS = (a_0 + 1 \cdot a_1) + (a_1 + 2 \cdot a_2) \cdot x + (a_2 + 3 \cdot a_3) \cdot x^2 + \cdots$

$RHS = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$

$a_1 = 1 \wedge a_2 = -\frac{1}{2} \wedge a_3 = \frac{1}{2 \cdot 3} \wedge a_4 = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \wedge \cdots$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval } as$

$f' = \text{eval } as'$

$$\begin{cases} as + as' = 1:0:0:\ldots\ldots \\ as = \text{integ } (f\,0)\ as' = \boxed{f\,0} : \text{zipWith } (/)\ as'\ [1..] \end{cases}$$

$as = \boxed{0} :$

$as' = \quad :$

$rhs = \boxed{1 \quad :}$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval } as$

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$$\begin{cases} \boxed{as + as' = 1:0:0:\ldots} \\ as = \text{integ } (f\,0)\ as' = \boxed{f\,0} : \underline{\text{zipWith } (/)\ as'\ [1..]} \end{cases}$$

$as = \boxed{0} :$

$as' = \boxed{1} :$

$rhs = \boxed{1} :$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

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$as = \boxed{0} : 1/1 : \frac{-1}{2}$

$as' = 1 : -1 :$

$rhs = 1 : 0 : 0$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval } as$

$f' = \text{eval } as'$

$$\begin{cases} \boxed{as + as' = 1:0:0:\ldots} \\ as = \text{integ } (f0)\ as' = \boxed{f0} : \text{zipWith } (/)\ as'\ [1..] \end{cases}$$

$as = \boxed{0} : 1/1 : \frac{-1}{2} : \frac{1}{6} : \frac{-1}{24}$

$as' = 1 : -1 : 1/2 : -1/6 :$

$rhs = 1 : 0 : 0 : 0 :$

Power series and ODEs: Transform

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as = integ (f 0) as' = f 0 : zipWith (/) as' [1..]

Background:

f = eval as

f' = eval as'

$I : \mathbb{R} \to (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$

$I\ c\ g\ x = c + \int_0^x g = c + \int_0^x g(t)\,dt$

Laws: $D\ (I\ c\ g) = g$

deriv (integ c as) = as

$I\ (f\ 0)\ (D\ f) = f$

$\boxed{\text{integ (head as) as' = as}}$

Semantics

Syntax

Transform example: $f' + 2 \cdot f = r \quad , f(0) = c_0$

let $f = \text{eval } as$; $f' = \text{eval } as'$; $r = \text{eval } rhs$

in $\begin{cases} as' = rhs - \text{scale } 2 \ as \\ as = \text{integ } c_0 \ as' \end{cases}$

Then solve the system starting from c_0

$as = c_0 : \quad : \quad :$

$as' = \quad : \quad : \quad :$

$rhs = \quad : \quad : \quad :$

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Examples: $e(x) - e'(x) = 0$, $e(0) = 1$

$$\Rightarrow \begin{cases} as' = as \\ as = \text{integ } 1 \; as' \end{cases}$$

$$\Rightarrow \quad as = \text{integ } 1 \; as$$

$$as = 1 : \quad : \quad : \quad : \quad :$$

$$e(x) = \text{eval } as \; x = 1 + \quad + \quad + \quad + \quad + \cdots$$

$$=$$

Power series and ODEs

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Examples: $e(x) - e'(x) = 0 \;, \quad e(0) = 1$

$$\Rightarrow \begin{cases} as' = as \\ as = \text{integ } 1 \; as' \end{cases} \Rightarrow as = \text{integ } 1 \; as$$

$$as = 1 : \frac{1}{1} : \frac{1}{2} : \frac{1}{6} : \frac{1}{24} : \ldots\ldots : \frac{1}{n!} :$$

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Examples: $s(x) + s''(x) = 0\ ,\ s(0) = 0,\ s'(0) = 1$

$$\begin{cases} as'' = -\,as \\ as' = \text{integ}\ 1\ as'' \\ as = \text{integ}\ 0\ as' \end{cases}$$

$$\begin{cases} as'' = \\ as' = \\ as = \end{cases}$$

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Examples: $s(x) + s''(x) = 0\ ,\ s(0) = 0, s'(0) = 1$

$$\begin{cases} as'' = -\,as \\ as' = \underline{integ\ 1\ as''} \\ as = \underline{integ\ 0\ as'} \end{cases}$$

$$\begin{cases} as'' = 0 : -1 : \\ as' = 1 : 0 : \\ as = 0 : 1 : 0 \end{cases}$$

$$(/1) \quad (/2) \quad (/3) \quad (/4)$$

Power series and ODEs

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Examples: $s(x) + s''(x) = 0\ ,\ s(0) = 0, s'(0) = 1$

$$\begin{cases} as'' = -\,as \\ as' = \underline{\text{integ } 1\ as''} \\ as = \underline{\text{integ } 0\ as'} \end{cases}$$

$s(x) = \text{eval } as\ x$

$\approx x - \frac{x^3}{6} + \frac{x^5}{120}$

$$\begin{cases} as'' = 0 : -1 : 0 : \frac{1}{6} : 0 \\ as' = 1 : 0 : -\frac{1}{2} : 0 : \frac{1}{24} : \\ as = 0 : 1 : 0 : -\frac{1}{6} : 0 : \frac{1}{120} \end{cases}$$

$(/1)\quad (/2)\quad (/3)\quad (/4)\quad (/5)$

Taylor & Maclaurin

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$$f\ x \approx f\,0 + f'\,0 \cdot x + \frac{f''\,0}{2} \cdot x^2 + \cdots + \frac{f^{(n)}\,0}{n!} \cdot x^n + \cdots$$

"Taylor series at 0" or "Maclaurin series"

$$f \xmapsto{\text{DAll}} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots] \quad : \mathrm{DS}\,(\mathbb{R}\to\mathbb{R})$$

$$\downarrow \text{map (apply 0)}$$

$$[f\,0, f'\,0, f''\,0, \ldots, f^{(n)}\,0, \ldots] \quad : \mathrm{DS}\,\mathbb{R}$$

$$\downarrow \backslash ds \to \text{zipWith } (/)\ ds\ \text{factorials}$$

$$[f\,0, f'\,0, \frac{f''\,0}{2}, \quad , f^{(n)}\,0/n!, \ldots] \quad : \mathrm{PS}\,\mathbb{R}$$

Taylor & Maclaurin

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$$f\ x \simeq f\ 0 + f'\ 0 \cdot x + \frac{f''\ 0}{2} \cdot x^2 + \cdots + \frac{f^{(n)}\ 0}{n!} \cdot x^n + \cdots$$

"Taylor series at 0" or "Maclaurin series"

$$f \xmapsto{DAll} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots] \quad : DS\ (\mathbb{R}\to\mathbb{R})$$

$$DAll : (\mathbb{R}\to\mathbb{R}) \to DS\ (\mathbb{R}\to\mathbb{R}) \qquad DS\ a \overset{\sim}{=} [a]$$

$$\text{eval} \uparrow \qquad\qquad \uparrow \text{map eval}$$

$$derAll : FunExp \to DS\ FunExp$$

Taylor & Maclaurin

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Note: $\mathrm{DAll}\ (D f) = \mathrm{tail}\ (\mathrm{DAll}\ f)$

$f = \mathrm{head}\ (\mathrm{DAll}\ f)$

$H_1(\mathrm{DAll}, D, \mathrm{tail})$

$$f \xmapsto{\mathrm{DAll}} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots] \quad : \mathrm{DS}\ (\mathbb{R}\to\mathbb{R})$$

$$\mathrm{DAll} : (\mathbb{R}\to\mathbb{R}) \to \mathrm{DS}\ (\mathbb{R}\to\mathbb{R}) \qquad \mathrm{DS}\ a = [a]$$

$$\begin{array}{ccc} & \uparrow \mathrm{eval} & & \uparrow \mathrm{map\ eval} \end{array}$$

$$\mathrm{derAll} : \mathrm{FunExp} \to \mathrm{DS}\ \mathrm{FunExp}$$

Taylor & Maclaurin

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Note: $\text{DAll}\ (D\,f) = \text{tail}\ (\text{DAll}\ f)$

$f = \text{head}\ (\text{DAll}\ f)$

Examples: $\text{DAll id} = \text{id} : \backslash\_ \to 1 : \text{zero}$

$:: DS\ (\mathbb{R} \to \mathbb{R})$

$\text{DAll sin} = \text{sin} : \text{cos} : -\text{sin} : -\text{cos} :$

$\text{DAll exp} = \text{exp} : \text{exp} : \text{exp} : \ldots\ldots$

Taylor & Maclaurin

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Note:  DAll (D f) = tail (DAll f)
              f   = head (DAll f)
                     map (apply 0)

Examples:  DAll id  = id  : \_ -> 1 : zero
                      0   : 1       : 0    : 0 : 0 : ....

           DAll sin = sin : cos : -sin : -cos :
                      0   : 1   : 0    : -1   :

           DAll exp = exp : exp : exp : ......
                      1   : 1   : 1   : 1 :
```

$$\frac{1}{1-x} \approx 1:1:1::$$