Laplace transform

Patrik Jansson

$L f s = I\,(f \cdot g_s)\,\infty$

$$= \int_0^{x} f(t) \cdot e^{-s \cdot t}\, dt$$

$g_s\, t = e^{-st}$

Assume $f x \cdot g_s\, x \to 0$ as $x \to \infty$

$V \cong \mathbb{R} \to \mathbb{R}$

$W \cong \mathbb{C} \to \mathbb{C}$

$L\,(D f)\, s = -f\,0 + s \cdot L\, f\, s$

$L : \quad V \to W$

$\mathit{LinTrans}\,(L, V, W)$ (exercise!)

Laplace transform

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$$L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L (D f) s = - f 0 + s \cdot L f s$$

Compute L for some example "vectors": exp, sin, cos

We know $\boxed{D \exp = \exp} \wedge \exp 0 = 1$

$$L (D \exp) s = - \exp 0 + s \cdot L \exp s$$

$$\underbrace{L \exp s}_{k} = -1 + s \cdot \underbrace{L \exp s}_{k}$$

$$k_s = -1 + s \cdot k \qquad (s-1) \cdot k_s = 1$$

$$k_s = \frac{1}{s-1}$$

Laplace transform

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$\cdot \ L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt$

$L (D f) s = - f 0 + s \cdot L f s$

Compute L for some example "vectors": exp, sin, cos

We know $D \exp = \exp \ \wedge \ \exp 0 = 1$

$$L (D \exp) s = L \exp s = - \exp 0 + s \cdot L \exp s$$

$$(1 - s) \cdot L \exp s = -1$$

$$L \exp s = \frac{1}{s-1}$$

$$L \exp = \backslash s \to \frac{1}{s-1}$$

Laplace transform

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$$L\,f\,s = \int_0^\infty f(t)\cdot e^{-s\cdot t}\,dt \qquad L\,(D f)\,s = -f\,0 + s\cdot L\,f\,s$$

Compute L for some example "vectors": exp, sin, cos

$$Df = D\,(\backslash t \to e^{\alpha t}) = \backslash t \to \alpha\cdot e^{\alpha\cdot t} = \text{scale}\,\alpha\,(\backslash t \to e^{\alpha\cdot t}) = \alpha \odot f$$

$$L\,(Df)\,s = L\,(\alpha\odot f)\,s = \alpha\cdot L\,f\,s = -f\,0 + s\cdot L\,f\,s = -1 + s\cdot L\,f\,s$$

$$(\alpha - s)\cdot L\,f\,s = -1 \qquad L\,(\backslash t \to e^{0\cdot t})\,s = \frac{1}{s}$$

$$L\,f\,s = \frac{1}{s-\alpha} \qquad L\ \text{one}\ s$$

$$\text{one} = \text{const}\,1$$

Laplace transform

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$$\cdot \; L f s = \int_0^x f(t) \cdot e^{-s \cdot t} \, dt \qquad L(Df)s = -f0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f0 = 0$, $f'0 = 1$

Laplace transform

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$$L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L(Df)s = -f0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f0 = 0$, $f'0 = 1$

$$L(f'' + 2 \cdot f)s = L(3 \cdot f')s$$

$$\underline{L f'' s} + 2 \cdot L f s = 3 \cdot \underline{L f' s}$$

Laplace transform

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$$Lfs = \int_0^x f(t)\cdot e^{-s\cdot t}\,dt \qquad L(Df)s = -f0 + s\cdot Lfs$$

$$f'' = Df'$$

Solve $f'' + 2\cdot f = 3\cdot f'$, $f0 = 0$, $f'0 = 1$

$$L(f'' + 2\cdot f)s = L(3\cdot f')s$$

$$\boxed{Lf's = -0 + s\cdot Lfs}$$

$$Lf''s + 2\cdot Lfs = 3\cdot Lf's$$

$$\|$$

$$\boxed{-f'0 + s\cdot Lf's = -1 + s^2\cdot Lfs}$$

Laplace transform

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$$Lfs = \int_0^{\infty} f(t) \cdot e^{-s \cdot t}\, dt \qquad L(Df)s = -f0 + s \cdot Lfs$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f0 = 0$, $f'0 = 1$

$$L(f'' + 2 \cdot f)s = L(3 \cdot f')s$$

$$\boxed{Lf's = -0 + s \cdot Lfs}$$

$$Lf''s + 2 \cdot Lfs = 3 \cdot \underline{Lf's}$$

let $F = Lf$

$$-1 + s^2 \cdot Fs + 2 \cdot Fs - 3 \cdot s \cdot Fs = 0$$

$$(s^2 - 3 \cdot s + 2) \cdot Fs = 1$$

$$Fs = \frac{1}{(s-1) \cdot (s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Ansatz

Laplace transform

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$$\mathcal{L} f s = \int_0^{\infty} f(t)\cdot e^{-s\cdot t}\, dt \qquad \mathcal{L}(Df)s = -f0 + s\cdot \mathcal{L} f s$$

Solve $f'' + 2\cdot f = 3\cdot f'$, $f0 = 0$, $f'0 = 1$, $F = \mathcal{L} f$

$$Fs = \frac{1}{(s-1)\cdot(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = -\frac{1}{s-1} + \frac{1}{s-2}$$

$$1 = A\cdot(s-2) + B\cdot(s-1)$$

$s=1$: $1 = A\cdot(1-2) + \cancel{B\cdot(1-1)} = -A$

$s=2$: $1 = B\cdot(2-1) = B$

$A = -1$, $B = 1$

$$\Downarrow$$

$$f t = -e^{t} + e^{2\cdot t}$$

$$f' t = -e^{t} + 2\cdot e^{2\cdot t}$$

$$f'' t = -e^{t} + 4\cdot e^{2\cdot t}$$

Laplace transform

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$$\cdot\ L f s = \int_0^{\infty} f(t) \cdot e^{-s \cdot t}\, dt \qquad L\,(D f)\, s = -f\,0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f\,0 = 0$, $f'\,0 = 1$, $F = L f$

$$F s = \frac{1}{(s-1)\cdot(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = -\frac{1}{s-1} + \frac{1}{s-2}$$

$$f\,t = -e^{t} + e^{2 \cdot t}$$

$$f'\,t = -e^{t} + 2 \cdot e^{2 \cdot t}$$

$$f''\,t = -e^{t} + 4 \cdot e^{2 \cdot t}$$

$$LHS = (f'' + 2 \cdot f)\,t = -e^{t} + 4e^{2t} - 2e^{t} + 2e^{2t}$$

$$RHS = 3 \cdot f'\,t = -3e^{t} + 6e^{2t}$$

$$f\,0 =$$

$$f'\,0 =$$

Laplace summary

$$L : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{C} \to \mathbb{C})$$

$$L\, f\, s = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

$$\mathrm{LinTran}\,(L)$$

$$L\,(\lambda t \to e^{\alpha t})\, s = \frac{1}{s-\alpha}$$

$$L\, f'\, s = -f\, 0 + s \cdot L\, f\, s$$

$$L\,(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

Solve ODE by:

- transform
- solve rational expr.
- partial fraction decomposition
- transform back by "pattern matching"

$\boxed{\text{+ Check}}$

Laplace sin & cos

$$\begin{cases} D\sin = \cos & , \sin 0 = 0 \\ D\cos = -\sin & , \cos 0 = 1 \end{cases}$$

$$L(\lambda t \to e^{\alpha t})s = \frac{1}{s-\alpha}$$

$$L f' s = -f0 + s \cdot L f s$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$\begin{cases} L \sin' s = -\sin 0 + s \cdot L \sin s \\ L \cos' s = -\cos 0 + s \cdot L \cos s \end{cases}$$

let $S = L\sin$

$C = L\cos$

$$L\cos s = s \cdot L \sin s$$

$$-L \sin s = -1 + s \cdot L \cos s$$

$$S s = \frac{1}{s^2+1}$$

$$\begin{cases} C s = s \cdot S s \\ S s = 1 - s \cdot C s \end{cases}$$

$$S s = 1 - s \cdot s \cdot S s$$

$$(1+s^2) \cdot S s = 1$$

$$C s = \frac{s}{s^2+1}$$

Laplace sin & cos

$$D \sin = \cos \quad , \sin 0 = 0$$

$$D \cos = -\sin \quad , \cos 0 = 1$$

let $S = L \sin$, $C = L \cos$

$$L(\lambda t \mapsto e^{\alpha t})s = \frac{1}{s-\alpha}$$

$$L f' s = -f 0 + s \cdot L f s$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$\begin{cases} L \sin' s = -\sin 0 + s \cdot L \sin s \\ L \cos' s = -\cos 0 + s \cdot L \cos s \end{cases}$$

$$\begin{cases} C s = s \cdot S s \\ -S s = -1 + s \cdot C s \end{cases}$$

Laplace sin & cos

$D\sin = \cos \quad , \sin 0 = 0$

$D\cos = -\sin \quad , \cos 0 = 1$

let $S = L\sin$, $C = L\cos$

$$L(\lambda t \to e^{\alpha t})s = \frac{1}{s-\alpha}$$

$$L f' s = -f0 + s \cdot L f s$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$\begin{cases} L \sin' s = -\sin 0 + s \cdot L \sin s \\ L \cos' s = -\cos 0 + s \cdot L \cos s \end{cases}$$

$$\begin{cases} C s = s \cdot S s \\ -S s = -1 + s \cdot C s = -1 + s^2 \cdot S s \end{cases}$$

$$S s = \frac{1}{s^2+1} \quad , \quad C s = \frac{s}{s^2+1} = \frac{s}{(s-i)\cdot(s+i)} = \frac{A}{s-i} + \frac{B}{s+i}$$

$A \cdot L(\lambda t \to e^{it})s$

$\|$ $\quad B \cdot L(\lambda t \to e^{-it})s$

$\|$

Laplace foscaleα

$$L\,(\backslash t \to f(\alpha \cdot t))\,s = ?$$

$$\underbrace{\backslash t \to f(\alpha \cdot t)}_{g}$$

$$Dg = \alpha \cdot Df \quad , \quad g0 = f0$$

Assume $F = Lf$

then $Gs = \frac{1}{\alpha} \cdot F\left(\frac{s}{\alpha}\right)$

$$L\,(\backslash t \to e^{\alpha t})\,s = \frac{1}{s-\alpha}$$

$$Lf's = -f0 + s \cdot Lfs$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot Lf + \beta \cdot Lg$$

$$L \sin s = 1/(s^2+1)$$

$$L \cos s = s/(s^2+1)$$

Laplace f∘scale α

$$\mathcal{L}\,\underbrace{(\backslash t \to f(\alpha\cdot t))}_{g}\,s = \,?$$

$$D\,g\,t = \alpha \cdot D\,f\,t$$

$$D\,g = \text{scale}\,\alpha\,(D\,f)$$

$$g = f \circ \text{scale}\,\alpha$$

$$\vdots$$

$$\mathcal{L}\,g\,s = \frac{1}{\alpha}\cdot \mathcal{L}\,f\left(\frac{s}{\alpha}\right)$$

$$\mathcal{L}\,(\backslash t \to e^{\alpha t})\,s = \frac{1}{s-\alpha}$$

$$\mathcal{L}\,f'\,s = -f\,0 + s\cdot \mathcal{L}\,f\,s$$

$$\mathcal{L}\,(\alpha\cdot f + \beta\cdot g) = \alpha\cdot \mathcal{L}\,f + \beta\cdot \mathcal{L}\,g$$

$$\mathcal{L}\,\sin\,s = 1/(s^2+1)$$

$$\mathcal{L}\,\cos\,s = s/(s^2+1)$$