Domain-Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Results	Announced within 15 workdays
Exam check	2022-09-09, 12.15-13.00 in EDIT 6452
Aids	One textbook of your choice (Domain-Specific Languages of Mathematics, or Beta - Mathematics Handbook, or Rudin, or Adams and Essex, or ...). No printouts, no lecture notes, no notebooks, etc.
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain-Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure**: Group (lightly edited from the Wikipedia entry)

   A group is a set, G, together with a binary operation on G, here denoted "$\cdot$", that combines any two elements a and b to form another element of G, denoted $a \cdot b$, such that the following three requirements, known as *group axioms* are satisfied:

   - **Associativity**: For all a, b, c in G, one has $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - **Identity element**: There exists an element e in G such that, for every a in G, one has $e \cdot a = a$ and $a \cdot e = a$.
   - **Inverse element**: For each a in G, there exists an element b in G, such that $a \cdot b = e$ and $b \cdot a = e$, where e is the identity element. For each a, the element b is unique; it is called *the inverse* of a and is commonly denoted a^{-1}.

   (a) Define a type class *Group* that corresponds to the group structure.

   (b) Define a datatype $G\ v$ for the language of group expressions (with variables of type v) and define a *Group* instance for it. (These are expressions formed from applying the group operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Group* class.

   (d) Give a type signature for, and define, a general evaluator for $G\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Group* instances defined in (1c). Take three group expressions of type G *String*, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**: Consider the following differential equation:

$$f''(t) = 2 * f(t) - f'(t), \quad f(0) = a, \quad f'(0) = b$$

   where a and b are real numbers.

   (a) [10pts] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs*$'$, and *fs*$''$. What are the first three coefficients of *fs* (expressed in terms of a and b)? Explain how you compute them.

   (b) [15pts] Solve the equation using the Laplace transform. You should need this formula (and the rules for linearity + derivative):

$$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

   Show that your solution does indeed satisfy the three requirements.

3. [25p] **Typing/proof: polynomials, remainders, and roots**

Consider the following mathematical text from Adams and Essex:

> If A_m and B_n are polynomials having degrees m and n, respectively, and if $m > n$, then we can express the rational function A_m/B_n (in a unique way) as the sum of a quotient polynomial Q_{m-n} of degree $m-n$ and another rational function R_k/B_n where the numerator polynomial R_k (the remainder in the division) is either zero or has degree $k < n$:
>
> $$\frac{A_m(x)}{B_n(x)} = Q_{m-n}(x) + \frac{R_k(x)}{B_n(x)} \quad \text{(The Division Algorithm)}$$
>
> ...
>
> A number r is called a **root** of the polynomial P if $P(r) = 0$.
>
> ...
>
> In our study of calculus we will often find it useful to factor polynomials into products of polynomials of lower degree, especially degree 1 or 2 (linear or quadratic polynomials).
>
> **The Factor Theorem** The number r is a root of the polynomial P of degree not less than 1 if and only if $x - r$ is a factor of $P(x)$.

(a) [7p] Give, and explain, the types of A, m, B, n, Q, x, R, k, P, r.

(b) [8p] Compute $Q(x) = q_0 + q_1 * x + q_2 * x^2$ and $R(x) = r_0$ if $A(x) = x^3 - 4x$ and $B(x) = x - a$ for some arbitrary real number a. Check that for a being -2, 0, and 2 the polynomial R is zero.

(c) [10p] Assume there is a function $\mathit{roots} :: \mathit{Poly}^{\geq 1}\ \mathbb{R} \to [\mathbb{R}]$ which computes all the real roots of a polynomial (of degree at least 1) in increasing order. Prove that $\exists\ \mathit{mul}.\ H_2\ (\mathit{roots}, (*), \mathit{mul})$, where $(*)$ is multiplication of polynomials. That is, give mul a type and a definition, and prove that roots is a homomorphism from polynomial multiplication to mul.

4. [25p] **Type/proof: differentiability in terms of continuity**

Consider the following text (adapted from [Pickert, 1969])

> **Definition 1** *Let $f : X \to \mathbb{R}$ and $a : X$. If there exists a function $\phi_f : X \to X \to \mathbb{R}$ such that, for all $x : X$*
>
> $$f\ x = f\ a + (x - a) * \phi_f\ a\ x$$
>
> *such that $\phi_f\ a : X \to \mathbb{R}$ is continuous at a, then f is **differentiable** at a. The value $\phi_f\ a\ a$ is called the **derivative** of f at a and is denoted $f'\ a$.*

(a) [5p] The definition above can be adapted to work for X containing vectors or matrices, when division is not available, but if $X \subseteq \mathbb{R}$ there is a natural candidate for defining ϕ_f — what is it? (Be careful to avoid division by zero.)

(b) [5p] Use the definition above to calculate the derivative of $s\ x = x^2$.

(c) [15p] Assume that f and g are both differentiable at a. Use equational reasoning to show that $f * g$ is also differentiable at a and compute its value. (The function $h = f * g$ is defined as $h\ x = f\ x * g\ x$.)