Fixing "almost" homomorphisms

Patrik Jansson

$$\begin{tikzcd} \mathit{Syn} \arrow[r, "\mathit{eval}"] \arrow[d, "\mathit{derive}"'] \arrow[dr, "\mathit{eval}'"] & \mathit{Sem} \arrow[d, "D"] \\ \mathit{Syn} \arrow[r] & \mathit{Sem} \end{tikzcd}$$

$\mathit{Syn} = \mathit{FunExp}$

$\mathit{Sem} = \mathbb{R} \to \mathbb{R}$

$$\boxed{\text{eval}' \text{ is not a homomorphism}}$$

$$\mathit{eval}' = D \circ \mathit{eval} = \mathit{eval} \circ \mathit{derive}$$

For example: $\neg \exists\, \mathit{mul}.\ H_2(\mathit{eval}', \mathit{Mul}, \mathit{mul})$

Fixing "almost" homomorphisms by tupling

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$$eval : \begin{tikzcd} Syn \arrow[r] \arrow[d, "derive"'] \arrow[dr, "eval'"] & Sem \arrow[d, "D"] \\ Syn \arrow[r] & Sem \end{tikzcd}$$

$Syn = FunExp$

$Sem = \mathbb{R} \to \mathbb{R}$

$eval'$ is not a homomorphism

$$eval' = D \circ eval = eval \circ derive$$

$$evalD\ f = (eval\ f,\ eval'\ f)$$

$$evalD : Syn \to \underbrace{Sem \times Sem}_{\text{a pair type}}$$

$evalD$ is a homomorphism

$$eval' = snd \circ evalD$$

Fixing "almost" homomorphisms : degree

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First try
$$\begin{cases} \text{degree} : \text{Poly} \to \mathbb{N} \\ \text{degree } [\,] = z \\ \text{degree } (a:as) = \text{length } as \end{cases}$$

$\text{Poly} = [R]$ (Without trailing 0:s)

We want $H_2(\text{degree}, \text{mulP}, (+)) = \forall as, bs.$

$$\underbrace{\text{degree}(\text{mulP } as\ bs)}_{\text{degree}([\,]) \,=\, z} == \underbrace{\text{degree } as}_{=\,z} + \underbrace{\text{degree } bs}_{=\,n}$$

let $as = [\,]$

$bs = \underbrace{[0,0,0\ldots,1]}_{n+1}$

$\forall n.\ z = z + n \Rightarrow z = z+1$

Fixing "almost" homomorphisms : degree

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First try
$$\begin{cases} degree : Poly \to \mathbb{N} \\ degree\ [\,] = z \\ degree\ (a:as) = length\ as \end{cases}$$

find

$Poly = [R]$ (Without trailing 0:s)

We want $H_2(degree, mulP, (+)) \equiv$
$\forall as, bs.\ degree\ (mulP\ as\ bs) = degree\ as + degree\ bs$

Fixing "almost" homomorphisms : degree

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First try
$$\begin{cases} degree : Poly \to \mathbb{N} \\ degree\ [\,] = z \\ degree\ (a:as) = length\ as \end{cases}$$

find

$Poly = [R]$ (Without trailing 0:s)

We want $H_2(degree, mulP, (+)) \equiv$

$\forall as, bs.\ degree\ (mulP\ as\ bs) = degree\ as + degree\ bs$

let $as = [\,]$; $bs = x^n$

$$degree\ [\,] = degree\ [\,] + degree\ (x^n)$$
$$z = z + n$$

Contradiction!

Fixing "almost" homomorphisms : deg

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First try $\begin{cases} \text{degree} : \text{Poly} \to \mathbb{N} \\ \text{degree } [\,] = z \\ \text{degree } (a:as) = \text{length } as \end{cases}$

find

let $as = [\,]$;
$bs = x^n$;
in $z = z + n$

$z \stackrel{?}{=} -\infty$

deg : Poly → M ?

data M where
  NegInf : M
  Num : $\mathbb{N} \to$ M

deg [ ] = NegInf
deg as = Num (degree as)

Fixing "almost" homomorphisms : deg

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First try degree : Poly → ℕ
degree (a:as) = length as

let as = [ ];
bs = x^n;
in z = z ⊕ n

z : Mℕ
z = Nothing

deg [ ] = Nothing
deg as = Just (degree as)

deg : Poly → Maybe ℕ

data Maybe a where
Nothing : Maybe a
Just : a → Maybe a

⊕ : Mℕ → Mℕ → Mℕ
⊕ = opMaybe (+)

Maybe ℕ

opMaybe : (a → a → a) → (Maybe a → Maybe a → Maybe a)
opMaybe op (Just x) (Just y) = Just (op x y)
opMaybe _ _ _ = Nothing

Fixing "almost" homomorphisms : Maybe

Patrik Jansson

First try $degree : Poly \to \mathbb{N}$

$degree\ (a:as) = length\ as$

let $as = [\,]$;

$bs = x^n$;

in $z = z \oplus n$

$z : M\mathbb{N}$

$z = Nothing$

$deg\ [\,] = Nothing$

$deg\ as = Just\ (degree\ as)$

$deg : Poly \to Maybe\ \mathbb{N}$

data Maybe a where

Nothing : Maybe a

Just : a → Maybe a

$\oplus : \underbrace{M\mathbb{N}}_{} \to M\mathbb{N} \to M\mathbb{N}$

Maybe $\mathbb{N}$

$\oplus = opMaybe\ (+)$

opMaybe : (a → a → a) → (Maybe a → Maybe a → Maybe a)

opMaybe op (Just x) (Just y) = Just (op x y)

opMaybe op _ _ = Nothing

Fixing "almost" homomorphisms : Maybe

Patrik Jansson

data Maybe a where
  Nothing : Maybe a
  Just : a → Maybe a

Model partial functions

A → B

A → Maybe B

invent a new value

Define operator for new value

opMaybe : (a → a → a) → (Maybe a → Maybe a → Maybe a)
opMaybe op (Just x) (Just y) = Just (op x y)
opMaybe op _ _ = Nothing

(Must also check the laws.)

Polynomial division $\longrightarrow$ Power Series

Patrik Jansson

? $\exists$ bs : Poly. mul as bs = one

If so, we can define recip as

recip $x = \frac{1}{x}$

Motivation: $x/y \overset{def}{=} x \cdot \frac{1}{y} = x \cdot \text{recip } y$

Specification: recip as = bs

$\Leftrightarrow$ mul as bs = one

Polynomial division → Power Series

Patrik Jansson

? ∃ bs : Poly. mul as bs = one

If so, we can define recip as

recip $x = \frac{1}{x}$

Specification: recip (a:as) = b:bs

mul (a:as) (b:bs) = one : zero

1 : 0 : 0 : ....

recip [] = ?

mul [] bs = [] ≠ one

Polynomial division ⟶ Power Series

Patrik Jansson

recip [ ] = error "Division by zero"

Specification: recip (a:as) = b:bs

⟺ mul (a:as) (b:bs) = one : zero

$$\frac{1}{1-x}$$

Try a simple case: a:as = 1:[-1] (= 1 − x)

Polynomial division → Power Series

Patrik Jansson

recip [ ] = error "Division by zero"

Specification: recip (a:as) = b:bs

$\Leftrightarrow$ mul (a:as) (b:bs) = one : zero

Try a simple case: a:as = 1:[-1]

thus eval (a:as) x = 1 - x

let eval (b:bs) x = q

We know H_2(eval, mul, *)

eval (mul (a:as) (b:bs)) x == eval (a:as) x * eval (b:bs) x

== (1-x) * q == q - x*q

$q - x \cdot q = 1$

eval (one:zero) x = $1 + x \cdot 0 = 1$

Polynomial division → Power Series

Patrik Jansson

$$\text{recip } [\,] = \text{error "Division by zero"}$$

Specification: $\text{recip } (a:as) = b:bs$

$\Leftrightarrow \quad \text{mul } (a:as)\ (b:bs) = \text{one} : \text{zero}$

$$\boxed{\begin{array}{l} \text{eval } (a:as)\ x = 1- \\ \text{eval } (b:bs)\ x = q \end{array}}$$

$$\text{eval } (\text{mul } (a:as)\ (b:bs))\ x == \text{eval } (a:as)\ x * \text{eval } (b:bs)\ x$$

$$== (1-x) * q == q - x*q == \text{eval } (\text{one} : \text{zero})\ x == 1$$

Identify terms:

$$\left.\begin{array}{l} b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ \quad - b_0 x - b_1 x^2 - b_2 x^3 - \cdots \\ == 1 + 0 \cdot x + 0 \cdot x^2 + 0\, x^3 + \cdots \end{array}\right\} \begin{cases} b_0 = 1 \\ b_1 - b_0 = 0 \\ b_2 - b_1 = 0 \\ \vdots \quad \vdots \end{cases} \begin{cases} b_0 = 1 \\ b_1 = 1 \\ b_2 = 1 \\ \vdots \end{cases}$$

Polynomial division → Power Series

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We computed recip [1, -1] = [1, 1, 1, 1, ...

it is not a polynomial, but a "formal power series".

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} b_i \cdot x^i = \sum_{i=0}^{\infty} 1 \cdot x^i = \sum_{i=0}^{\infty} x^i$$

Note: the sum may not converge (depending on x)

$$x=2 \qquad \frac{1}{1-2} = \frac{1}{-1} = -1 \neq \sum_{i=0}^{\infty} 2^i = 1+2+4+8+\cdots$$

Polynomial division → Power Series

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If so, we can define recip as

recip $x = \frac{1}{x}$

Specification: recip (a:as) = b:bs

$\Leftrightarrow$ mul (a:as) (b:bs) = one

mul (a:as) (b:bs) = (a·b) : (scale a bs + mul as (b:bs))

= 1 : zero . . . . .

Polynomial division → Power Series

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? $\exists$ bs $\in$ Poly. mul as bs = one

Specification: recip (a:as) = b:bs

$\Leftrightarrow$ mul (a:as) (b:bs) = one

If so, we can define recip as

$\text{recip } x = \frac{1}{x}$

mul (a:as) (b:bs) = (a·b) : (scale a bs + mul as (b:bs))

= 1 : zero

$\Leftrightarrow$ a·b = 1 $\wedge$ scale a bs = − mul as (b:bs)

$\Leftrightarrow$ b = 1/a $\wedge$ scale b (scale a bs) = $\boxed{\text{scale (-b) (mul as (b:bs))}}$

$\Leftrightarrow$ b = recip a $\wedge$ scale (b·a) bs = scale 1 bs = $\boxed{\text{bs}}$

Polynomial division → Power Series

Patrik Jansson

? $\exists$ bs $\in$ Poly. mul as bs = one

If so, we can define recip as

$\text{recip } x = \frac{1}{x}$

Specification: recip (a:as) = b:bs

$\Leftrightarrow$ mul (a:as) (b:bs) = one

mul (a:as) (b:bs) = (a·b) : (scale a bs + mul as (b:bs))

= 1 : zero

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recip (a:as) = b:bs
  where b = recip a
        bs = scale (negate b) (mul as (b:bs))
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