

# Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \rightarrow W$$

"linear transform"

$$\text{LinTran}(h, V, W) = \begin{aligned} & H_0(h, \text{zero}_V, \text{zero}_W) \\ & \wedge H_2(h, (+_V), (+_W)) \\ & \wedge \forall c. H_1(h, \text{scale}_V^c, \text{scale}_W^c) \end{aligned}$$

# Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \rightarrow W$$

"linear transform"

$$\text{LinTran}(h, V, W) = H_0(h, \text{zero}_V, \text{zero}_W) \\ \wedge H_2(h, (+_V), (+_W)) \\ \wedge \forall c. H_1(h, \text{scale}_V^c, \text{scale}_W^c)$$

Examples:

- $\forall c. \text{LinTran}(\text{scale}_V^c, V, V)$
- $\forall c. \text{LinTran}(\text{apply}_c, a \rightarrow s, s)$

"projections"

# Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \rightarrow W$$

$$\text{LinTran}(h, V, W) = H_0(h, \text{zero}_V, \text{zero}_W)$$

$$\wedge H_2(h, (+_V), (+_W))$$

$$\wedge \forall c. H(h, \text{scale}_V c, \text{scale}_W c)$$

$$v : V = G \rightarrow S$$

$$hv =$$

$$w = hv : W = G' \rightarrow S$$

# Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \rightarrow W$$

$$\text{LinTran}(h, V, W) = \text{H}_0(h, \text{zero}_V, \text{zero}_W)$$

$$\wedge \text{H}_2(h, (+_V), (+_W))$$

$$\wedge \forall c. H(h, \text{scale}_V c, \text{scale}_W c)$$

$$v : V = G \rightarrow S$$

$$\begin{aligned} hv &= h\left(\sum_i^V \text{scale}_V v_i e_i\right) = \sum_i^W h(\text{scale}_V v_i e_i) \\ &= \sum_i^W \text{scale}_W v_i (he_i) \end{aligned}$$

$$w = hv : W = G' \rightarrow S$$

$$M_h = \begin{pmatrix} | & | & | \\ he_0 & he_1 & he_2 \dots \\ | & | & | \end{pmatrix}$$

# Linear Algebra & Homomorphisms

Patrik Jansson

$$h: V \rightarrow W$$

$$h v = h \left( \sum_i \text{scale}_V v_i e_i \right) = \sum_i \text{scale}_W h(v_i) e_i = \\ = \sum_i \text{scale}_W v_i (h e_i)$$

Syntax = Matrix  $s$

$$M_h = \begin{pmatrix} | & | & | \\ he_0 & he_1 & he_2 & \dots \\ | & | & | \end{pmatrix}$$

Semantics =  $V \rightarrow W$

$$\text{eval}_{MV} m v = \sum_i \text{scale}_W (v_i) (m_i)$$

# Linear Algebra & Homomorphisms

Patrik Jansson

Example:  $\text{der} : P_3 \rightarrow P_2$  "polynomial derivative"

$P_n = \{\text{polynomials of degree } \leq n\} = \{0..n\} \rightarrow R$  (coeff.)

$\text{eval}_P : P_n \rightarrow R \rightarrow R$

$\text{eval}_P a = \sum_i \text{scale } a_i \cdot p_i$

$p_i : x^i = x^i$

$\text{der } p_i = \text{scale } i \cdot p_{i-1}$

$$\begin{array}{c|c|c|c} x^0 & x^1 & x^2 & x^3 \\ \hline & & & \\ & & & \end{array}$$

$$\begin{array}{c} x^0 \\ x^1 \\ x^2 \\ x^3 \end{array}$$

DER =  
(matrix version)  
of der

$\text{eval}_{MV} : M \rightarrow (V \rightarrow W)$

# Linear Algebra & Homomorphisms

Patrik Jansson

Example:  $\text{der} : P_3 \rightarrow P_2$

"polynomial derivative"

$P_n = \{\text{polynomials of degree } \leq n\} = \{0..n\} \rightarrow \mathbb{R}$  (coeff.)

$\text{eval}_P : P_n \rightarrow \mathbb{R} \rightarrow \mathbb{R}$

$\text{eval}_P a = \sum_i \text{scale } a_i \cdot p_i$

$p_i(x) = x^i$

$\text{der } p_i = \text{scale } i \cdot p_{i-1}$

$$\text{DER} = \left( \begin{array}{c|c|c|c}
x^0 & x^1 & x^2 & x^3 \\
\hline
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{array} \right)$$

$x^0 \quad x^1 \quad x^2 \quad x^3$

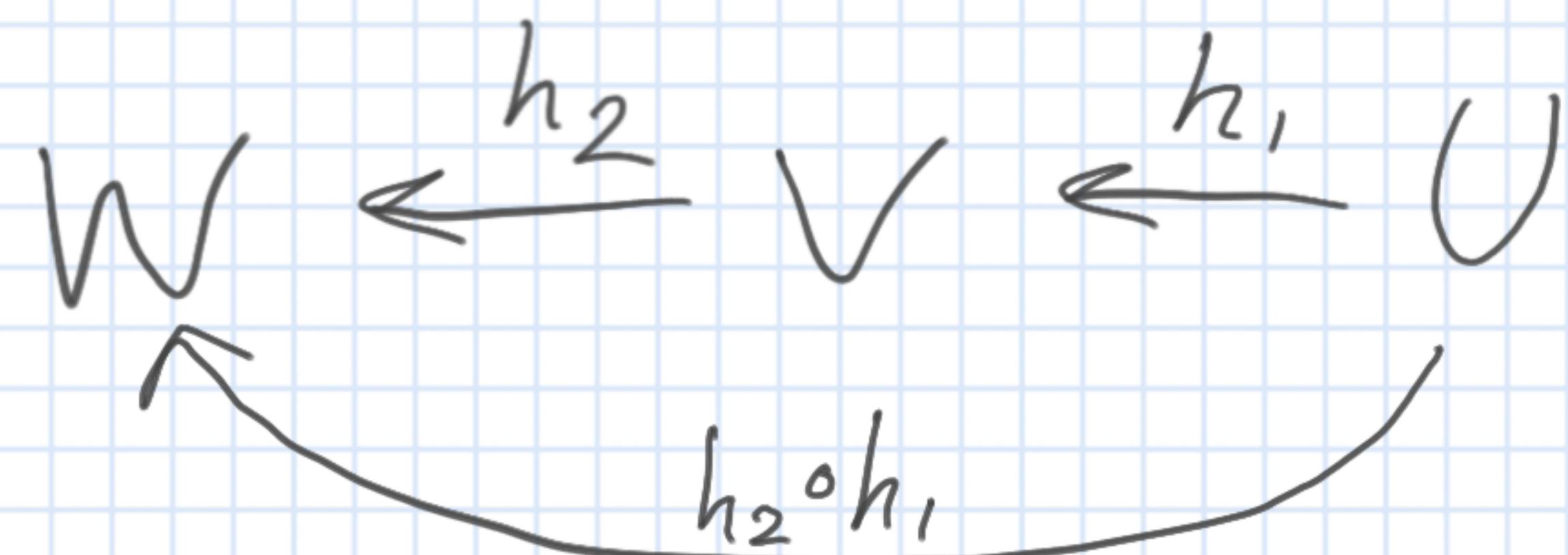
$\text{eval}_{MV} : M \rightarrow V \rightarrow W$

$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$

$\downarrow$

$\text{LinTran}(h_2 \circ h_1, U, W)$

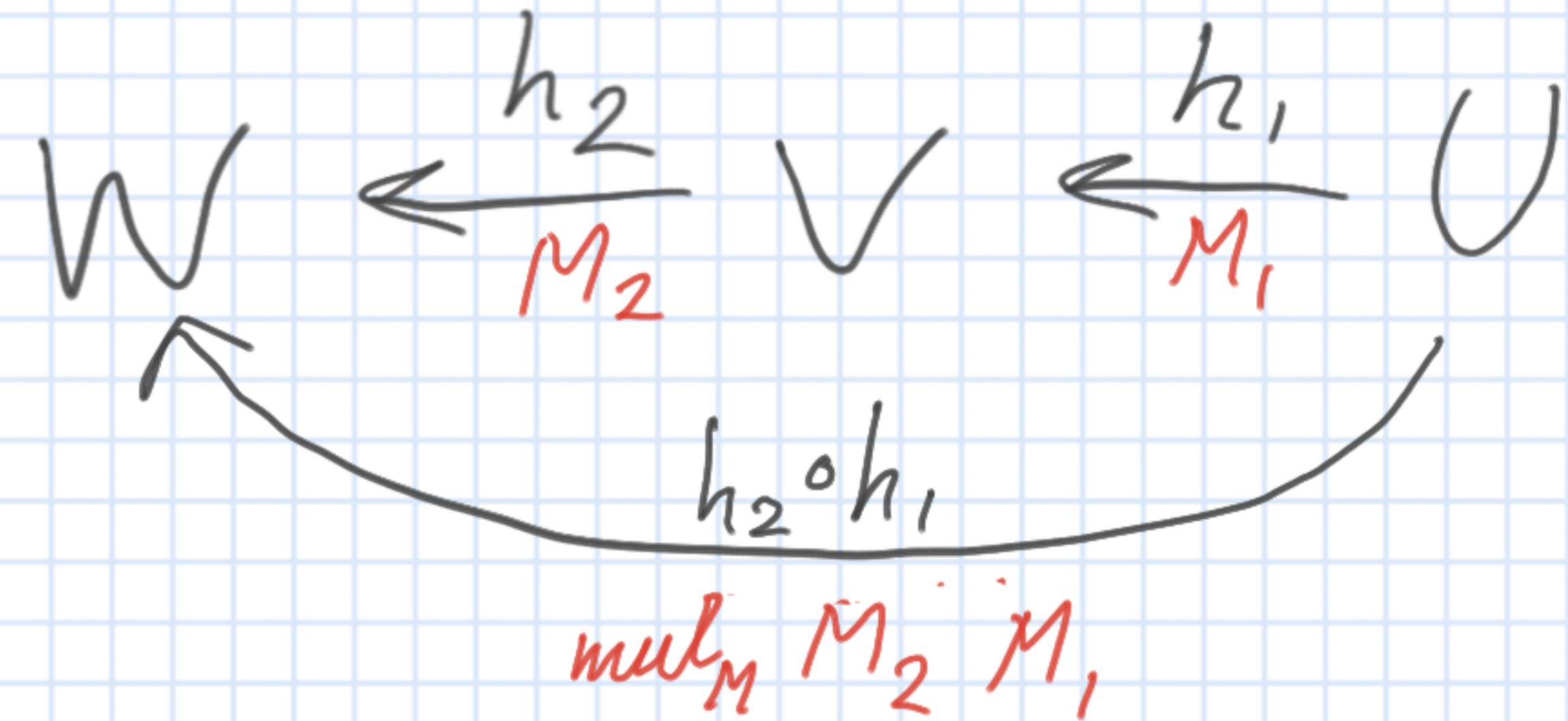


$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$



$\text{LinTran}(h_2 \circ h_1, U, W)$



$$\text{eval}_{MV} M_1 = h_1$$

$$\text{eval}_{MV} M_2 = h_2$$

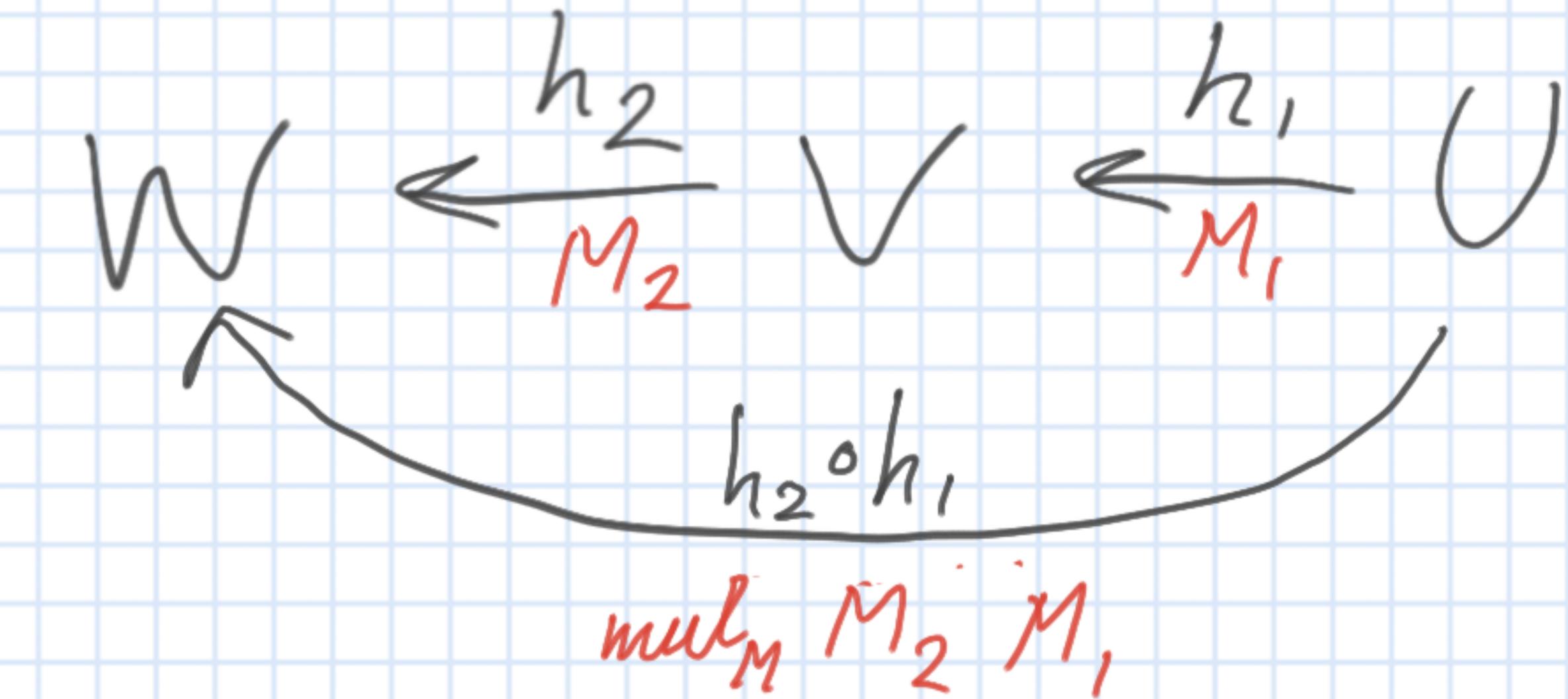
$$\text{eval}_{MV} (\text{mult}_M M_2 M_1) = h_2 \circ h_1$$

$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$



$\text{LinTran}(h_2 \circ h_1, U, W)$



$$\begin{aligned} \text{eval}_{MV} M_1 &= h_1 \\ \text{eval}_{MV} M_2 &= h_2 \end{aligned} \quad \Rightarrow$$

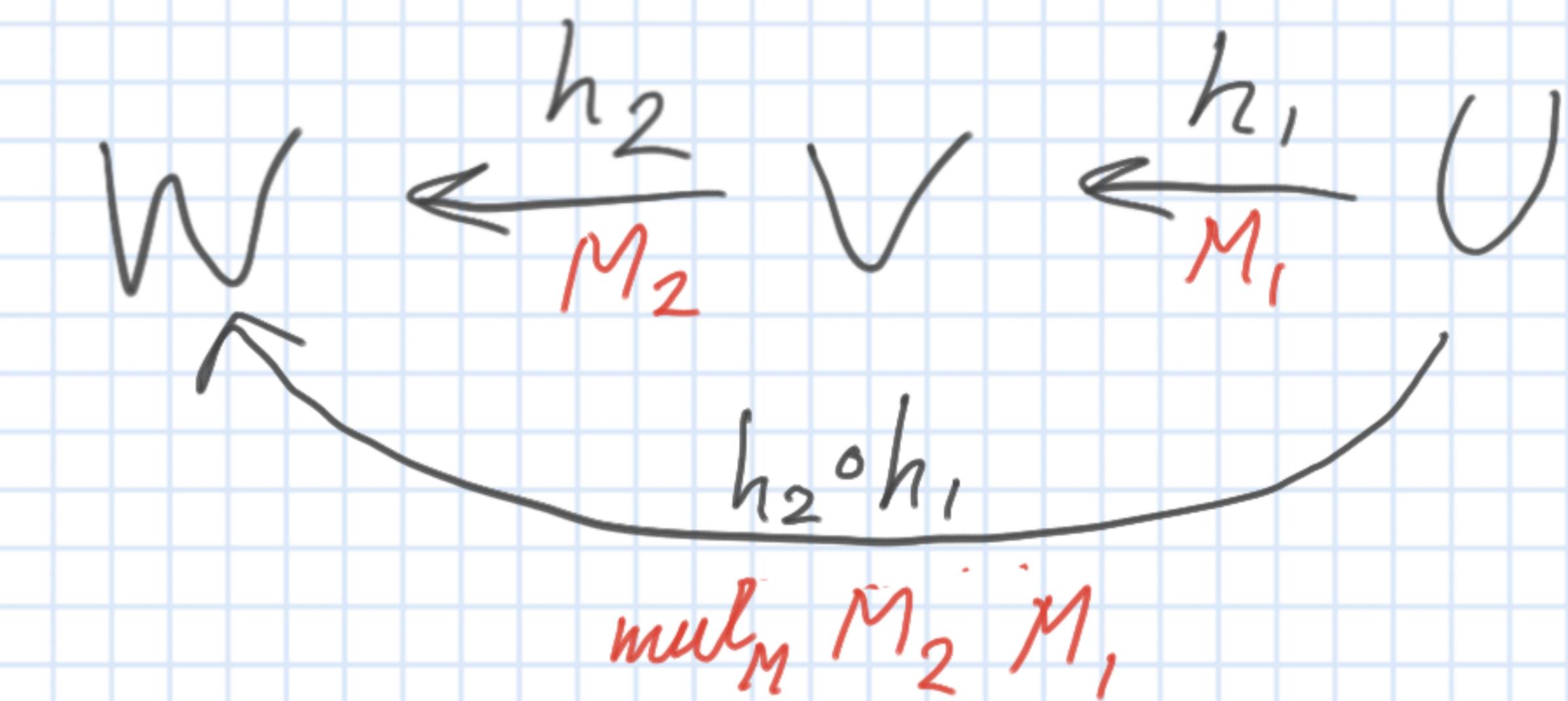
$$\begin{aligned} h_2 \circ h_1 &= \text{eval}_{MV} M_2 \circ \text{eval}_{MV} M_1 \\ h_2 \circ h_1 &= \text{eval}_{MV} (\text{mul}_M M_2 M_1) \end{aligned}$$

$\text{LinTran}(h_1, U, V) \wedge$

$\text{LinTran}(h_2, V, W)$

$\downarrow$

$\text{LinTran}(h_2 \circ h_1, U, W)$



$$\begin{aligned} \text{eval}_{MV} M_1 &= h_1 \\ \text{eval}_{MV} M_2 &= h_2 \end{aligned} \quad \Rightarrow \quad h_2 \circ h_1 = \text{eval}_{MV} M_2 \circ \text{eval}_{MV} M_1$$

$$= \text{eval}_{MV} (\text{mul}_M M_2 M_1)$$

$H_2(\text{eval}_{MV}, \underline{\text{mul}}_{M_1}(\circ))$

Specification, but what  
is the implementation?

type  $Vsa = a \rightarrow s$

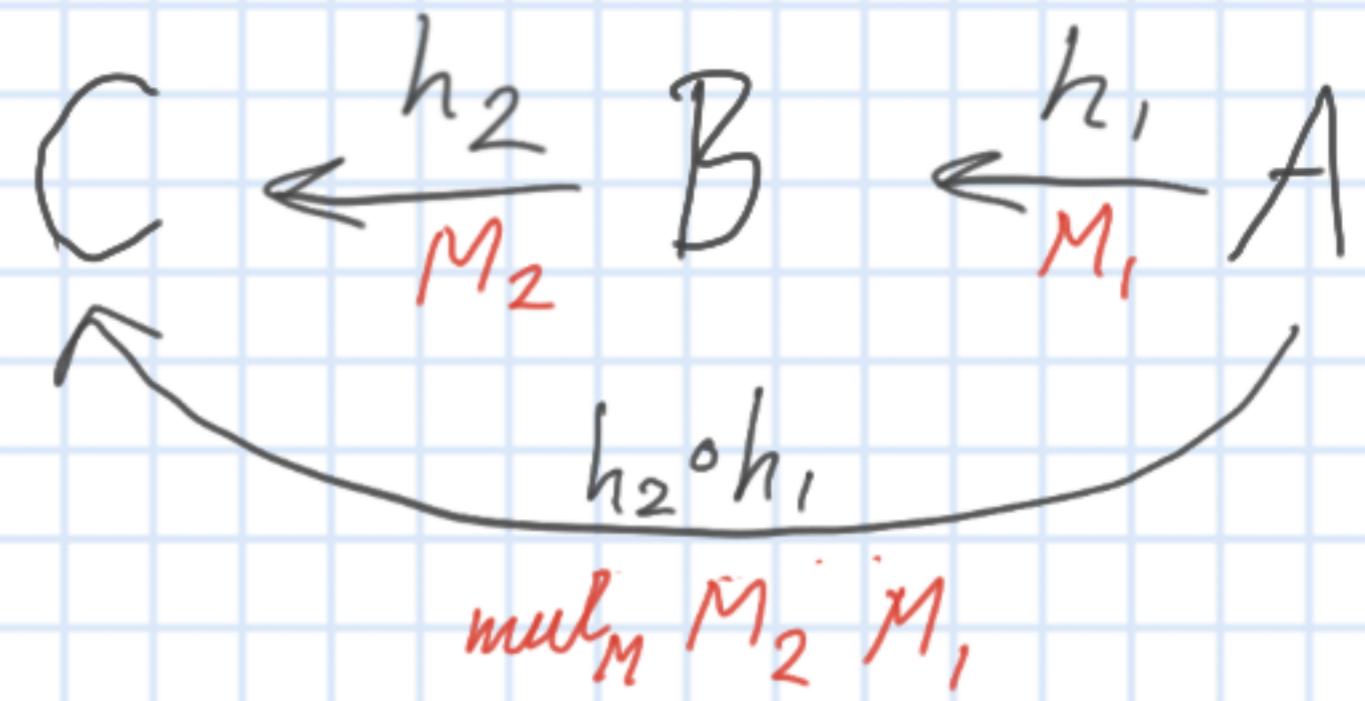
type  $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vs a$

transpose :  $M_{sab} \rightarrow M_{sba}$

transpose  $m = \lambda j i \rightarrow m^{ij} = \text{flip } m$

getCol :  $M_{sab} \rightarrow a \rightarrow Vs b$

getCol = flip = transpose



$$\begin{pmatrix} & & & \\ & | & | & | \\ & he_0 & he_1 & he_2 \\ & | & | & | \end{pmatrix}$$

type  $Vsa = a \rightarrow s$

type  $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vs a$

transpose :  $M_{sab} \rightarrow M_{sba}$

transpose  $m = \lambda j \ i \rightarrow m^{ij} = \text{flip } m$

getCol :  $M_{sab} \rightarrow a \rightarrow Vs b$

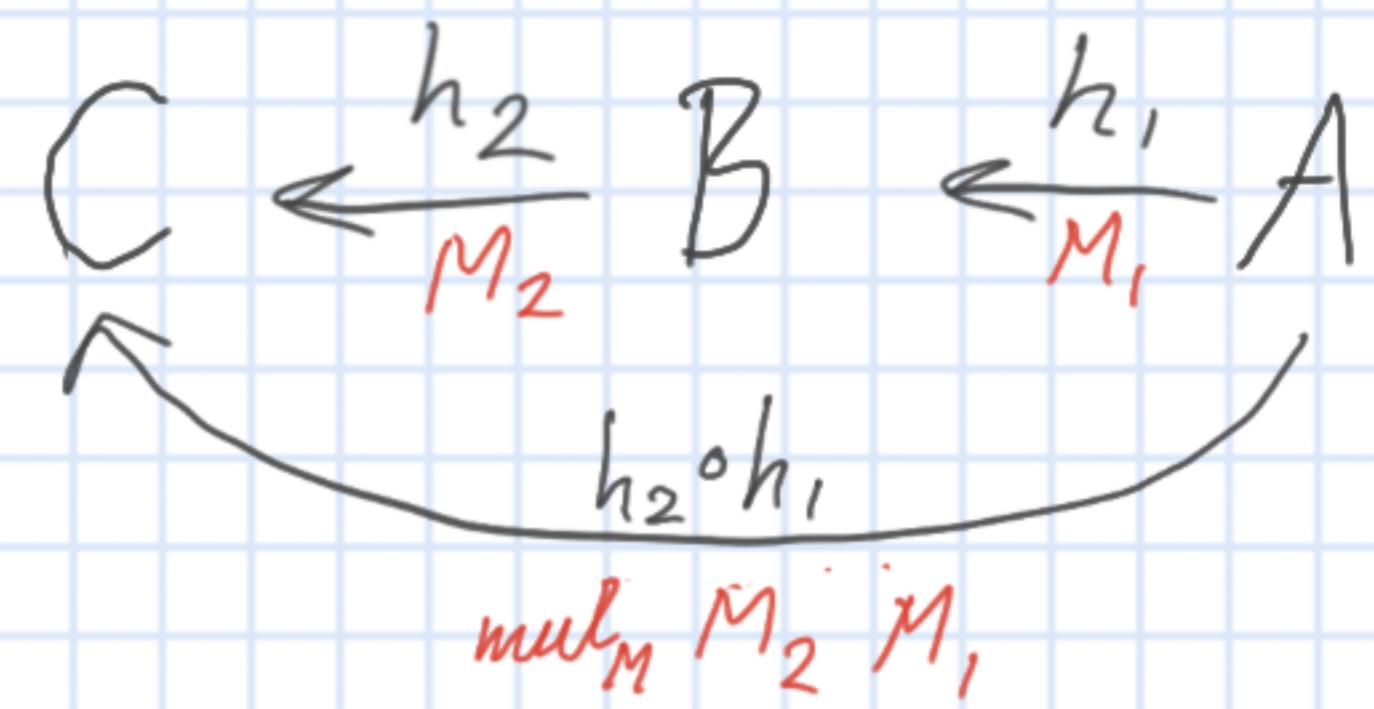
getCol = flip = transpose

Example:  $a = \{X, Y\}$ ,  $b = \{0, 1, 2\}$ ,  $c = \{1\}$

$$M_1 = \begin{pmatrix} 1 & | & 1 \\ h_1 e_X & h_1 e_Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

$h_2 = \text{apply } 1$  (projection)



$$e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

type  $Vsa = a \rightarrow s$

type  $Msab = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

transpose :  $Msab \rightarrow Msba$

transpose  $m = \lambda j i \rightarrow m[ij] = flip m$

getCol :  $Msab \rightarrow a \rightarrow Vs b$

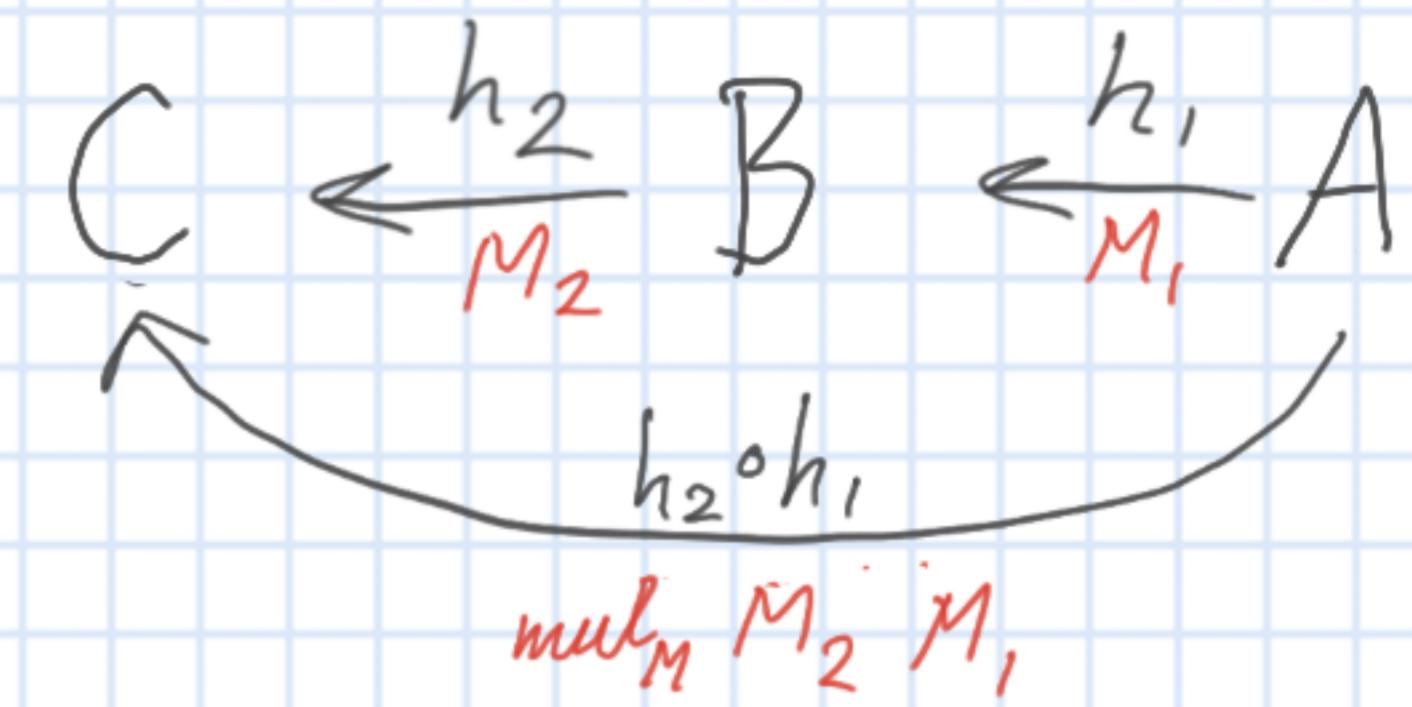
getCol = flip = transpose

Spec. of eval<sub>MV</sub>: getCol  $m[i] = eval_{MV} m (e[i])$

getCol  $m = eval_{MV} m \circ e$

flip(flip m) = m = flip(eval<sub>MV</sub> m  $\circ e$ )

$m = flip(h \circ e)$



We know  $flip \circ flip = id$

type  $Vsa = a \rightarrow s$

type  $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip :  $M_{sab} \rightarrow M_{sba}$

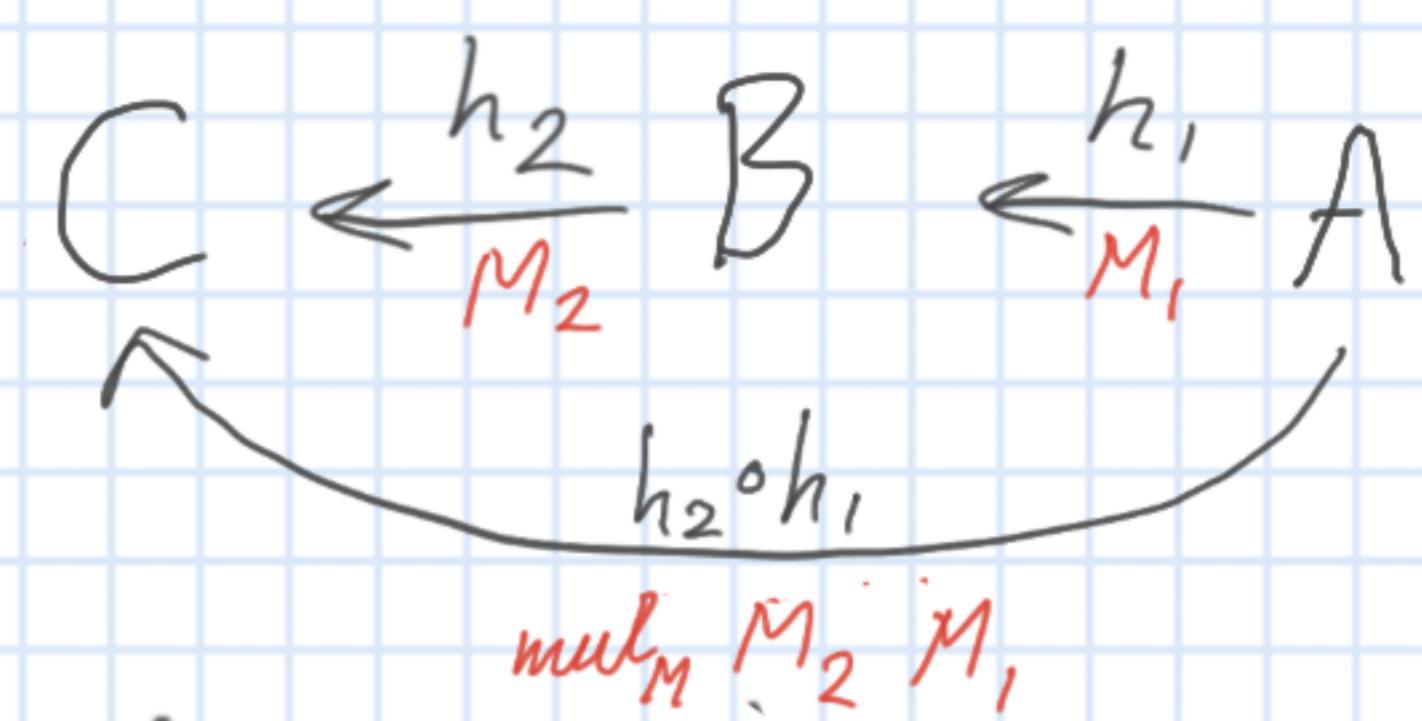
Spec. of  $\text{eval}_{MV}$ :  $m = \text{flip}(\text{eval}_{MV} m \circ e)$

$\text{getCol}(\text{mul}_M m_2 m_1)_i = h_2(h_1 e_i)$

$\text{eval}_{MV} m_2$

$h_1 e_i$

$\text{getCol } m_1, i$



type  $Vsa = a \rightarrow s$

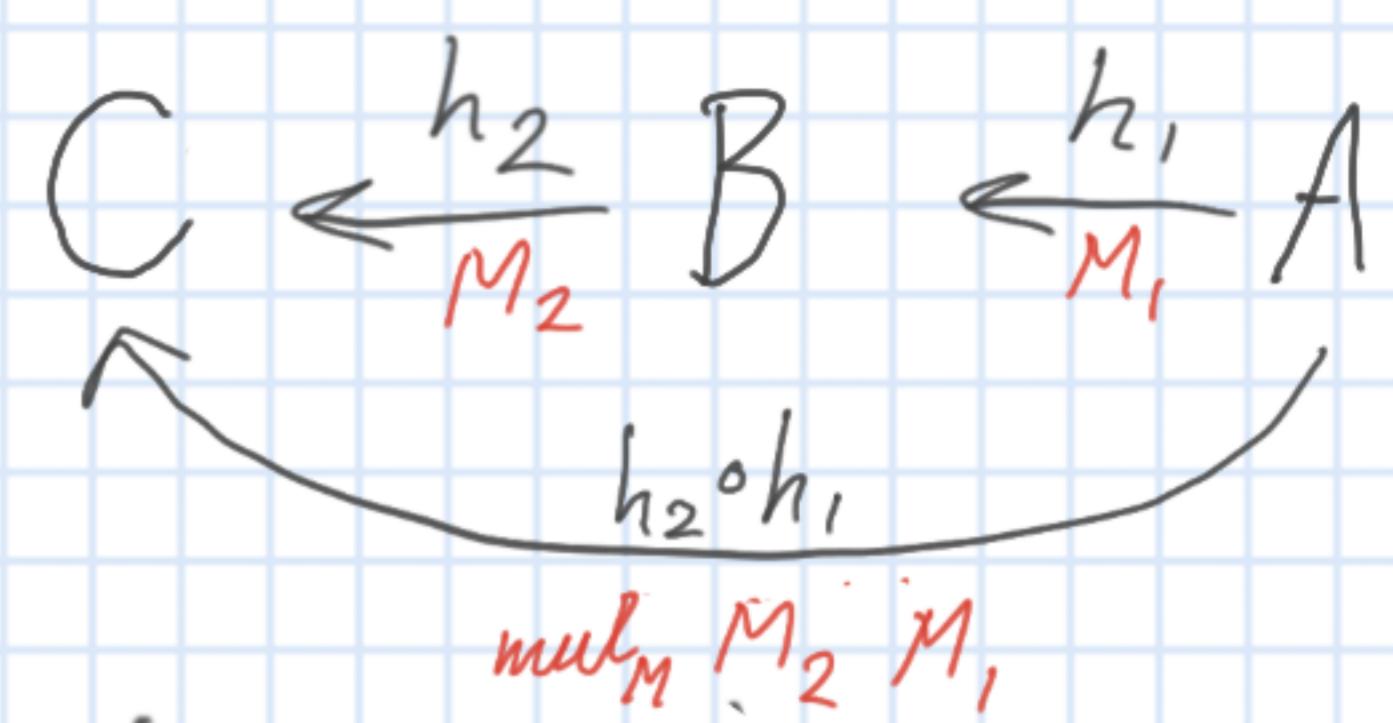
type  $Msab = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip :  $Msab \rightarrow Msba$

Spec. of  $\text{eval}_{MV}$ :  $m = \text{flip}(\text{eval}_{MV} m \circ e)$

$\text{getCol}(\text{mul}_M m_2 m_1)_i = h_2(h_1 e_i) =$

$= \text{eval}_{MV} m_2 (\text{getCol } m_1, i)$



type  $Vsa = a \rightarrow s$

type  $Msab = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

flip :  $Msab \rightarrow Msba$

Spec. of  $\text{eval}_{MV}$ :  $m = \text{flip}(\text{eval}_{MV} m \circ e)$

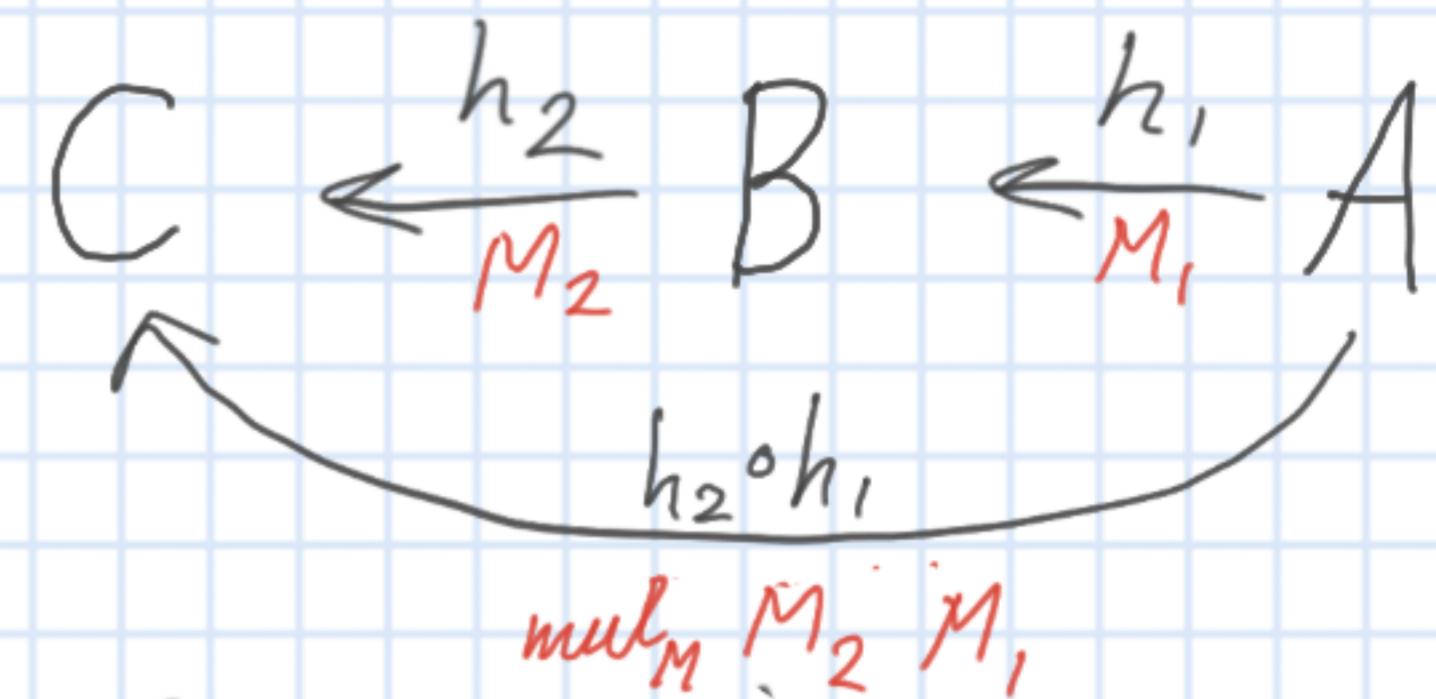
$\text{getCol}(\text{mul}_M m_2 m_1)_i = h_2(h_1 e_i) =$

$= \text{eval}_{MV} m_2 (\text{getCol } m_1, i)$

$\text{getCol}(\text{mul}_M m_2 m_1) = \text{eval}_{MV} m_2 \circ \text{getCol } m_1$

$\boxed{\text{mul}_M m_2 m_1 = \text{flip}(\text{eval}_{MV} m_2 \circ \text{flip } m_1)}$

where  $\text{eval}_{MV} m v = \sum_i \text{scale } v_i m_i$



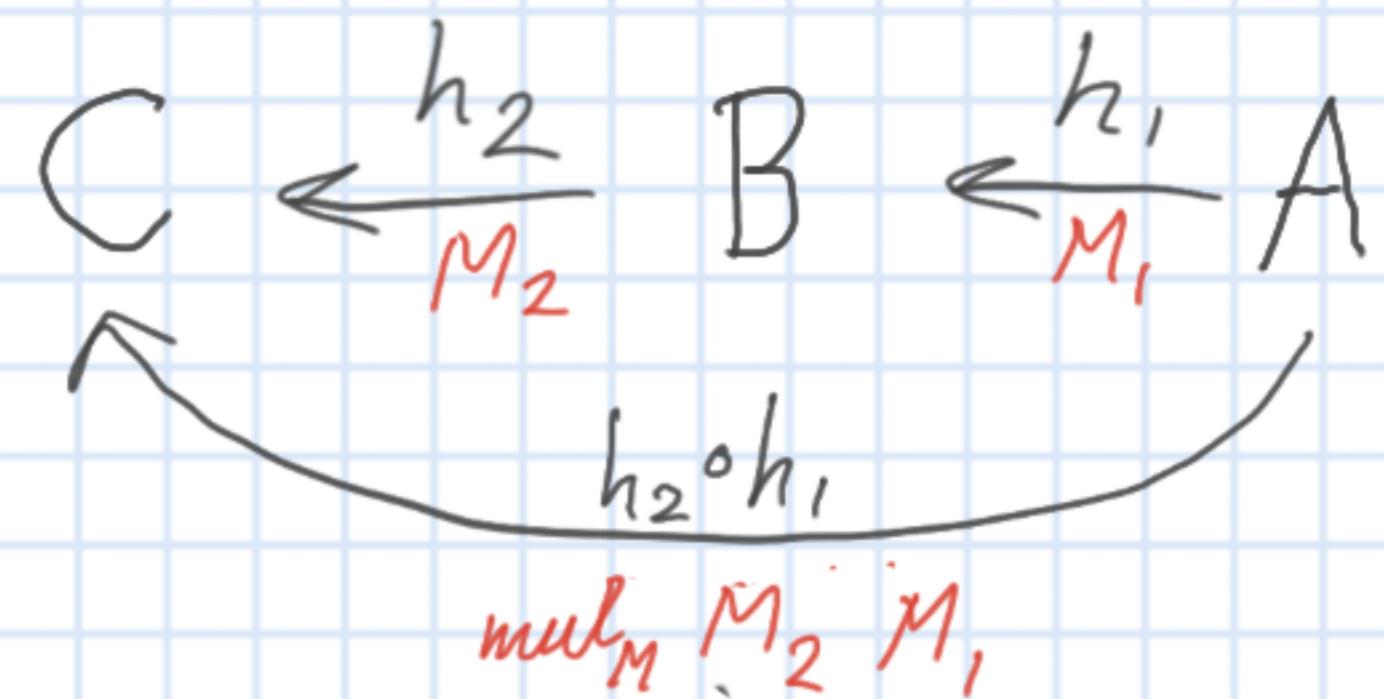
Det. of  
 $\text{mul}_M$

type  $Vsa = a \rightarrow s$

type  $M_{sab} = b \rightarrow a \rightarrow s = b \rightarrow Vsa$

$\text{flip} : M_{sab} \rightarrow M_{sba}$

$\text{flip} = \text{transpose} = \text{getCol}$



$$\boxed{\text{mul}_M m_2 m_1 = \text{flip} (\text{eval}_{MV} m_2 \circ \text{flip} m_1)} \quad \begin{array}{l} A = Vsa \\ B = Vs_b \\ C = Vs_c \end{array}$$

where  $\text{eval}_{MV} m v = \sum_i \text{scale } v_i m_i$

$\text{mul}_M : M_{sab} \rightarrow M_{sbc} \rightarrow M_{sac}$

$\text{eval}_{MV} = \text{mul}_{MV} : M_{sab} \rightarrow Vs_a \rightarrow Vs_b$



