Domain-Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Contact	Patrik Jansson, 0729852033.
Results	Announced within 15 workdays
Exam check	(by appointment via email)
Aids	One textbook of your choice (Domain-Specific Languages of Mathematics, or Beta - Mathematics Handbook, or Rudin, or Adams and Essex, or . . .). No printouts, no lecture notes, no notebooks, etc.
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain-Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure**: A DSL for Loop

   Consider the following mathematical definition (adapted from Quasigroup, Wikipedia):

   > A Loop $(Q, e, (\otimes), (//), (\backslash\backslash))$ is a set L equipped with an identity element e and three binary operators (called multiplication, right- and left-division) satisfying the following identities for all x and y in L:
   >
   > $$\begin{aligned} &y = (y \otimes x) \,//\, x \\ &y = (y \,//\, x) \otimes x \\ &y = x \,\backslash\backslash\, (x \otimes y) \\ &y = x \otimes (x \,\backslash\backslash\, y) \\ &x \otimes e = x \\ &e \otimes x = x \end{aligned}$$

   (a) Define a type class *Loop l* that corresponds to the Loop structure.

   (b) Define a datatype $L\ v$ for the language of loop expressions (with variables of type v) and define a Loop instance for it. (These are expressions formed from applying the loop operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Integers form an instance of *Loop* with $(\otimes) = (+)$ and $e = 0$. Calculate the definitions of $(//)$ and $(\backslash\backslash)$ from the above identities and implement the instance of the Loop class.

   (d) Give a type signature for, and define, a general evaluator for $L\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the *Loop* instance defined in (1c). Define three loop expressions of type *L String*. Give appropriate assignments and compute the results of evaluating the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**

   Consider the following differential equation:

   $$f'' = \frac{3 * f' - f}{2}, \quad f(0) = 1, \quad f'(0) = \frac{5}{2}$$

   (a) [10p] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs'*, *fs''*. What are the first three coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be a complex number) and the rules for linearity + derivative:

   $$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

   Show that your solution does indeed satisfy the three requirements.

3. [20p] **Proofs of homomorphisms**

   The predicate $H_2(h, Op, op)$ states that $h : A \to B$ is a homomorphism from $Op : A \to A \to A$ to $op : B \to B \to B$. In first-order logic this is expressed as:

   $$H_2(h, Op, op) = \forall\ x.\ \ \forall\ y.\ \ h\ (Op\ x\ y) == op\ (h\ x)\ (h\ y)$$

   Consider the following simple functions:

   $$f : \mathbb{Z} \to \mathbb{N}$$
   $$f\ x = x^2$$
   $$g : \mathbb{Z} \to (\mathbb{Z}, \mathbb{N})$$
   $$g\ x = (-x, x^2)$$

   (a) [10p] Consider the claim $\exists\ op.\ \ H_2(f, (+), op)$. What is the type of op? Prove or disprove the claim.

   (b) [10p] Consider the claim $\exists\ op.\ \ H_2(g, (-), op)$. What is the type of op? Prove or disprove the claim.

4. [30p] **Typing maths: Uniform convergence**

   Consider the following (lightly edited) quote from Conway [2017]:

   > If $\{f_n\}$ is a sequence of functions from a subset X of $\mathbb{R}$ into $\mathbb{R}$, say that $\{f_n\}$ *converges uniformly* to a function g if for every $\epsilon > 0$ there is an N such that $|f_n(x) - g(x)| < \epsilon$ for all x in X and all $n \geq N$. In symbols this is written as $f_n \to_u g$ on X.

   (a) [7p] Define the first-order logic predicate $ConUni\ (X, f, g)$ encoding the property of uniform convergence $f_n \to_u g$ on X.

   (b) [8p] Give the types of X, n, f, g, ϵ, and N. Explain your reasoning.

   (c) [5p] Simplify the predicate $\neg\ ConUni\ (X, f, g)$ by pushing the negation through all the way.

   (d) [10p] Prove $\neg\ ConUni\ (\mathbb{R}, sq, zero)$ where $sq\ n\ x = (x\ /\ n)^2$ and $zero\ x = 0$.