Power series and ODEs

Patrik Jansson

Ordinary Differential Equation (ODE)

Examples:

$$f(x) + f'(x) = 1 \quad , \qquad f(0) = 0$$

$$e(x) - e'(x) = 0 \quad , \qquad e(0) = 1$$

$$s(x) + s''(x) = 0 \;,\; s(0) = 0 \,,\, s'(0) = 1$$

Power series and ODEs:

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Then $f'(x) = \sum_{i=1}^{\infty} a_i \cdot i \cdot x^{i-1}$

$f(0) = a_0 = 0$

$a_0 = 0$
$a_1 = 1$
$a_2 = -\frac{1}{2}$
$a_3 = \frac{1}{6}$
$a_4 =$

$LHS = (a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots) +$
$(a_1 + a_2 \cdot 2 \cdot x + a_3 \cdot 3 \cdot x^3 + \cdots)$

$RHS = 1 + 0 \cdot x + \cdots$

$a_0 + a_1 = 1 \Rightarrow a_1 = 1$ $\quad a_{i-1} = -i \cdot a_i$

$a_1 + 2a_2 = 0 \Rightarrow a_2 = -\frac{1}{2}$ $\quad a_3 = -\frac{a_2}{3}$

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Example: $f(x) + f'(x) = 1$, $f(0) = 0$

Idea: solve using power series. Assume $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Then $f'(x) = \sum_{i=1}^{\infty} i \cdot a_i \cdot x^{i-1}$

$f(0) = a_0 = 0$

$f(x) \approx 0 + x \cdot \left(1 - \frac{1}{2}x + \frac{x^2}{6} - \frac{x^3}{24}\right)$

$$LHS = (a_0 + 1 \cdot a_1) + (a_1 + 2 \cdot a_2) \cdot x + (a_2 + 3 \cdot a_3) \cdot x^2 + \cdots$$

$$RHS = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$$

$$a_1 = 1 \land a_2 = -\frac{1}{2} \land a_3 = \frac{1}{2 \cdot 3} \land a_4 = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \land \cdots$$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series. $f = \text{eval } as$

$f' = \text{eval } as'$

$$\begin{cases} as + as' = 1:0:0:\ldots \\ as = \text{integ } (f\,0)\ as' = \boxed{f\,0} : \underline{\text{zipWith } (/)\ as'\ [1..]} \end{cases}$$

$as = \boxed{0} : 1 : -\tfrac{1}{2} : \tfrac{1}{6} :$

$as' = 1 : -1 : \tfrac{1}{2} : \quad :$

$rhs = \boxed{1 : 0 : 0 : 0 : \ldots}$

$f(x) \approx 0 + 1\cdot x - \frac{x^2}{2} + \frac{x^3}{6} \ldots$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval as}$

$f' = \text{eval as}'$

$$\begin{cases} \boxed{as + as' = 1 : 0 : 0 : \ldots} \\ as = \text{integ } (f\,0)\ as' = \boxed{f\,0} : \underline{\text{zipWith } (/)\ as'\ [1..]} \end{cases}$$

$as = \boxed{0} :$

$as' = \boxed{1} :$

$rhs = \boxed{1} :$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval } as$

$f' = \text{eval } as'$

$$\begin{cases} \boxed{as + as' = 1:0:0: \ldots} \\ as = \text{integ } (f0)\ as' = \boxed{f0} : \text{zipWith } (/)\ as'\ [1..] \end{cases}$$

$as = \boxed{0} : 1/1 : \frac{-1}{2}$

$as' = 1 : -1 :$

$rhs = 1 : 0 : 0$

Power series and ODEs: Transform

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Example: $f(x) + f'(x) = \boxed{1}$, $\boxed{f(0) = 0}$

Idea: solve using power series.

$f = \text{eval } as$

$f' = \text{eval } as'$

$$\begin{cases} \boxed{as + as' = 1 : 0 : 0 : \ldots} \\ as = \text{integ } (f0)\ as' = \boxed{f0} : \text{zipWith } (/)\ as'\ [1..] \end{cases}$$

$as = \boxed{0} : 1/1 : \frac{-1}{2} : \frac{1}{6} : \frac{-1}{24}$

$as' = 1 : -1 : 1/2 : -1/6 :$

$rhs = 1 : 0 : 0 : 0 :$

Power series and ODEs: Transform

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as = integ (f 0) as' = f 0 : zipWith (/) as' [1..]

f = eval as

f' = eval as'

Background:

$I : \mathbb{R} \to (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$

$I\ c\ g\ x = c + \int_0^x g = c + \int_0^x g(t)\,dt$

Laws: $D\ (I\ c\ g) = g$

$I\ (f\ 0)\ (D\ f) = f$

deriv (integ c as) = as

$\boxed{\text{integ (head as) as' = as}}$

Semantics

Syntax

Power series and ODEs: Transform

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Transform example: $f' + 2 \cdot f = r \quad , f(0) = c_0$

let $f = \text{eval } as$; $f' = \text{eval } as'$; $r = \text{eval } rhs$

in $\begin{cases} as' = rhs - \text{scale } 2 \; as \\ as = \text{integ } c_0 \; as' \end{cases}$

Then solve the system starting from c_0

$as = c_0 : 2 : 4 : 6$

$as' = 1 : 3 : 5 :$

$rhs = \; : \; : \; :$

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Examples: $e(x) - e'(x) = 0 \quad , \quad e(0) = 1$

$$\Rightarrow \begin{cases} as' = as \\ as = \text{integ } 1 \; as' \end{cases}$$

$$\Rightarrow \quad as = \text{integ } 1 \; as$$

$$as = 1 : 1 : 1/2 : 1/{2 \cdot 3} : 1/{2 \cdot 3 \cdot 4} :$$

$$f(x) = \text{eval } as \; x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$= e^x$$

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Examples: $e(x) - e'(x) = 0 \; , \quad e(0) = 1$

$$\Rightarrow \begin{cases} as' = as \\ as = \text{integ } 1 \; as' \end{cases} \Rightarrow as = \text{integ } 1 \; as$$

$$as = 1 : \frac{1}{1} : \frac{1}{2} : \frac{1}{6} : \frac{1}{24} : \ldots\ldots : \frac{1}{n!} :$$

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Examples: $s(x) + s''(x) = 0\ ,\ s(0) = 0, s'(0) = 1$

$$\begin{cases} as'' = -\, as \\ as' = \underline{\text{integ } 1\ as''} \\ as = \underline{\text{integ } 0\ as'} \end{cases}$$

$$\begin{cases} as'' = -0 : \\ as' = 1 : 0 \\ as = 0 : 1 : 0 \end{cases}$$

$s(x) \approx 0 + x + 0 \cdot x^2 +$

Power series and ODEs

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Examples: $s(x) + s''(x) = 0\ ,\ s(0) = 0, s'(0) = 1$

$$\begin{cases} as'' = -\,as \\ as' = \text{integ}\ 1\ as'' \\ as = \text{integ}\ 0\ as' \end{cases}$$

$$\begin{cases} as'' = 0 : -1 : \\ as' = 1 : 0 : -\frac{1}{2} \\ as = 0 : 1 : 0 : -\frac{1}{6} \end{cases}$$

$(/1)\quad(/2)\quad(/3)\quad(/4)$

Power series and ODEs

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Examples: $s(x) + s''(x) = 0 \;,\; s(0) = 0 , s'(0) = 1$

$$\begin{cases} as'' = -\,as \\ as' = \underline{\text{integ } 1 \; as''} \\ as = \underline{\text{integ } 0 \; as'} \end{cases}$$

$s(x) = \text{eval } as \; x$

$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120}$

$$\begin{cases} as'' = 0 : -1 : 0 : \frac{1}{6} : 0 \\ as' = 1 : 0 : -\frac{1}{2} : 0 : \frac{1}{24} : \\ as = 0 : 1 : 0 : -\frac{1}{6} : 0 : \frac{1}{120} \end{cases}$$

$(/1) \quad (/2) \quad (/3) \quad (/4) \quad (/5)$

Taylor & Maclaurin

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$$f\,x \approx f\,0 + f'\,0 \cdot x + \frac{f''\,0}{2} \cdot x^2 + \cdots + \frac{f^{(n)}\,0}{n!} \cdot x^n + \cdots$$

"Taylor series at 0" or "Maclaurin series"

$$f \xmapsto{DAll} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots] : DS\,(\mathbb{R} \to \mathbb{R})$$

$\downarrow$ map (apply 0)

$$[f\,0, f'\,0, f''\,0, \ldots, f^{(n)}\,0, \ldots] : DS\,\mathbb{R}$$

$\downarrow$ \ds -> zipWith (/) ds factorials

$$[f\,0, f'\,0, \frac{f''\,0}{2}, \ \ , f^{(n)}\,0/n!, \ldots] : PS\,\mathbb{R}$$

Taylor & Maclaurin

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$$f\ x \simeq f\,0 + f'\,0 \cdot x + \frac{f''\,0}{2} \cdot x^2 + \cdots + \frac{f^{(n)}\,0}{n!} \cdot x^n + \cdots$$

"Taylor series at 0" or "Maclaurin series"

$$f \xmapsto{DAll} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots] \quad : DS\ (\mathbb{R}\to\mathbb{R})$$

$$DAll : (\mathbb{R}\to\mathbb{R}) \to DS\ (\mathbb{R}\to\mathbb{R}) \qquad DS\ a \simeq [a]$$

$$\uparrow eval \qquad\qquad \uparrow map\ eval$$

$$derAll : FunExp \to DS\ FunExp$$

Taylor & Maclaurin

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Note: $DAll\ (D f) = tail\ (DAll\ f)$

$\qquad\qquad\qquad\quad f = head\ (DAll\ f)$

$H_1(DAll, D, tail)$

$f \overset{DAll}{\longmapsto} [f, f', f'', \ldots, f^{(n)}, \ldots\ldots]$ $\quad : DS\ (\mathbb{R}\to\mathbb{R})$

$DAll : (\mathbb{R}\to\mathbb{R}) \to DS\ (\mathbb{R}\to\mathbb{R})$ $\qquad DS\ a = [a]$

$\qquad\quad eval \uparrow \qquad\qquad\qquad\qquad \uparrow map\ eval$

$derAll : FunExp \to DS\ FunExp$

Taylor & Maclaurin

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Note: $\text{DAll}\ (D\, f) = \text{tail}\ (\text{DAll}\ f)$

$\qquad\qquad\qquad f = \text{head}\ (\text{DAll}\ f)$

Examples: $\text{DAll id} = \text{id} : \backslash\_ \to 1 : \underline{\text{zero}}$ $:: DS\ (\mathbb{R}\to\mathbb{R})$

$\text{DAll sin} = \text{sin} : \text{cos} : -\text{sin} : -\text{cos} :$

$\text{DAll exp} = \text{exp} : \text{exp} : \text{exp} : \ldots\ldots$

Taylor & Maclaurin

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Note: `DAll (D f) = tail (DAll f)`

`f = head (DAll f)`

`map (apply 0)`

Examples: `DAll id = id : \_ -> 1 : zero`

`0 : 1 : 0 : 0 : 0 : ....`

`DAll sin = sin : cos : -sin : -cos :`

`0 : 1 : 0 : -1 :`

`DAll exp = exp : exp : exp : ......`

`1 : 1 : 1 : 1 :`

$:: DS\ \mathbb{R} \stackrel{?}{=} PS\ \mathbb{R}$

$\frac{1}{1-x} \approx 1:1:1:$