

DSLs of Mathematics: limit of functions

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2026-02-03

Math book quote: The limit of a function

We say that $f(x)$ **approaches the limit L** as x **approaches a** , and we write

$$\lim_{x \rightarrow a} f(x) = L,$$

if the following condition is satisfied:

for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε , such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

Limit of a function – continued: fragments translated to logic

$$\lim_{x \rightarrow a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \text{Dom } f \wedge |f(x) - L| < \varepsilon$$

Limit of a function – continued

First attempt at translation:

$$\lim a f L = \forall \epsilon > 0. \exists \delta > 0. P \epsilon \delta$$

$$\text{where } P \epsilon \delta = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f x - L| < \epsilon)$$

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Scope check:

- a, f, L bound in the def. of \lim .

Limit of a function – continued

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- a, f, L bound in the def. of \lim .
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- Anything missing?

Limit of a function – continued

Finally (after adding a binding for x):

$$\lim a f L = \forall \epsilon > 0. \exists \delta > 0. P \epsilon \delta$$

$$\text{where } P \epsilon \delta = \forall x. Q \epsilon \delta x$$

$$Q \epsilon \delta x = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f x - L| < \epsilon)$$

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Lesson learned: be careful with scope and binding (of x in this case).

Variants of *lim* and some properties

- Typing: let $X \subseteq \mathbb{R}$; $Y \subseteq \mathbb{R}$; $a : X$; $L : Y$; and $f : X \rightarrow Y$
Version “limProp” $\underline{\text{lim}}$: $X \rightarrow (X \rightarrow Y) \rightarrow Y \rightarrow \text{Prop}$
Version “limMaybe” $\underline{\text{lim}}$: $X \rightarrow (X \rightarrow Y) \rightarrow \text{Maybe } Y$
Version “limSloppy” lim : $X \rightarrow (X \rightarrow Y) \rightarrow Y$

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- $\underline{\text{lim}}$ can be used as a partial function $\underline{\text{lim}}$, or lim :

$$\forall a, f, L_1, L_2. (\underline{\text{lim}} a f L_1 \wedge \underline{\text{lim}} a f L_2) \Rightarrow L_1 == L_2$$

$$\forall a, f, L. (\underline{\text{lim}} a f L) \Rightarrow (\underline{\text{lim}} a f = \text{Just } L) \wedge (\text{lim } a f = L)$$

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- lim is linear:

$$\text{lim } a (f \oplus g) = \text{lim } a f + \text{lim } a g$$

$$\text{lim } a (c \triangleleft f) = c * (\text{lim } a f)$$

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Variants of \lim and some properties

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$$\lim a (f \oplus g) = \lim a f + \lim a g$$

$$\lim a (c \triangleleft f) = c * (\lim a f)$$

- What is $(\oplus) : (X \rightarrow Y) \rightarrow (X \rightarrow Y) \rightarrow (X \rightarrow Y)$?

$$f \oplus g = \lambda x \rightarrow f x + g x$$

Example 2: derivative

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x .

We can write

$$D f x = \lim_{h \rightarrow 0} g(h) \quad \text{where} \quad g(h) = \frac{f(x+h) - f(x)}{h}; \quad g :: H \rightarrow Y; \text{type } H = \mathbb{R}^{\neq 0}$$

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$$D f x = \lim_{h \rightarrow 0} (\varphi x) \quad \text{where} \quad \varphi x h = \frac{f(x+h) - f(x)}{h}; \quad \varphi :: X \rightarrow (H \rightarrow Y)$$

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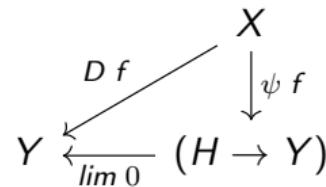
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$$D f x = \lim_0 (\varphi x) \quad \text{where} \quad \varphi x h = \frac{f(x+h) - f(x)}{h}; \quad \varphi :: X \rightarrow (H \rightarrow Y)$$

$$D f = \lim_0 \circ \psi f \quad \text{where} \quad \psi f x h = \frac{f(x+h) - f(x)}{h};$$



Derivatives, cont.

Examples:

$$D : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R})$$

$$sq\ x = x^2$$

$$double\ x = 2 * x$$

$$c_2\ x = 2$$

$$sq' \Leftarrow D\ sq \Leftarrow D\ (\lambda x \rightarrow x^2) \Leftarrow D\ (^2) \Leftarrow (2*) \Leftarrow double$$

$$sq'' \Leftarrow D\ sq' \Leftarrow D\ double \Leftarrow c_2 \Leftarrow const\ 2$$

Note: we cannot *implement* D (of this type) in Haskell.

Given only $f : \mathbb{R} \rightarrow \mathbb{R}$ as a “black box” we cannot compute the actual derivative $f' : \mathbb{R} \rightarrow \mathbb{R}$.

We need the “source code” of f to apply rules from calculus.