Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \longrightarrow W \qquad \text{"linear transform"}$$

$$\begin{aligned} LinTran(h, V, W) = \; & H_0(h, zero_V, zero_W) \\ \wedge \; & H_2(h, (+_V), (+_W)) \\ \wedge \; & \forall s.\ H_1(h, scale_V\, s, scale_W\, s) \end{aligned}$$

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$$h : V \longrightarrow W \qquad \text{"linear transform"}$$

$$\begin{aligned}
\mathrm{LinTran}(h, V, W) = \; & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\
\wedge \; & H_2(h, (+_V), (+_W)) \\
\wedge \; & \forall s.\ H_1(h, \mathit{scale}_V\, s, \mathit{scale}_W\, s)
\end{aligned}$$

Examples: $\forall s.\ \mathrm{LinTran}(\mathit{scale}_V\, s, V, V)$

$\forall c.\ \mathrm{LinTran}(\mathit{apply}\ c, a \to s, s)$

"projections"

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$$h : V \to W$$

"linear transform"

$$\begin{aligned}
\mathit{LinTran}(h, V, W) = \; & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\
\wedge \; & H_2(h, (+_V), (+_W)) \\
\wedge \; & \forall s.\ H_1(h, \mathit{scale}_V\, s, \mathit{scale}_W\, s)
\end{aligned}$$

$$v : V = G \to s$$

$$\begin{aligned}
h\,v = h\Big(\sum_i{}_V \mathit{scale}_V\, v_i\, e_i\Big) &= \sum_i{}_W h(\mathit{scale}_V\, v_i\, e_i) \\
&= \sum_i{}_W \mathit{scale}_W\, v_i\, (h\, e_i)
\end{aligned}$$

$$w = h\,v : W = G' \to s$$

$$M_h = \begin{pmatrix} | & | & | & \\ h e_0 & h e_1 & h e_2 & \cdots \\ | & | & | & \end{pmatrix}$$

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$$h : V \to W$$

"linear transform"

$$\begin{aligned}\mathrm{LinTran}(h, V, W) = {} & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\ \wedge\ & H_2(h, (+_V), (+_W)) \\ \wedge\ & \forall s.\ H_1(h, \mathit{scale}_V\, s, \mathit{scale}_W\, s)\end{aligned}$$

$$v : V = G \to s$$

$$h\, v = h\left(\sum_{i\,V} \mathit{scale}_V\, v_i\ e_i\right) = \sum_{i\,W} h\,(\mathit{scale}_V\, v_i\, e_i) =$$

$$= \sum_i \mathit{scale}_W\, v_i\, (h\, e_i)$$

$$w = h\,v : W = G' \to s$$

$$M_h = \begin{pmatrix} | & | & | & \\ h\,e_0 & h\,e_1 & h\,e_2 & \cdots \\ | & | & | & \end{pmatrix}$$

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$h : V \longrightarrow W$

$h\ v = h\left(\sum_i{}_V \text{scale}\ v_i\ e_i\right) = \sum_i{}_W h\left(\text{scale}_V\ v_i\ e_i\right) =$

$= \sum_i \text{scale}_W\ v_i\ (h\ e_i)$

$M_h = \begin{pmatrix} | & | & | & \\ h e_0 & h e_1 & h e_2 & \cdots \\ | & | & | & \end{pmatrix}$

Syntax = Matrix s

Semantics = $V \to W$

$\text{eval}_{MV}\ m\ v = \sum_i \text{scale}_W\ (v\ i)\ (m\ i)$

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Example: $\mathrm{der} : \boxed{P_3} \to \boxed{P_2}$ "polynomial derivative"

$P_n = \{\text{polynomials of degree} \le n\} = \{0..n\} \to \mathbb{R}$ (coeff.)

$\mathrm{eval}_P : P_n \to \mathbb{R} \to \mathbb{R}$

$p_i\ x = x^i$

$\mathrm{eval}_P\ a = \sum_i \mathrm{scale}\ a_i\ p_i$

$\mathrm{der}\ p_i = \mathrm{scale}\ i\ p_{i-1}$

$$\mathrm{DER} = \begin{array}{cc} & \begin{array}{cccc} x^0 & x^1 & x^2 & x^3 \end{array} \\ & \left(\begin{array}{c|c|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right) \begin{array}{c} x^0 \\ x^1 \\ x^2 \end{array} \end{array}$$

$\mathrm{eval}_{MV} : M \to V \to V$

$$\mathrm{LinTran}(h_1, U, V) \wedge$$
$$\mathrm{LinTran}(h_2, V, W)$$
$$\Downarrow$$
$$\mathrm{LinTran}(h_2 \circ h_1, U, W)$$

$$\begin{tikzcd}
W & V \arrow[l, "h_2"'] & U \arrow[l, "h_1"'] \arrow[ll, bend left, "h_2 \circ h_1"]
\end{tikzcd}$$

$$\mathit{LinTran}(h_1, U, V) \land \mathit{LinTran}(h_2, V, W)$$

$$\Downarrow$$

$$\mathit{LinTran}(h_2 \circ h_1, U, W)$$

$$
\begin{tikzcd}
W & & V \arrow[ll, "h_2", "M_2"'] & & U \arrow[ll, "h_1", "M_1"'] \arrow[llll, bend left, "h_2 \circ h_1", "\mathit{mul}_M\ M_2\ M_1"']
\end{tikzcd}
$$

$$\left.\begin{aligned} \mathit{eval}_{MV}\ M_1 &= h_1 \\ \mathit{eval}_{MV}\ M_2 &= h_2 \end{aligned}\right\} \Longrightarrow \begin{aligned} h_2 \circ h_1 &= \mathit{eval}_{MV}\ M_2 \circ \mathit{eval}_{MV}\ M_1 \\ &= \mathit{eval}_{MV}\ (\mathit{mul}_M\ M_2\ M_1) \end{aligned}$$

$$\boxed{H_2(\mathit{eval}_{MV}, \underline{\mathit{mul}_M}, (\circ))} \longleftarrow$$

specification, but what is the implementation?

```
type V s a = a -> s
type M s a b = b -> a -> s = b -> V s a
transpose : M s a b -> M s b a
transpose m = \j i -> m i j   = flip m
getCol : M s a b -> a -> V s b
getCol = flip = transpose
```

$$\begin{tikzcd} C & B \arrow[l, "h_2", "M_2"'] & A \arrow[l, "h_1", "M_1"'] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\ M_2\ M_1"'] \end{tikzcd}$$

$$\begin{pmatrix} | & | & | \\ h e_0 & h e_1 & h e_2 \\ | & | & | \end{pmatrix}$$

$$\texttt{type } V\, s\, a = a \to s$$

$$\texttt{type } M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$$

$$\begin{tikzcd} C & B \arrow[l, "h_2", "M_2"'] & A \arrow[l, "h_1", "M_1"'] \arrow[ll, bend left, "h_2 \circ h_1"', "\mathrm{mul}_M\ M_2\ M_1"] \end{tikzcd}$$

$$\text{transpose} : M\, s\, a\, b \to M\, s\, b\, a$$

$$\text{transpose } m = \backslash j\ i \to m\ i\ j \quad = \text{flip } m$$

$$\text{getCol} : M\, s\, a\, b \to a \to V\, s\, b$$

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{getCol} = \text{flip} = \text{transpose}$$

$$e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example: $a = \{X, Y\}$, $b = \{0, 1, 2\}$, $c = \{1\}$

$$M_1 = \begin{pmatrix} | & | \\ h_1 e_X & h_1 e_Y \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

$$h_2 = \text{apply } 1 \qquad \text{(projection)}$$

type $V\ s\ a = a \to s$

type $M\ s\ a\ b = b \to a \to s = b \to V\ s\ a$

$$\begin{tikzcd} C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1"', "mul_M\ M_2\ M_1"] \end{tikzcd}$$

transpose : $M\ s\ a\ b \to M\ s\ b\ a$

transpose $m = \backslash j\ i \to m\ i\ j$ = flip m

getCol : $M\ s\ a\ b \to a \to V\ s\ b$

getCol = flip = transpose

$\boxed{\text{We know } flip \circ flip = id}$

Spec. of $eval_{MV}$: $getCol\ m\ i = eval_{MV}\ m\ (e\ i)$

$getCol\ m = eval_{MV}\ m \circ e$

$\underline{flip\ (flip\ m) = m} = flip\ (\underbrace{eval_{MV}\ m}_{h} \circ e) = flip\ (h \circ e)$

$$\text{type } V\, s\, a = a \to s$$

$$\text{type } M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$$

$$\text{flip} : M\, s\, a\, b \to M\, s\, b\, a$$

$$
\begin{tikzcd}
C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "\text{mul}_M\, M_2\, M_1"']
\end{tikzcd}
$$

Spec. of eval_{MV}: $m = \text{flip}\,(\text{eval}_{MV}\, m \circ e)$

$$\text{getCol}\,(\text{mul}_M\; m_2\; m_1)\; i = \underbrace{h_2}_{\text{eval}_{MV}\, m_2}\,(\underbrace{h_1\; e_i}_{\text{getCol}\; m_1\; i})$$

type $V\ s\ a = a \to s$

type $M\ s\ a\ b = b \to a \to s = b \to V\ s\ a$

$$\mathit{flip} : M\ s\ a\ b \to M\ s\ b\ a$$

$$\begin{tikzcd}
C & B \arrow[l, "h_2", "\color{red}{M_2}"'] & A \arrow[l, "h_1", "\color{red}{M_1}"'] \arrow[ll, bend left, "h_2 \circ h_1", "\color{red}{\mathit{mul}_M\ M_2\ M_1}"']
\end{tikzcd}$$

Spec. of eval_{MV}: $m = \mathit{flip}\,(\mathit{eval}_{MV}\ m \circ e)$

$$\mathit{getCol}\ (\mathit{mul}_M\ m_2\ m_1)\ i = h_2\,(h_1\ e_i) =$$

$$= \mathit{eval}_{MV}\ m_2\ (\mathit{getCol}\ m_1\ i)$$

$\text{type } V\, s\, a = a \to s$

$\text{type } M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$

$\quad \mathit{flip} : M\, s\, a\, b \to M\, s\, b\, a$

$$\begin{tikzcd} C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "\mathit{mul}_M\, M_2\, M_1"'] \end{tikzcd}$$

Spec. of eval_{MV}: $\; m = \mathit{flip}\,(\mathit{eval}_{MV}\, m \circ e)$

Def. of mul_M

$\mathit{getCol}\,(\mathit{mul}_M\; m_2\; m_1)\; i = h_2\,(h_1\, e_i) =$

$\qquad = \mathit{eval}_{MV}\, m_2\, (\mathit{getCol}\; m_1\; i)$

$\mathit{getCol}\,(\mathit{mul}_M\; m_2\; m_1) = \mathit{eval}_{MV}\; m_2 \circ \mathit{getCol}\; m_1$

$$\boxed{\mathit{mul}_M\; m_2\; m_1 \;=\; \mathit{flip}\,(\mathit{eval}_{MV}\; m_2 \circ \mathit{flip}\; m_1)}$$

where $\mathit{eval}_{MV}\; m\;\; v = \sum_i \mathit{scale}\; v_i\; m_i$

$$\text{type } V\,s\,a = a \to s$$

$$\text{type } M\,s\,a\,b = b \to a \to s = b \to V\,s\,a$$

$$\begin{tikzcd} C & B \arrow[l, "h_2", "M_2"'] & A \arrow[l, "h_1", "M_1"'] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\ M_2\ M_1"'] \end{tikzcd}$$

$$flip : M\,s\,a\,b \to M\,s\,b\,a$$

$$flip = transpose = getCol$$

$$\boxed{mul_M\ m_2\ m_1 = flip\ (eval_{MV}\ m_2 \circ flip\ m_1)}$$

$$A = V\,s\,a$$

$$B = V\,s\,b$$

$$C = V\,s\,c$$

$$\text{where } eval_{MV}\ m\ v = \sum_i scale\ v_i\ m_i$$

$$mul_M : M\,s\,a\,b \to M\,s\,b\,c \to M\,s\,a\,c$$

$$eval_{MV} = mul_{MV} : M\,s\,a\,b \to V\,s\,a \to V\,s\,b$$