Domain-Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Results	Announced within 15 workdays
Exam check	2023-04-03, 12.15-13.00 in EDIT 6452
Aids	One textbook of your choice (Domain-Specific Languages of Mathematics, or Beta - Mathematics Handbook, or Rudin, or Adams and Essex, or . . .). No printouts, no lecture notes, no notebooks, etc.
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain-Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure**: Consider the Wikipedia definition of a *field*:

   Formally, a field is a set F together with two operations called addition and multiplication. [...] These operations are required to satisfy the following properties, referred to as field axioms. In the following definitions, a, b and c are arbitrary elements of the field F.

   - Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
   - Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a$.
   - Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that $a + 0 = a$ and $a \cdot 1 = a$.
   - Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called additive inverse of a, such that $a + (-a) = 0$.
   - Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1}, $1/a$, or $\frac{1}{a}$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.
   - Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

   (a) Define a type class *Field* that corresponds to the field structure (with operations *add*, *mul*, *zero*, *one*, *negate*, *recip*).

   (b) Define a datatype $F\ v$ for the language of field expressions (with variables of type v) and define a *Field* instance for it. (These are expressions formed from applying the field operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Field* class.

   (d) Give a type signature for, and define, a general evaluator for $F\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Field* instances defined in (1c). Take three field expressions of type F *String*, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**

   Consider the following differential equation:

$$f'' - f = sin, \quad f(0) = 0, \quad f'(0) = -\frac{1}{2}$$

   (a) [10p] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs*′, *fs*″. What are the first four coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be a complex number) and the rules for linearity + derivative:

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

   You may assume the following equation holds for all complex t:

$$\sin t = \frac{1}{2i}(e^{i*t} - e^{-i*t})$$

   Show that your solution does indeed satisfy the three requirements.

3. [20p] **Proofs: Fixing polynomials**

   Consider the following **buggy** implementation of the ring of polynomials:

```haskell
import qualified Prelude
import Prelude (map)
import DSLsofMath.Algebra
type Poly a = [a]
instance Additive a => Additive       (Poly a) where zero = zeroL; (+) = addL
instance Ring a     => Multiplicative (Poly a) where one  = oneL;  (*) = mulL
zeroL :: [a];            zeroL = []
oneL  :: Ring a => [a];  oneL  = [one]
addL :: Additive a => [a] -> [a] -> [a]
addL []     bs     = []
addL as     []     = []
addL (a:as) (b:bs) = (a + b) : addL as bs
mulL :: Ring a => [a] -> [a] -> [a]
mulL []     bs = []
mulL as     [] = []
mulL (a:as) bs = addL (scaleL a bs) (mulL as bs)
scaleL :: Ring a => a -> [a] -> [a]
scaleL c = map (c*)
```

   Note: The learning outcome tested here is mainly "perform calculational proofs", thus you are expected to use equational reasoning and motivate your steps.

   (a) [5p] Prove that $[1,7,3] + [8] == [9]$ by equational reasoning with the buggy *addL*.

   (b) [5p] Implement a correct *addL*. Explain your reasoning.

   (c) [5p] Assuming a correct *addL*, prove that $[1,7] * one == [8]$ by equational reasoning with the buggy *mulL*.

   (d) [5p] Implement a correct *mulL*. Explain your reasoning.

4. [30p] **Typing maths: Isolated Points and Accumulation Points**

   Consider the following definition (edited Wikipedia entries for "isolated point" and "accumulation point")

   > A real number x is called an *isolated point* of a subset S of the real numbers, if x is an element of S and there exists an $\varepsilon > 0$ such that any $y \in \mathbb{R}$ with $|x - y| < \varepsilon$ is not a member of S or $x = y$.
   >
   > A real number x is called an *accumulation point* of $S \subseteq \mathbb{R}$, if x is an element of S and for all $\varepsilon > 0$, there exists an element y of S, different from x, such that $|x - y| < \varepsilon$.

   (a) [4p] Provide types for x, S, y and ε. Explain your reasoning.

   (b) [8p] Define the first-order logic predicates $I(S, x)$ and $A(S, x)$ encoding the properties "isolated point" and "accumulation point" of S.

   (c) [8p] Let $S := \{\frac{1}{n} \mid n \in \mathbb{N}, n > 0\} \cup \{0\}$. Show $I(S, 1)$ and $A(S, 0)$.

   (d) [10p] Show that for any subset S of the reals and any $x \in S$, $\neg I(S, x)$ is equivalent to $A(S, x)$.