DSLs of Mathematics: limit of functions

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2023-08-30

Math book quote: The limit of a function

*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

Limit of a function – continued

$$\lim_{x \to a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

Limit of a function – continued

First attempt at translation:

$$
\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow \\
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Scope check:

- a, f, L bound in the def. of *lim*.

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- a, f, L bound in the def. of *lim*.
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- Anything missing?

Limit of a function – continued

Finally (after adding a binding for x):

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Lesson learned: be careful with scope and binding (of x in this case).

Variants of *lim* and some properties

- Typing: let $X \subseteq \mathbb{R}$; $Y \subseteq \mathbb{R}$; $a : X$; $L : Y$; and $f : X \to Y$

  Version "limProp" $\quad \underline{lim} \quad : \quad X \to (X \to Y) \to Y \to Prop$

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  Version "limSloppy" $\quad lim \quad : \quad X \to (X \to Y) \to Y$

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- $\underline{\mathit{lim}}$ can be used as a partial function $\underline{\mathit{lim}}$, or lim:

$$\forall\ a, f, L_1, L_2.\ \ (\underline{\mathit{lim}}\ a\ f\ L_1 \land \underline{\mathit{lim}}\ a\ f\ L_2) \Rightarrow L_1 == L_2$$

$$\forall\ a, f, L.\ \ (\underline{\mathit{lim}}\ a\ f\ L) \Rightarrow (\underline{\mathit{lim}}\ a\ f = \mathit{Just}\ L) \land (\mathit{lim}\ a\ f = L)$$

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$$\begin{aligned} lim\ a\ (f \oplus g) &= lim\ a\ f + lim\ a\ g \\ lim\ a\ (c \triangleleft f) &= c * (lim\ a\ f) \end{aligned}$$

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$$f \oplus g = \lambda x \to f\ x + g\ x$$

Example 2: derivative

*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}; \qquad g :: \quad H \to Y; \text{type } H = \mathbb{R}^{\neq 0}$$

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$$D\ f\ x = \mathit{lim}\ 0\ (\varphi\ x) \text{ where } \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}; \quad \varphi :: X \to (H \to Y)$$

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$$D\ f \quad = \mathit{lim}\ 0 \circ \psi\ f\ \text{ where } \psi\ f\ \ x\ h = \tfrac{f\ (x+h) - f\ x}{h};$$

$$\begin{tikzcd}
 & X \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\
Y & (H \to Y) \arrow[l, "\mathit{lim}\ 0"]
\end{tikzcd}$$

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&\mathit{sq}\ x \quad\quad = x\hat{}2 \\
&\mathit{double}\ x = 2 * x \\
&c_2\ x \quad\quad = 2 \\
&\mathit{sq}' \ \texttt{==}\ D\ \mathit{sq}\ \ \texttt{==}\ D\ (\lambda x \to x\hat{}2)\ \texttt{==}\ D\ (\hat{}2)\ \texttt{==}\ (2*)\ \texttt{==}\ \mathit{double} \\
&\mathit{sq}''\ \texttt{==}\ D\ \mathit{sq}'\ \texttt{==}\ D\ \mathit{double}\ \texttt{==}\ c_2\ \texttt{==}\ \mathit{const}\ 2
\end{aligned}
$$

Note: we cannot *implement* D (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a “black box” we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the “source code” of f to apply rules from calculus.