

DSLsofMath: Typing Mathematics (Week 3)

the Lagrangian Equations

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Starting point: a maths quote (the Lagrangian)

From [Sussman and Wisdom, 2013]:

A mechanical system is described by a Lagrangian function of the system state (time, coordinates, and velocities). A motion of the system is described by a path that gives the coordinates for each moment of time. A path is allowed if and only if it satisfies the Lagrange equations. Traditionally, the Lagrange equations are written

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

What could this expression possibly mean?

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The use of notation for “partial derivative”, $\partial L / \partial q$, suggests that L is a function of at least a pair of arguments:

$$L : \mathbb{R}^i \rightarrow \mathbb{R}, i \geq 2$$

This is consistent with the description: “Lagrangian function of the system state (time, coordinates, and velocities)”. So, if we let “coordinates” be just one coordinate, we can take $i = 3$:

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}$$

The “system state” here is a triple, of type $S = T \times Q \times V$, and we can call the three components $t : T$ for time, $q : Q$ for coordinate, and $v : V$ for velocity. ($T = Q = V = \mathbb{R}$.)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Looking again at $\partial L / \partial q$, q is the name of a variable, one of the 3 args to L . In the context, which we do not have, we would expect to find somewhere the definition of the Lagrangian as

$$\begin{aligned} L : T \times Q \times V &\rightarrow \mathbb{R} \\ L(t, q, v) &= \dots \end{aligned}$$

- therefore, $\partial L / \partial q$ should also be a function of the same triple:

$$(\partial L / \partial q) : T \times Q \times V \rightarrow \mathbb{R}$$

It follows that the equation expresses a relation between *functions*, therefore the 0 on the right-hand side is *not* the real number 0, but rather the constant function *const 0*:

$$\begin{aligned} \text{const } 0 : T \times Q \times V &\rightarrow \mathbb{R} \\ \text{const } 0(t, q, v) &= 0 \end{aligned}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- We now have a problem: d / dt can only be applied to functions of *one* real argument t , and the result is a function of one real argument:

$$(d / dt) (\partial L / \partial \dot{q}) : T \rightarrow \mathbb{R}$$

Since we subtract from this the function $\partial L / \partial q$, it follows that this, too, must be of type $T \rightarrow \mathbb{R}$. But we already typed it as $T \times Q \times V \rightarrow \mathbb{R}$, contradiction!

- The expression $\partial L / \partial \dot{q}$ appears to also be malformed. We would expect a variable name where we find \dot{q} , but \dot{q} is the same as dq/dt , a function.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The only immediate candidate for an application of d/dt is “a path that gives the coordinates for each moment of time”. Thus, the path is a function of time, let us say

$$w : T \rightarrow Q \quad \text{-- with } T \text{ for time and } Q \text{ for coords } (q : Q)$$

We can now guess that the use of the plural form “equations” might have something to do with the use of “coordinates”. In an n -dim. space, a position is given by n coordinates. A path would then be

$$w : T \rightarrow Q \quad \text{-- with } Q = \mathbb{R}^n$$

which is equivalent to n functions of type $T \rightarrow \mathbb{R}$, each computing one coordinate as a function of time. We would then have an equation for each of them. We will use $n = 1$ for the rest of this example.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Now that we have a path, the coordinates at any time are given by the path. And as the time derivative of a coordinate is a velocity, we can actually compute the trajectory of the full system state $T \times Q \times V$ starting from just the path.

$$q : T \rightarrow Q$$

$$q = w \quad \text{-- or, equivalently, } q(t) = w(t)$$

$$\dot{q} : T \rightarrow V$$

$$\dot{q} = D w \quad \text{-- or, equivalently, } \dot{q} t = dw(t) / dt$$

We combine these in the “combinator” *expand*, given by

$$expand : (T \rightarrow Q) \rightarrow (T \rightarrow T \times Q \times V)$$

$$expand w t = (t, w t, D w t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- With *expand* in our toolbox we can fix the typing problem.

$$(\partial L / \partial q) \circ (\text{expand } w) : T \rightarrow \mathbb{R}$$

- We now move to using D for d / dt , D_2 for $\partial / \partial q$, and D_3 for $\partial / \partial \dot{q}$. In combination with *expand* w we find these type correct combinations for the two terms in the equation:

$$\begin{aligned} D ((D_3 L) \circ (\text{expand } w)) &: T \rightarrow \mathbb{R} \\ (D_2 L) \circ (\text{expand } w) &: T \rightarrow \mathbb{R} \end{aligned}$$

The equation becomes

$$D ((D_3 L) \circ (\text{expand } w)) - (D_2 L) \circ (\text{expand } w) = \text{const } 0$$

or, after simplification:

$$D (D_3 L \circ \text{expand } w) = D_2 L \circ \text{expand } w$$

Case 3: Lagrangian, summary

“A path is allowed if and only if it satisfies the Lagrange equations” means that this equation is a predicate on paths:

$$\text{Lagrange}(L, w) = D(D_3 L \circ \text{expand } w) == D_2 L \circ \text{expand } w$$

Thus: If we can describe a mechanical system in terms of “a Lagrangian” ($L : S \rightarrow \mathbb{R}$), then we can use the predicate to check if a particular candidate path $w : T \rightarrow \mathbb{R}$ qualifies as a “motion of the system” or not. The unknown of the equation is the path w , and the equation is an example of a partial differential equation (a PDE).

G. J. Sussman and J. Wisdom. *Functional Differential Geometry*. MIT Press, 2013.

Domain-Specific Languages of Mathematics, BSc level course at Chalmers and GU,
<https://github.com/DSLsofMath/>