DSLsofMath: Typing Mathematics (Week 3)

the Lagrangian Equations

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Starting point: a maths quote (the Lagrangian)

From [Sussman and Wisdom, 2013]:

> *A mechanical system is described by a Lagrangian function of the system state (time, coordinates, and velocities). A motion of the system is described by a path that gives the coordinates for each moment of time. A path is allowed if and only if it satisfies the Lagrange equations. Traditionally, the Lagrange equations are written*
>
> $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$
>
> *What could this expression possibly mean?*

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The use of notation for "partial derivative", $\partial L / \partial q$, suggests that L is a function of at least a pair of arguments:

  $$L : \mathbb{R}^i \to \mathbb{R}, i \geq 2$$

  This is consistent with the description: "Lagrangian function of the system state (time, coordinates, and velocities)". So, if we let "coordinates" be just one coordinate, we can take $i = 3$:

  $$L : \mathbb{R}^3 \to \mathbb{R}$$

  The "system state" here is a triple, of type $S = T \times Q \times V$, and we can call the three components $t : T$ for time, $q : Q$ for coordinate, and $v : V$ for velocity. ($T = Q = V = \mathbb{R}$.)

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Looking again at $\partial L / \partial q$, q is the name of a variable, one of the 3 args to L. In the context, which we do not have, we would expect to find somewhere the definition of the Lagrangian as

$$\begin{aligned} L &: T \times Q \times V \to \mathbb{R} \\ L\ (t, q, v) &= \ldots \end{aligned}$$

- therefore, $\partial L / \partial q$ should also be a function of the same triple:

$$(\partial L \,/\, \partial q) : T \times Q \times V \to \mathbb{R}$$

It follows that the equation expresses a relation between *functions*, therefore the 0 on the right-hand side is *not* the real number 0, but rather the constant function $\mathit{const}\ 0$:

$$\begin{aligned} \mathit{const}\ 0 &: T \times Q \times V \to \mathbb{R} \\ \mathit{const}\ 0\ \ (t, q, v) &= 0 \end{aligned}$$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- We now have a problem: $d \mathbin{/} dt$ can only be applied to functions of *one* real argument t, and the result is a function of one real argument:

  $$(d \mathbin{/} dt)\,(\partial L \mathbin{/} \partial \dot{q}) : T \to \mathbb{R}$$

  Since we subtract from this the function $\partial L / \partial q$, it follows that this, too, must be of type $T \to \mathbb{R}$. But we already typed it as $T \times Q \times V \to \mathbb{R}$, contradiction!
- The expression $\partial L / \partial \dot{q}$ appears to also be malformed. We would expect a variable name where we find $\dot{q}$, but $\dot{q}$ is the same as dq/dt, a function.

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The only immediate candidate for an application of d/dt is "a path that gives the coordinates for each moment of time". Thus, the path is a function of time, let us say

  $w : T \to Q$ -- with T for time and Q for coords $(q : Q)$

  We can now guess that the use of the plural form "equations" might have something to do with the use of "coordinates". In an n-dim. space, a position is given by n coordinates. A path would then be

  $w : T \to Q$ -- with $Q = \mathbb{R}^n$

  which is equivalent to n functions of type $T \to \mathbb{R}$, each computing one coordinate as a function of time. We would then have an equation for each of them. We will use $n = 1$ for the rest of this example.

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Now that we have a path, the coordinates at any time are given by the path. And as the time derivative of a coordinate is a velocity, we can actually compute the trajectory of the full system state $T \times Q \times V$ starting from just the path.

$$
\begin{aligned}
&q \;:\; T \to Q \\
&q = w \qquad \text{-- or, equivalently, } q(t) = w(t) \\
&\dot{q} : T \to V \\
&\dot{q} = D\ w \quad \text{-- or, equivalently, } \dot{q}(t) = dw(t)\ /\ dt
\end{aligned}
$$

We combine these in the "combinator" *expand*, given by

$$
\begin{aligned}
&expand : (T \to Q) \to (T \to T \times Q \times V) \\
&expand\ w\ t = (t, w\ t, D\ w\ t)
\end{aligned}
$$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- With *expand* in our toolbox we can fix the typing problem.

$$(\partial L \mathbin{/} \partial q) \circ (\mathit{expand}\ w) : T \to \mathbb{R}$$

- We now move to using D for $d \mathbin{/} dt$, D_2 for $\partial \mathbin{/} \partial q$, and D_3 for $\partial \mathbin{/} \partial \dot{q}$. In combination with $\mathit{expand}\ w$ we find these type correct combinations for the two terms in the equation:

$$\begin{aligned} D\ ((D_3\ L) \circ (\mathit{expand}\ w)) &: T \to \mathbb{R} \\ (D_2\ L) \circ (\mathit{expand}\ w)\ \ &: T \to \mathbb{R} \end{aligned}$$

  The equation becomes

$$D\ ((D_3\ L) \circ (\mathit{expand}\ w)) - (D_2\ L) \circ (\mathit{expand}\ w) = \mathit{const}\ 0$$

  or, after simplification:

$$D\ (D_3\ L \circ \mathit{expand}\ w) = D_2\ L \circ \mathit{expand}\ w$$

Case 3: Lagrangian, summary

"A path is allowed if and only if it satisfies the Lagrange equations" means that this equation is a predicate on paths:

$$\mathit{Lagrange}\ (L, w) = \qquad D\ (D_3\ L \circ \mathit{expand}\ w) == D_2\ L \circ \mathit{expand}\ w$$

Thus: If we can describe a mechanical system in terms of "a Lagrangian" ($L : S \to \mathbb{R}$), then we can use the predicate to check if a particular candidate path $w : T \to \mathbb{R}$ qualifies as a "motion of the system" or not. The unknown of the equation is the path w, and the equation is an example of a partial differential equation (a PDE).

Bibliography

G. J. Sussman and J. Wisdom. *Functional Differential Geometry*. MIT Press, 2013.

Domain-Specific Languages of Mathematics, BSc level course at Chalmers and GU, `https://github.com/DSLsofMath/`