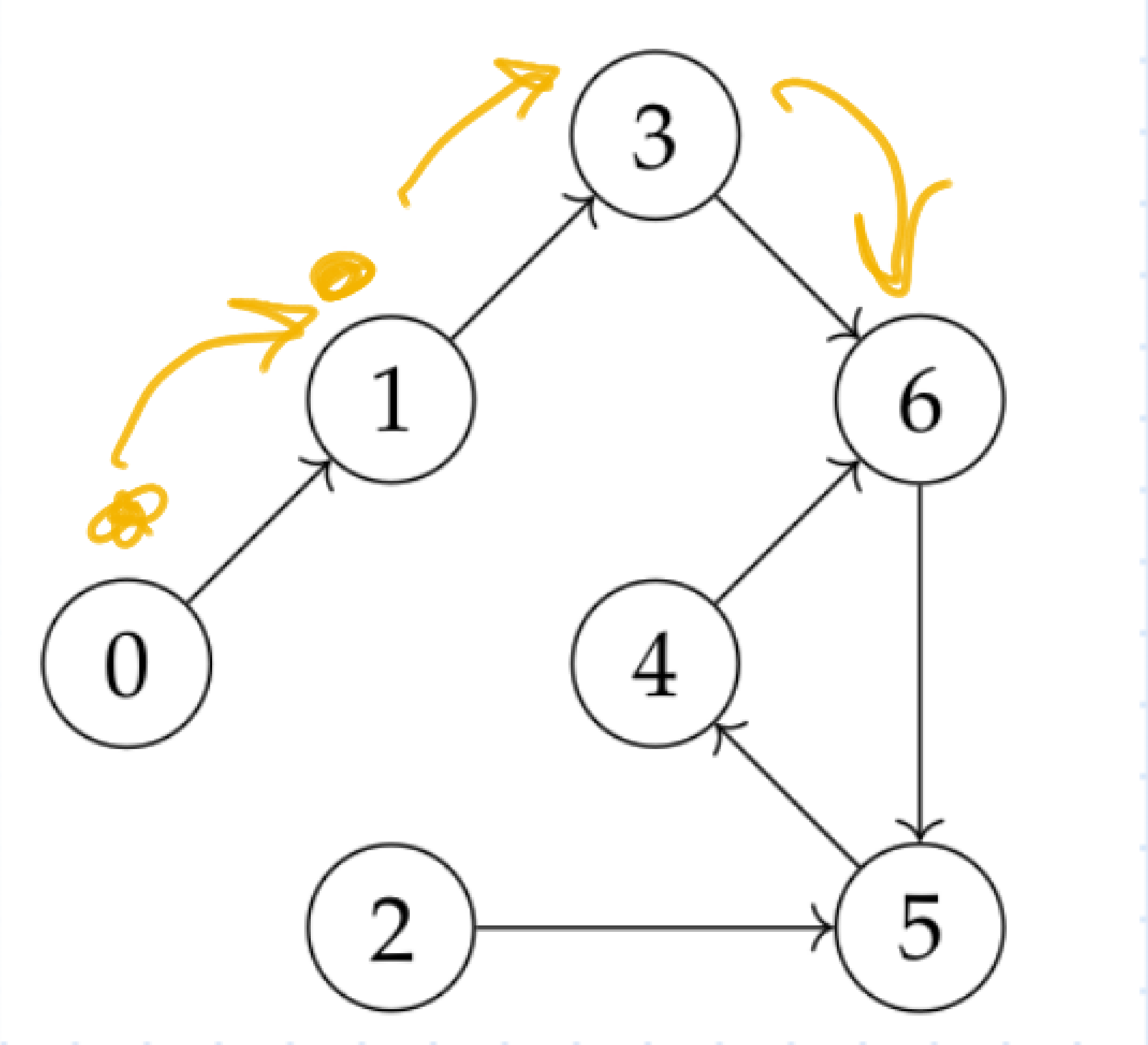
Linear Algebra & Dynamical Systems

Patrik Jansson



Deterministic dyn. sys.

$G = \{0..6\}$

$V = G \to \mathbb{R}$ "stocks"

$next : G \to G$

$next\ 0 = 1;\ next\ 1 = 3;\ \ldots$

$e_0 : V$ = "1 item at node 0"

$h : V \to V$ ← spec. of h

$h\ (e\ i) = e\ (next\ i)$

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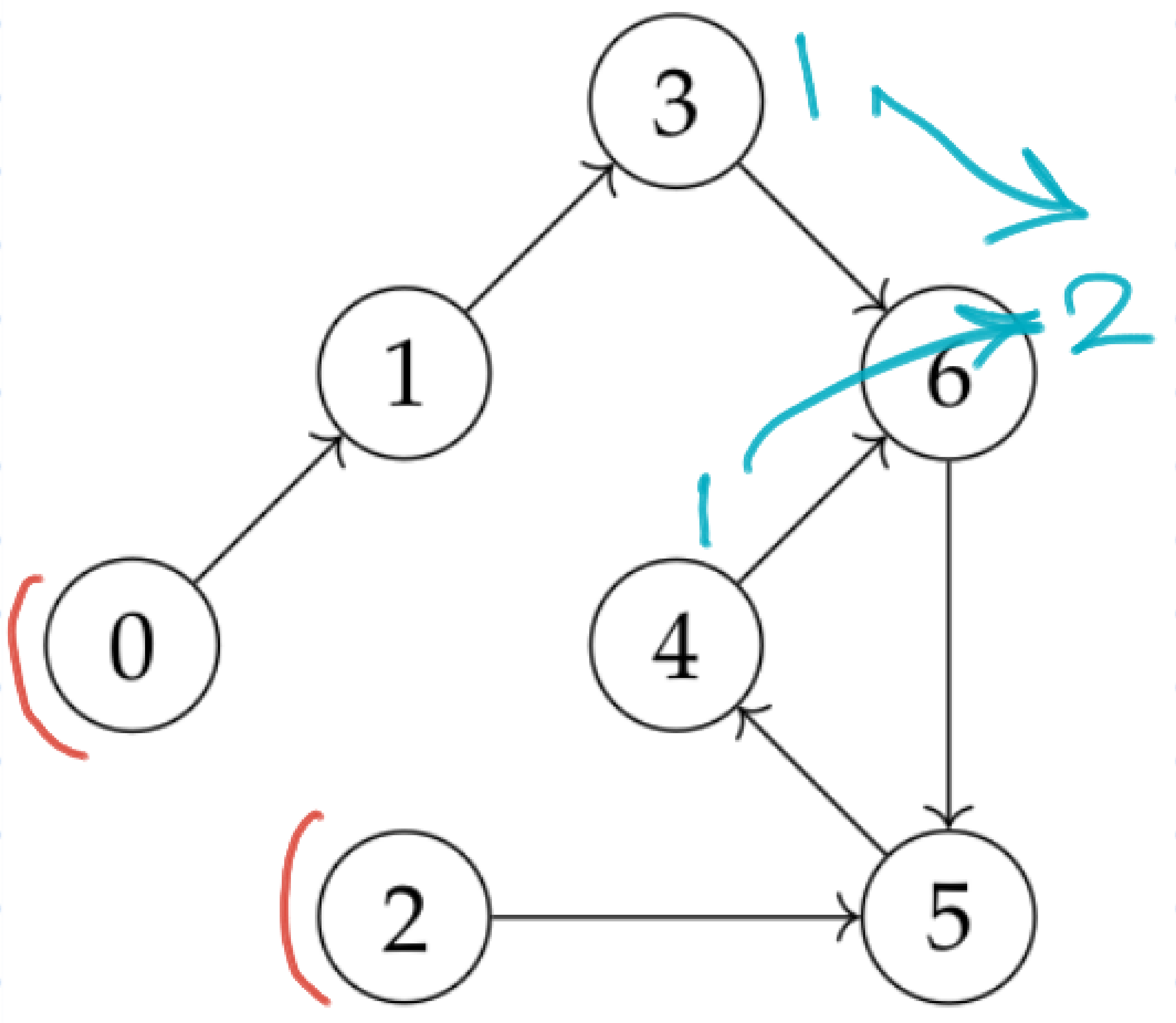

$$f :: Vector\ \mathbb{R}\ G \to Vector\ \mathbb{R}\ G$$
$$f\ (e\ i) = e\ (next\ i)$$

$$M = \begin{array}{c c c}
 & \begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} & \\
\begin{array}{c} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} &
\begin{array}{l} \leftarrow \text{zero} \\ 1 \\ \leftarrow \\ 1 \\ 1 \\ 2 \\ 2 \end{array} \\
 & \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} &
\end{array}$$

Deterministic
dyn. sys.

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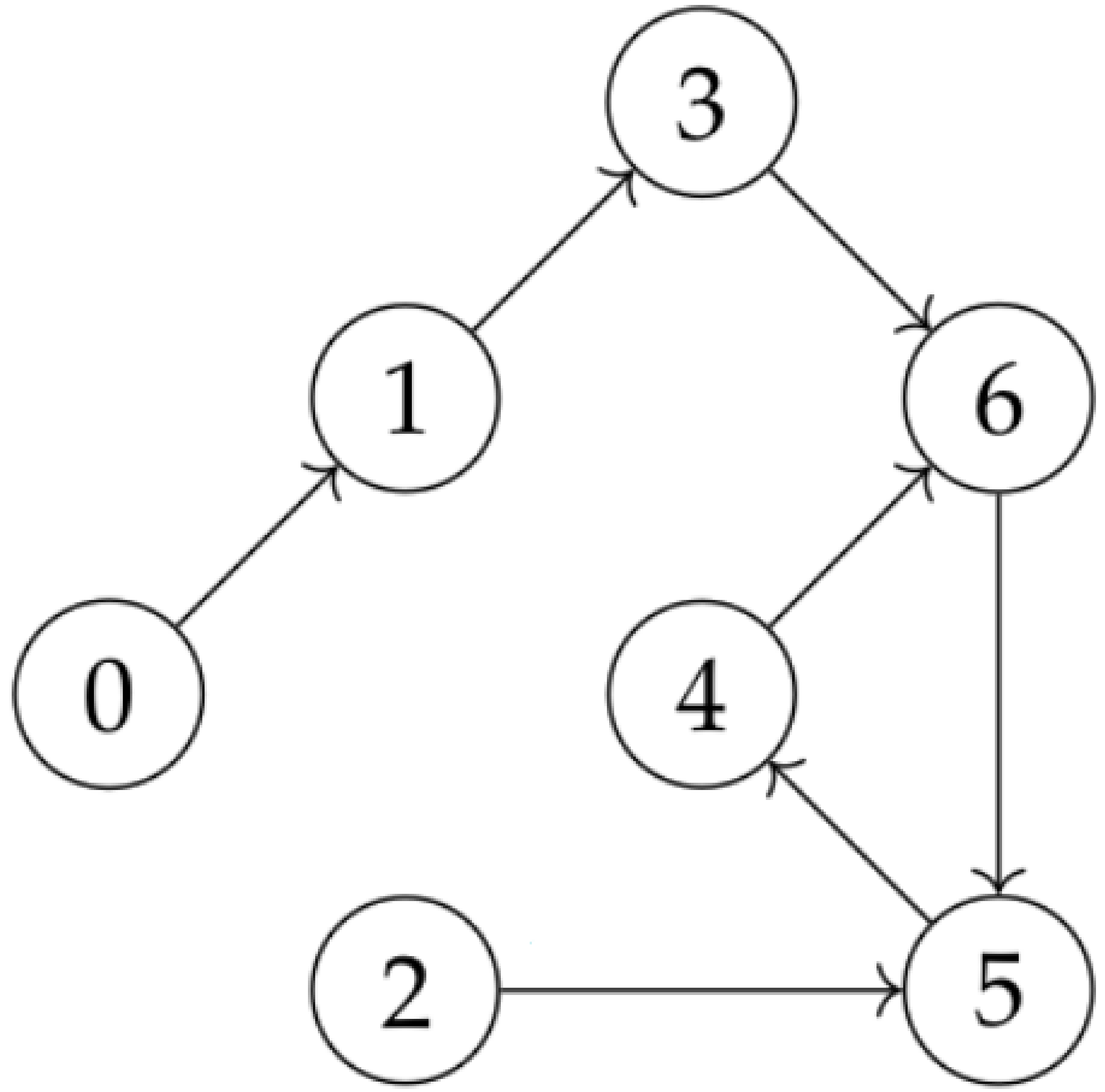

$M^2 = M \cdot M$ = take two steps

$X = 1$
$0 = 0$

$$M = \begin{array}{c c c} & \begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} & \\ \begin{array}{c} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow \text{zero} \\ \\ \leftarrow \text{zero} \\ \\ \\ \\ \\ \end{array} \end{array}$$

Deterministic
dyn. sys.

DSL → δσλ
DSLsofMath

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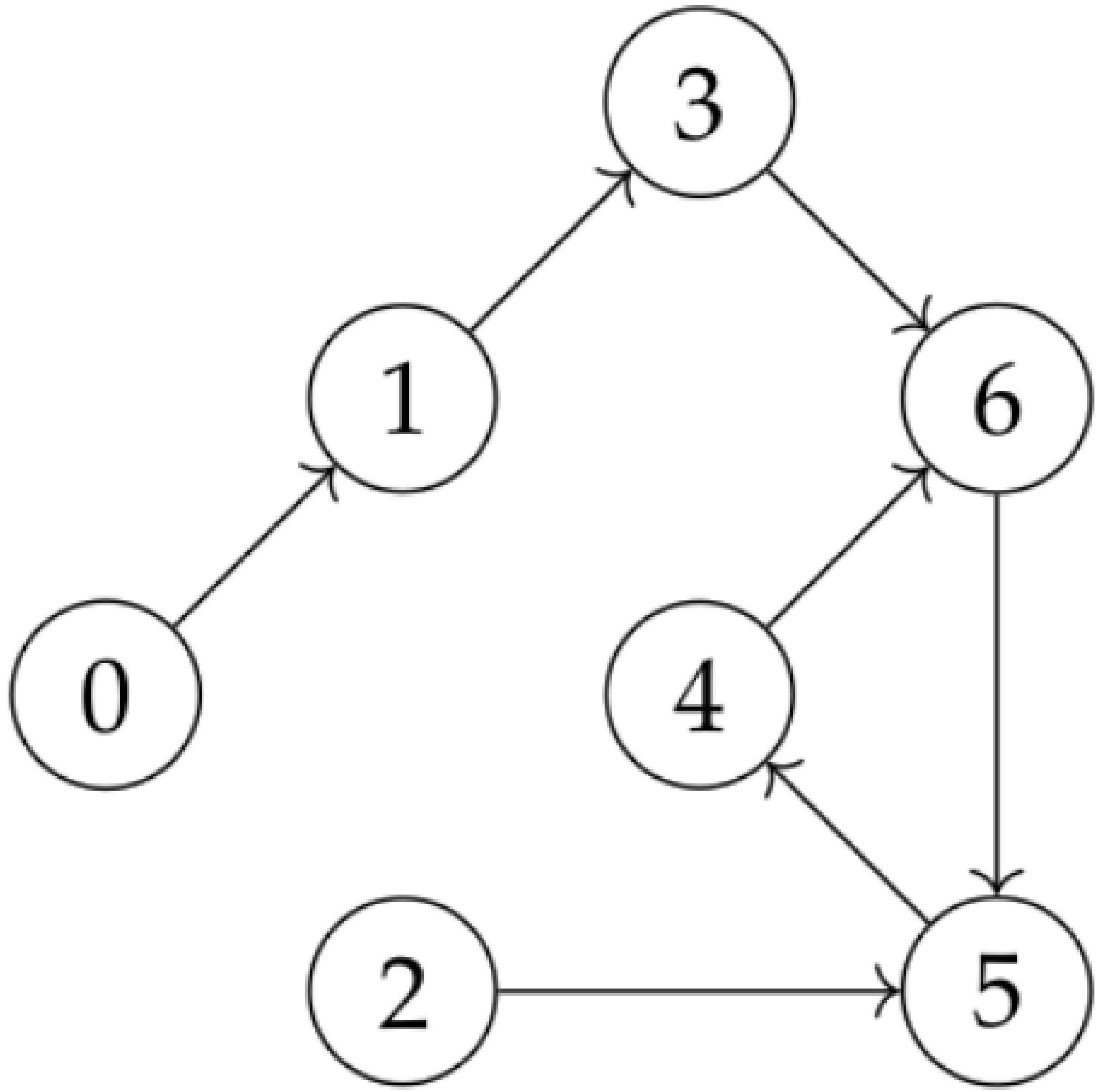

$M^3 =$ take three steps

$x = 1$

$0 = 0$

Deterministic dyn. sys.

$$M = \begin{array}{c} \\ r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

DSL → δσλ

DSLsofMath

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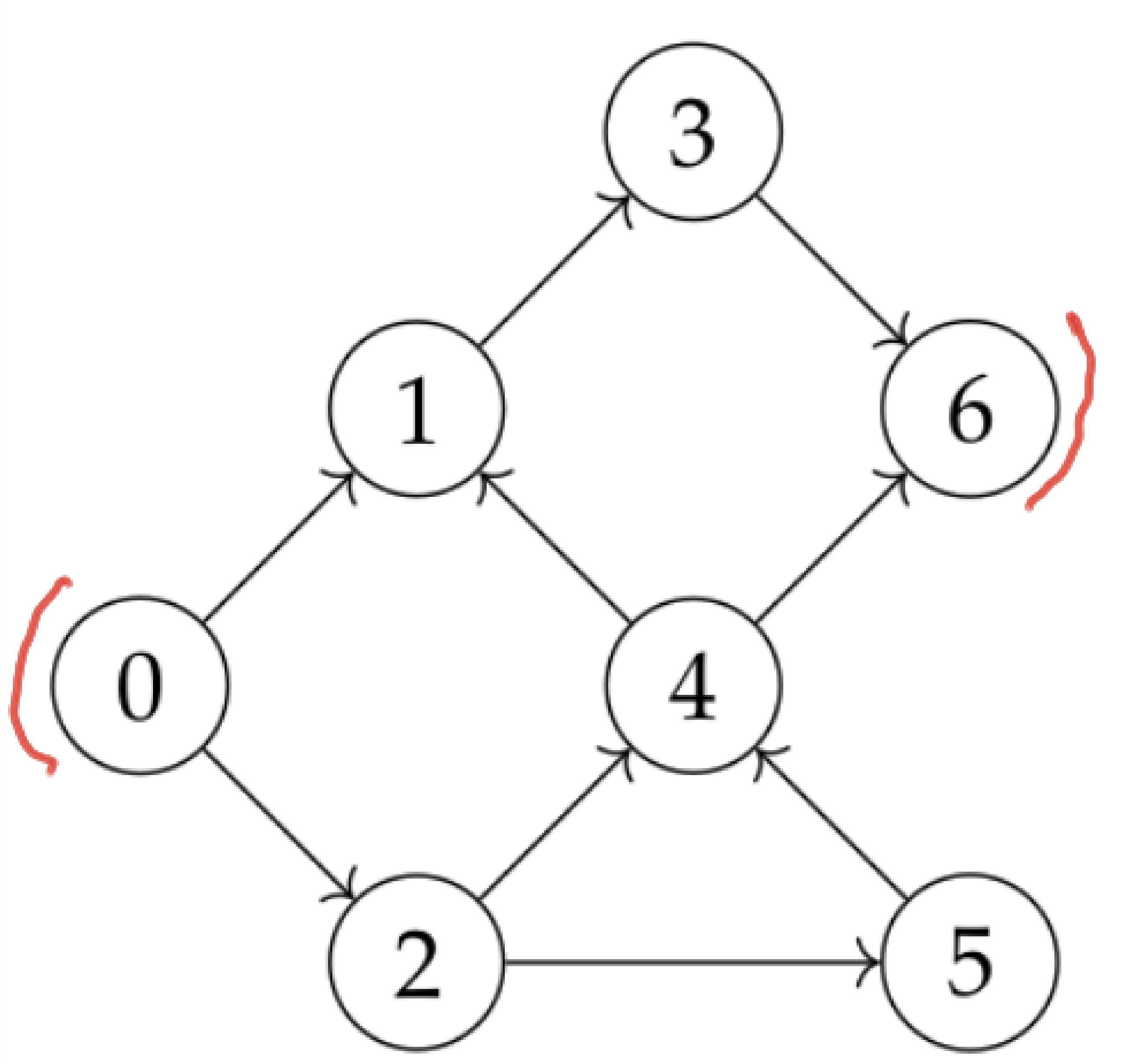

$\text{next} : G \to [G]$

$\text{next}\ 0 = [1,2]$

$\vdots$

$\text{next}\ 6 = []$

zero

$$M = \begin{array}{c} \\ r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

zero

Nondeterministic

dyn. sys.

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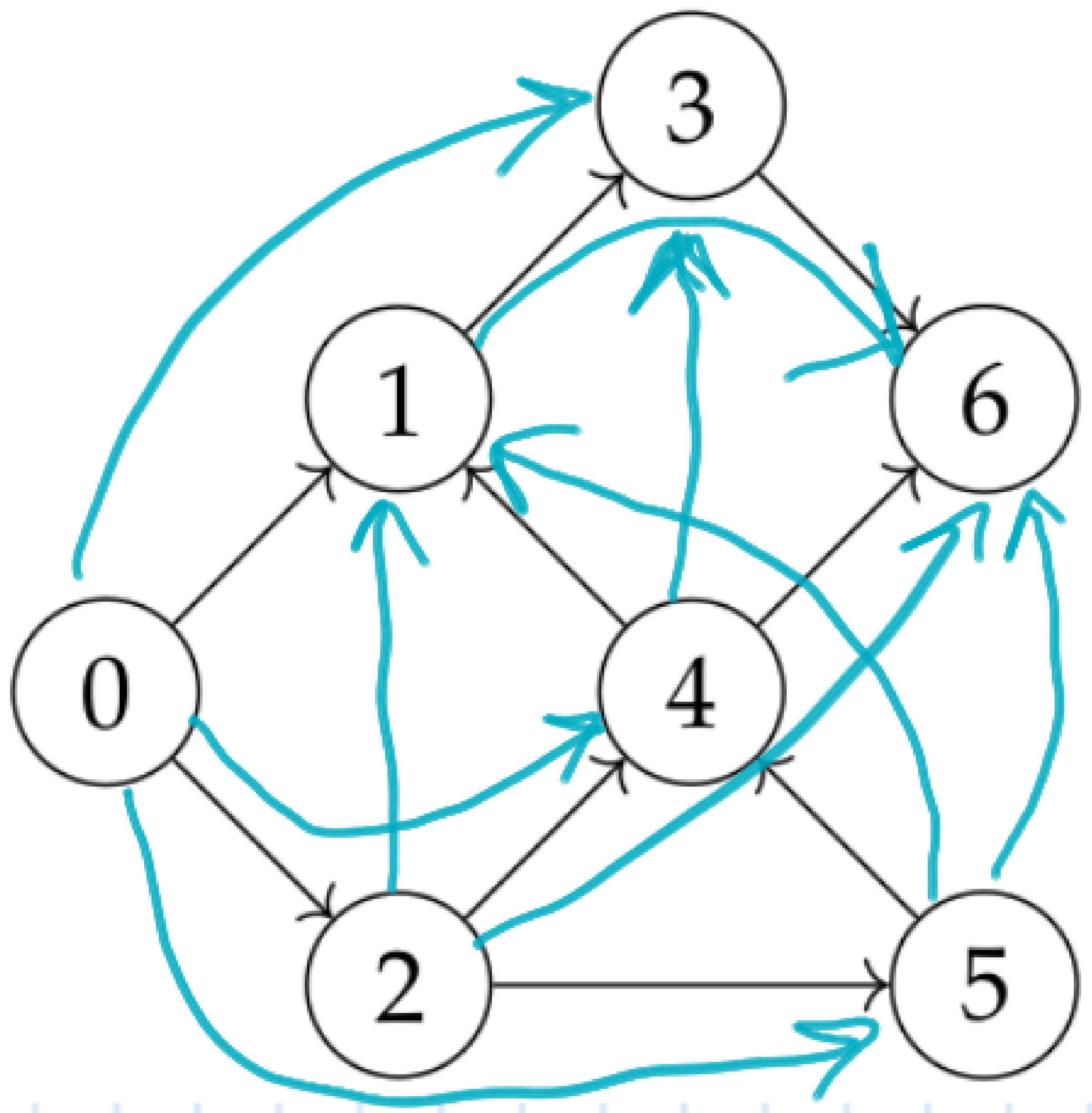

M^2 = take two steps

M^6 = all zeroes

Nondeterministic dyn. sys.

$$
M = \begin{array}{c} \\ r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{array}
\Bigg\} \text{zero}
$$

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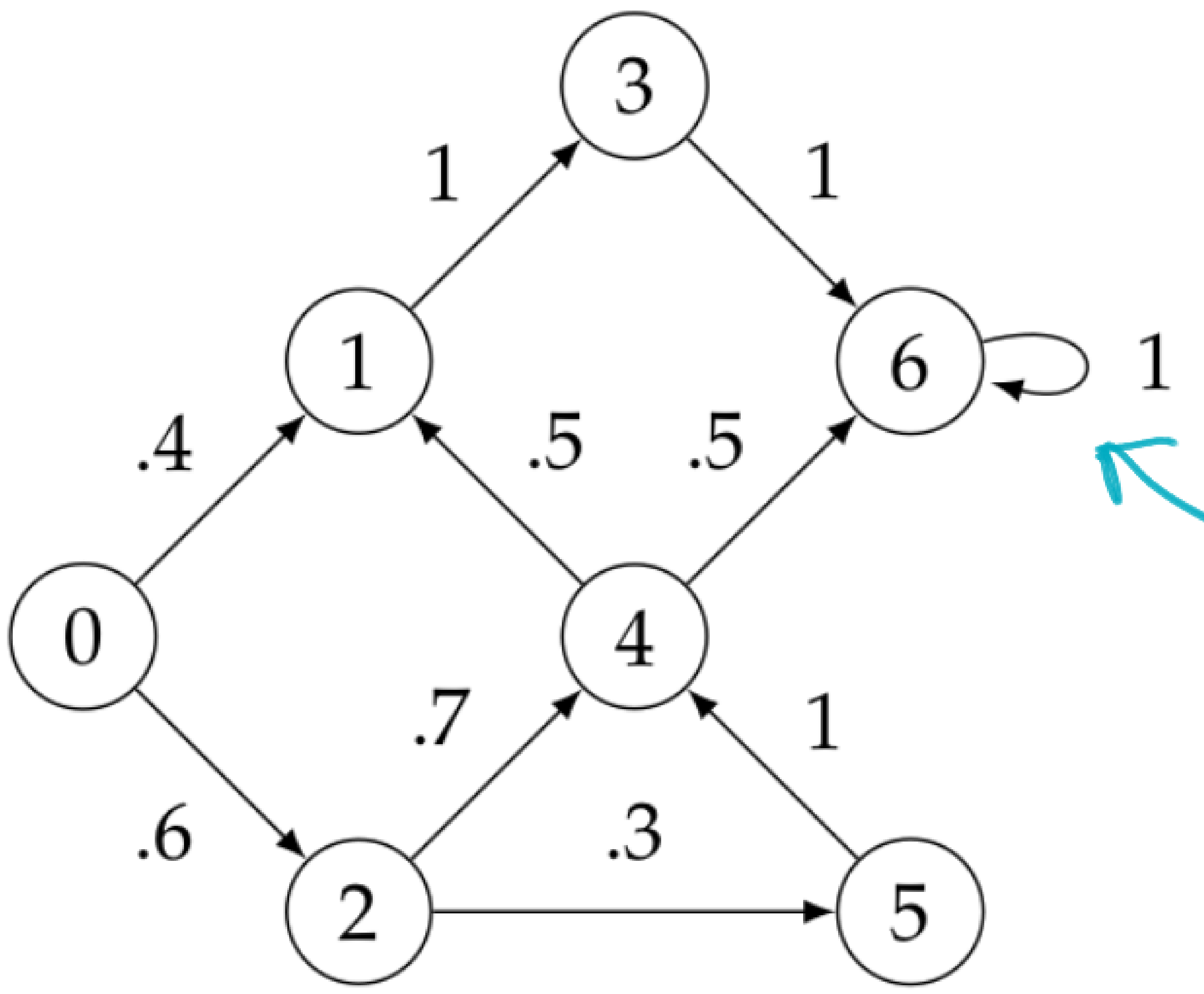

$$V = \{ p : G \to \text{Prob} \mid \sum_i p_i = 1 \}$$

Probabilistic / stochastic dyn. sys.

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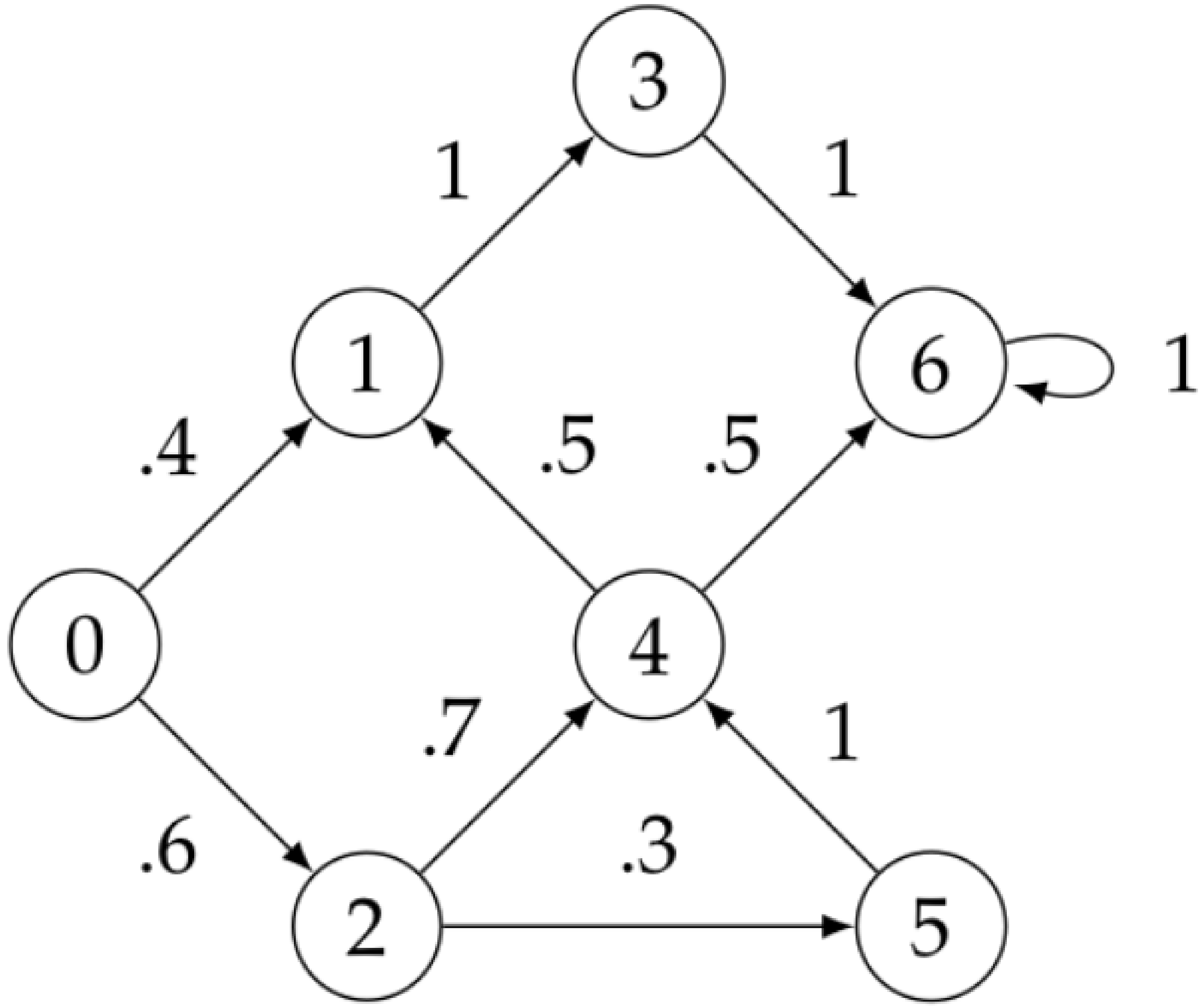

$$V = \{ p : G \to \mathrm{Prob} \mid \sum_i p_i = 1 \}$$

... columns sum to 1 = 100%

$$M = \begin{array}{c} \\ r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .4 & 0 & 0 & 0 & .5 & 0 & 0 \\ .6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .7 & 0 & 0 & 1 & 0 \\ 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .5 & 0 & 1 \end{pmatrix} \end{array}$$

Probabilistic / stochastic dyn. sys.

[For more, see Markov chains.]

DSL → δσλ
DSLsofMath

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Deterministic
Non-deterministic
Stochastic / Probabilistic

⋮

transition matrices

[More: Monadic Dynamic Systems]

Next:

- Live coding vectors & matrices

Laplace

$$L : V \to W$$

$$L(\alpha \cdot f + \beta \cdot g) =$$
$$\alpha \cdot Lf + \beta \cdot Lg$$

$$L(f + 2 \cdot f' + 3 \cdot f'') =$$
$$Lf + 2 \cdot Lf' + 3 \cdot Lf''$$

$$V \cong \mathbb{R} \to \mathbb{R}$$
$$W \cong \mathbb{C} \to \mathbb{C}$$
$$\alpha, \beta : \mathbb{R}$$
$$f, g : V = \mathbb{R} \to \mathbb{R}$$
$$(\cdot) = scale : \mathbb{R} \to V \to V$$

Laplace

$$L : V \to W$$

$$L(\alpha \cdot f + \beta \cdot g) =$$
$$\alpha \cdot Lf + \beta \cdot Lg$$

$$L[f(t)] = F(s)$$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L \sin s = \frac{1}{s^2+1}$$

$$L \cos s = \frac{s}{s^2+1}$$

$(\cdot) = \text{scale}$

$$f'' + 4 \cdot f' + f = 6 \cdot \cos \quad , \quad f\,0 = 0 \quad , \quad f'\,0 = 0$$

$$L\,f'\,s = -f\,0 + s \cdot L\,f\,s$$

$$L\,f\,s = 6 \cdot \frac{s}{(s^2+1)\cdot(s^2+4s+1)}$$

$$L\,(f'' + 4 \cdot f' + f)\,s = L\,(6 \cdot \cos)\,s$$

$$L\,f''\,s + 4 \cdot L\,f'\,s + L\,f\,s = 6 \cdot L\,\cos\,s$$

$$s^2 \cdot L\,f\,s + 4 \cdot s \cdot L\,f\,s + L\,f\,s = 6 \cdot \frac{s}{s^2+1}$$

$$(s^2+4s+1) \cdot L\,f\,s = \text{—''—}$$

$$f'' + 4\cdot f' + f = 6\cdot\cos\ ,\quad f\,0 = 0\ ,\quad f'\,0 = 0$$

$$L\,f'\,s = -f\,0 + s\cdot L\,f\,s$$

$$L\,f\,s = 6\cdot\frac{s}{(s^2+1)\cdot(s^2+4s+1)} = \frac{3}{2}\cdot\Big(\overbrace{\frac{1}{s^2+1} - \frac{1}{s^2+4s+1}}^{4s}\Big)$$

$$s^2+4s+1 = (s-s_1)\cdot(s-s_2)$$

$$s_1 = -2+\sqrt{3}\ ,\quad s_2 = -2-\sqrt{3}$$

$$\frac{1}{p\cdot q} = \frac{A}{p} + \frac{B}{q}$$

$$L\sin s \longrightarrow \frac{1}{s^2+1}$$

$$f'' + 4\cdot f' + f = 6\cdot\cos\ ,\ f\,0 = 0\ ,\ f'\,0 = 0$$

$$L\,f'\,s = -f\,0 + s\cdot L\,f\,s$$

$$\frac{3}{2}\cdot\frac{1}{s^2+4s+1} = \frac{A}{s-s_1} + \frac{B}{s-s_2} \qquad s_1 = -2+\sqrt{3}\ ,\ s_2 = -2-\sqrt{3}$$

$$3 = 2\cdot A\cdot(s-s_2) + 2\cdot B\cdot(s-s_1)$$

$$s=s_1: \quad 3 = 2\cdot A\cdot(s_1-s_2) = 2\cdot A\cdot 2\cdot\sqrt{3} \Rightarrow A = \frac{\sqrt{3}}{4}$$

$$s=s_2: \quad 3 = 2\cdot B\cdot(s_2-s_1) = 2\cdot B\cdot(-2\sqrt{3}) \Rightarrow B = -A$$

$$L\,f\,s = \frac{3}{2}\cdot L\sin s - \frac{\sqrt{3}}{4}\cdot\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

$$f'' + 4 \cdot f' + f = 6 \cdot \cos \;, \quad f\,0 = 0 \;, \quad f'\,0 = 0$$

$$L\,f'\,s = -f\,0 + s \cdot L\,f\,s$$

$s_1 = -2 + \sqrt{3} \;, \; s_2 = -2 - \sqrt{3}$

$\approx -0{,}3 \qquad \approx -3{,}7$

$$L\,f\,s = \frac{3}{2} \cdot L\sin s - \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{s - s_1} - \frac{1}{s - s_2} \right)$$

$$f\,t = \frac{3}{2} \cdot \sin t - \frac{\sqrt{3}}{4} \cdot \left(\exp(s_1 \cdot t) - \exp(s_2 \cdot t) \right)$$

$$f'\,t = \frac{3}{2} \cdot \cos t - \frac{\sqrt{3}}{4} \cdot \left(s_1 \cdot \qquad - s_2 \cdot \qquad \right)$$

$$f''\,t = -\frac{3}{2} \sin t - \frac{\sqrt{3}}{4} \left(s_1^2 \cdot \qquad - s_2^2 \cdot \qquad \right)$$

$$f'' + 4\cdot f' + f = 6\cdot\cos, \quad f\,0 = 0, \quad f'0 = 0$$

$s_1 = -2+\sqrt{3}, \quad s_2 = -2-\sqrt{3}$

$$f\,t = \frac{3}{2}\cdot\sin t - \frac{\sqrt{3}}{4}\cdot\left(\exp(s_1\cdot t) - \exp(s_2\cdot t)\right)$$

$$f'\,t = \frac{3}{2}\cdot\cos t - \frac{\sqrt{3}}{4}\cdot\left(s_1\cdot\exp(s_1\cdot t) - s_2\cdot\exp(s_2\cdot t)\right)$$

$$f''\,t = -\frac{3}{2}\cdot\sin t - \frac{\sqrt{3}}{4}\cdot\left(s_1^2\cdot\exp(s_1\cdot t) - s_2^2\cdot\exp(s_2\cdot t)\right)$$

$$\mathrm{dot}(\;)(1,4,1) = 4\cdot\frac{3}{2}\cdot\cos t = 6\cdot\cos t = \mathrm{RHS}$$

$$\mathrm{dot}\;(1,4,1) = (1+4\cdot s_1 + s_1^2)\cdot\exp(s_1\cdot t) = 0$$

$$\mathrm{dot}\;(1,4,1) = (1+4 s_2 + s_2^2)\cdot\exp(s_2\cdot t) = 0$$

OK!

$$f'' + 4 \cdot f' + f = 6 \cdot \cos \ , \ f\,0 = 0 \ , \ f'\,0 = 0$$

$$L\,f'\,s = -f\,0 + s \cdot L\,f\,s \qquad s_1 = -2 + \sqrt{3} \ , \ s_2 = -2 - \sqrt{3}$$

$$L\,f\,s = \frac{3}{2} \cdot L\,\sin\,s - \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{s - s_1} - \frac{1}{s - s_2} \right)$$

$$f\,t = \frac{3}{2} \cdot \sin t - \frac{\sqrt{3}}{4} \cdot \left(\exp(s_1 \cdot t) - \exp(s_2 \cdot t) \right)$$

$$f'\,t = \frac{3}{2} \cdot \cos t - \frac{\sqrt{3}}{4} \cdot \left(s_1 \cdot \exp(s_1 \cdot t) - s_2 \cdot \exp(s_2 \cdot t) \right)$$

$$f''\,t = -\frac{3}{2} \cdot \sin t - \frac{\sqrt{3}}{4} \cdot \left(s_1^2 \cdot \exp(s_1 \cdot t) - s_2^2 \cdot \exp(s_2 \cdot t) \right)$$

$$\left.\begin{array}{l} f\,0 = 0 - \frac{\sqrt{3}}{4} \cdot (1 - 1) = 0 \\ f'\,0 = \frac{3}{2} \cdot 1 - \frac{\sqrt{3}}{4} \cdot (s_1 \cdot 1 - s_2 \cdot 1) = \frac{3}{2} - \frac{\sqrt{3}}{4} \cdot 2 \cdot \sqrt{3} = \frac{3}{2} - \frac{3}{2} = 0 \end{array}\right\} OK$$