Partial derivatives

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tions. Our example here is by Mac Lane [1986, page 169], where we read

> [...] a function $\boxed{z = f(x,y)}$ for all points (x,y) in some open set U of the cartesian (x,y)-plane. [...] If one holds y fixed, the quantity z remains just a function of x; its derivative, when it exists, is called the *partial derivative* with respect to x. Thus at a point (x,y) in U this derivative for $h \neq 0$ is

$$= \frac{\partial f(x,y)}{\partial x}$$

$$\partial z/\partial x = f'_x(x,y) = \overbrace{\lim_{h\to 0}(\underbrace{f(x+h,y) - f(x,y))/h}_{\text{1-Arg expr. in } h}}^{\mathbb{R}}$$

Types: $f : \underset{x}{\mathbb{R}} \times \underset{y}{\mathbb{R}} \to \underset{z}{\mathbb{R}}$ $\qquad f : U \to \mathbb{R}$

$\frac{\partial z}{\partial x} : \mathbb{R}$ $\qquad f'_x : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\qquad h : \mathbb{R} \setminus \{0\}$

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[...] a function $z = f(x,y)$ for all points (x,y) in some open set U of the cartesian (x,y)-plane. [...] If one holds y fixed, the quantity z remains just a function of x; its derivative, when it exists, is called the *partial derivative* with respect to x. Thus at a point (x,y) in U this derivative for $h \neq 0$ is

$\boxed{D_1 f}$ $\qquad (D_1 f)(x,y) \neq D_1(f(x,y))$

$$\partial z/\partial x = \boxed{f'_x}(x,y) = \lim_{h\to 0}(f(x+h,y) - f(x,y))/h$$

$D_1 : (\mathbb{R}\times\mathbb{R}\to\mathbb{R}) \to (\mathbb{R}\times\mathbb{R}\to\mathbb{R})$

$(D_1\ f)(x,y) = \lim 0\ (\Psi_1\ f\ (x,y))$

$\Psi_1 : (\mathbb{R}\times\mathbb{R}\to\mathbb{R}) \to \mathbb{R}\times\mathbb{R} \to \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$\Psi_1\ f\ (x,y)\ h = (f(x+h,y) - f(x,y))/h$

$\Psi_2\ f\ (x,y)\ h = (f(x,y+h) - f(x,y))/h$

$q\ f = \lambda p \to e$

$\Updownarrow$

$q\ f\ p = e$

e *partial derivative* with respect to x. Thus at a point (x, y) in U this
erivative for $h \neq 0$ is

$z = f(x, y)$

$$\partial z / \partial x = f'_x(x, y) = \lim_{h \to 0} (f(x + h, y) - f(x, y))/h$$

$: \mathbb{R}$

$D_1, D_2 : (\mathbb{R} \times \mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \times \mathbb{R} \to \mathbb{R})$

Here $D_1 = \frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial y}$ where $f(x, y) = \cdots\cdots$

Matching "by name"

Be careful with scoping
& the difference between
expressions & functions

$\frac{\partial f}{\partial x} : \mathbb{R}^2 \to \mathbb{R}$

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$$z = f(x,y) = x^2 + yx$$

$$\frac{\partial z}{\partial x} \{at\ (a,b)\} = 2a + b$$

$$\frac{\partial z}{\partial y} \{at\ (x_0,y_0)\} = x_0$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}\left((2t)^2 + (1-t)\cdot 2t\right) =$$

$$= \frac{\partial}{\partial t}\left(4t^2 + 2t - 2t^2\right) = 4t + 2$$

$x = 2t$, $y = 1 - t$

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DSLsofMath: Typing Mathematics (Week 3)

the Lagrangian Equations

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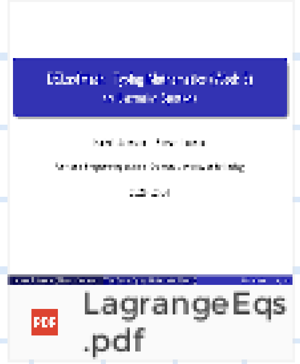
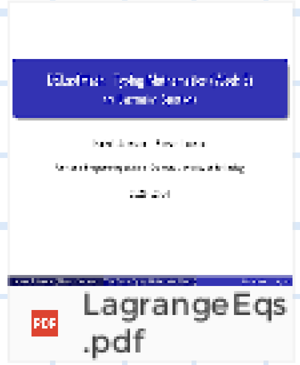


Starting point: a math quote (the Lagrangian)



L

From [Sussman and Wisdom, 2013]:

> *A mechanical system is described by a Lagrangian function of the system state (time, coordinates, and velocities). A motion of the system is described by a path that gives the coordinates for each moment of time. A path is allowed if and only if it satisfies the Lagrange equations. Traditionally, the Lagrange equations are written*
>
> $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$
>
> *What could this expression possibly mean?*

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

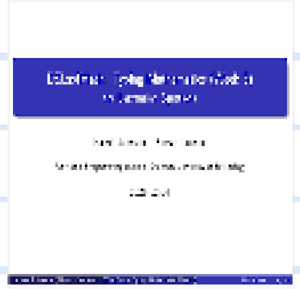
- The use of notation for "partial derivative", $\partial L / \partial q$, suggests that L is a function of at least a pair of arguments:

  $$L : \mathbb{R}^{\mathrm{i}} \to \mathbb{R}, i \geqslant 2$$

  This is consistent with the description: "Lagrangian function of the system state (time, coordinates, and velocities)". So, if we let "coordinates" be just one coordinate, we can take $i = 3$:

  $$L : \mathbb{R}^3 \to \mathbb{R}$$

  The "system state" here is a triple, of type $S = T \times Q \times V$, and we can call the three components $t : T$ for time, $q : Q$ for coordinate, and $v : V$ for velocity. ($T = Q = V = \mathbb{R}$.)



LagrangeEqs .pdf

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Looking again at $\partial L / \partial q$, q is the name of a variable, one of the 3 args to L. In the context, which we do not have, we would expect to find somewhere the definition of the Lagrangian as

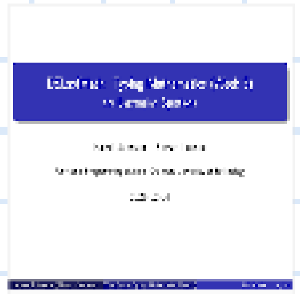
$$\begin{array}{ll} L : T \times Q \times V \to & \mathbb{R} \\ L \ \ (t, q, v) & = \ \ldots \end{array}$$

- therefore, $\partial L / \partial q$ should also be a function of the same triple:

$$(\partial L \, / \, \partial q) : T \times Q \times V \to \mathbb{R}$$

It follows that the equation expresses a relation between *functions*, therefore the 0 on the right-hand side is *not* the real number 0, but rather the constant function *const* 0:

$$\begin{array}{ll} const\ 0 : T \times Q \times V \to \mathbb{R} & \\ const\ 0 \ \ (t, q, v) & = 0 \end{array}$$



LagrangeEqs .pdf

Lagrangian, cont.

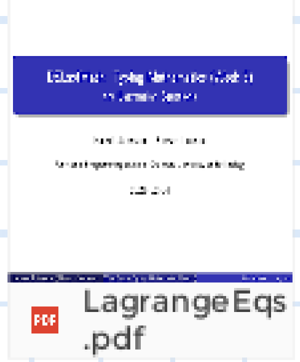
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- We now have a problem: $d \,/\, dt$ can only be applied to functions of *one* real argument t, and the result is a function of one real argument:

  $$(d \,/\, dt)\,(\partial L \,/\, \partial \dot{q}) : T \to \mathbb{R}$$

  Since we subtract from this the function $\partial L / \partial q$, it follows that this, too, must be of type $T \to \mathbb{R}$. But we already typed it as $T \times Q \times V \to \mathbb{R}$, contradiction!
- The expression $\partial L / \partial \dot{q}$ appears to also be malformed. We would expect a variable name where we find $\dot{q}$, but $\dot{q}$ is the same as dq/dt, a function.

$L(t, q, v) = \ldots$



$$\frac{d}{dt} = D : (T \to \mathbb{R}) \to (T \to \mathbb{R})$$

$$f' = \dot{f} = \frac{df}{dt}$$

$$\frac{\partial L}{\partial \left(\frac{dq}{dt}\right)} = \frac{\partial L}{\partial v} = D_3$$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

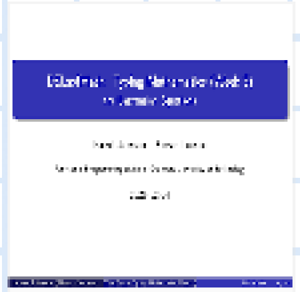
- The only immediate candidate for an application of d/dt is "a path that gives the coordinates for each moment of time". Thus, the path is a function of time, let us say

  $w : T \to Q$ -- with T for time and Q for coords $(q : Q)$

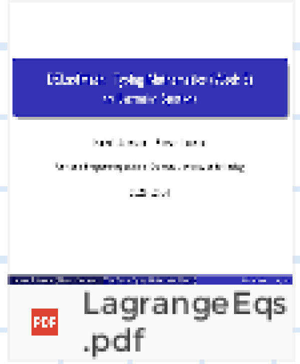
  We can now guess that the use of the plural form "equations" might have something to do with the use of "coordinates". In an n-dim. space, a position is given by n coordinates. A path would then be

  $w : T \to Q$ -- with $Q = \mathbb{R}^n$

  which is equivalent to n functions of type $T \to \mathbb{R}$, each computing one coordinate as a function of time. We would then have an equation for each of them. We will use $n = 1$ for the rest of this example.



LagrangeEqs.pdf



$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Now that we have a path, the coordinates at any time are given by the path. And as the time derivative of a coordinate is a velocity, we can actually compute the trajectory of the full system state $T \times Q \times V$ starting from just the path.

$$q \;:\; T \to Q$$
$$q = w \qquad \text{-- or, equivalently, } q\,(t) = w\,(t)$$
$$\dot{q} : T \to V$$
$$V = \quad \dot{q} = D\,w \qquad \text{-- or, equivalently, } \dot{q}\;t = dw\,(t)\;/\;dt$$

We combine these in the "combinator" *expand*, given by

Path

$$expand : (T \to Q) \to (T \to T \times Q \times V)$$
$$expand\;w\;t = (t, w\;t, D\;w\;t)$$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- With *expand* in our toolbox we can fix the typing problem.

$$(\partial L \,/\, \partial q) \circ (expand\ w) : T \to \mathbb{R}$$

- We now move to using D for $d \,/\, dt$, D_2 for $\partial \,/\, \partial q$, and D_3 for $\partial \,/\, \partial \dot{q}$. In combination with *expand w* we find these type correct combinations for the two terms in the equation:

$$\begin{aligned} D\,((D_3\ L) \circ (expand\ w)) &: T \to \mathbb{R} \\ (D_2\ L) \circ (expand\ w)\ &: T \to \mathbb{R} \end{aligned}$$

  The equation becomes

$$D\,((D_3\ L) \circ (expand\ w)) - (D_2\ L) \circ (expand\ w) = const\ 0$$

  or, after simplification:

$$D\,(D_3\ L \circ expand\ w) = D_2\ L \circ expand\ w$$

$$\begin{tikzcd} T \arrow[r, "ex\ w"] \arrow[d] & T \times Q \times V \arrow[dl, "\frac{\partial L}{\partial q}"] \\ \mathbb{R} & \end{tikzcd}$$

$$\frac{\partial}{\partial \dot{q}} = \frac{\partial}{\partial v} = D_3$$

Case 3: Lagrangian, summary

"A path is allowed if and only if it satisfies the Lagrange equations" means that this equation is a predicate on paths:

$$Lagrange\ (L, w) = \qquad D\ (D_3\ L \circ expand\ w) == D_2\ L \circ expand\ w$$

Thus: If we can describe a mechanical system in terms of "a Lagrangian" ($L : S \to \mathbb{R}$), then we can use the predicate to check if a particular candidate path $w : T \to \mathbb{R}$ qualifies as a "motion of the system" or not. The unknown of the equation is the path w, and the equation is an example of a partial differential equation (a PDE).

Example of Lagrange eq.s

$$L : T \times Q \times V \longrightarrow \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$D_3 L : T \times Q \times V \longrightarrow \mathbb{R}$$

$$D_3 L (t, x, v) = m \cdot v$$

$$D_2 L (t, x, v) = -m \cdot g$$

$$\mathit{Lagrange}\,(L, w) = \quad D\,(D_3\, L \circ \mathit{expand}\ w) == D_2\, L \circ \mathit{expand}\ w$$

Example of Lagrange eq.s

$$L : T\times Q\times V \to \mathbb{R}$$

$$L(t, x, v) = -m\cdot g\cdot x + \frac{m\cdot v^2}{2}$$

$$ex : (T \to Q) \to (T \to T\times Q\times V)$$

$$ex\ w\ t = (t, w\,t, D\,w\,t)$$

$$\begin{tikzcd} \mathbb{R} & T\times Q\times V \arrow[l] & T \arrow[l] \arrow[ll, bend right, "\alpha"'] \end{tikzcd}$$

$$\mathsf{Lagrange}\,(L, w) = \quad D\,(\underbrace{D_3\,L \circ \mathsf{expand}\ w}_{\alpha}) \mathrel{==} \underbrace{D_2\,L \circ \mathsf{expand}\ w}_{\beta}$$

Example of Lagrange eq.s

$L : T \times Q \times V \to \mathbb{R}$

$L\,(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$

$D_3 L\,(\_, \_, v) = m \cdot v$

$D_2 L\,(\_, \_, \_) = -m \cdot g$

$\alpha\ t = D_3 L\ (ex\ w\ t\ ) = 2 \cdot m \cdot B \cdot t$

$D \alpha\ t = \underbrace{2 \cdot m \cdot B}$

$D\alpha = const$

$w : T \to Q$

$w\ t = A + B \cdot t^2$

$ex\ w\ t =$
$(t, A + B t^2, 2 \cdot B \cdot t)$

$\beta\ t = -m \cdot g$

$\beta = const\,(-m \cdot g)$

$2 m \cdot B == -mg$

$const\ (2 \cdot m \cdot B) == const\,(-m \cdot g)$

$$Lagrange\,(L, w) = \quad D\,(\underbrace{D_3\ L \circ expand\ w}_{\alpha}) == \underbrace{D_2\ L \circ expand\ w}_{\beta}$$

Example of Lagrange eq.s

$L : T \times Q \times V \to \mathbb{R}$

$L\,(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$

$D_3 L\,(\_, \_, v) = m \cdot v$

$D_2 L\,(\_, \_, \_) = -m \cdot g$

$\alpha\ t = D_3 L\ (ex\ w\ t) = 2 \cdot B \cdot m \cdot t$

$D\alpha\ t = 2 \cdot B \cdot m$

$w : T \to Q$

$w\ t = A + B \cdot t^2$

$ex\ w\ t = (t, A + B \cdot t^2, 2 \cdot B \cdot t)$

$\beta\ t = -m \cdot g$

$m = 0$ or $B = -g/2$

$const\ (2 \cdot B \cdot m) == const\ (-m \cdot g)$

$$Lagrange\ (L, w) = D\,(\underbrace{D_3\,L \circ expand\ w}_{\alpha}) == \underbrace{D_2\,L \circ expand\ w}_{\beta}$$

Example of Lagrange eq.s

$$L : T \times Q \times V \to \mathbb{R}$$

$$L(t, x, v) = -m \cdot g \cdot x + \frac{m \cdot v^2}{2}$$

$$w : T \to Q$$

$$w\,t = A + B \cdot t^2$$

$$\mathit{Lagrange}\ (L, w) = \quad D\,(D_3\, L \circ \mathit{expand}\ w) == D_2\, L \circ \mathit{expand}\ w$$

$\Rightarrow$ $m = 0$ with A, B free

or $w\,t = A - \dfrac{g \cdot t^2}{2}$ with A free

(free fall with acc. $-g$ from height A)