Linear Algebra & Homomorphisms

Patrik Jansson

$$h : V \longrightarrow W \qquad \text{"linear transform"}$$

$$
\begin{aligned}
\mathrm{LinTran}(h, V, W) = \quad & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\
\wedge\ & H_2(h, (+_V), (+_W)) \\
\wedge\ & \forall c.\ H_1(h, \mathit{scale}_V\, c, \mathit{scale}_W\, c)
\end{aligned}
$$

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$$h : V \longrightarrow W$$

"linear transform"

$$\begin{aligned}
\mathit{LinTran}(h, V, W) = \; & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\
\wedge \; & H_2(h, (+_V), (+_W)) \\
\wedge \; & \forall c.\; H_1(h, \mathit{scale}_V\, c, \mathit{scale}_W\, c)
\end{aligned}$$

Examples: $\forall c.\; \mathit{LinTran}(\mathit{scale}_V\, c, V, V)$

$\forall c.\; \mathit{LinTran}(\mathit{apply}\, c, a \to s, s)$

"projections"

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"linear transform"

$$\begin{aligned}\mathrm{LinTran}(h, V, W) = \quad & H_0(h, \mathit{zero}_V, \mathit{zero}_W) \\ \wedge\ & H_2(h, (+_V), (+_W)) \\ \wedge\ & \forall c.\ H_1(h, \mathit{scale}_V\, c, \mathit{scale}_W\, c)\end{aligned}$$

$$v : V = G \to s$$

$$h\, v =$$

$$w = h\, v : W = G' \to s$$

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$$h : V \to W$$

"linear transform"

$$\begin{aligned}
\mathrm{LinTran}(h, V, W) = {} & H_0(h, zero_V, zero_W) \\
\wedge {} & H_2(h, (+_V), (+_W)) \\
\wedge {} & \forall c.\ H_1(h, scale_V\, c, scale_W\, c)
\end{aligned}$$

$$v : V = G \to s$$

$$h\,v = h\left(\Sigma_{V\,i}\ scale_V\, v_i\, e_i\right) = \Sigma_{W\,i}\ h(scale_V\, v_i\, e_i)$$

$$= \Sigma_{W\,i}\ scale_W\, v_i\, (h\, e_i)$$

$$w = h\,v : W = G' \to s$$

$$M_h = \begin{pmatrix} | & | & | & \\ h e_0 & h e_1 & h e_2 & \cdots \\ | & | & | & \end{pmatrix}$$

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$h : V \longrightarrow W$

$h\, v = h\left(\sum_i{}_V \, scale\; v_i\; e_i\right) = \sum_i{}_W \, h\,(scale_V\; v_i\; e_i) =$

$= \sum_i scale_W\; v_i\; (h\, e_i)$

$M_h = \begin{pmatrix} | & | & | & \\ h e_0 & h e_1 & h e_2 & \cdots \\ | & | & | & \end{pmatrix}$

$Syntax = Matrix\; s$

$Semantics = V \to W$

$eval_{MV}\; m\; v = \sum_i scale_W\; (v\; i)\; (m\; i)$

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Linear Algebra & Homomorphisms

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Example: $\mathrm{der} : \boxed{P_3} \to \boxed{P_2}$ "polynomial derivative"

$P_n = \{\text{polynomials of degree} \leq n\} = \{0..n\} \to \mathbb{R}$ (coeff.)

$\mathrm{eval}_P : P_n \to \mathbb{R} \to \mathbb{R}$

$p_i \; x = x^i$

$\mathrm{eval}_P \; a = \sum_i \mathrm{scale} \; a_i \; p_i$

$\mathrm{der} \; p_i = \mathrm{scale} \; i \; p_{i-1}$

$$\mathrm{DER} = \begin{array}{c|c|c|c|c} & x^0 & x^1 & x^2 & x^3 \\ \hline x^0 & & & & \\ x^1 & & & & \\ x^2 & & & & \end{array}$$

(matrix version of der)

$\mathrm{eval}_{MV} : M \to (V \to W)$

Linear Algebra & Homomorphisms

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Example: der : $\boxed{P_3} \to \boxed{P_2}$ "polynomial derivative"

$P_n = \{\text{polynomials of degree} \le n\} = \{0..n\} \to \mathbb{R}$ (coeff.)

$eval_P : P_n \to \mathbb{R} \to \mathbb{R}$

$eval_P\ a = \sum_i \text{scale}\ a_i\ p_i$

$p_i\ x = x^i$

$\text{der}\ p_i = \text{scale}\ i\ p_{i-1}$

$$DER = \begin{array}{c|c|c|c|c} x^0 & x^1 & x^2 & x^3 & \\ \hline \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 2 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 3 \end{matrix} & \begin{matrix} x^0 \\ x^1 \\ x^2 \end{matrix} \end{array}$$

$$DER = \begin{array}{cccc} x^0 & x^1 & x^2 & x^3 \\ \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{matrix} x^0 \\ x^1 \\ x^2 \end{matrix}$$

$eval_{MV} : M \to V \to W$

$$\mathrm{LinTran}(h_1, U, V) \wedge \mathrm{LinTran}(h_2, V, W)$$

$$\Downarrow$$

$$\mathrm{LinTran}(h_2 \circ h_1, U, W)$$

$$\begin{tikzcd}
W & V \arrow[l, "h_2"'] & U \arrow[l, "h_1"'] \arrow[ll, bend left, "h_2 \circ h_1"]
\end{tikzcd}$$

$$\mathrm{LinTran}\,(h_1, U, V) \wedge \mathrm{LinTran}\,(h_2, V, W)$$

$$\Downarrow$$

$$\mathrm{LinTran}\,(h_2 \circ h_1, U, W)$$

$$\begin{tikzcd}
W & & V \arrow[ll, "h_2", "M_2"'] & & U \arrow[ll, "h_1", "M_1"'] \arrow[llll, bend left, "h_2 \circ h_1", "\mathrm{mul}_M\ M_2\ M_1"']
\end{tikzcd}$$

$$\mathrm{eval}_{MV}\ M_1 = h_1$$

$$\mathrm{eval}_{MV}\ M_2 = h_2$$

$$\mathrm{eval}_{MV}\ (\mathrm{mul}_M\ M_2\ M_1) = h_2 \circ h_1$$

$$\mathrm{LinTran}\,(h_1, U, V) \wedge \mathrm{LinTran}\,(h_2, V, W)$$

$$\Downarrow$$

$$\mathrm{LinTran}\,(h_2 \circ h_1, U, W)$$

$$\begin{tikzcd}
W & & V \arrow[ll, "h_2"', "M_2"] & & U \arrow[ll, "h_1"', "M_1"] \arrow[llll, bend left, "h_2 \circ h_1"', "\mathrm{mul}_M\ M_2\ M_1"]
\end{tikzcd}$$

$$\left.\begin{array}{l} \mathrm{eval}_{MV}\ M_1 = h_1 \\ \mathrm{eval}_{MV}\ M_2 = h_2 \end{array}\right\} \Longrightarrow \begin{array}{l} h_2 \circ h_1 = \mathrm{eval}_{MV}\ M_2 \circ \mathrm{eval}_{MV}\ M_1 \\ h_2 \circ h_1 = \mathrm{eval}_{MV}\ (\mathrm{mul}_M\ M_2\ M_1) \end{array}$$

$$\mathrm{LinTran}(h_1, U, V) \wedge$$
$$\mathrm{LinTran}(h_2, V, W)$$
$$\Downarrow$$
$$\mathrm{LinTran}(h_2 \circ h_1, U, W)$$

$$\begin{tikzcd} W & V \arrow[l, "h_2"', "M_2"] & U \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "\mathrm{mul}_M\ M_2\ M_1"'] \end{tikzcd}$$

$$\left.\begin{aligned} \mathrm{eval}_{MV}\ M_1 &= h_1 \\ \mathrm{eval}_{MV}\ M_2 &= h_2 \end{aligned}\right\} \Longrightarrow \begin{aligned} h_2 \circ h_1 &= \mathrm{eval}_{MV}\ M_2 \circ \mathrm{eval}_{MV}\ M_1 \\ &= \mathrm{eval}_{MV}\ (\mathrm{mul}_M\ M_2\ M_1) \end{aligned}$$

$$\boxed{H_2(\mathrm{eval}_{MV}, \underline{\mathrm{mul}_M}, (\circ))}$$ ⟵ specification, but what is the implementation?

```
type V s a = a -> s
type M s a b = b -> a -> s = b -> V s a
transpose : M s a b -> M s b a
transpose m = \j i -> m i j    = flip m
getCol : M s a b -> a -> V s b
getCol = flip = transpose
```

$$\begin{tikzcd} C & B \arrow[l, "h_2", "M_2"'] & A \arrow[l, "h_1", "M_1"'] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\, M_2\, M_1"'] \end{tikzcd}$$

$$\begin{pmatrix} | & | & | \\ h e_0 & h e_1 & h e_2 \\ | & | & | \end{pmatrix}$$

```
type V s a = a -> s
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```

$$
\begin{tikzcd}
C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "\mathrm{mul}_M\, M_2\, M_1"']
\end{tikzcd}
$$

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example: $a = \{X, Y\}$, $b = \{0, 1, 2\}$, $c = \{1\}$

$$M_1 = \begin{pmatrix} | & | \\ h_1 e_X & h_1 e_Y \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

$h_2 = \text{apply } 1$ (projection)

$$\begin{tikzcd} C & B \arrow[l, "h_2", "M_2"'] & A \arrow[l, "h_1", "M_1"'] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\, M_2\, M_1"'] \end{tikzcd}$$

type $V\, s\, a = a \to s$

type $M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$

transpose : $M\, s\, a\, b \to M\, s\, b\, a$

transpose $m = \backslash j\ i \to m\ i\ j \;=\; \mathrm{flip}\ m$

getCol : $M\, s\, a\, b \to a \to V\, s\, b$

getCol = flip = transpose

$\boxed{\text{We know } \mathrm{flip} \circ \mathrm{flip} = id}$

Spec. of $eval_{MV}$: $\quad \mathrm{getCol}\ m\ i = eval_{MV}\ m\ (e\ i)$

$$\mathrm{getCol}\ m = eval_{MV}\ m \circ e$$

$$\underline{\mathrm{flip}\,(\mathrm{flip}\ m) = m} \;=\; \mathrm{flip}\,(\underbrace{eval_{MV}\ m}_{h} \circ e)$$

$$m = \mathrm{flip}\,(h \circ e)$$

$$\text{type } V\,s\,a = a \to s$$

$$\text{type } M\,s\,a\,b = b \to a \to s = b \to V\,s\,a$$

$$\text{flip} : M\,s\,a\,b \to M\,s\,b\,a$$

$$\begin{tikzcd} C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1"', "\text{mul}_M\ M_2\ M_1"] \end{tikzcd}$$

Spec. of eval_{MV}: $m = \text{flip}\,(\text{eval}_{MV}\ m \circ e)$

$$\text{getCol}\,(\text{mul}_M\ m_2\ m_1)\ i = \underbrace{h_2}_{\text{eval}_{MV}\ m_2}\,(\underbrace{h_1\ e_i}_{\text{getCol}\ m_1\ i})$$

$$\text{type } V\, s\, a = a \to s$$

$$\text{type } M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$$

$$\text{flip} : M\, s\, a\, b \to M\, s\, b\, a$$

$$\begin{tikzcd}
C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1"', "\mathrm{mul}_M\, M_2\, M_1"]
\end{tikzcd}$$

Spec. of eval_{MV}: $\quad m = \text{flip}\,(\mathrm{eval}_{MV}\, m \circ e)$

$$\text{getCol}\,(\mathrm{mul}_M\, m_2\, m_1)\, i = h_2\,(h_1\, e_i) =$$

$$= \mathrm{eval}_{MV}\, m_2\,(\text{getCol}\, m_1\, i)$$

$\text{type } V\,s\,a = a \to s$

$\text{type } M\,s\,a\,b = b \to a \to s = b \to V\,s\,a$

$\quad flip : M\,s\,a\,b \to M\,s\,b\,a$

$$\begin{tikzcd} C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\ M_2\ M_1"'] \end{tikzcd}$$

Spec. of $eval_{MV}$: $\quad m = flip\,(eval_{MV}\ m \circ e)$

$getCol\ (mul_M\ m_2\ m_1)\ i = h_2\,(h_1\ e_i) =$

$\qquad = eval_{MV}\ m_2\ (getCol\ m_1\ i)$

$getCol\ (mul_M\ m_2\ m_1) = eval_{MV}\ m_2 \circ getCol\ m_1$

Def. of mul_M

$$\boxed{mul_M\ m_2\ m_1 = flip\ (eval_{MV}\ m_2 \circ flip\ m_1)}$$

where $eval_{MV}\ m\ v = \sum_i scale\ v_i\ m_i$

$$\text{type } V\, s\, a = a \to s$$

$$\text{type } M\, s\, a\, b = b \to a \to s = b \to V\, s\, a$$

$$flip : M\, s\, a\, b \to M\, s\, b\, a$$

$$flip = transpose = getCol$$

$$\begin{tikzcd}
C & B \arrow[l, "h_2"', "M_2"] & A \arrow[l, "h_1"', "M_1"] \arrow[ll, bend left, "h_2 \circ h_1", "mul_M\ M_2\ M_1"']
\end{tikzcd}$$

$$A = V\, s\, a$$

$$B = V\, s\, b$$

$$C = V\, s\, c$$

$$\boxed{mul_M\ m_2\ m_1 = flip\ (eval_{MV}\ m_2 \circ flip\ m_1)}$$

$$\text{where } eval_{MV}\ m\ v = \sum_i scale\ v_i\ m_i$$

$$mul_M : M\, s\, a\, b \to M\, s\, b\, c \to M\, s\, a\, c$$

$$eval_{MV} = mul_{MV} : M\, s\, a\, b \to V\, s\, a \to V\, s\, b$$