FLABloM: Functional Linear Algebra with Block Matrices

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Functional linear algebra with block matrices

- Inspired by work on parallel parsing by Bernardy & Jansson
- Matrices in Agda
- Reflexive, transitive closure of matrices

$$\boxed{\begin{array}{cc}
\boxed{\begin{array}{cc} 1 & \boxed{\begin{array}{cc} 0 & 1 \end{array}} \\ \boxed{\begin{array}{c} 0 \\ 0 \end{array}} & \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} \end{array}} &
\boxed{\begin{array}{cc} 0 & \boxed{\begin{array}{cc} 1 & 2 \end{array}} \\ \boxed{\begin{array}{c} 0 \\ 0 \end{array}} & \boxed{\begin{array}{cc} 1 & 7 \\ 3 & 8 \end{array}} \end{array}} \\
\boxed{\begin{array}{cc} 2 & \boxed{\begin{array}{cc} 1 & 0 \end{array}} \\ \boxed{\begin{array}{c} 1 \\ 0 \end{array}} & \boxed{\begin{array}{cc} 9 & 8 \\ 7 & 6 \end{array}} \end{array}} &
\boxed{\begin{array}{cc} 1 & \boxed{\begin{array}{cc} 0 & 1 \end{array}} \\ \boxed{\begin{array}{c} 0 \\ 0 \end{array}} & \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} \end{array}}
\end{array}}$$

Towards a datatype for matrices

Desirable:

- Easy to program with
- Easy to write proofs with

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Possibilities:

- Vectors of vectors: $Vec\ (Vec\ a\ n)\ m$
- Functions from indices: $Fin\ m \to Fin\ n \to a$
- ...

Matrix "shapes"

A type for shapes (generalisation of natural numbers):

$$
\begin{array}{l}
\textbf{data}\ \mathit{Shape}\ :\ \mathit{Set}\ \textbf{where} \\
\quad L\ :\ \mathit{Shape} \\
\quad B\ :\ \mathit{Shape} \to \mathit{Shape} \to \mathit{Shape}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{two} & = & B\ L\ L \\
\mathit{three} & = & B\ \mathit{two}\ L \\
\mathit{three}' & = & B\ L\ \mathit{two}
\end{array}
$$

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\end{aligned}$$

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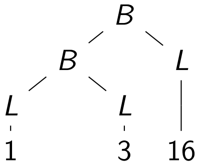
\mathit{three} &= B\ \mathit{two}\ L\\
\mathit{three}' &= B\ L\ \mathit{two}
\end{aligned}$$

Shapes for one dimension: (a vector/row matrix)



Matrices: building blocks

Matrices are indexed by two shapes:

$$\textbf{data}\ M\ (a\ :\ Set)\ :\ (rows\ cols\ :\ Shape) \to Set$$

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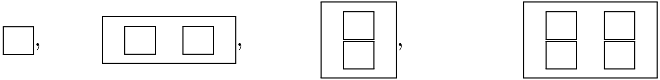

$M\ a\ L\ L$ $\quad M\ a\ L\ (B\ c_1\ c_2)$ $\quad M\ a\ (B\ r_1\ r_2)\ L$ $\quad M\ a\ (B\ r_1\ r_2)\ (B\ c_1\ c_2)$

Matrices: a datatype

$$
\begin{array}{lll}
\multicolumn{3}{l}{\textbf{data}\ M\ (a\ :\ Set)\ :\ (rows\ cols\ :\ Shape) \to Set\ \textbf{where}} \\
\quad One & : & a \to M\ a\ L\ L \\
\quad Col & : & \{r_1\ r_2\ :\ Shape\} \to \\
 & & M\ a\ r_1\ L \to M\ a\ r_2\ L \to M\ a\ (B\ r_1\ r_2)\ L \\
\quad Row & : & \{c_1\ c_2\ :\ Shape\} \to \\
 & & M\ a\ L\ c_1 \to M\ a\ L\ c_2 \to M\ a\ L\ (B\ c_1\ c_2) \\
\quad Q & : & \{r_1\ r_2\ c_1\ c_2\ :\ Shape\} \to \\
 & & M\ a\ r_1\ c_1 \to M\ a\ r_1\ c_2 \to \\
 & & M\ a\ r_2\ c_1 \to M\ a\ r_2\ c_2 \to \\
 & & M\ a\ (B\ r_1\ r_2)\ (B\ c_1\ c_2)
\end{array}
$$

Rings

A hierarchy of rings as Agda records:

- *SemiNearRing*
  $\simeq$, $+$, $\cdot$, *0* (+ is associative and commutes, 0 identity of $+$ and zero of $\cdot$, $\cdot$ distributes over $+$)
- *SemiRing*
  1 (1 identity of $\cdot$, $\cdot$ is associative)
- *ClosedSemiRing*
  an operation * with $w^* \simeq 1 + w \cdot w^*$.

Lifting matrices

We take a semi-(near)-ring and lift it to square matrices.
A lifting function *Square* for each *Shape* and ring structure.

$$\begin{aligned}
\mathit{Square} &: \mathit{Shape} \to \mathit{SemiNearRing} &&\to \mathit{SemiNearRing}\\
\mathit{Square}' &: \mathit{Shape} \to \mathit{SemiRing} &&\to \mathit{SemiRing}\\
\mathit{Square}'' &: \mathit{Shape} \to \mathit{ClosedSemiRing} &&\to \mathit{ClosedSemiRing}
\end{aligned}$$

Lifting matrices

(Parts of) lifted equivalence:

$$
\begin{array}{lllll}
\_\simeq_S\_ & : \forall \{r\ c\} \to M\ s\ r\ c \to M\ s\ r\ c \to Set \\
(One\ x) & \simeq_S\ (One\ x_1) & = & x\ \simeq_s\ x_1 \\
(Row\ m\ m_1) & \simeq_S\ (Row\ n\ n_1) & = & (m\ \simeq_S\ n) \times (m_1\ \simeq_S\ n_1)
\end{array}
$$

(Parts of) lifted multiplication:

$$
\begin{array}{lllll}
\_\cdot_S\_ & : \forall \{r\ m\ c\} \to M\ s\ r\ m \to M\ s\ m\ c \to M\ s\ r\ c \\
One\ x & \cdot_S\ One\ y & = & One\ (x\ \cdot_s\ y) \\
Row\ m_0\ m_1 & \cdot_S\ Col\ n_0\ n_1 & = & m_0\ \cdot_S\ n_0\ +_S\ m_1\ \cdot_S\ n_1 \\
Col\ m_0\ m_1 & \cdot_S\ Row\ n_0\ n_1 & = & Q\ (m_0\ \cdot_S\ n_0)\ (m_0\ \cdot_S\ n_1) \\
 & & & \quad\ (m_1\ \cdot_S\ n_0)\ (m_1\ \cdot_S\ n_1)
\end{array}
$$

Closure for matrices

Computing the reflexive, transitive closure:

$$\boxed{a}^* = \boxed{a^*}$$

$$\boxed{\begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix}}^* = \boxed{\begin{matrix} A_{11}^* + A_{11}^* \cdot A_{12} \cdot \Delta^* \cdot A_{21} \cdot A_{11}^* & A_{11}^* \cdot A_{12} \cdot \Delta^* \\ \Delta^* \cdot A_{21} \cdot A_{11}^* & \Delta^* \end{matrix}}$$

(with $\Delta = A_{22} + A_{21} \cdot A_{11}^* \cdot A_{12}$)

with proof that it satisfies $w^* \simeq 1 + w \cdot w^*$

Reachability example

$$\begin{tikzcd} 1 & 2 \arrow[d] \\ 3 \arrow[ur] & 4 \end{tikzcd}$$

$$\boxed{\begin{matrix} \boxed{\begin{matrix}0&0\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\0&1\end{matrix}} \\ \boxed{\begin{matrix}0&1\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\0&0\end{matrix}} \end{matrix}}^{*}$$

Reachability example

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1 & 2 \arrow[d] \\
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$$\boxed{\begin{matrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{matrix}}^{*} = \boxed{\begin{matrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{matrix}}$$

$$\begin{tikzcd}
1 \arrow[loop above] & 2 \arrow[loop above] \arrow[d] \\
3 \arrow[loop below] \arrow[ur] \arrow[r] & 4 \arrow[loop below]
\end{tikzcd}$$

Wrapping up

Conclusions, further work, etc.

- This matrix definition is not the final word
- A more flexible matrix definition: sparse? fewer constructors?
- Automation (of proofs)!
- Generalisation to closed semi-near-ring for parsing applications.
- Agda development available at `https://github.com/DSLsofMath/FLABloM`.