module *TheThreeBridgesProblem*

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1 The 3 bridges problem [1]

A fugitive can escape a pursuer by crossing a river through one of three bridges. One bridge is safe, the other two are dangerous by different degrees:

- Bridge 1: safe, zero chances of death
- Bridge 2: unsafe, 10% chances of death
- Bridge 3: unsafe, 20% chances of death

The opponent pursues the fugitive at the other side of the river. He also has to pick-up a bridge. If fugitive and chaser select the same bridge, the fugitive is killed.

$Y = \{1, 2, 3\}$ -- the controls for the fugitive and the pursuer
$X = \{D, A\}$ -- the possible outcomes for the fugitive

$\sigma : Y \times Y \to SP\ X$ -- the transition function
$\sigma\ (1,1) = \sigma\ (2,2) = \sigma\ (3,3) = \{(D, 1.0)\}$
$\sigma\ (1,2) = \sigma\ (2,3) \qquad\qquad = \{(A, 1.0)\}$
$\sigma\ (2,1) = \sigma\ (2,3) \qquad\qquad = \{(D, 0.1), (A, 0.9)\}$
$\sigma\ (3,1) = \sigma\ (3,2) \qquad\qquad = \{(D, 0.2), (A, 0.8)\}$

The fugitive only cares about surviving, no matter how. His reward function is

$r : X \to \{0, 1\}$
$r\ D = 0$
$r\ A = 1$

Similarly, the pursuer only cares about the fugitive ending up dead, no matter how. His payoffs are exactly opposite to those of the fugitive:

$r' : X \to \{0, 1\}$
$r'\ D = 1$
$r'\ A = 0$

The value function for the fugitive

$val : Y \times Y \to SP\ \{0, 1\}$
$val = map\ r \circ \sigma$

can be represented by the table

	1	2	3
1	$\{(0, 1.0)\}$	$\{(1, 1.0)\}$	$\{(1, 1.0)\}$
2	$\{(0, 0.1), (1, 0.9)\}$	$\{(0, 1.0)\}$	$\{(0, 0.1), (1, 0.9)\}$
3	$\{(0, 0.2), (1, 0.8)\}$	$\{(0, 0.2), (1, 0.8)\}$	$\{(0, 1.0)\}$

The entries of the table are probability distributions. The table for the pursuer is obtained from that for the fugitive by flipping the 0 and 1 values in all probability distributions. If both the fugitive and the pursuer measures uncertainty with the expected value measure, they end up with the payoffs

	1	2	3
1	(0.0, 1.0)	(1.0, 0.0)	(1.0, 0.0)
2	(0.9, 0.1)	(0.0, 1.0)	(0.9, 0.1)
3	(0.8, 0.2)	(0.8, 0.2)	(0.0, 1.0)

Here, each entry represents the expected payoff for the fugitive and for the pursuer for the corresponding pair of bridges. If they measure uncertainty with the worst case measure, they end up with the payoffs

	1	2	3
1	(0, 1)	(1, 0)	(1, 0)
2	(0, 0)	(0, 1)	(0, 0)
3	(0, 0)	(0, 0)	(0, 1)

Notice that the worst case measure seems to suggest that the fugitive would be better off by selecting bridge number 1. But, by the same argument, the pursuer could come to the conclusion that it would be better for him to wait for the fugitive at bridge 1. But this would be a very strong reason for the fugitive to cross the river at bridge 2 or perhaps at bridge 3. But then the pursuer ...

The analysis suggests that under *anticipation*, the best that the fugitive can do is to select a bridge randomly. Instead of a bridge, he seek a probability distribution over bridges.

Given probability distributions over bridges for the fugitive and for the pursuer, one can compute the correspondent probability distribution over pairs of bridges and the value function for the fugitive

$$\begin{aligned}
&\mathit{val}' \;:\; (\mathit{SP}\ \ Y) \times (\mathit{SP}\ \ Y) \;\to\; \mathit{SP}\ \{0,1\} \\
&\mathit{val}'\ (\mathit{xs}, \mathit{ys}) = (\mathit{xs} \otimes \mathit{ys}) \ggg\!= \mathit{val} = \mathit{join}\ (\mathit{map}\ \mathit{val}\ (\mathit{xs} \otimes \mathit{ys}))
\end{aligned}$$

and, by flipping 0 and 1, for the pursuer. Here $\otimes$ is

$$\begin{aligned}
&(\otimes) \;:\; \mathit{SP}\ A \;\to\; \mathit{SP}\ B \;\to\; \mathit{SP}\ (A \times B) \\
&\mathit{as} \otimes \mathit{bs} = \{\, ((a, b), \mathrm{prob}\ \ \mathit{as}\ a * \mathrm{prob}\ \ \mathit{bs}\ b) \mid a \leftarrow \mathit{supp}\ \mathit{as}, b \leftarrow \mathit{supp}\ \mathit{bs} \,\}
\end{aligned}$$

, prob $\mathit{xs}\ x$ denotes the probability of x according to the distribution xs and $\mathit{supp}\ \mathit{xs}$ denotes the support of xs.

But which probability distribution is *best* for the fugitive and for the pursuer?

If all three bridges were safe, one would expect both the fugitive and the pursuer to select

$$\mathit{xs} = \mathit{ys} = \{\, (1, 1\,/\,3), (2, 1\,/\,3), (3, 1\,/\,3) \,\}$$

In this case, the value of $(\mathit{xs}, \mathit{ys})$ for the fugitive would be

$$
\begin{aligned}
& val'\ (xs, ys) \\
& \quad = \\
& join\ (map\ val\ (xs \otimes ys)) \\
& \quad = \\
& join\ (map\ val\ \{((x,y), 1\ /\ 9) \mid x \leftarrow \{1,2,3\}, y \leftarrow \{1,2,3\}\}) \\
& \quad = \\
& join\ \{(\{(0,1.0)\}, 1\ /\ 9), (\{(1,1.0)\}, 1\ /\ 9), (\{(1,1.0)\}, 1\ /\ 9), \\
& \qquad\ (\{(1,1.0)\}, 1\ /\ 9), (\{(0,1.0)\}, 1\ /\ 9), (\{(1,1.0)\}, 1\ /\ 9), \\
& \qquad\ (\{(1,1.0)\}, 1\ /\ 9), (\{(1,1.0)\}, 1\ /\ 9), (\{(0,1.0)\}, 1\ /\ 9)\} \\
& \quad = \\
& \{(0, 1\ /\ 3), (1, 2\ /\ 3)\}
\end{aligned}
$$

as one would expect. Symmetrically, the value of picking up a random bridge with uniform probability for the pursuer would be $\{(1, 1\ /\ 3), (0, 2\ /\ 3)\}$ and the corresponding expected values for the fugitive and for the pursuer would be 2/3 and 1/3, respectively.

Notice that, in this situation, neither the fugitive, nor the pursuer can increase their expected values by selecting a different probability distribution. For instance, if the fugitive would select

$$xs' = \{(1, 1\ /\ 2), (2, 1\ /\ 2)\}$$

instead of xs, he would end up with

$$
\begin{aligned}
& val'\ (xs', ys) \\
& \quad = \\
& join\ (map\ val\ (xs' \otimes ys)) \\
& \quad = \\
& join\ (map\ val\ \{((x,y), \mathrm{prob}\ \ xs'\ x * \mathrm{prob}\ \ ys\ y) \mid x \leftarrow \{1,2\}, y \leftarrow \{1,2,3\}\}) \\
& \quad = \\
& join\ (map\ val\ \{((1,1), 1\ /\ 6), ((1,2), 1\ /\ 6), ((1,3), 1\ /\ 6), \\
& \qquad\qquad\qquad\quad ((2,1), 1\ /\ 6), ((2,2), 1\ /\ 6), ((2,3), 1\ /\ 6)\}) \\
& \quad = \\
& join\ \{(\{(0,1.0)\}, 1\ /\ 6), (\{(1,1.0)\}, 1\ /\ 6), (\{(1,1.0)\}, 1\ /\ 6), \\
& \qquad\ (\{(1,1.0)\}, 1\ /\ 6), (\{(0,1.0)\}, 1\ /\ 6), (\{(1,1.0)\}, 1\ /\ 6)\} \\
& \quad = \\
& \{(0, 1\ /\ 3), (1, 2\ /\ 3)\} = val'\ (xs, ys)
\end{aligned}
$$

The same holds if the fugitive selects $xs' = \{(3, 1.0)\}$ or, indeed, any other probability distribution. In other words, any $xs\ :\ SP\ \{1,2,3\}$ is in Nash equilibrium with $ys = \{(1, 1/3), (2, 1/3), (3, 1/3)\}$. The same holds for any ys given $xs = \{(1, 1\ /\ 3), (2, 1\ /\ 3), (3, 1\ /\ 3)\}$.

The crucial observation here is that, given the choice ys of the pursuer, any xs' is exactly as good as xs!

Selecting a probability distribution that makes the choice of the opponent irrelevant is a guiding principle in decision making under anticipation.

Let's consider the original problem from the pursuer point of view. He seeks a distribution

$$ys\ :\ SP\ \{1,2,3\}$$

such that for any $xs, xs'\ :\ SP\ \{1,2,3\}$

$$ev\ (val'\ (xs, ys)) = ev\ (val'\ (xs', ys))$$

Here ev denotes the expected value measure. Let p, q denote the probabilities of bridge 1 and 2 in ys that is

$$ys = \{(1, p), (2, q), (3, 1 - p - q)\}$$

Consider

$$\begin{aligned} xs &= \{(1, 1.0)\} \\ xs' &= \{(2, 1.0)\} \\ xs'' &= \{(3, 1.0)\} \end{aligned}$$

We have

$$ev\ (val'\ (xs, ys)) = ev\ (val'\ (xs', ys)) = ev\ (val'\ (xs'', ys))$$
$$=$$
$$ev\ (join\ (map\ val\ (xs \otimes ys)))$$
$$=$$
$$ev\ (join\ (map\ val\ (xs' \otimes ys)))$$
$$=$$
$$ev\ (join\ (map\ val\ (xs'' \otimes ys)))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((x, y), \text{prob}\ \ xs\ x * \text{prob}\ \ ys\ y) \mid x \leftarrow \{1\}, y \leftarrow \{1, 2, 3\}\}))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((x, y), \text{prob}\ \ xs'\ x * \text{prob}\ \ ys\ y) \mid x \leftarrow \{2\}, y \leftarrow \{1, 2, 3\}\}))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((x, y), \text{prob}\ \ xs''\ x * \text{prob}\ \ ys\ y) \mid x \leftarrow \{3\}, y \leftarrow \{1, 2, 3\}\}))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((1, 1), p), ((1, 2), q), ((1, 3), 1 - p - q)\}))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((2, 1), p), ((2, 2), q), ((2, 3), 1 - p - q)\}))$$
$$=$$
$$ev\ (join\ (map\ val\ \{((3, 1), p), ((3, 2), q), ((3, 3), 1 - p - q)\}))$$
$$=$$
$$ev\ (join\ \{(\{(0, 1.0)\}, p), (\{(1, 1.0)\}, q), (\{(1, 1.0)\}, 1 - p - q)\})$$
$$=$$
$$ev\ (join\ \{(\{(0, 0.1), (1, 0.9)\}, p), (\{(0, 1.0)\}, q), (\{(0, 0.1), (1, 0.9)\}, 1 - p - q)\})$$
$$=$$
$$ev\ (join\ \{(\{(0, 0.2), (1, 0.8)\}, p), (\{(0, 0.2), (1, 0.8)\}, q), (\{(0, 1.0)\}, 1 - p - q)\})$$
$$=$$
$$ev\ \{(0, p), (1, 1 - p)\}$$
$$=$$
$$ev\ \{(0, 0.1 * p + q + 0.1 * (1 - p - q)), (1, 0.9 * p + 0.9 * (1 - p - q))\}$$
$$=$$
$$ev\ \{(0, 0.2 * p + 0.2 * q + 1 - p - q), (1, 0.8 * p + 0.8 * q)\}$$
$$=$$
$$1 - p = 0.9 * (1 - q) = 0.8 * (p + q)$$
$$=$$
$$\begin{aligned} &p \quad\quad - 0.9 * q = 0.1 \\ &1.8 * p + 0.8 * q = 1 \end{aligned}$$
$$=$$
$$\begin{aligned} &p = 49\ /\ 121 \\ &q = 41\ /\ 121 \end{aligned}$$

A similar computation yields the “optimal” probability distribution for the pursuer.

The three bridges problem raises a number of obvious questions:

- How to decide if a decision problem is one in which a player is better off by making random choices?
- Are there games that are not *zero sum* but require the players to make random choices?
- Under which conditions (e.g., on the uncertainty measure) can players compute “optimal” random choices in games under anticipation? When are these choices unique?

References

[1] Don Ross. Game theory. In *Stanford Encyclopedia of Philosophy*. 2008.