PROOF We set up the Newton quotient for fg and then add 0 to the numerator in a way that enables us to involve the Newton quotients for f and g separately:

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

To get the last line we have used the fact that f and g are differentiable and the fact that g is therefore continuous (Theorem 1), as well as limit rules from Theorem 2 of Section 1.2. A graphical proof of the Product Rule is suggested by Figure 2.19.

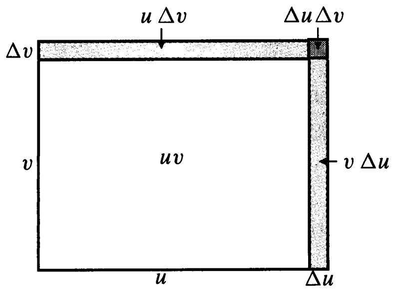

Figure 2.19

A graphical proof of the Product Rule

Here $u = f(x)$ and $v = g(x)$, so that the rectangular area uv represents $f(x)g(x)$. If x changes by an amount Δx, the corresponding increments in u and v are Δu and Δv. The change in the area of the rectangle is

$$\begin{aligned}&\Delta(uv) \\ &= (u + \Delta u)(v + \Delta v) - uv \\ &= (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v),\end{aligned}$$

the sum of the three shaded areas. Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

$$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right) v + u \left(\frac{dv}{dx}\right),$$

since

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \Delta v = \frac{du}{dx} \times 0 = 0$$

EXAMPLE 3 Find the derivative of $(x^2 + 1)(x^3 + 4)$ using and without using the Product Rule.

Solution Using the Product Rule with $f(x) = x^2+1$ and $g(x) = x^3+4$, we calculate

$$\frac{d}{dx}\big((x^2+1)(x^3+4)\big) = 2x(x^3+4) + (x^2+1)(3x^2) = 5x^4 + 3x^2 + 8x.$$

On the other hand, we can calculate the derivative by first multiplying the two binomials and then differentiating the resulting polynomial:

$$\frac{d}{dx}\big((x^2+1)(x^3+4)\big) = \frac{d}{dx}(x^5 + x^3 + 4x^2 + 4) = 5x^4 + 3x^2 + 8x.$$

EXAMPLE 4 Find $\dfrac{dy}{dx}$ if $y = \left(2\sqrt{x} + \dfrac{3}{x}\right)\left(3\sqrt{x} - \dfrac{2}{x}\right)$.

Solution Applying the Product Rule with f and g being the two functions enclosed in the large parentheses, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{1}{\sqrt{x}} - \frac{3}{x^2}\right)\left(3\sqrt{x} - \frac{2}{x}\right) + \left(2\sqrt{x} + \frac{3}{x}\right)\left(\frac{3}{2\sqrt{x}} + \frac{2}{x^2}\right) \\ &= 6 - \frac{5}{2x^{3/2}} + \frac{12}{x^3}.\end{aligned}$$

EXAMPLE 5 Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if $u(2) = 2$, $u'(2) = -5$, $v(2) = 1$, and $v'(2) = 3$.

Solution From the Product Rule we have

$$y' = (uv)' = u'v + uv'.$$

Therefore,

$$y'(2) = u'(2)v(2) + u(2)v'(2) = (-5)(1) + (2)(3) = -5 + 6 = 1.$$

EXAMPLE 6 Use mathematical induction to verify the formula $\dfrac{d}{dx}x^n = n\,x^{n-1}$ for all positive integers n.