NEW CONGESTION CONTROL SCHEMES OVER WIRELESS NETWORKS: STABILITY ANALYSIS [1]

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Abstract: The objective of this work is to introduce two original flow control schemes for wireless networks. The mathematical underpinnings lie on the recently-developed congestion control models for TCP-like schemes; more precisely, the model proposed by Kelly for the wired case will be taken as a template, and properly extended to the more subtle wireless setting. We shall propose two ways to modify one of the entities of the model; these changes are justified by the literature on wireless networks. The first will be through a static law, while the second via a dynamic one. In both cases, we shall prove the global stability of the schemes. Furthermore, a convergence rate study and a stochastic analysis will be presented. The completeness of the study advocates the real applicability of the control scheme and its generality encourages the extensibility to different communication protocols and networks.

Keywords: Communication networks and protocols, flow control, large-scale systems, stability robustness, delay analysis.

1. INTRODUCTION

Congestion Schemes for Communication Networks have proven to be of the utmost importance when applied to key applications such as the Internet; for instance, the TCP protocol is widely regarded as the most known and employed scheme for the exchange of digital information (Jacobson, 1998).

A network is described via two of its entities, the users and the links. The idea behind TCP [2], which regulates the packets sent by the users in the network, is to avoid channel congestions by increasing the window size, that is the number of packets sent at a time, only when no packet is lost during the previous round trip time, while halving it in the opposite instance; therefore, it assumes that lost packets are symptomatic of congestion. The measure of these losses is described via loss probability variables at the links. The update mechanisms for both rates and losses are distributed, that is based on just local information, and can be either static or dynamic.

The analysis of this and other similar protocols has dealt with many issues: modeling (Kelly, 2003)-(Kelly *et al.*, Dec 1999)-(Alpcan and Basar, Dec 2003)-(Alpcan and Basar, Dec 2002)-(Kunniyur and Srikant, Mar 2000)-(Floyd, Oct 1994)-(Floyd and Jacobson, Aug 1993), stability

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[2] Along these lines, we shall refer to the Reno case, for simplicity sake.

(Kunniyur and Srikant, Mar 2001)-(Kunniyur and Srikant, Oct 2003)-(Johari and Tan, Dec 2001)-(Paganini *et al.*, n.d.)-(Wang and Paganini, Dec 2003), robustness to delays (Vinnicombe, 2001)-(Vinnicombe, 2002), assessment of the utility functions for the underlying optimization problem (Low and Lapsley, 1956), duality interpretation (Low, Aug 2003). All these efforts have focused on the wired case.

Congestion Control over Wireless Networks poses additional challenges than the wired case; in fact, in this new framework packet loss is due not just to congestion at the link, but possibly also to physical channel error. As a consequence, the network could be underutilized, which happens a lot in reality. A recent work on MULTFRC, (Chen and Zakhor, 2004), proposes an original dynamic scheme to improve the performance. In a paper by the same authors, (Chen *et al.*, 2005) [3], a static scheme and a dynamic one are proposed; the global stability of both schemes are proved in generality, and the study of delay sensitivity of the first one is also tackled. This work aims at completing the former paper, and will be presented as follows. First, the mathematical framework will be introduced and the two models proposed. We shall then quote the stability results for both schemes. Then, the rate of convergence of the two schemes will be derived, and a stochastic analysis will be developed. Some final considerations on the proposed scheme and the discussion of future work will close up the paper.

Another paper by the same authors, (Abate *et al.*, 2005), the delay sensitivity analysis for the static and the dynamic case is studied [4]. Sufficient structural conditions on the dynamic scheme are introduced that ensure (local) stability to delays in the system; the oscillations of the solutions are qualitatively analyzed.

2. PROBLEM SETTING

2.1 The model for the wired case

A network is described via its J *resources*, its links, and its R *users* (sender-receiver pairs), which can also be conceived as subsets of J, the *routes*. Each link j has a finite capacity C_j. The connections of the network are described via a matrix $A = (a_{jr}, j \in J, r \in R)$, where $a_{jr} = 1$ if $j \in r$. Every user is endowed with a sending rate $x_r \geq 0$ and a utility function $U_r(x_r)$, assumed to be increasingly, strictly concave and $\mathbb{C}^1$. Kelly was the first to interpret the flow control as the solution of the following concave maximization problem, dependent on the aggregate utility functions for the rates and on some costs on the links:

$$\max \sum_{r\in R} U_r(x_r) - \sum_{j\in J} P_j \left(\sum_{s:j\in s} x_s \right), \qquad (1)$$

where the cost functions $P_j(\cdot)$ are defined as:

$$P_j(y) = \int_0^y p_j(z)\, dz. \qquad (2)$$

Here $p_j(y)$'s are the prices at the links and are assumed to be non-negative, continuous and increasing functions; moreover, they are expected to depend on the aggregate rate passing through the link. Throughout this paper we shall stick to the following form, the "packet loss rate",

$$p_j(y) = \frac{(y - C_j)^+}{y}. \qquad (3)$$

The end-to-end packet loss rate for user r is $1 - \prod_{j\in r} p_j(\sum_{s:j\in s} x_s)$, which is approximately $\sum_{j\in r} p_j(\sum_{s:j\in s} x_s)$ when $p_j(\sum_{s:j\in s} x_s)$ is small (we shall assume this in the following). We will consider the following primal scheme, which is a more general, continuous-time version of the TCP-like additive increase, multiplicative decrease algorithm:

$$\frac{d}{dt} x_r(t) = k_r \left(w_r^o - x_r(t) \sum_{j\in r} \mu_j(t) \right), \quad r \in R \qquad (4)$$

with k_r being a positive scale factor affecting the adaptation rate; the congestion signal is generated at a link j as

$$\mu_j(t) = p_j \left(\sum_{s:j\in s} x_s(t) \right). \qquad (5)$$

With this primal scheme (4)-(5), the unique, globally asymptotically stable points of the entire network, denoted by $x^o = (x_r^o, r \in R)$ [5], are given by

$$x_r^o = \frac{w_r^o}{\sum_{j\in r} p_j \left(\sum_{s:j\in s} x_s^o \right)}, \quad r \in R; \qquad (6)$$

This unique solution is also optimal in the sense that the network bottlenecks are fully utilized, the total net utility is maximized, and the users are proportionally fair to each other (Kelly *et al.*, Dec 1999).

[3] The paper is currently under review; the reviewers can access it via the following link: http://www-video.eecs.berkeley.edu/∼minghua/papers/infocom.2005.pdf

[4] The paper has been submitted to the IFAC05 conference and is therefore currently under review. It is complementary to this one; the reviewers are invited to access it via the following link: http://www-video.eecs.berkeley.edu/∼minghua/papers/ifac.2005b.pdf

[5] In order to keep the notation light, throughout the whole paper users or links variables with no subscript will directly denote vectorial quantities. We burden the reader with the easy task to identify them.

2.2 The wireless case

One of the main differences between the wired case and the wireless one is the presence, in this latter case, of physical channel errors; these affect, in the setting of our model, the packet loss rate, which in the wired case depended just on the congestion measure. Say every link j is affected by the error ϵ_j; then, the new price function ν_j is:

$$\nu_j(t) = p_j\left(\sum_{s:j\in s} x_s(t)\right) + \left(1 - p_j\left(\sum_{s:j\in s} x_s(t)\right)\right)\epsilon_j$$
$$\doteq q_j\left(\sum_{s:j\in s} x_s(t)\right) \geq p_j\left(\sum_{s:j\in s} x_s(t)\right). \quad (7)$$

The primal scheme (4) then will adapt according to this new price functions q_j, which have the same structural properties as the old p_j; the equilibrium points of the system will therefore change accordingly. A further analysis will show how the new equilibria are always less than or equal to those obtained with the wired model; from an optimization prospective, they are a suboptimal solution for the network. Exploiting duality arguments, this also means that some of the bottleneck links may be under utilized.

The task of this work is to address and solve this issue. Contrary to existing approaches, which try to provide the user with the feedback of the exact price $\sum_{j\in r}\mu_j(t)$, or at least with an estimate of it, the idea has been to act on the term w_r, which physically corresponds to modifying the number of connections the user has to the network (Chen and Zakhor, 2004) (Chen *et al.*, 2005). In the first approach, w_r is instantaneously adjusted using a static function with respect to $\nu_j(t)$ and $\mu_j(t)$; in the second approach, w_r is gradually adjusted by a dynamic update law. Both of them are end-to-end application layer based schemes, since changing w_r can be implemented by changing the number of connections opened by one user. Therefor they require no modification to either the network infrastructure or to the network protocols.

3. TWO NEW CONTROL SCHEMES

3.1 Static Update

Assume the term ω_r is time dependent, $w_r(t)$, and is adjusted according to the following law:

$$w_r(t) = w_r^o \frac{\sum_{j\in r}\nu_j(t)}{\sum_{j\in r}\mu_j(t)}. \quad (8)$$

Then, the source rate for user r then is given by:

$$\frac{d}{dt}x_r(t) = k_r\left(w_r(t) - x_r(t)\sum_{j\in r}\nu_j(t)\right). \quad (9)$$

A rapid calculation shows how, under this change, the equilibrium of the system is again x^o. Intuitively, as can be seen from (8), if the noise is large, i.e. $v_j(t) > u_j(t)$, an increase in $w_r(t)$ counteracts it.

3.2 Dynamic Update

Rather than a simultaneous adaptation rule, we advance a dynamic update for w_r:

$$\frac{d}{dt}w_r(t) = c_r\left(w_r^o - w_r(t)\frac{\sum_{j\in r} p_j(\sum_{s:j\in s} x_s(t))}{\sum_{j\in r} q_j(\sum_{s:j\in s} x_s(t))}\right). \quad (10)$$

The equilibrium points of the new, extended system are composed by a first part given by the vector x^o and a second, for the new dynamics, given by $w_r^o\frac{\sum_{j\in r}\nu_j(t)}{\sum_{j\in r}\mu_j(t)}$. The system of coupled equations (4)-(5)-(10) is strongly nonlinear and asymmetric.

4. GLOBAL STABILITY

Stability is probably the first requirement one wants to check upon a dynamical system. This concept has proven to be quite hard to handle within large-scale, non-linear, primal-dual, coupled systems such as the two we have introduced. We state here two facts, the proofs of which can be found on the paper referenced in the introduction. Also, simulations are discussed there.

4.1 Static Update

Theorem 1. System (4)-(5)-(8) is globally asymptotically stable with the following Lyapunov function:

$$V(x) = \sum_{r\in R} w_r^o \log x_r - \sum_{j\in J}\int_0^{\sum_{s:j\in s} x_s} p_j(y)dy. \quad (11)$$

All trajectories converge to the equilibrium point x^o in (6) that maximizes $V(x)$.

4.2 Dynamic Update

The key assumption that we make in this section is that *the dynamics corresponding to (4)-(5) and (10) evolve in two different time scales; the first in a faster one, while the second in a slower one.* This two relations can be vectorized as [6] :

$$\varepsilon\dot{x}(t) = kw(t) - diag(kx(t))A^T(p\circ Ax(t)) = F(x); \quad (12)$$

[6] The vectors kw and kx are built through the componentwise products of the vectors k, w and k, x respectively.

$$\dot{w}(t) = G(w, x). \qquad (13)$$

The composition operation on p is intended to be performed componentwise. In this case [7], the following holds:

Theorem 2. For the overall system (12)-(13), featuring prices that are functions of only the aggregate rates, e.g. (3) and (7), the following is true:

- Equation (12) has a region of equilibrium points $F(x) = 0$ that identifies a manifold $w = h(x)$;
- Within this manifold, Equation (13), also known as the *reduced system*, has a unique global equilibrium which is the solution of $G(h(x), x) = 0$;
- The functions F, G, h and their partial derivatives are bounded near the global equilibrium;
- The equilibrium manifold for the *boundary-layer system* [8] is exponentially stable;
- The global equilibrium of the reduced system $\dot{w} = \frac{\partial h}{\partial x}\dot{x} = G(x, h(x))$ is asymptotically stable.

Then, there exists an ε^* such that, $\forall \varepsilon \leq \varepsilon^*$, the equilibrium point of the composite system is asymptotically stable [9].

5. RATE OF CONVERGENCE AND STOCHASTIC ANALYSIS

5.1 Rate of convergence for the Static Update

To carry out the rate of convergence analysis, let $l_r = k_r \sum_{j\in r} q_j / \sum_{j\in r} p_j$ and $x_r(t) = x_r^o + (l_r x_r^o)^{1/2}\, y_r(t)$; we linearize the static system (9) around the equilibrium point $x^o = [x_r^o, r \in R]$:

$$\dot{y_r}(t) = -\big(l_r y_r(t) \sum_{j\in r} p_j + (l_r x_r^o)^{1/2} \sum_{j\in r} p'_j \sum_{s:j\in s} y_s(t)\,(l_s x_s^o)^{1/2}\big) \qquad (14)$$

We may write it in matrix form as:

$$\dot{Y}(t) = -\Gamma^T \Phi \Gamma\, Y(t), \qquad (15)$$

where Γ is an orthogonal matrix, $\Gamma^T\Gamma = I$, and $\Phi = diag(\phi_r, r \in R)$ is the matrix of eigenvalues, necessarily positive, of the following real, symmetric, positive definite matrix

$$\Gamma^T \Phi \Gamma = diag\{\sum_{j\in r} p_j\} L + L^{1/2} X^{1/2} A^T P' A X^{1/2} L^{1/2}, \qquad (16)$$

where $X = diag\{x_r^o, r \in R\}$, $P' = diag\{p'_j, j \in J\}$, $L = diag\{l_r\}$, and A is the connectivity matrix.

Hence the rate of convergence to the equilibrium point is determined by the smallest eigenvalue, $\phi_r, r \in R$, of the matrix (16).

5.2 Stochastic analysis for the Static Update

Here we consider the stochastic perturbation of the linearized equation (15). The perturbations can be caused by the random nature of packet loss. Let

$$dY(t) = -\Gamma^T \Phi \Gamma\, Y(t) dt - F dB(t), \qquad (17)$$

where F is an arbitrary $R \times I$ matrix and $B(t) = (B_i(t), i \in I)$ is a collection of independent standard Brownian motions, extended to $-\infty < t < \infty$. Following the similar procedure for the stochastic analysis (Kelly *et al.*, Dec 1999), we conclude that the stationary solution to the system (17) has a multivariate normal distribution, $Y(t) \sim N(0, \Sigma)$, where

$$\Sigma = E[Y(t)Y(t)^T] = \Gamma^T [\Gamma F; \Phi] \Gamma,$$

where the symmetric matrix $[\Gamma F; \Phi]$ is given by

$$[\Gamma F; \Phi]_{rs} = \left[\int_{-\infty}^{0} e^{\tau\Phi} \Gamma F F^T \Gamma^T e^{\tau\Phi}\right]_{rs} = \frac{[\Gamma F F^T \Gamma^T]_{rs}}{\phi_r + \phi_s}.$$

5.3 Rate of convergence for the Dynamic Update

To carry out the the rate of convergence analysis, we first linearize the entire system around the equilibrium point (x^o, w); then we apply the two timescale decomposition to decouple the analysis of $x(t)$ and $w(t)$.

Let $x_r(t) = x_r^o + (x_r^o)^{1/2} y_r(t)$, and $w_r(t) = w_r + (x_r^o)^{1/2} z_r(t)$. Linearizing system (4)-(5)-(10) around equilibrium point (x^o, w) results in

$$\dot{y_r}(t) = k_r\big(z_r(t) - y_r(t)\sum_{j\in r} q_j - (x_r^o)^{1/2} \sum_{j\in r} q'_j \sum_{s:j\in s} y_s(t)(x_s^o)^{1/2}\big), \qquad (18)$$

and

$$\dot{z_r}(t) = -c_r\big(z_r(t) \frac{\sum_{j\in r} p_j}{\sum_{j\in r} q_j} + (x_r^o)^{1/2} \sum_{j\in r} p'_j \sum_{s:j\in s} y_s(t)(x_s^o)^{1/2} - (x_r^o)^{1/2} \cdot \frac{\sum_{j\in r} p_j}{\sum_{j\in r} q_j} \sum_{j\in r} q'_j \sum_{s:j\in s} y_s(t)(x_s^o)^{1/2}\big). \qquad (19)$$

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[7] For simplicity, we shall consider the simplified case $c = 1$.

[8] This system is obtained through a rescaling of time, $\tau = t/\epsilon$ and letting $\epsilon \to 0$.

[9] Refer to literature on stability for singularly-perturbed, non-linear systems in (Sastry, 1999) and (Khalil, 2001).

This is a coupled, multivariate linear system; we apply two timescale decomposition to decouple the system into boundary system and reduced system to fit into the classical singular perturbation framework, as we did in the global asymptotic convergence analysis.

The boundary system is described by (18) with $z_r(t)$ to be fixed. We may write boundary system in matrix form as

$$\dot{Y}(t) = -KB\,Y(t) + KZ, \tag{20}$$

where

$$B = diag\{\sum_{j\in r} q_j\} + X^{1/2}A^T Q' A X^{1/2}$$

is a symmetric, positive definite matrix, $Q' = diag\{q'_j, j \in J\}$, and $K = diag\{k_r, r \in R\}$. The product KB is a diagonalizable matrix with all eigenvalues to be positive (Horn and Johnson, 1985); hence, the linearized system (20) converges to a manifold, i.e. its equilibrium $Y(t) = B^{-1}Z$, with the rate of convergence determined by the smallest eigenvalue of matrix KB.

On the larger timescale, we analyze the rate of convergence for the reduced system, described by (19) on the manifold $Y(t) = B^{-1}Z(t)$. We may write the reduced system in matrix form as

$$\dot{Z}(t) = -C\,D\,Y(t) = -C\,D\,B^{-1}Z(t), \tag{21}$$

where

$$D = diag\{\sum_{j\in r} p_j\} + X^{1/2}A^T P' A X^{1/2},$$

and $C = diag\{c_r, r \in R\}$.

Again, since D and B^{-1} are positive definite matrices, the product of them DB^{-1} is a diagonalizable matrix with positive eigenvalues. Hence in the case where $c_r = c, r \in R$ [10], the linearized reduced system (21) converges to the equilibrium point (x^o, w), with the rate of convergence determined by the smallest eigenvalue of matrix DB^{-1}.

In summary, the rate of convergence of the entire system depends on how it converges on large timescale, i.e. the rate of convergence of the reduced system, and hence is determined by the smallest eigenvalue of the matrix DB^{-1}.

5.4 Stochastic analysis for the Dynamic Update

5.4.1. Boundary layer system Here we first consider the stochastic perturbations of the linearized equation (20) as follows:

$$dY(t) = -KB\,Y(t) + KZ - F_1 dG(t), \tag{22}$$

where F_1 is an arbitrary $R \times I$ matrix and $G(t) = (G_i(t), i \in I)$ is a collection of independent standard Brownian motions, extended to $-\infty < t < \infty$. Following the similar procedure of stochastic analysis part in Section 5.2, we conclude that the stationary solution to the system (22) is

$$Y(t) = -K\int_{-\infty}^{t} e^{-(t-\tau)B\,K} F_1 dG(\tau) + B^{-1}Z, \tag{23}$$

this solution contains a linear transformation of the Gaussian process $(dG(\tau), \tau < t)$ and a constant vector $B^{-1}Z$; hence $Y(t)$ has a multivariate normal distribution, $Y(t) \sim N(B^{-1}Z, \Sigma_1)$. Let the diagonalizable matrix $B\,K = S_1^{-1}\Phi S_1$, then

$$\begin{aligned}\Sigma_1 &= E[(Y(t) - B^{-1}Z)(Y(t) - B^{-1}Z)^T] \\ &= KS_1^{-1}\left[\int_{-\infty}^{0} e^{\tau\Phi} S_1 F_1 F_1^T S_1^{-1} e^{\tau\Phi} d\tau\right] S_1 K.\end{aligned}$$

Define matrix $[S_1F_1;\Phi]$ as

$$\begin{aligned}[S_1F_1;\Phi]_{rs} &= \left[\int_{-\infty}^{0} e^{\tau\Phi} S_1 F_1 F_1^T S_1^{-1} e^{\tau\Phi} d\tau\right]_{rs} \\ &= \frac{[S_1F_1F_1^TS_1^{-1}]_{rs}}{\phi_r + \phi_s},\end{aligned}$$

then we can rewrite Σ_1 as

$$\Sigma_1 = KS_1^{-1}[S_1F_1;\Phi]S_1K. \tag{24}$$

5.4.2. Reduced system Next we start to consider the stochastic perturbations of the linearized equation (21), under the assumption $c_r = c, r \in R$, as follows:

$$\dot{Z}(t) = -c\,D\,B^{-1}Z(t) - F_2 dE(t), \tag{25}$$

where F_2 is an arbitrary $R \times I$ matrix and $E(t) = (E_i(t), i \in I)$ is a collection of independent standard Brownian motions, extended to $-\infty < t < \infty$. Following the similar procedure of stochastic analysis part in Section 5.2, we conclude that the stationary solution to the system (17) is

$$Z(t) = -c\int_{-\infty}^{t} e^{-(t-\tau)DB^{-1}} F_2 dE(\tau), \tag{26}$$

again, the solution is a linear transformation of the Gaussian process $(dE(\tau), \tau < t)$; hence $Z(t)$ has a multivariate normal distribution, $Z(t) \sim N(0, \Sigma_2)$. Let the diagonalizable matrix $DB^{-1} = S_2^{-1}\Phi S_2$, then

$$\Sigma_2 = S_2^{-1}[S_2F_2;\Phi]S_2, \tag{27}$$

where matrix $[S_2F_2;\Phi]$ is given by

$$\begin{aligned}[S_2F_2;\Phi]_{rs} &= \left[\int_{-\infty}^{0} e^{\tau\Phi} S_2 F_2 F_2^T S_2^{-1} e^{\tau\Phi} d\tau\right]_{rs} \\ &= \frac{[S_2F_2F_2^TS_2^{-1}]_{rs}}{\phi_r + \phi_s}.\end{aligned}$$

[10] In the general case with different values for $c_r, r \in R$, we expect the eigenvalues of the product CDB^{-1} to be all positive to ensure the system is stable. However, at this moment, this is nothing more than a conjecture.

In summary, stochastic perturbations introduce additive multivariate normal distributions with zero mean and variances given in (24) and (27) to the equilibrium solutions in boundary system and reduced system. Since both systems are robust to stochastic perturbations around the equilibrium point (x^o, w), and the entire system is then robust to stochastic perturbations around (x^o, w).

6. CONCLUSIONS

In this paper we proposed two new flow control schemes over wireless networks, a static and a dynamic one. We continued the analysis of the structural properties of the schemes started in (Chen *et al.*, 2005), and focused on the stochastic study of stability, and on the computation of the rate of convergence of the proposed schemes. The study of delay sensitivity and of the quality of the oscillations due to delays in the system has been investigated in (Abate *et al.*, 2005); this second work also contains simulations.

This complete study advocated the applicability of the algorithms in real wireless networks. Moreover, the generality of the proof suggests that these ideas can be tailored to the analysis and the control of similar networks. Future work will address these two important issues.

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