Flow Control over Wireless Network and Application Layer Implementation

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I. Proof of Theorem 1

First note (x^*, n^*) is the equilibrium of the system in (14), and x^* is also the solution for the optimization problem in (19), which can be seen by setting the derivative to zero. Now we show the equilibrium exists and is unique. By the definition of $h_r(z)$ and $g_j(z)$, it is not difficult to see the objective function in (19) is a concave function. Note the constrain set of the optimization problem shown in (19) is convex, we can see the optimization problem is in fact a concave optimization problem. Hence it has a unique solution, which is in fact x^*. Since the equilibrium must lie on the equilibrium manifold in (17), we know the unique x^* lead to a unique n^*. Therefore, the equilibrium (x^*, n^*) exists and is unique.

II. Proof of theorem 2

Proof: On the equilibrium manifold, we can have the relation between $\dot{x}$ and $\dot{n}$ as

$$\left\{ diag\{\frac{x_r T_r^2}{2S^2}\sum_{j\in r}[\epsilon_j + g_j(y_j)]\} + \frac{1}{4S^2} diag\{x_r^2 T_r^2\} A^T diag\{g_j'(y_j)\} A \right\} \dot{x} = diag\{n_r\}\dot{n}.$$

Let $G(x) = diag(\frac{(x_r)^2 T_r^2}{2S^2})$ and

$$D(x) = diag\{\frac{1}{x_r}\sum_{j\in r}[\epsilon_j + g_j(y_j)]\} + \frac{1}{2} A^T diag\{g_j'(y_j)\} A,$$

also combine the dynamics of n in the reduced system, we can rewrite the above equation as

$$\dot{x} = D^{-1}(x) G^{-1}(x) diag\{n_r\}\dot{n} = c D^{-1}(x) G^{-1}(x) \{1 - \frac{T_r^2}{2S^2} x_r^2 \sum_{j\in r}[\epsilon_j + g_j(y_j)] f_r(\sum_{j\in r} g_j(y_j))\}.$$

This work was supported by AFOSR contract F49620-00-1-0327.

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Around the equilibrium of the reduced system, let $x_r(t) = x_r^* + z_r(t)$, denote $D(x^*)$ as $\tilde{D}$ and $G(x^*)$ as $\tilde{G}$; after linearization, we have that, $\forall r \in R$,

$$\begin{aligned}
\dot{z}(t) &= c\tilde{D}^{-1}\tilde{G}^{-1}\Bigg[\frac{x_r^* T_r^2}{S^2} f_r(g_j(y_j^*)) \sum_{j\in r} \left(\epsilon_j + g_j(y_j^*)\right) z_r(t) \\
&\quad + \frac{(x_r^*)^2 T_r^2}{2S^2} \sum_{j\in r} \left(\epsilon_j + g_j(y_j^*)\right) f_r'(g_j(y_j^*)) \sum_{j\in r} g_j'(y_j^*) \sum_{s:j\in s} z_s(t) \\
&\quad + \frac{(x_r^*)^2 T_r^2}{2S^2} f_r(g_j(y_j^*)) \sum_{j\in r} g_j'(y_j^*) \sum_{s:j\in s} z_s(t)\Bigg] \\
&= -c\tilde{D}^{-1}\tilde{G}^{-1}\Bigg[2diag\left(f_r(g_j(y_j^*))\right)\tilde{G}\tilde{D} + \tilde{G}\cdot diag\big(\sum_{j\in r}\left(\epsilon_j + g_j(y_j^*)\right)\big) \\
&\quad \cdot diag\left(f_r'(g_j(y_j^*))\right) A^T diag(g_j'(y_j^*))A\Bigg] z(t), \qquad (1)
\end{aligned}$$

where

$$g_j'(y_j^*) = \frac{C_j}{(y_j^*)^2}\frac{e^{\beta\frac{y_j^*-C_j}{y_j^*}}}{1+e^{\beta\frac{y_j^*-C_j}{y_j^*}}} > 0, \quad j \in J.$$

Denote $E = 2diag\Big(f_r(g_j(y_j^*))\Big)\tilde{G}\tilde{D} + \tilde{G}\cdot diag\big(\sum_{j\in r}\Big(\epsilon_j + g_j(y_j^*)\Big)\big)diag\Big(f_r'(g_j(y_j^*))\Big)A^T diag(g_j'(y_j^*))A$. Then by simple arguments, the system in (1) is stable if and only if $\tilde{D}^{-1}\tilde{G}^{-1}E$ has all positive eigenvalues. We now show that this requirement is verified.

First note that this is equivalent to show $E\tilde{D}^{-1}\tilde{G}^{-1}$ has all eigenvalues be positive since $E\tilde{D}^{-1}\tilde{G}^{-1}$ is similar to $\tilde{D}^{-1}\tilde{G}^{-1}E$.

$$E\tilde{D}^{-1}\tilde{G}^{-1} = \tilde{G}\cdot diag\left(f_r'(g_j(y_j^*))\sum_{j\in r}(\epsilon_j + g_j(y_j^*)\right)\left\{2\cdot diag\left(\frac{f_r(g_j(y_j^*))}{f_r'(g_j(y_j^*))\sum_{j\in r}(\epsilon_j + g_j(y_j^*)}\right) + A^T diag(g_j'(y_j^*))A\tilde{D}^{-1}\right\}\tilde{G}^{-1},$$

where $A^T diag(g_j'(y_j^*))A\tilde{D}^{-1}$ is a product of a positive definite matrix and a (semi)-positive definite matrix, hence it has all eigenvalues be nonnegative. On the other hand, $A^T diag(g_j'(y_j^*))A = 2(\tilde{D} - diag\{\frac{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}{x_r^*}\})$; hence

$$A^T diag(g_j'(y_j^*))A\tilde{D}^{-1} = 2\cdot diag\{\frac{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}{x_r^*}\}\left[diag\{\frac{x_r^*}{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}\} - \tilde{D}^{-1}\right].$$

Now we claim $diag\{\frac{x_r^*}{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}\} - \tilde{D}^{-1} \succeq 0$. This can be shown by contradiction. Suppose the contrary, then left multiple the above equation by $\left(diag\{\frac{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}{x_r^*}\}\right)^{-1/2}$, and right multiple the above equation by $\left(diag\{\frac{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}{x_r^*}\}\right)^{1/2}$, then we will have the left hand side has all eigenvalues be nonnegative, while the right hand side is a non-semi-positive definite matrix with at least one eigenvalue being negative, resulting in an contradiction.

Therefore, :

$$\begin{aligned}
E\tilde{D}^{-1}\tilde{G}^{-1} &= 2\tilde{G}\cdot diag\left(f_r'(g_j(y_j^*))\frac{[\sum_{j\in r}(\epsilon_j + g_j(y_j^*)]^2}{x_r^*}\right)\cdot \\
&\quad \left\{diag\left(\frac{x_r^* f_r(g_j(y_j^*))}{f_r'(g_j(y_j^*))[\sum_{j\in r}(\epsilon_j + g_j(y_j^*)]^2}\right) + diag\{\frac{x_r^*}{\sum_{j\in r}[\epsilon_j + g_j(y_j^*)]}\} - \tilde{D}^{-1}\right\}\tilde{G}^{-1}.
\end{aligned}$$

Define the terms inside the brackets as B, we claim $E\tilde{D}^{-1}\tilde{G}^{-1}$ has all eigenvalues be positive, due to the following three facts:

- $B \succ 0$ since it is a sum of two positive definite matrices.
- $diag\left(f_r'(g_j(y_j^*))\frac{[\sum_{j\in r}(\epsilon_j+g_j(y_j^*)]^2}{x_r^*}\right)B$ has all its eigenvalues to be positive, because it is the product of two positive definite matrices;
- $E\tilde{D}^{-1}\tilde{G}^{-1}$ has all eigenvalues be positive, because it is similar to $diag\left(f_r'(g_j(y_j^*))\frac{[\sum_{j\in r}(\epsilon_j+g_j(y_j^*)]^2}{x_r^*}\right)B$.

Eventually, $\tilde{D}^{-1}\tilde{G}^{-1}E$ has all eigenvalues be positive and hence the reduced system in (1) is locally exponentially stable for arbitrary $\beta > 0$.

Hence combining the fact that the boundary system is locally exponentially stable, we conclude the entire system is locally exponentially stable, following the same argument and proof for the singular perturbation system in [1], [2]. ∎

III. Proof of theorem 3

The logic behind the proof is as follows. First we show the reduced system converges to a manifold, using Lasalle principle [1]. Then we show on the manifold, the system further converges to the unique equilibrium, using another time of Lasalle principle. Therefore the reduced order system globally asymptotically converges to the unique equilibrium. Hence combining the fact that the boundary system is semi-globally exponentially stable, we conclude the entire system is globally asymptotically stable, following the same argument and proof in [1], [2].

Now we show the reduced system converges to a manifold $n_r(t) = const, r \in R$. First note as $f_1(\sum_{j\in 1} g_j(y_j(t))) = f_r(\sum_{j\in r} g_j(y_j(t))), r \in R$, it follows that all the users must share the same bottleneck. As only bottleneck cause congestion based packet loss, we assign g function to the bottleneck, then we write $\sum_{j\in 1} g_j(y_j(t)) = \sum_{j\in r} g_j(y_j(t)) = g(\sum_{r\in R} x_r(t)), r \in R$; hence. Let

$$V_1(n) = \sum_{r\in R}\left[n_1^2 - n_r^2\right]^2,$$

then

$$\dot{V}_1 = 4\sum_{r\in R}\left[n_1^2 - n_r^2\right](n_1\dot{n}_1 - n_r\dot{n}_r) = -4f_1(g(\sum_{r\in R} x_r(t)))\sum_{r\in R}\left[n_1^2 - n_r^2\right]^2 \le 0.$$

Therefore, by Lasalle principle, the reduced system converges to the manifold $\dot{V}_1 = 0$, i.e. $n_r(t) = const, \dot{n}_r(t) = const, r \in R$.

Along the manifold, consider

$$V_2(n) = \sum_{r\in R}\left[n_r\dot{n}_r\right]^2 = \sum_{r\in R}\left[1 - n_r^2 f_r(g(\sum_{r\in R} x_r(t)))\right]^2,$$

we have

$$\dot{V}_2 = -2\sum_{r\in R}\{2n_r^2(\dot{n}_r)^2 f_r(g(\sum_{r\in R} x_r(t))) + n_r\dot{n}_r f_1'(g(\sum_{r\in R} x_r(t)))g'(\sum_{r\in R} x_r(t))\sum_{r\in R}\dot{x}_r\} \leq 0.$$

In the above derivation, we use the fact that along the manifold, $\dot{n}_r \sum_{r\in R}\dot{x}_r \geq 0$, i.e. the change in the number of connections and the change of aggregate rate passing through the bottleneck are of the same direction.

Therefore, by Lasalle principle, along the manifold $n_r(t) = const, \dot{n}_r(t) = const, r \in R$, the reduced system converges to the invariant set $\dot{V}_2 = 0$, which contains only the equilibrium (x^*, n^*). Therefore the reduced order system globally asymptotically converges to the unique equilibrium. Hence combining the fact that the boundary system is semi-globally exponentially stable, we conclude the entire system is globally asymptotically stable, following the same argument and proof in [1], [2].

References

[1] S. Sastry, *Non Linear Systems, Analysis, Stability and Control*. New York, NY: Springer Verlag, 1999.

[2] H. Khalil, *Nonlinear Systems (3rd edition)*. Prentice Hall, 2001.