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A Distribution-free Control Chart for the Joint Monitoring of Location and Scale

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Traditional statistical process control for variables data often involves the use of a separate mean and a standard deviation chart. Several proposals have been published recently, where a single (combination) chart that is simpler and may have performance advantages, is used. The assumption of normality is crucial for the validity of these charts. In this article, a single distribution-free Shewhart-type chart is proposed for monitoring the location and the scale parameters of a continuous distribution when both of these parameters are unknown. The plotting statistic combines two popular nonparametric test statistics: the Wilcoxon rank sum test for location and the Ansari–Bradley test for scale. Being nonparametric, all in-control properties of the proposed chart remain the same and known for all continuous distributions. Control limits are tabulated for implementation in practice. The in-control and the out-of-control performance properties of the chart are investigated in simulation studies in terms of the mean, the standard deviation, the median, and some percentiles of the run length distribution. The influence of the reference sample size is examined. A numerical example is given for illustration. Summary and conclusions are offered. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

Control charts continue to play a transformative role in all walks of life in the 21st century. The mean and the variance of a process are the two parameters most often monitored, and the traditional approach has been to use two separate control charts, one for the mean and one for the variance (standard deviation) or scale. If either chart gives a signal, action is contemplated to diagnose and solve any special cause issues. Many authors have considered the problem of joint monitoring of the mean and the variance of a normal distribution with two charts. Among these are Jones and Case[1], Saniga[2], and Rahim and Costa[3]. These charts are the Shewhart-type charts and are generally good for detecting larger shifts. Reynolds and Stoumbos[4,5] studied joint monitoring by using several combinations of charts, including an exponentially weighted moving average (EWMA) chart for the mean and an EWMA chart for the variance/standard deviation.

It is recognized that using two charts can sometimes be difficult in practice for the interpretation of signals because the effects of changes in one of the parameters (charts) can affect the changes in the other one. As an alternative, researchers have put forward the idea of using a single chart, using a single plotting statistic, for joint monitoring and suggested that these may be more appealing from a practical point of view. Gan[6] noted that, "The current practice of a combined scheme consisting of a mean chart and a variance chart is basically looking at a bivariate problem using two univariate procedures." He went on to develop new single charts for the joint monitoring of the mean and the variance of a normally distributed process when the process mean and the variance are both known or specified. The semicircle chart of Chao and Cheng[7] and the Max chart of Chen and Cheng[8] are two Shewhart-based single chart schemes. Chen *et al.*[9] proposed the MaxEWMA chart that combines two EWMA charts into one and showed that their chart is effective in detecting both increases and decreases in the mean and/or the variance. Memar and Niaki[10] recently proposed a generalization of the MaxEWMA chart for individual observations. Costa and Rahim[11] also considered a single chart, called the *noncentral chi-square chart*, for the joint monitoring of mean and variance. Zhang *et al.*[12] considered monitoring the mean and the variability on the basis of a likelihood ratio test. Cheng and Thaga[13] gave a thorough overview of the literature on joint monitoring and favored the one-chart approach. Hawkins and Deng[14] proposed two single (combination) charts on the basis of the likelihood ratio test and the so-called *Fisher approach* on the basis of combining the *P* values corresponding to separate tests for the mean and the standard deviation, respectively. They also considered the cumulative sum variants of these charts.

Although these are important advances, there are two important issues with the existing approaches. First, there are situations in practice where the form of the underlying distribution is either not sufficiently known or is at least known to be

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nonnormal. It is known that many control charts for the mean or the variance are not robust to nonnormality, which can diminish their value in practice because the false alarm rate may be unknown and often inflated. Second, there are situations where along with the distribution, in-control (IC) mean and variance values may not be known and need to be estimated before process monitoring can begin. In this article, we propose a nonparametric or a distribution-free control chart (we use the two terms interchangeably) that does not require a distributional assumption, and we construct the chart on the basis of a reference (IC sample) obtained in a phase I analysis.

Nonparametric control charts have received a lot of attention in the recent quality control literature. To conserve space, the reader is referred to Chakraborti *et al.*[15] and the references therein for a comprehensive overview of the area. The purpose of the present work is to develop a phase II nonparametric control chart to simultaneously monitor the unknown location and scale parameters of an unknown underlying continuous distribution. We assume that a reference (IC) sample of a size m is available from a phase I study at the beginning of phase II monitoring. Note that control charting in phase I, including nonparametric control charting, has also received considerable attention recently. The interested reader is referred to the overview by Chakraborti *et al.*[15]; we do not discuss this issue here. For simplicity, we consider the Shewhart-type nonparametric charts in this article. These charts are known to be effective in detecting larger shifts; refinements and enhancements to these charts will be studied elsewhere. At each stage of phase II monitoring, as a test sample is obtained, we calculate a Lepage[16] statistic (details provided in the next section) as the plotting statistic, which is a combination of the Wilcoxon rank sum (WRS) test for location and the Ansari–Bradley (AB) test for scale, using the test and the reference sample. The resulting control chart is called the Shewhart–Lepage (SL) control chart. For a general introduction to nonparametric tests for location and scale, the reader is referred to Gibbons and Chakraborti[17]. If the plotting statistic falls outside the control limits, an out-of-control (OOC) signal is produced, and a search for assignable causes is begun to detect the source of the signal. A follow-up procedure is proposed, which can aid to this end by diagnosing whether the signal is due to a shift in the location parameter or in the scale parameter or in both. On the other hand, if the plotting statistic does not fall above the control limit, the process is deemed to be IC, and monitoring continues to the next test sample.

The rest of the article is organized as follows. Section 2 provides a brief background along with a review of some of the nonparametric control charting literature. Statistical framework and preliminaries are presented in Section 3. The proposed SL control chart is introduced in Section 4. Section 5 is devoted to the practical implementation of the chart. The performance of the chart is studied in Section 6 on the basis of various run length distribution characteristics obtained in a simulation study. In Section 7, we examine the influence of the reference sample size on the performance of the chart. The charting procedure is illustrated in Section 8 with a data set from Montgomery[18]. Section 9 ends with a summary and conclusions.

2. Background

Nonparametric control charts may be the way to go in practice when it is not reasonable to assume a specific parametric distribution, such as the normal, for the underlying process distribution. This may happen when the distributional assumption is untenable because of the theory, lack of knowledge or available information or the data (for an early overview of nonparametric control charts, see Chakraborti *et al.*[19] and Chakraborti and Graham[20]). These charts have the advantage that their IC performance (such as the IC average run length, the false alarm rate, etc.) is completely known and remains the same for all continuous distributions. Several nonparametric control charts have been proposed for the settings when (i) the process parameters are assumed known or specified, that is, the *standards known* case, and (ii) the process parameters are unknown or unspecified, that is, the *standards unknown* case (e.g. see Bakir and Reynolds[21], Amin and Searcy[22], Amin *et al.*[23], Amin and Widmaier[24], Bakir[25], Chakraborti and Eryilmaz[26], Human *et al.*[27], which focus on the standards known case, whereas Hackl and Ledolter[28,29] and Bakir[30] focus on the standards unknown case). Other recent work on nonparametric control charts includes Chakraborti and Van de Wiel[31], who considered a Shewhart-type chart adapting the Mann–Whitney statistic for monitoring location in the standards unknown case. Along the same lines, Li *et al.*[32] considered cumulative sum (CUSUM) and EWMA-type charts. Also, see Albers and Kallenberg[33,34] and Hawkins and Deng[35]. For an extensive overview of the published literature through 2010, see Chakraborti *et al.*[17]

Although the previous nonparametric control charts have been mostly for monitoring the location, the problem of the simultaneous monitoring of location and scale parameters has not been addressed. Assuming that the location and scale parameters are both unknown, we adapt a two-sample nonparametric test for location and scale developed by Lepage[16] to propose a Shewhart-type control chart.

3. Statistical framework and preliminaries

Let $U_1, U_2, \ldots, U_M$ and $V_1, V_2, \ldots, V_n$ be the independent random samples from two populations with continuous distribution functions $F(x)$ and $G(y) = F(\delta y+\theta)$, $\delta>0, -\infty<\theta<\infty$), respectively, where F is some unknown continuous distribution function. The constants θ and δ represent the unknown location and scale parameter, respectively. We define an indicator variable $Z_k = 1$ when the kth order statistic of the combined N (= $m + n$) observations is a V; otherwise, $Z_k = 0$. To test the equality of the two location parameters, we use one of the popular nonparametric tests used in the literature, the WRS test. The WRS test statistic, say T_1, is defined as

$$T_1 = \sum_{k=1}^{N} kZ_k \quad (1)$$

Among the nonparametric tests for the equality of the two scale parameters, the AB test is a popular choice. The AB test statistic, say T_2, is defined as

$$T_2 = \sum_{k=1}^{N} \left| k - \frac{1}{2}(N+1) \right| Z_k \quad (2)$$

Both the WRS and the AB tests are linear rank tests and are powerful in a variety of situations; interested readers may see Gibbons and Chakraborti[17] for further details.

In the process monitoring context, let the U denote the phase I reference data and let the V denote the phase II (test) observations under monitoring. In phase II, the process is said to be IC when $F = G$, that is, when $\theta = 0$ and $\delta = 1$. Thus, we monitor both the location and the scale parameters, simultaneously.

It is well known (see Gibbons and Chakraborti[17]) that

$$E(T_1|IC) = \mu_1 = \frac{1}{2}n(N+1) \text{ and } \mathrm{Var}(T_1|IC) = {\sigma_1}^2 = \frac{1}{12}mn(N+1) \quad (3)$$

Moreover,

$$E(T_2|IC) = \mu_2 = \begin{cases} \dfrac{nN}{4} & \text{when } N \text{ is even} \\ \dfrac{n(N^2-1)}{4N} & \text{when } N \text{ is odd} \end{cases} \text{ and } Var(T_2|IC) = \sigma_2^2 = \begin{cases} \dfrac{1}{48} mn \dfrac{(N^2-4)}{N-1} & \text{when } N \text{ is even} \\ \dfrac{1}{48} \dfrac{mn(N+1)(N^2+3)}{N^2} & \text{when } N \text{ is odd} \end{cases} \quad (4)$$

We describe the proposed control charting procedure in the next section.

4. Proposed charting procedure

The proposed Shewhart-Laplace (SL) control chart is constructed as follows.

Step 1: collect a reference sample of size m from an IC process,

$$\boldsymbol{X}_m = (X_1, X_2, \ldots, X_m)$$

Step 2: let $\boldsymbol{Y}_{i.n}=(Y_{i1},Y_{i2},\ldots,Y_{in})$ be the ith phase II (test) sample of size n, $i=1,2,\ldots$

Step 3: identify the U's with the X's and the V's with the Y's, respectively. Calculate the WRS statistic T_{1i} and the AB statistic T_{2i} between the ith test sample and the reference sample using steps 1 and 2 and their means and standard deviations using steps 3 and 4, respectively, according to whether $N = m + n$ is even or odd.

Step 4: calculate the standardized WRS and AB statistics $S_{1i} = \frac{T_{1i}-\mu_1}{\sigma_1}$ and $S_{2i} = \frac{T_{2i}-\mu_2}{\sigma_2}$, respectively.

Step 5: calculate the SL plotting statistic, $S_i^2 = S_{1t}^2 + S_{2i}^2, \quad i = 1, 2, \ldots$

Step 6: plot S_i^2 against an upper control limit H. The lower control limit is 0. Note that $S_i^2 \geq 0$ by definition, and larger values of S_i^2 suggest an OOC process.

Step 7: if S_i^2 exceeds H, the process is declared OOC at the ith test sample. If not, the process is thought to be IC, and testing continues to the next test sample.

Step 8 (follow-up): when the process is declared OOC at the ith test sample, compare each of S_{1i}^2 and S_{2i}^2 with specified constants H_1 and H_2, (H_1,$H_2 < H$), respectively.

(i) If only S_{1i}^2 exceeds H_1, a shift in location is indicated.
(ii) If only S_{2i}^2 exceeds H_2, a shift in scale is implied.
(iii) If S_{1i}^2 and S_{2i}^2 exceed H_1 and H_2, respectively, a shift in both location and scale is indicated.

Thus, the proposed distribution-free char can not only indicate a shift or a change (an OOC situation) in the process but can also help diagnose the type of the change. The determination of the constants H, H_1, and H_2 is discussed next.

5. Practical implementation

To implement the SL chart, we need the charting constants H, H_1, and H_2. First, we discuss the determination of H.

5.1. Determination of H

Using an analytical form of the run length distribution, we can find an expression for the IC average run length (ARL_0) (for details, see Appendix). Then, by setting the ARL_0 equal to some desired (nominal) value ARL_0^*, we can determine the constant H by solving

$$\mathrm{ARL}_0^* = \int_{\boldsymbol{X}_m} \frac{1}{1-\Psi_{F=G}(\boldsymbol{X}_m, H)} dF(\boldsymbol{X}_m)$$

This formulation, although direct in principle, is difficult to be implemented in practice because many integrals are involved. An alternative is to use simulations, and we use this approach here. Because the chart is distribution-free, data are generated from the standard normal distribution for the IC sample as well as the test sample. We use the R.2.11.1[36] software and 50,000 replicates that provide a stable estimate of the ARL_0 and other percentiles of IC run length distribution. We chose m = 30, 50, 100, and 150 for the reference sample size and n = 5, 11, and 25 as the test sample size. The results are shown in Table I. For any given pair of (m, n) values, a search is conducted with different values of H, and that value of H is obtained for which the ARL_0 is approximately 500. The third column of Table I gives the required H values. Thus, for example, when 30 reference samples and test samples of size 5 are available and an ARL_0 of 500 is desired, the upper control limit H for the SL chart is given by 9.4. Further, in Figures 1 and 2, we showed the IC ARL values (profiles) for various H values for the same values of m (30, 50, 100, and 150) and n (5, 11, and 25). Figures 1 and 2 can be used to find the necessary H value for other target ARL_0 values such as 250 and 370, respectively. These are discussed in the next section.

Figures 1 and 2 show that the ARL_0 profiles become steeper when n increases for a given m, and therefore for a given number of reference samples m, we need a smaller H value to attain a specified ARL_0 value for increasing n. However, H increases with m when both ARL_0 and n are held fixed. It was observed in the simulations that the H values become more stable when both m and n get large. Figures 1 and 2 can help in finding the chart constant H for other desired ARL_0 values that are not 500 and hence not within the range of Table I. For example, for an ARL_0 = 250 with m = 30 and n = 25, we need H = 7.6, whereas for m = 50 and n = 25, we need H = 8.5.

5.2. Determination of H_1 and H_2

If the proposed chart signals an OOC process at the rth phase II sample, we propose a follow-up procedure to determine whether the change is in the location or in the scale or both. For this purpose, the constants H_1 and H_2 are necessary. The following result is helpful to this end:

Result: There exist nonnegative constants H_1 and H_2 (= $H - H_1$) both less than or equal to H such that for a given [R=r], the event $[S_r{}^2 > H]$ can be partitioned into three mutually exclusive events as A: $[S_{1r}^2 > H_1, S_{2r}^2 \le H_2 | R = r]$; B : $[S_{1r}^2 \le H_1, S_{2r}^2 > H_2 | R = r]$; and C : $[S_{1r}^2 > H_1, S_{2r}^2 > H_2 | R = r]$.

The proof of this result is given in the Appendix. Now let $P[S_{1r}^2 > H_1 | R = r, IC] = \gamma_1$ and $P[S_{2r}^2 > H_2 | R = r, IC] = \gamma_2$. Then, because S_{1r}^2 and S_{2r}^2 are independent when the process is IC, as shown in Lepage (1971), we have $P[S_{1r}^2 > H_1, S_{2r}^2 \le H_2 | R = r, IC] = \gamma_1(1-\gamma_2)$, $P[S_{1r}^2 \le H_1, S_{2r}^2 > H_2 | R = r, IC] = \gamma_2(1-\gamma_1)$,and $P[S_{1r}^2 > H_1, S_{2r}^2 > H_2 | R = r, IC] = \gamma_2\gamma_1$.

where γ_1 and γ_2 must satisfy the following equation:

$$\gamma_1 + \gamma_2 - \gamma_1\gamma_2 = \alpha, \tag{4}$$

and α is the overall false alarm probability, that is, $\alpha = P(S_i^2 > H | IC)$.

To find H_1 and H_2 we assumed $\gamma_1 = \gamma_2$, which means that the events A (a location shift) and B (a scale shift) are deemed equally likely when the process is IC. Thus, we calculated H_1 and H_2 by setting $0 < \gamma_1 = \gamma_2 \cong 1 - \sqrt{1-\alpha} = \gamma(\alpha) < 1$, which satisfies Equation (4), and by applying the monotonic property of the distribution functions of S_{1r}^2 and S_{2r}^2. Note that the sign "$\cong$" is used to accommodate the descreteness of S_{1r} or S_{2r} for any r. Now, H_1 is calculated from $P[S_{1r}^2 > H_1 | R = r, IC] \cong 1 - \sqrt{1-\alpha}$, where $\alpha = P(S_i^2 > H | IC)$. The latter probability can be approximated for large sample sizes by using the chi-square distribution with 2 degrees of freedom (Lepage, 1971). Accordingly, for large sample sizes, H is approximately the $(1-\alpha)^{\text{th}}$ quantile of the chi-square distribution with 2 degrees of freedom, and hence H_1 is approximately the $(1-\gamma(\alpha))^{\text{th}}$ quantile of the chi-square distribution with 1 degree of freedom.

Note that as mentioned earlier, having found an OOC signal, we concluded a location shift if $[S_{1r}^2 > H_1, S_{2r}^2 \le H_2]$, a scale shift if $[S_{1r}^2 \le H_1, S_{2r}^2 > H_2]$, or a shift in both location and scale if $[S_{1r}^2 > H_1, S_{2r}^2 > H_2]$. From the result, it is evident that the probability of each of these events, being a partition, is less than that of the event $[S_r^2 > H]$ given [R=r]. Therefore, the individual false alarm probabilities while concluding a location shift only, a scale shift only, or a shift in both will not exceed the overall probability of a false alarm α. In fact, under the suggested choices, these probabilities will be approximately $\sqrt{1-\alpha} - (1-\alpha)$, $\sqrt{1-\alpha} - (1-\alpha)$ and $\left(1 - \sqrt{1-\alpha}\right)^2$ respectively.

Table I. The charting constants H, H_1, and H_2 for the SL chart for the various values of m and n with ARL_0 = 500

m	n	H	H_1	H_2	M	n	H	H_1	H_2
30	5	9.40	5.75	3.65	50	5	10.32	6.52	3.80
30	11	9.24	5.00	4.24	50	11	10.10	6.10	4.00
30	25	8.40	4.30	4.10	50	25	9.50	5.00	4.50
100	5	11.25	7.25	4.00	150	5	11.50	7.65	3.85
100	11	11.07	6.35	4.72	150	11	11.45	6.80	4.65
100	25	10.74	5.40	5.34	150	25	11.17	5.61	5.56

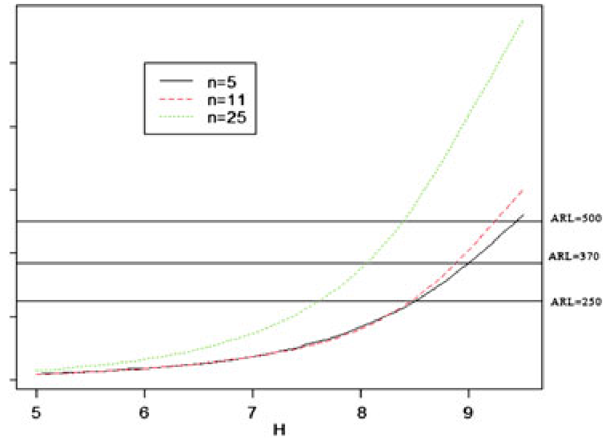

Figure 1. ARL_0 profiles for the different values of n and H when $m = 30$

Preliminary investigations suggest that $m > 50$ and $n > 30$ are needed for the chi-square approximations to be accurate. Otherwise, the constants can be found by simulation as described in the next paragraph. For practical implementation, we provided the values of H_1 and H_2 for various combinations of m and n values for which the H value was obtained so that the ARL_0 is close to 500 (see Table 1). Now, H_1 (and $H_2 = H - H_1$) is determined by 50,000 replicates with given m and n values under the IC setting and out of the number of cases, say b, where $S_i^2 > H$. Out of these b cases where the chart signals (an OOC situation), the joint distribution of S_{1i}^2 and S_{2i}^2 is examined because H_1 and H_2 are needed only when an OOC signal is given by the overall statistic S_i^2. Thus, H_1 (and hence H_2) is determined conditionally from this joint distribution so that $P\left[S_{1i}^2 > H_1, S_{2i}^2 \le H_2 | IC\right] \approx P\left[S_{1i}^2 \le H_1, S_{2i}^2 > H_2 | IC\right]$.

For example, for the first line in Table 1, H_1 (and $H_2 = H - H_1$) is determined in 50,000 simulations under the IC setting with $m = 30$ and $n = 5$. First, we found the number of cases (simulations) in which $S_i^2 > 9.4$ (i.e. the H value for which the ARL_0 is 500). Out of these cases, the empirical joint distribution of S_{1i}^2 and S_{2i}^2 is studied, and it is found by using a search method so that $P\left[S_{1i}^2 > 5.4, S_{2i}^2 \le 4\right] \cong P\left[S_{1i}^2 \le 5.4, S_{2i}^2 > 4\right]$. Thus, $H_1 = 5.4$ (and hence $H_2 = 4$), Other values of H_1 and H_2 are found in a similar manner.

6. Performance of the SL chart

The performance of a control chart is measured in terms of the run length distribution, and because the run length distribution is skewed to the right, it is useful to look at various summary measures such as the mean, the standard deviation, and several percentiles including the first and the third quartiles to characterize the distribution. We considered both the IC and the OOC setup. For the IC setup, we simulated both the reference and the test sample from the standard normal distribution. We chose m = 30, 50, 100, and 150 and n = 5, 11, and 25. The different choices of H are considered for a given pair of (m, n), and we found H by searching, for which ARL_0 is approximately 500. The findings under the IC setup are shown in Table II.

Table II shows that the IC run length distribution is highly skewed to the right; the mean 500 is much larger than the medians. As the reference sample size m increases for a fixed n, all the percentiles (including the median) increase and the standard deviation

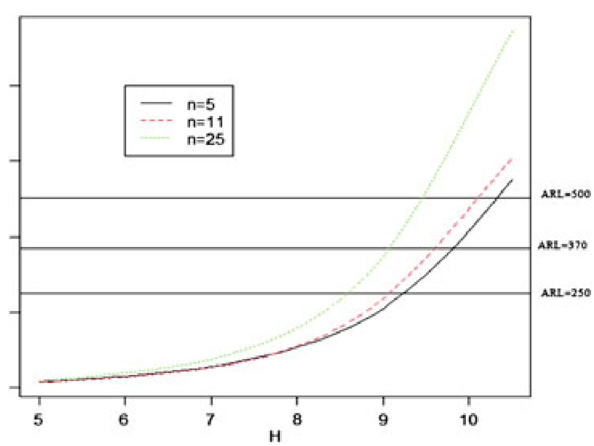

Figure 2. ARL_0 profiles for the different values of n and H when $m = 50$

Table II. IC performance characteristics of the SL chart for $ARL_0 = 500$

m	n	H	H_1	SD	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile
30	5	9.4	5.4	1216.59	9	59	176	486	1956
30	11	9.24	5.0	978.05	8	58	187	533	1978
30	25	8.4	4.3	1027.58	5	38	148	503	2175
50	5	10.32	7.1	918.88	13	78	215	534	1886
50	11	10.1	6.1	860.88	11	73	219	572	1936
50	25	9.5	5.0	890.65	8	57	190	556	2000
100	5	11.25	7.25	690.00	21	108	274	606	1710
100	11	11.07	6.35	703.58	18	99	281	611	1742
100	25	10.74	5.40	703.81	13	89	255	648	1837
150	5	11.50	7.65	692.79	18	113	287	632	1633
150	11	11.45	6.80	627.38	20	113	291	659	1675
150	25	11.17	5.60	660.91	16	95	272	645	1712

decreases. However, interestingly, for a fixed reference sample size m, the percentiles decrease with increasing n, except the 3rd quartile. It is seen that the 3rd quartiles are closer to the nominal IC ARL (500). Table II is useful to determine the critical values H, H_1, and H_2 ($=H$-H_1).

Next, we examined the performance characteristics of the run length distribution when the IC sample is from $N(0,1)$ but the test samples are from a $N(\theta, \delta)$ distribution. To examine the effects of shifts in the mean and the variance, 32 combinations of (θ, δ) values are considered, θ = 0, 0.25, 0.5, 0.75, 1, 1.25, and 1.5 and δ = 1, 1.25, 1.5, 1.75, and 2.0, respectively. Of course, θ = 0 and δ=1 corresponds to the IC case. For brevity, only the results for n = 5 are presented in Table III. We also examined the performance characteristics for the heavier tailed and symmetric Laplace distribution. These results are shown in Table IV.

First, observe that the OOC run length distributions are also skewed to the right. Moreover, for a fixed m and n and a given ARL_0, the OOC ARL values as well as the percentiles decrease sharply with the increasing shift in the mean and also with the increasing shift in variance. This means the chart is sensitive to both, that is, it detects shifts in the mean and/or in the variance but not equally. The SL chart seems to react more quickly to a shift in the standard deviation than in the mean. For example, from Table III, it is easily seen that for a 25% increase in the mean when the standard deviation is IC, the ARL decreases by 26.3%, whereas for a 25% increase in the standard deviation when the mean is IC, the ARL decreases by 77%. Finally, when both the mean and the standard deviation increase by 25%, the ARL decreases by 85%. The pattern is the same for the standard deviation of the run length (SDRL) distribution; it decreases for an increase in the shift in both parameters but decreases more for a shift in the process standard deviation. For example, when m = 30, For a 25% increase in the mean, the standard deviation decreases by 19%, but for a 25% increase in the standard deviation, the SDRL decreases by 83%.

Although nonparametric charts are IC robust by definition, that is, their IC performance remains the same for all continuous distributions, it is of interest to study the effect of the tail heaviness of the underlying distribution on the chart performance. Heavy-tailed symmetric distributions such as a Laplace distribution often arise in applications where more extreme values can occur with higher probability. We repeated the simulation study with data from the Laplace distribution. The performance characteristics of the run length distribution were evaluated when the IC sample is from a Laplace (0, 0.71) distribution that has a mean of 0 and a variance of 1, but the test samples are from a Laplace (θ, δ) distribution, where θ denotes the mean and δ denotes the standard deviation. Note that $\delta = \sqrt{2}b$, where b is the scale parameter of a Laplace distribution. To examine the effects of a shift in the location and scale, as in the normal case, we studied the same 32 combinations of (θ, δ) values.

It is seen that for the Laplace distribution, the general patterns remain the same as in the case of the normal distribution, but the OOC ARL values are increased for a shift in the mean or in the variance, indicating a slightly slower detection of shifts under the heavier-tailed distribution. For example, when m = 30 and the mean increases 25% to 0.25 and the standard deviation increases 50% to 1.5, the ARL increases to 60.87 compared with 32.96 in the normal case. Moreover, the percentiles as well as the SDRL all increase under the Laplace distribution. This is interesting as the distribution-free tests are generally more efficient under the heavier-tailed distributions.

As far as we are aware, there are no other distribution-free charts in the literature for the joint monitoring of location and scale and as such a comparison is not feasible. Also, because our chart is for the case of unknown parameters, most of the known normal theory charts are also not directly comparable.

7. Influence of the reference sample

The proposed control chart is calibrated to achieve an overall IC ARL equal to a target, for example, 500. However, users in practice may be interested in the IC ARL depending (conditional) on the phase I sample they have. The attained IC ARL will vary, although the chart may have been designed for a nominal IC ARL of 500 and the phase I sample is from an IC process. To examine this issue, we studied the conditional IC ARL and the SDRL of the SL chart for a given IC phase I sample, with H determined as before for m = 100 and 150, respectively, and n = 5. To this end, 100 different phase I samples each of size m are generated from a standard normal

Table III. Performance characteristics of the run length distribution of the SL chart for the normal (θ,δ) distribution with $ARL_0 = 500$

		$m = 30, n = 5, H = 9.4$							$m = 50, n = 5, H = 10.32$						
θ	δ	ARL	SD	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile	ARL	SD	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile
0	1	500.79	1216.59	9	59	176	486	1956	499.62	918.88	13	78	215	534.25	1886
0.25	1	369.08	984.81	5	34	107	323	1500.1	292.69	641.30	7	40	116	308.25	1124.05
0.5	1	145.18	474.79	2	12	37	113	571.05	94.69	253.87	2	12	34	91	351
0.75	1	42.49	134.36	1	5	13	34	152.05	26.95	61.14	1	5	12	28	96
1	1	13.09	34.62	1	2	5	12	46	9.09	14.29	1	2	5	10	31
1.25	1	5.45	10.97	1	1	3	6	17.05	4.13	4.94	1	1	3	5	13
1.5	1	2.67	3.24	1	1	2	3	8	2.32	2.18	1	1	2	3	6
0	1.25	114.11	210.38	4	18	50	125	420	106.21	197.81	4	21	54	124	369.05
0.25	1.25	86.72	171.41	3	13	36	92	332	73.59	116.14	3	14	37	86	261
0.5	1.25	45.57	89.61	2	7	18	47	173.05	35.4	55.45	2	7	18	42	123
0.75	1.25	19.9	41.60	1	4	9	21	70	15.17	22.45	1	4	8	18	51
1	1.25	9.27	18.47	1	2	5	10	30	7.43	9.11	1	2	4	9	24
1.25	1.25	4.79	6.36	1	1	3	6	15	4.19	4.63	1	1	3	5	12
1.5	1.25	2.89	3.08	1	1	2	3	8	2.54	2.30	1	1	2	3	7
0	1.5	39.54	59.82	2	8	21	46	138	36.82	46.98	2	9	22	47	118
0.25	1.5	32.96	54.04	2	7	17	39	115	30.64	38.81	2	7	18	39	102
0.5	1.5	21.63	36.20	1	5	11	25	73	19	24.77	1	5	11	24	64
0.75	1.5	12.19	17.80	1	3	7	15	41	10.78	12.87	1	3	7	14	34
1	1.5	7.28	9.79	1	2	4	9	23.05	6.31	6.92	1	2	4	8	19
1.25	1.5	4.5	5.47	1	1	3	6	13.05	4.03	4.00	1	1	3	5	12
1.5	1.5	3	2.97	1	1	2	4	9	2.8	2.50	1	1	2	4	8
0	1.75	18.7	24.52	1	5	11	23	61	18.48	20.74	1	5	11	24	59
0.25	1.75	16.81	22.46	1	4	10	21	55	16.67	20.38	1	4	11	22	53
0.5	1.75	13.07	17.37	1	3	8	16	43	12.07	13.85	1	3	8	16	37
0.75	1.75	8.8	10.99	1	2	5	11	28	8.28	8.98	1	2	5	11	25
1	1.75	6.04	7.27	1	2	4	8	19	5.48	5.52	1	2	4	7	16
1.25	1.75	4.23	4.51	1	1	3	5	12	3.85	3.60	1	1	3	5	11
1.5	1.75	3.07	2.99	1	1	2	4	8	2.83	2.47	1	1	2	4	8
0	2	11.2	13.46	1	3	7	14	36	11.26	12.25	1	3	7	15	34
0.25	2	10.47	12.39	1	3	6	13	33	10.31	11.04	1	3	7	14	32
0.5	2	8.87	10.31	1	2	6	11	28	8.52	9.04	1	3	6	11	25
0.75	2	6.94	7.82	1	2	4	9	21.05	6.48	6.41	1	2	4	9	19
1	2	5.17	5.34	1	2	3	7	15	4.83	4.76	1	2	3	6	14
1.25	2	3.97	4.03	1	1	3	5	11	3.68	3.34	1	1	3	5	10
1.5	2	3.05	2.77	1	1	2	4	8	2.87	2.51	1	1	2	4	8

Table IV. Performance characteristics of the run length distribution of the SL chart for the Laplace (θ,δ) distribution[a] with $ARL_0 = 500$

		$m = 30, n = 5, H = 9.4$							$m = 50, n = 5, H = 10.32$						
θ	δ	ARL	SD	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile	ARL	SD	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile
0	1	487.67	939.52	10	61	182	504	1942	495.25	859.69	14	79	217	550	1813
0.25	1	394.53	896.41	5	34	116	356	1630	330.07	699.29	6	36	114	324	1342
0.5	1	207.50	736.13	2	10	33	120	818	114.91	408.76	2	9	29	87	426
0.75	1	71.90	381.40	1	3	9	31	244	28.81	88.58	1	3	8	22	106
1	1	25.56	245.16	1	1	3	9	55	7.93	30.18	1	1	3	7	25
1.25	1	7.78	150.72	1	1	2	4	16	3.19	7.22	1	1	2	3	9
1.5	1	2.36	6.67	1	1	1	2	6	1.80	6.03	1	1	1	2	4
0	1.25	171.96	363.68	4	25	68	183	670	155.88	260.80	6	30	78	181	553
0.25	1.25	140.08	312.71	3	16	48	133	539	111.78	201.46	4	18	50	124	415
0.5	1.25	81.19	322.74	1	7	20	60	307	46.89	116.34	2	7	27	44	164
0.75	1.25	29.86	136.21	1	3	7	20	106	16.24	35.09	1	3	7	16	57
1	1.25	10.15	28.48	1	2	3	8	38	6.15	11.77	1	1	3	6	20
1.25	1.25	4.31	9.67	1	1	2	4	13	3.12	3.92	1	1	2	4	9
1.5	1.25	2.41	4.16	1	1	1	2	7	1.93	1.83	1	1	1	2	5
0	1.5	76.75	157.58	2	13	34	83	283	66.31	94.34	3	14	35	82	234
0.25	1.5	60.87	109.86	2	10	26	69	227	50.32	80.76	2	10	26	61	175
0.5	1.5	36.65	90.13	1	5	13	35	138	27.36	45.11	1	5	13	31	101
0.75	1.5	17.32	51.04	1	3	6	16	64	11.44	17.83	1	2	6	13	41
1	1.5	7.90	21.54	1	2	3	7	27	5.30	7.58	1	2	3	6	17
1.25	1.5	3.92	7.02	1	1	2	4	13	3.16	3.68	1	1	2	4	9
1.5	1.5	2.47	2.29	1	1	1	3	7	2.06	2.20	1	1	1	2	5
0	1.75	39.46	64.96	2	8	19	46	143	36.04	51.27	2	8	20	44	120
0.25	1.75	34.26	61.31	2	6	16	37	120	29.08	40.03	2	7	16	36	97
0.5	1.75	22.24	40.57	1	4	10	24	79	17.82	24.97	1	4	10	26	64
0.75	1.75	11.68	24.28	1	2	5	12	39	9.90	14.40	1	2	5	12	33
1	1.75	6.70	12.43	1	2	3	7	23	5.07	6.77	1	1	3	6	16
1.25	1.75	3.77	5.68	1	1	2	4	11	3.12	3.53	1	1	2	4	9
1.5	1.75	2.53	3.65	1	1	2	3	7	2.14	1.84	1	1	1	3	6
0	2	22.19	30.67	1	5	13	27	75	21.64	26.04	1	6	14	28	70
0.25	2	20.24	30.92	1	4	10	24	71	19.20	23.40	1	5	11	25	64
0.5	2	14.66	23.09	1	3	7	17	50	13.07	16.76	1	3	8	16	42
0.75	2	9.30	16.81	1	2	5	10	32	7.84	10.35	1	2	5	10	25
1	2	5.73	10.85	1	1	3	6	19	4.75	5.77	1	1	3	6	14
1.25	2	3.68	5.44	1	1	2	4	12	3.06	3.19	1	1	2	4	9
1.5	2	2.47	2.85	1	1	2	3	7	2.19	1.86	1	1	2	3	6

[a] Laplace (θ,δ) distribution has mean θ and standard deviation δ (scale parameter $\delta/\sqrt{2}$).

distribution, and the IC ARL of the proposed chart is calculated using 1000 simulations for each phase I sample. As expected, although the samples came from an IC process, they vary, and their differences are expressed in their means and variances. Such differences clearly alter the attained IC ARL. Depending on the values of the sample means and standard deviations, we classified them into four categories: (i) upward bias (values higher than the IC values) in sample mean and downward bias (values lower than the IC values) in sample standard deviation; (ii) downward bias in both sample mean and sample standard deviation; (iii) upward bias in both sample mean and sample standard deviation; and (iv) downward bias in sample mean and upward bias in sample standard deviation. The amount of bias is classified into three classes: (i) small (0% – 5%), (ii) moderate (5% – 10%), and (iii) high (>10%) in the sample mean and/or the standard deviation. A special class with only a marginal bias in the mean (0% – 1%) and almost no bias in the standard deviation (bias $< 0.01\%$) is shown separately.

The observed or the attained IC ARL and SDRL values are recorded in simulations under each of these cross classifications and are shown in various cells of Table V (for $m = 100$) and of Table VI (for $m = 150$). Recall that the nominal IC ARL value is 500. If there is a small bias (0%–5%) in both the sample mean and the standard deviation, it is seen that the attained IC ARL lies roughly in the range 275 to 700. This indicates that $m = 100$ gives reasonably satisfactory protection to high false alarm rate in practice when the phase I sample is a good representative of the true population. Further, we see that with $m = 100$, in no cases, the IC ARL falls below 100. On the upper side, when there is moderate downward bias in the sample mean accompanied by larger upward bias in the sample standard deviation, only in one case, the IC ARL exceeds 2000. The IC ARL exceeds 1000 only in 6 of the 100 cases, and most of these situations arise when there is moderate (5%–10%) or large bias (>10%) in the standard deviation. However, the IC ARL is found to be less than 200 in 13 cases when the sample standard deviation underestimates its population counterpart by more than 5% and also the sample mean deviates from the population mean by more than 5%. It seems reasonable to conclude that the bias in the sample standard deviation has a more adverse effect on the attained IC ARL of the chart, particularly when m is not large. Finally, in only approximately 1% of the cases, the attained IC ARL falls outside the interval 100 to 2000, whereas in 19% of the cases falls outside the interval 200 to 1000. When we repeat the simulation study with $m = 150$, we found that only 4% of the attained IC ARL lies less than 250, 4% exceeds 750, 4% lies outside 200 to 1000, and none falls outside 100 to 2000.

The values in Tables V and VI are summarized in Figure 3 and Table VII for a quick comparison over all the various classifications. The boxplots in Figure 3 are quite informative, showing, for example, the skewness of the attained run length distribution and how that is diminished for larger values of m. Table VI shows the summary statistics; again, it is seen that the variation among the phase I samples gets smaller as m increases. Thus, a larger number of phase I observations is preferable from a practitioner's point of view; at a minimum, 100–150 observations are recommended to have the attained IC ARL closer to the nominal ARL_0 for thin-tailed distributions such as the normal. For moderate to heavy-tailed distributions, preliminary calculations suggest that the required m is likely to be higher, closer to 400. Note that the desired number of phase I observations is roughly in the same range as for the normal theory Shewhart charts where 300 to 400 observations are recommended (e.g. see Quesenberry[37]).

8. Illustrative example

We illustrate the proposed nonparametric SL chart using the well-known piston ring data in Montgomery[18] (Table 5.1 and 5.2, respectively; Figure 3). Piston rings for an automotive engine are produced by a forging process. The goal is to establish a statistical control of the inside diameters of the rings manufactured by this process. Twenty-five samples each of size 5 (shown in Table 5.1 of Montgomery[18]) are collected. A phase I analysis in Montgomery[18] concluded that we may consider the data set with 125 observations as the set of reference data. Further, in Montgomery[18], 15 phase II samples (test samples) each of size 5 are given; that is, for the present purpose, $n = 5$. Using simulations, for a target ARL_0 of 250 and for $m = 125$ and $n = 5$, we found $H = 10.2$. The 15 SL plotting statistics are shown in Figure 4 along with the upper control limit of 10.2 and the lower control limit of 0.

The control chart shows that the process stays IC for the first 11 test samples and goes OOC for the first time at sample number 12. In fact, all three of the test samples, 12, 13, and 14, seem to come from an OOC process, indicating a shift in location, scale, or both. Note that for these three samples, we found $S^2_{12} = 13.3002$ with $S^2_{1.12} = 9.0143$ and $S^2_{2.12} = 4.2859$; $S^2_{13} = 15.5765$ with $S^2_{1.13} = 9.9459$ and $S^2_{2.13} = 5.6305$; and finally $S^2_{14} = 21.3902$ with $S^2_{1.14} = 12.1567$ and $S^2_{2.14} = 9.2335$.

Following the signal from the chart at sample 12, it is of interest to see if the signal is due to a shift in location, scale, or both. For this postsignal diagnostic stage, with $m = 125$, $n = 5$, and $H = 10.2$, we found $H_1 = 6.4$ and thus $H_2 = H - H_1 = 3.8$, using the simulations. Hence, we concluded that the process has shifted both in location and scale at test sample number 12. It may be interesting to note that for these data Montgomery[18](p220) (Figures 5-6), a shift in the mean was indicated at sample number 12, whereas the variance was thought to be IC on the basis of a three-sigma $\bar{X}$ and a three-sigma R chart, run separately. However, the false alarm rate (as well as the ARL_0) of this two-chart decision rule is suspect at best.

To summarize, the distribution-free SL chart monitors the location and the scale simultaneously on a single chart, maintains the nominal ARL_0 for all continuous distributions, provides an overall (some change in the process) and a follow-up decision (a shift in location, scale, or both), and yet *does not* require the practitioner to make the assumption of normality or any other distribution for the validity of the decision. Thus, the proposed chart can be useful in practical applications.

9. Summary and conclusions

A phase II Shewhart-type nonparametric control chart is proposed for simultaneously monitoring the location and the scale parameters of any continuous process. The chart uses a single plotting statistic and is based on the test statistic of Lepage[16]. Tables

Table V. Effect of phase I samples on the IC ARL and SDRL for m = 100, n = 5, and ARL_0 = 500

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD=1	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
Marginal bias in mean only (4 cases)	0.01101	0.999	508.80 (518.72)							−0.0219	1.007	290.54 (288.16)
	0.0204	0.997	558.88 (550.48)							−0.0380	1.008	473.61 (478.34)
Marginal bias in both mean and SD (18 cases)	0.0186	0.973	406.48 (400.52)	−0.0254	0.986	331.35 (346.05)	0.0110	1.017	604.22 (612.58)	−0.0003	1.040	674.51 (656.41)
	0.0249	0.960	284.40 (278.24)	−0.0282	0.969	350.54 (354.51)	0.0197	1.023	375.92 (368.03)	−0.0438	1.013	487.31 (479.41)
	0.0385	0.976	401.34 (412.97)	−0.0474	0.976	408.82 (402.83)	0.0461	1.027	551.19 (540.47)	−0.0237	1.024	692.64 (665.74)
				−0.0220	0.982	377.80 (380.78)	0.0419	1.0425	582.69 (605.31)	−0.0220	1.040	512.49 (542.036)
				−0.0187	0.987	417.31 (383.66)	0.0081	1.040	616.18 (642.35)			
							0.0007	1.043	669.90 (696.30)			
Marginal bias in mean and moderate bias in SD (11 cases)	0.0074	0.902	213.80 (200.80)	−0.0151	0.902	220.87 (225.85)	0.0009	1.065	788.86 (767.67)	−0.0267	1.079	1119.86 (1158.37)
	0.0258	0.944	237.61 (244.19)				0.0018	1.055	867.08 (824.41)	−0.0007	1.0567	559.70 (558.81)
	0.0207	0.941	287.05 (279.98)				0.0384	1.064	993.56 (990.906)			
	0.0065	0.919	232.70 (230.60)									
	0.0378	0.918	233.52									
			247.75									
	0.0042	0.920	340.05 (325.12)									
Marginal bias in mean and large bias in SD (5 cases)				−0.0032	0.886	211. 00 (207.38)				−0.0092	1.116	1042.80 (1027.17)
				−0.0171	0.843	126.33 (134.14)				−0.0084	1.147	1445.96 (1487.57)
				−0.0489	0.879	195.39 (195.79)						

(*Continues*)

Table V. *Continued.*

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD=1	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
Moderate bias in mean and marginal bias in SD (18 cases)	0.0905	0.983	490.45 (512.02)	−0.0807	0.999	329.03 (320.21)				−0.0990	1.021	466.85 (455.53)
	0.0969	0.991	422.73 (409.57)	−0.0971	0.990	447.57 (446.42)				−0.0742	1.043	653.64 (642.43)
	0.0879	0.972	303.71 (300.77)	−0.0783	0.996	(237.10) (242.41)				−0.0888	1.005	504.66 (561.30)
	0.0579	0.994	316.54 (314.26)	−0.0660	0.994	326.69 (319.57)				−0.0996	1.007	458.89 (495.037)
	0.0777	0.959	373.65 (381.12)	−0.0654	0.995	590.04 (574.31)						
	0.0973	0.994	463.89 (409.77)	−0.0701	0.954	389.54 (389.74)						
				−0.0737	0.973	299.41 (286.76)						
				−0.0578	0.954	362.19 (339.03)						
Moderate bias in both mean and SD (7 cases)	0.0684	0.905	252.64 (274.28)	−0.0878	0.939	315.08 (302.59)	0.0579	1.053	561.13 (590.37)	−0.0866	1.097	815.25 (824.54)
	0.0815	0.912	258.26 (275.89)				0.0754	1.076	1090.90 (1133.65)	−0.0780	1.060	883.91 (873.76)
Moderate bias in mean and large bias in SD (10 cases)	0.0857	0.893	131.10 (131.07)	−0.0675	0.858	147.90 (148.29)	0.0884	1.100	1215.75 (1167.78)	−0.0695	1.116	942.83 (932.51)
	0.0760	0.899	142.76 141.26	−0.0534	0.891	202.067 (200.13)	0.0575	1.126	969.15 (968.34)	−0.0570	1.121	2081.40 (2103.10)
				−0.0717	0.890	130.54 (134.76)						
				−0.0802	0.876	191.15 (180.25)						
Large bias in mean and marginal bias in SD (14 cases)	0.1075	0.983	355.80 (339.25)	−0.1760	0.963	262.37 (261.94)	0.1602	1.0148	336.99 (302.25)	−0.1572	1.002	282.11 (276.60)
	0.1216	0.962	253.75 (253.85)	−0.1591	0.982	208.24 (208.56)				−0.1292	1.016	401.18 (394.62)
	0.1734	0.964	156.86	−0.1820	0.994	373.8440				−0.1634	1.012	297.83

(*Continues*)

Table V. *Continued.*

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD=1	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
			(144.70)			(363.47)						(293.08)
	0.1887	0.971	213.26 (215.16)							−0.1860	1.006	318.25 (322.56)
	0.1604	0.989	402.38 (410.92)							−0.1814	1.037	235.89 (228.86)
Large bias in mean and moderate bias in SD (10 cases)	0.1056	0.939	241.52 (234.01)	−0.1440	0.926	290.40 (276.67)	0.1386	1.082	907.15 (897.16)	−0.1272	1.070	330.90 (344.06)
	0.1702	0.932	113.21 (107.22)	−0.1990	0.946	257.61 (248.14)				−0.1416	1.068	476.85 (475.81)
	0.1810	0.936	182.37 (172.43)							−0.2156	1.061	507.77 (515.36)
	0.1966	0.918	122.49 (106.79)									
Large bias in both mean and SD (3 cases)	0.1237	0.884	144.18 (146.86)				0.1568	1.148	797.07 (834.05)			
	0.2670	0.899	122.72 (129.59)									

Table VI. Effect of phase I samples on the IC ARL and SDRL for m = 150, n = 5, and ARL_0 = 500

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD =1	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
Marginal bias in mean only (2 cases)	0.0025	0.991	525.69 (598.94)									
	0.0090	0.999	511.00 (506.34)									
Marginal bias in both mean and SD (23 cases)	0.0369	0.992	444.30 (445.12)	−0.0041	0.981	449.93 (394.20)	0.0002	1.033	860.0280 (780.86)	−0.0061	1.012	405.09 (384.55)
	0.0266	0.955	387.86 (402.82)	−0.0476	0.967	340.38 (339.11)	0.0404	1.004	406.28 (381.20)	−0.0402	1.041	779.28 (859.78)
	0.0434	0.982	439.51 (417.34)	−0.0361	0.995	449.54 (451.92)	0.0144	1.010	623.42 (630.20)	−0.0053	1.020	628.34 (569.48)
	0.0140	0.987	469.52 (471.59)	−0.0180	0.987	479.77 (482.36)	0.0196	1.022	848.86 (950.54)	−0.0184	1.034	821.22 (868.74)
	0.0260	0.982	379.63 (379.99)	−0.0491	0.954	298.87 (296.01)	0.0214	1.018	590.84 (559.97)	−0.0238	1.029	649.90 (605.96)
	0.0434	0.982	439.51 (417.34)	−0.0072	0.974	474.06 (457.36)						
				−0.0240	0.974	417.54 (433.67)						
Marginal bias in mean and moderate bias in SD (13 cases)	0.0041	0.950	404.56 (365.36)	−0.0245	0.924	319.39 (269.40)	0.0162	1.085	779.50 (731.36)	−0.0078	1.064	683.64 (636.14)
				−0.0362	0.930	323.32 (288.25)	0.0071	1.064	896.14 (938.76)	−0.0301	1.050	751.88 (663.06)
				−0.0125	0.909	318.03 (309.77)	0.0169	1.052	730.14 (697.71)	−0.0482	1.068	829.47 (779.94)
				−0.0049	0.941	312.71 (328.35)				−0.0204	1.072	696.73 (642.08)
				−0.0557	0.924	267.30 (254.47)						
Marginal bias in mean and large bias	0.0313	0.855	133.49 (125.06)					0.0246	1.100	1276.05 (1272.22)		
	0.0499	0.892	215.13 (222.57)									

(*Continues*)

Table VI. *Continued.*

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD =1	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
in SD (3 cases)												
Moderate bias in mean and marginal bias in SD (22 cases)			0.0980	0.981	300.44	(310.12)	−0.0926	0.998	390.47 (393.87)	0.0983	1.004	413.53 (387.74)
			−0.0519	1.024		596.16 (596.81)						
			0.0859	0.996	335.02	(318.74)	−0.0672	0.991	425.62 (406.27)	0.0561	1.029	598.96 (707.88)
			−0.0633	1.044		746.00 (691.92)						
			0.0927	0.971	307.43	(363.48)	−0.0678	0.995	413.13 (361.25)	0.0793	1.034	652.12 (668.56)
			−0.0822	1.003		435.34 (482.83)						
			0.0895	0.979	340.03	(339.15)				0.0765	1.010	405.75 (387.14)
−0.0532	1.023	469.71	(461.02)									
0.0774	0.962	395.24 (403.38)	0.0526	1.002	515.48	(489.63)				−0.0777	1.026	614.26 (608.81)
0.0567	0.980	416.43 (480.90)	0.0664	1.004	553.14	(539.36)				−0.0611	1.040	541.46 (559.29)
			0.0811	1.014	423. 60	(470.75)						
Moderate bias in both mean and SD (11 cases)	0.0886	0.910	224.29 (204.62)	0.0612	1.050	799.85 (711.62)				−0.0651	1.076	880.11 (929.64)
	0.0588	0.932	365.13 (355.09)	0.0695	1.054	533.67 (497.05)				−0.0638	1.067	823.39 (820.57)
	0.0995	0.935	244.68 (263.53)	0.0727	1.062	752.560 (718.91)				−0.0649	1.095	880.90 (854.81)
				0.0846	1.077	816.74 (807.87)				−0.0729	1.069	1099.23 (1252.07)
Moderate bias in mean and							−0.0618	1.110	1071.02	(1172.35)		

(*Continues*)

Table VI. *Continued.*

Total 100 different phase I samples	Upward bias in sample mean and downward bias in sample SD in phase I sample			Downward bias in both sample mean and sample SD in phase I sample			Upward bias in both sample mean and sample SD in phase I sample			Downward bias in sample mean an upward bias in sample SD in phase I sample		
Target mean=0, target SD =1	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)	True phase I mean	True phase I SD	ARL_0 Attained ($SDRL_0$)
large bias in SD (1 cases)												
Large bias in mean and marginal bias in SD (17 cases)	0.1504	0.982	283.06 (263.13)	−0.1263	0.957	347.54 (351.70)	0.1680	1.028	305.56 (323.15)	−0.1151	1.031	520.95 (498.46)
	0.1478	0.981	291.16 (319.08)	−0.1266	0.970	290.72 (310.02)	0.1019	1.024	622.94 (627.92)	−0.1994	1.018	227.99 (246.05)
	0.1458	0.964	257.15 (251.84)	−0.1120	0.971	387.74 (382.73)	0.1135	1.028	363.24 (344.61)	−0.1540	1.020	168.96 (185.91)
				−0.2053	0.969	221.17 (224.66)	0.1286	1.0149	473.66 (469.83)	−0.1456	1.020	401.90 (375.86)
				−0.2267	0.989	272.45 (285.22)				−0.1309	1.001	500.80 (488.39)
Large bias in mean and moderate bias in SD (8 cases)				−0.1501	0.935	236.23 (229.93)	0.1362	1.068	751.44 (796.96)	−0.2038	1.050	495.98 (562.30)
							0.1850	1.062	464.98 (469.92)	−0.1196	1.097	1113.31 (1005.69)
							0.1438	1.059	441.74 (436.89)	−0.1657	1.054	480.93 (482.42)
										−0.1527	1.069	694.79 (610.74)

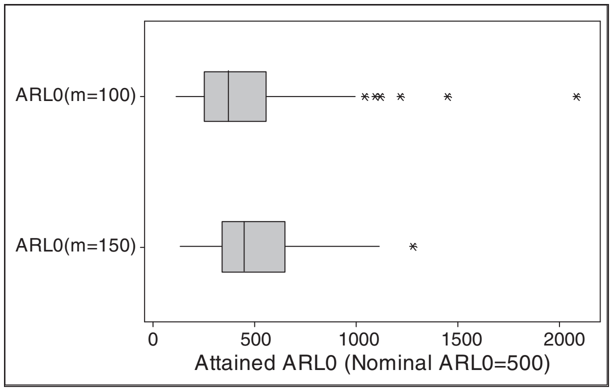

Figure 3. Boxplots for the attained IC run length distribution of the SL chart as a function of the number of phase I observations, m

Table VII. Summary statistics for the conditional IC run length distribution of the SL chart for 100 different phase I representative samples for m = 100 and 150

m	Mean	SD	Minimum	1st Quartile	Median	3rd Quartile	Maximum
100	455.0	315.9	113.2	252.6	373.7	558.9	2081.4
150	515.5	229.0	133.5	340.4	449.9	652.1	1276.1

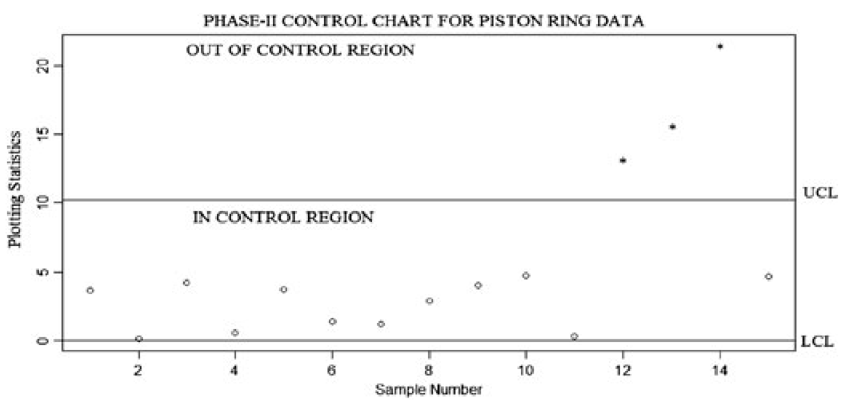

Figure 4. Nonparametric SL chart for the piston ring data

are provided for implementation along with the constants needed for a postsignal analysis of which parameter(s) may have shifted. The properties of the chart are studied in terms of the various characteristics of its run length distribution, including the mean, the median, and some percentiles. The influence of the phase I sample on the attained IC run length distribution is examined in terms of IC ARL and SDRL. It is seen that at least 100 to 150 reference observations are needed so that the attained IC ARL is close to the nominal value; the more, the better. A numerical illustration is given. Further enhancements and refinements, using CUSUM and EWMA statistics, will be considered elsewhere.

Appendix
Proof of result in Section 5

Result

There exist nonnegative constants H_1 and H_2 (= $H - H_1$) both less than or equal to H such that for a given [R=r], the event $[S_r{}^2 > H]$ can be partitioned into three mutually exclusive events as $A: \left[S_{1r}^2 > H_1, S_{2r}^2 \le H_2 | R = r\right]$, $B: \left[S_{1r}^2 \le H_1, S_{2r}^2 > H_2 | R = r\right]$, and C: $\left[S_{1r}^2 > H_1,\, S_{2r}^2 > H_2 | R = r\right]$.

Proof

Note that for any r, $0 \le S_{1r}^2 \le S_r^2$ and also $0 \le S_{2r}^2 \le S_r^2$. Therefore, the supremum of both S_{1r}^2 and S_{2r}^2 is H. This proves the existence of H_1 and H_2 being both less than or equal to H, which in turn provides the three mutually exclusive partitions of $[S_r^2 > H | R = r]$:

$[S_{1r}^2 > H_1, S_{2r}^2 \leq H_2 | R = r]$, $[S_{1r}^2 \leq H_1, S_{2r}^2 > H_2 | R = r]$, and $[S_{1r}^2 > H_1, S_{2r}^2 > H_2 | R = r]$. For the last part of the result, note that $H_1 + H_2$ must be equal to H. This is because, first, $H_1 + H_2$ cannot be less than H because in that case we may have $S_{1r}^2 > H_1$, $S_{2r}^2 > H_2$, but $S_r^2 \leq H$. This is a contradiction because H_2 and H_2 are only required when $S_r^2 > H$. Second, $H_1 + H_2$ cannot be greater than H because it may then happen that $S_r^2 > H$ but $S_{1r}^2 \leq H_1$ and $S_{2r}^2 \leq H_2$ Therefore, when we stop on the basis of H and search for a location and or a scale shift, we will find neither, which would be impractical. Hence, $H_1 + H_2$ must be equal to H. This completes the proof of the theorem.

Run length distribution and ARL

Let R be the random variable denoting the run length of the proposed SL control chart. It is clear that given the reference sample $\boldsymbol{X}_m$, the conditional run length distribution is geometric with success probability $\text{Prob}[S_1^2 > H | \boldsymbol{X}_m]$. Therefore, all moments, percentiles, and so forth, of the conditional run length distribution can be obtained directly from the properties of the geometric distribution. For example, the conditional ARL is given by $\left(\text{Prob}[S_1^2 > H | \boldsymbol{X}_m]\right)^{-1}$. Hence, all properties of the unconditional run length distribution can be obtained by averaging over the distribution of the reference sample. For example, the unconditional ARL can be found from $E\left(\text{Prob}[S_1^2 > H | \boldsymbol{X}_m]\right)^{-1}$. Both the conditional and unconditional run length distributions can provide useful information about the performance of the chart.

Writing $\psi(\boldsymbol{X}_m, H) = \text{Prob}[S_1^2 \leq H | \boldsymbol{X}_m]$, the unconditional run length distribution is given by

$$\begin{aligned}\text{Prob}[R = r] &= E\left[\left\{[\Psi(\boldsymbol{X}_m, H)]\right]^{r-1}[1 - \Psi(\boldsymbol{X}_m, H)]\right\} \\ &= E[\Psi(\boldsymbol{X}_m, H)]^{r-1} - E[\Psi(\boldsymbol{X}_m, H)]^r \quad \text{for } r = 1, 2, \ldots,\end{aligned}$$

Moments of the unconditional run length distribution can be calculated conditioning on the reference sample. For example, the unconditional ARL is

$$\text{ARL} = E\left[\frac{1}{1 - \Psi(\boldsymbol{X}_m, H)}\right] = \int_X \frac{1}{1 - \Psi(\boldsymbol{X}_m, H)} dF(\boldsymbol{X}_m)$$

The IC ARL can be obtained by substituting $F{=}G$

$$\text{ARL}_0 = E\left[\frac{1}{1 - \Psi(\boldsymbol{X}_m, H)}\right] = \int_X \frac{1}{1 - \Psi_{F=G}(\boldsymbol{X}_m, H)} dF(\boldsymbol{X}_m)$$

Other moments and characteristics (such as the percentiles) of the run length distribution can be calculated in a similar manner.

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