UTM
UNIVERSITI TEKNOLOGI MALAYSIA

# CHAPTER 2 (Part 1): LINEAR PROGRAMMING - BASIC


UTM JOHOR BAHRU

nzah@utm.my-20232024(S1)

# LINEAR PROGRAMMING (LP)

- In mathematics, **linear programming (LP)** is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints.

- Linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in each mathematical model and given some list of requirements represented as linear equations.

# Mathematical Formulation of LP Model

**Step 1**

- Study the given situation
- Find the key decision to be made
- Identify the **decision variables** of the problem

**Step 2**

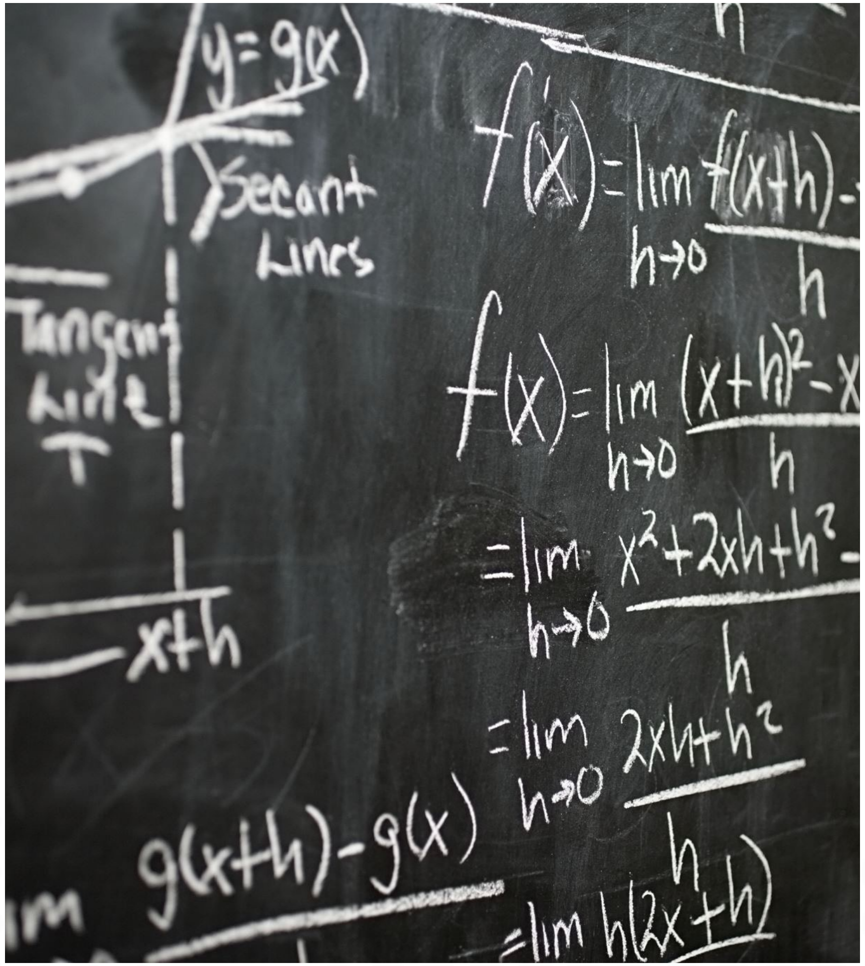
- Formulate the **objective function** to be optimized

**Step 3**

- Formulate the **constraints** of the problem

**Step 4**

- Add non-negativity restrictions or constraints.
- The objective function, the set of constraints and the non-negativity restrictions together form an LP model.



# Properties of LP Model

Linearity implies that the LP must satisfy **three basic properties**:

1) **Proportionality**

Contribution of each decision variable in both the objective function and constraints to be directly proportional to the value of the variable.

2) **Additivity**

Total contribution of all the variables in the objective function and in the constraints to be the direct sum of the individual contributions of each variable.

3) **Certainty**

All the objective and constraint coefficients of the LP model are deterministic (known constants).

LP coefficients are average-value approximations of the probabilistic distributions. If standard deviations of these distributions are sufficiently small, then the approximation is acceptable.

Large standard deviations can be accounted for directly by using stochastic LP algorithms or indirectly by applying sensitivity analysis to the optimum solution.

# Linear Programming: Two Variables

## EXAMPLE 1: THE GALAXY INDUSTRY PRODUCTION

- Galaxy manufactures two toy models:
  - i. Space Ray.
  - ii. Zapper.
- Resources are limited to:
  - i. 1200 pounds of special plastic.
  - ii. 40 hours of production time per week.
- Marketing requirement:
  - i. Total production cannot exceed 800 dozens.
  - ii. Number of dozens of Space Rays cannot exceed number of dozens of Zappers by more than 450.
- Technological input:
  - i. Space Rays requires 2 pounds of plastic and 3 minutes of labor per dozen.
  - ii. Zappers requires 1 pound of plastic and 4 minutes of labor per dozen.
- Current production plan calls for:
  - i. Producing as much as possible of the more profitable product, Space Ray (RM8 profit per dozen).
  - ii. Use resources left over to produce Zappers (RM5 profit per dozen).

# Linear Programming: Two Variables

## EXAMPLE 1 - Solution

- Decisions variables:
  - $X_1$ = Production level of Space Rays (in dozens per week).
  - $X_2$ = Production level of Zappers (in dozens per week).
- Objective Function:
  - Maximize $Z = 8X_1 + 5X_2$ (Weekly profit) - Weekly profit, to be maximized.
- Subject to:
  - $2X_1 + X_2 <= 1200$ (Plastic)
  - $3X_1 + 4X_2 <= 2400$ (Production Time)
  - $X_1 + X_2 <= 800$ (Total production)
  - $X_1 - X_2 <= 450$ (Mix)
- Add non-negativity:
  $X_1 >= 0; X_2 >= 0$

# Exercise #1

The Alex Garment Company manufactures men's shirts and women's blouses. The production process includes cutting, sewing, and packaging. The company employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hrs shift, 5 days a week. The following table gives the time requirements and profits per unit to produce the two garments. **Formulate the problem** to determine the optimal weekly production schedule for Alex Garment Company.

| | Minutes per unit | | | |
|---|---|---|---|---|
| Garment | *Cutting* | *Sewing* | *Packaging* | Unit profit ($) |
| Shirts | 20 | 70 | 12 | 8 |
| Blouses | 60 | 60 | 4 | 12 |

# Feasible Solutions for Linear Programming

- The set of all points that satisfy all the constraints of the model is called **Feasible Region.**
- Otherwise, the solution is infeasible.

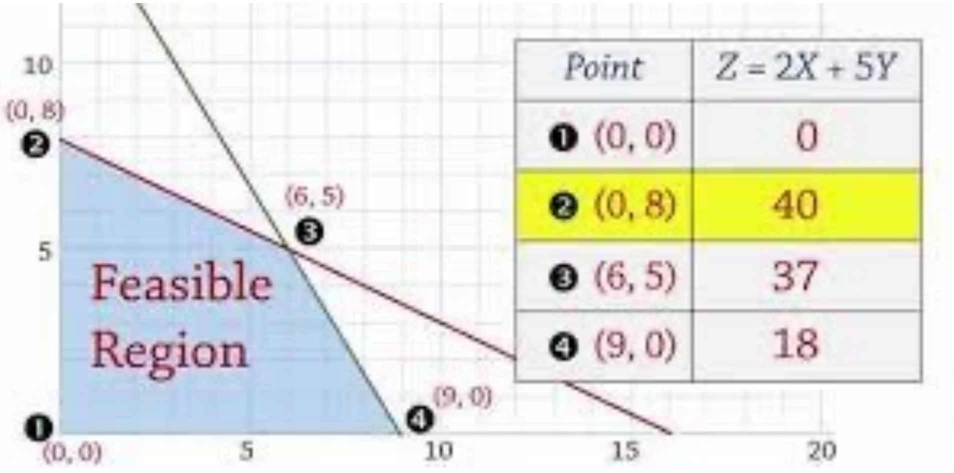



Figure - Example

# Graphical Presentation

- Using a graphical presentation, we can represent all the constraints, the objective function, and the three types of feasible points.
- The graphical solution includes two steps:
  1. Determination of the feasible solution space.
  2. Determination of the optimum solution from among all the points in the solution space.

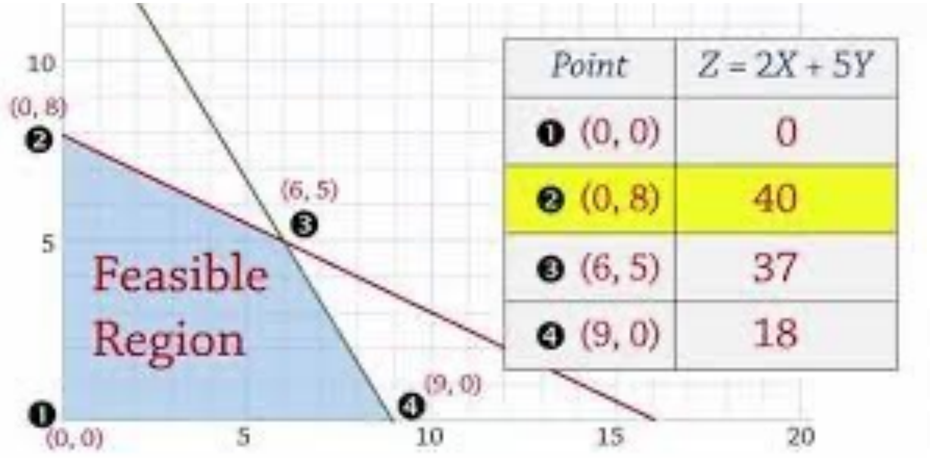
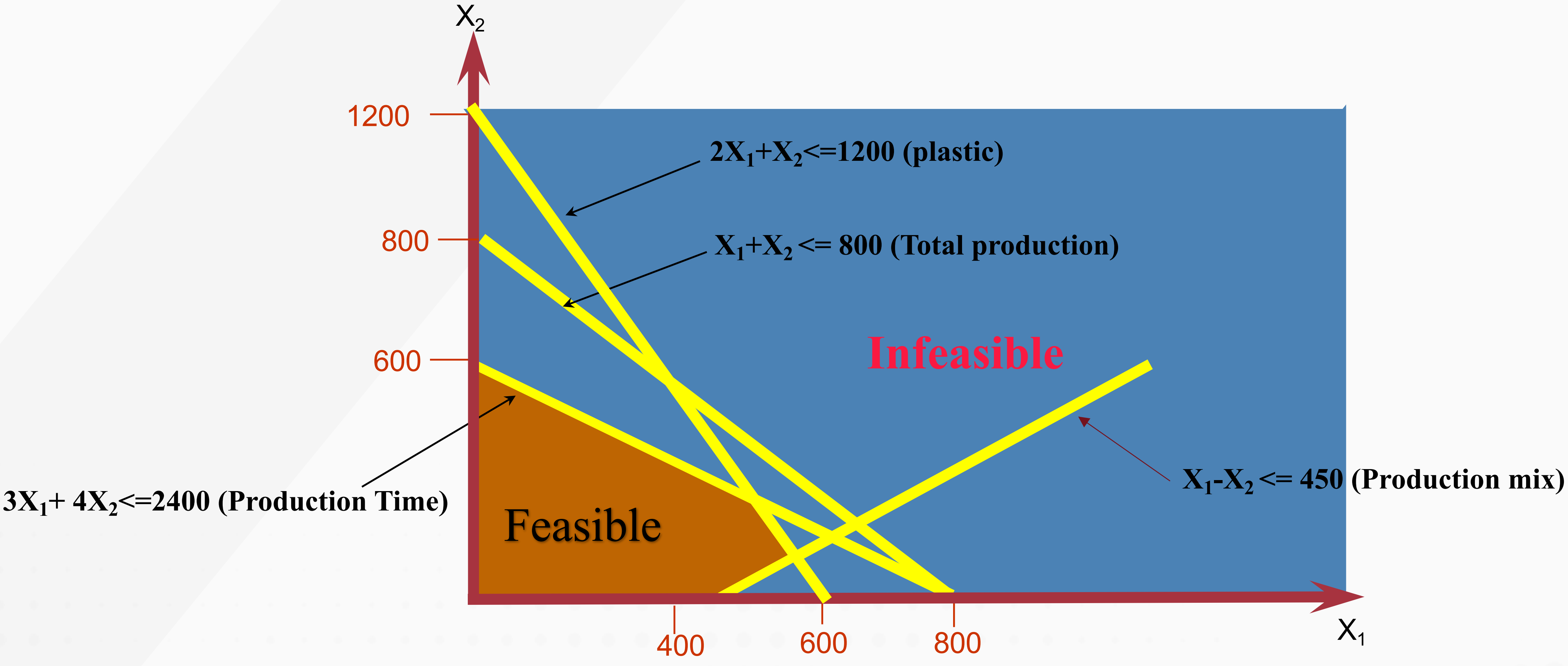



Figure - Example

## EXAMPLE 1: THE GALAXY INDUSTRY PRODUCTION




Find the coordinates for all the 4 equations of the restrictions (only take the equality sign)

# Type of Possible Points

**Interior point**:

Satisfies all constraint but non with equality.

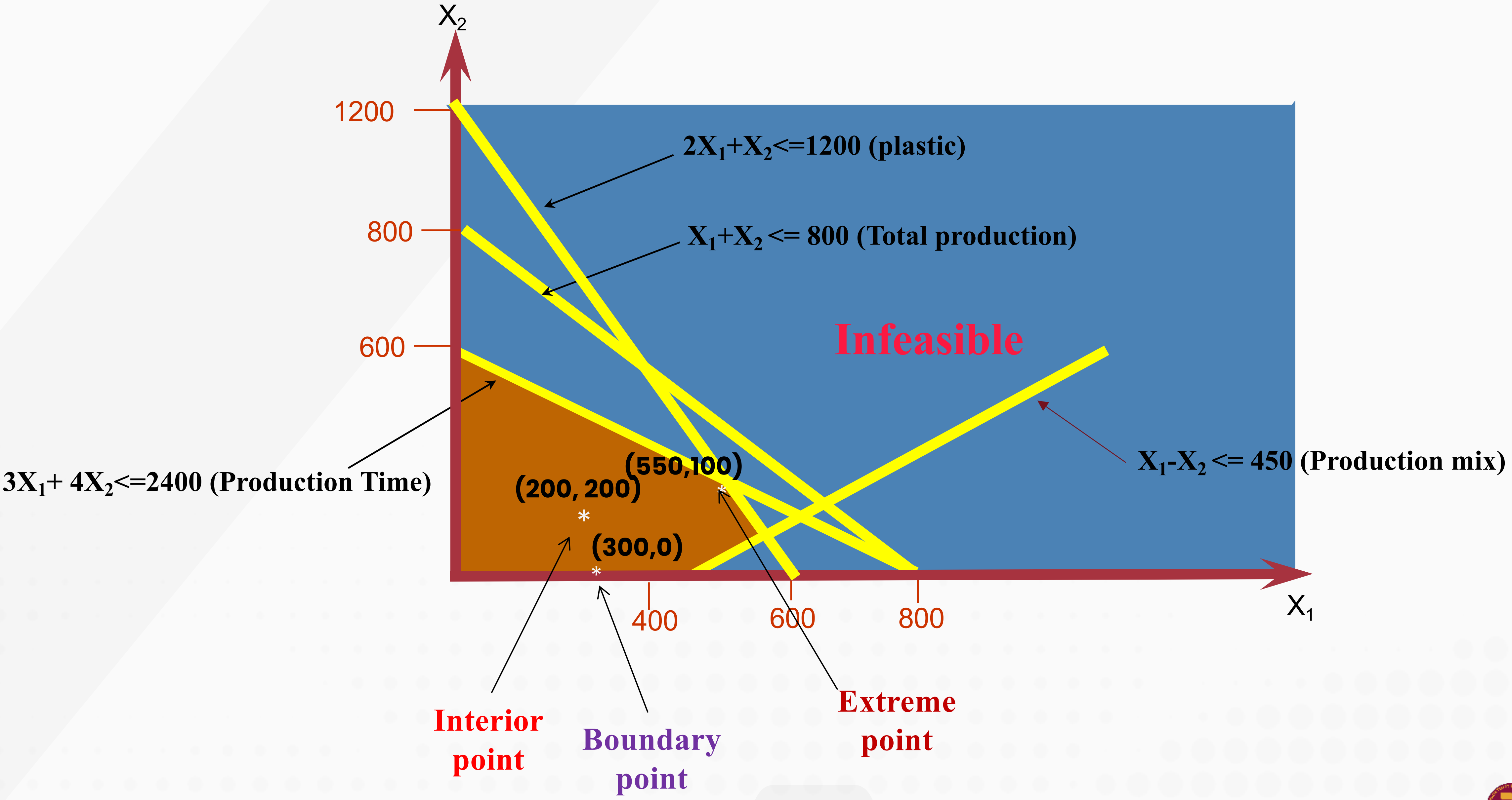
**Boundary points**:

Satisfies all constraints, at least one with equality.

**Extreme point**:

Satisfies all constraints, two with equality. If a linear programming has an optimal solution , an extreme point is optimal.

## EXAMPLE 1: THE GALAXY INDUSTRY PRODUCTION



## Solving Graphically for an Optimal Solution: Procedure

1. Graph the constraint to find the feasible point.
2. Set objective function equal to an arbitrary value so that line passes through the feasible region.
3. Move the objective function line parallel to itself until it touches the last point of the feasible region .
4. Solve for $X_1$ and $X_2$ by solving the two equation that intersect to determine this point.
5. Substitute these value into objective function to determine its optimal solution.

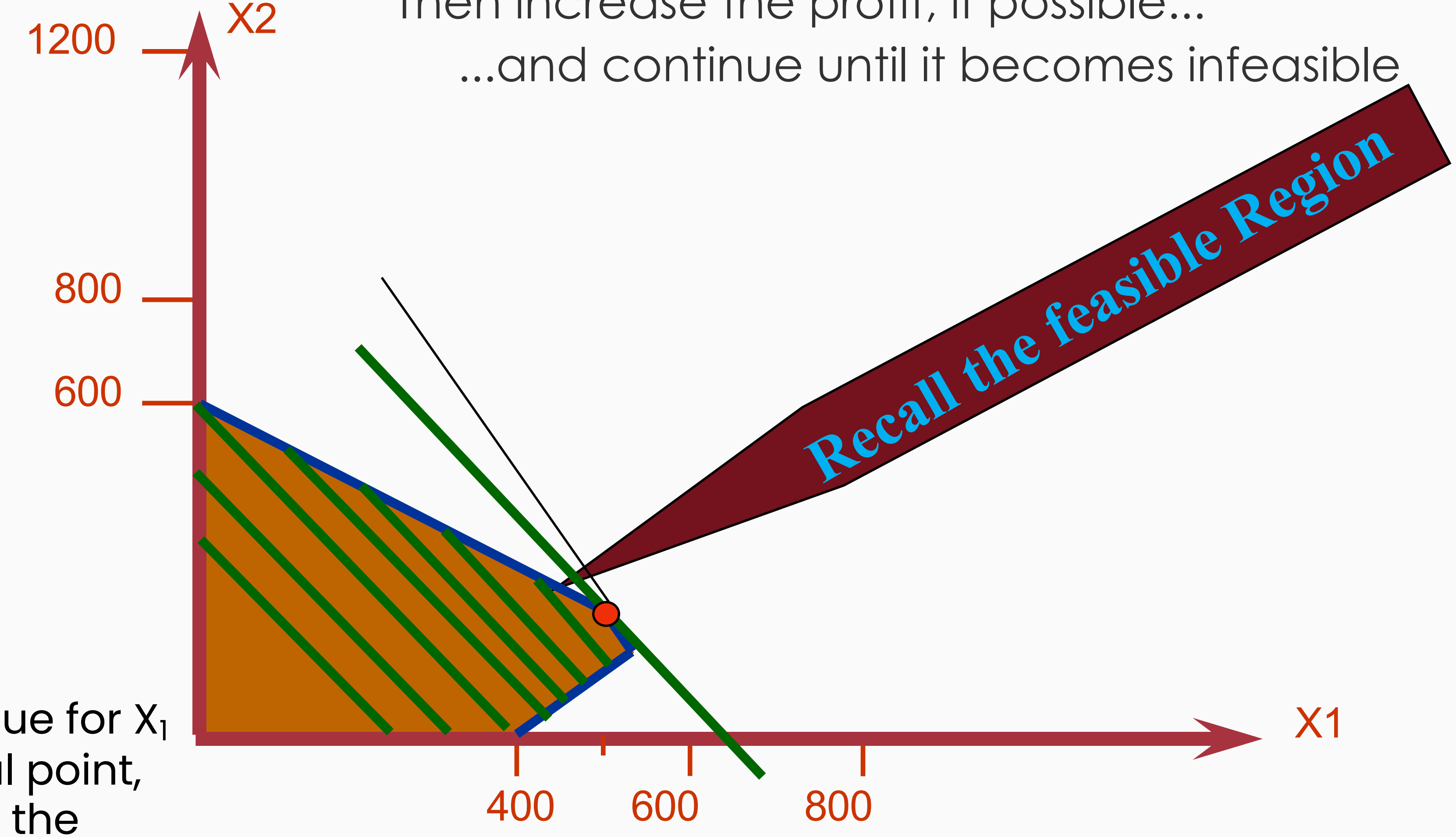
Note: The optimum solution of an LP, when it exists, is always associated with a corner point of the solution space

# Solving Graphically for an Optimal Solution: Procedure

Start at some arbitrary profit, say profit = $2,000...

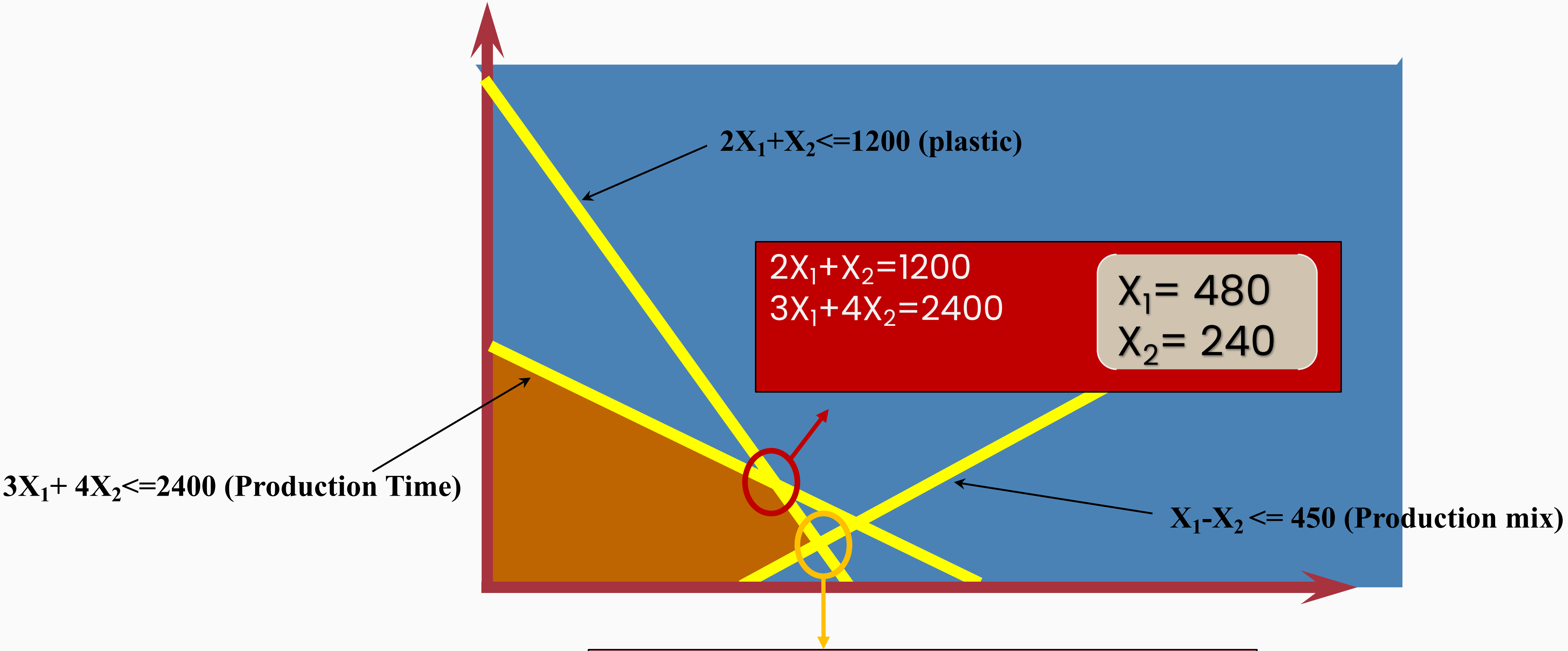
Then increase the profit, if possible...

...and continue until it becomes infeasible




To determine the value for $X_1$ and $X_2$ at the optimal point, the two equations of the binding constraint must be solved.

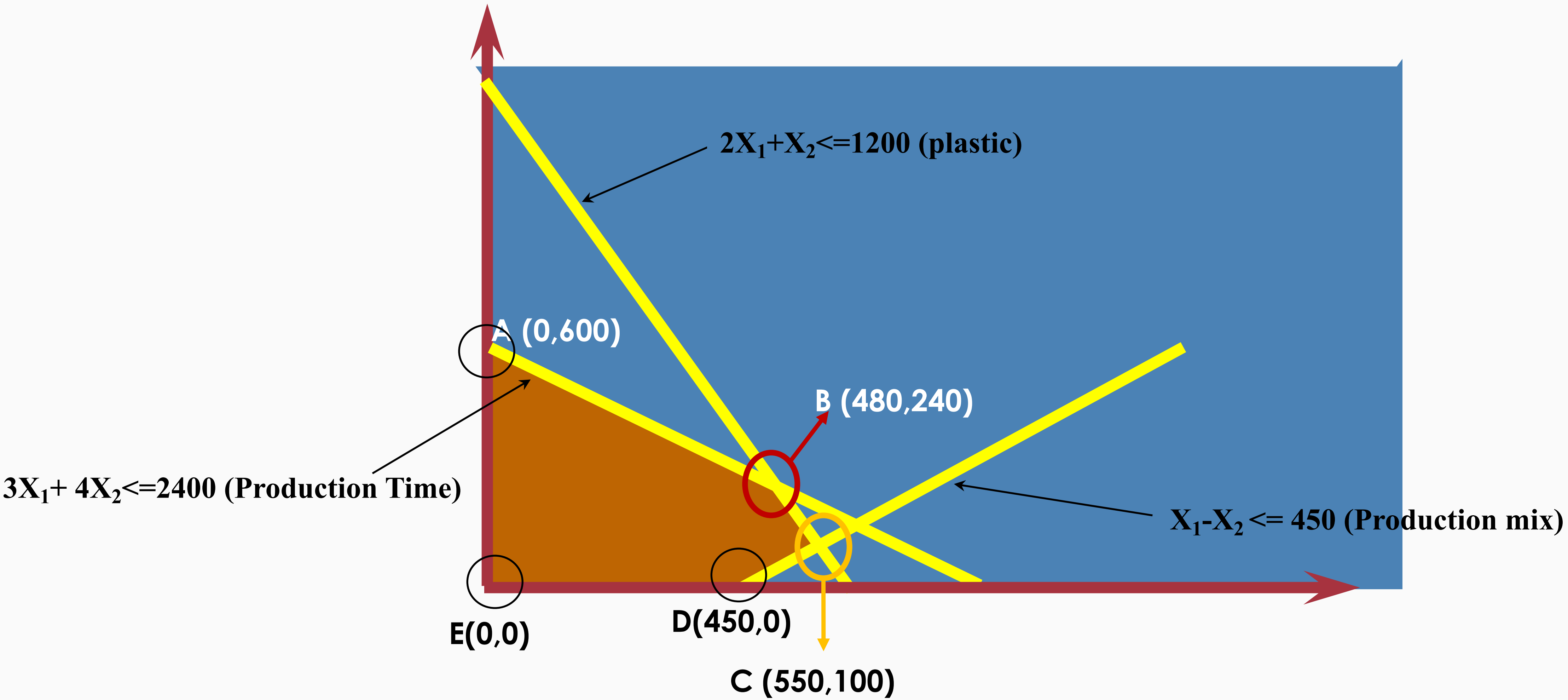
# Solving Graphically for an Optimal Solution: Procedure




$2X_1+X_2=1200$
$X_1-X_2=450$

$X_1 = 550$
$X_2 = 100$

# Solving Graphically for an Optimal Solution: Procedure



# Corner Point Method

By compensation on:

Max, $Z = 8X_1 + 5X_2$

| $(X_1, X_2)$ | Objective function |
|---|---|
| (0,0) | 0 |
| (450,0) | 3600 |
| **(480,240)** | **5040** |
| (550,100) | 4900 |
| (0,600) | 3000 |

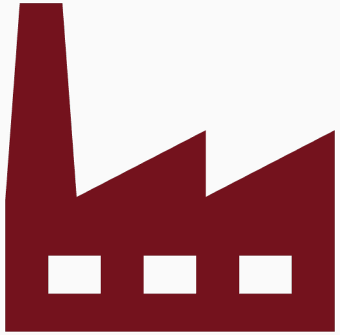
The maximum profit (5040) will be by producing:
Space Rays = 480 dozens, Zappers = 240 dozens

# Corner Point Method

**EXAMPLE 2: THE REDDY MIKKS COMPANY**

- Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. The following table provide the basic data of the problem:

| | Tons of raw material per ton of | | |
|---|---|---|---|
| | Exterior paint | Interior paint | Maximum daily availability(tons) |
| Raw material, M1 | 6 | 4 | 24 |
| Raw material, M2 | 1 | 2 | 6 |
| Profit per ton ($1000) | 5 | 4 | |



- The market survey restrict that maximum daily demand of interior paint is 2 tons.
- Daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton.
- Reddy  Mikks  wants to determine the optimum product mix of interior and  exterior paints that maximizes the total daily profit.

# Corner Point Method

## EXAMPLE 2 - Solution

**Variables:**

$X_1$ = tons produced daily of exterior paint

$X_2$ = tons produced daily of interior paint

$Z$ = total daily profit (in thousands of dollars)

**Objective:** Maximize, $Z = 5X_1 + 4X_2$

**Conditions:**

- Usage of a raw material by both paints $\leq$ Maximum raw material availability
- Usage of raw material M1 per day = $6X_1 + 4X_2$ tons
- Usage of raw material M2 per day = $X_1 + 2X_2$ tons
- Daily availability of raw material M1 is 24 tons
- Daily availability of raw material M2 is 6 tons



UTM UNIVERSITI TEKNOLOGI MALAYSIA

# Corner Point Method

**EXAMPLE 2 – Solution (cont'd)**

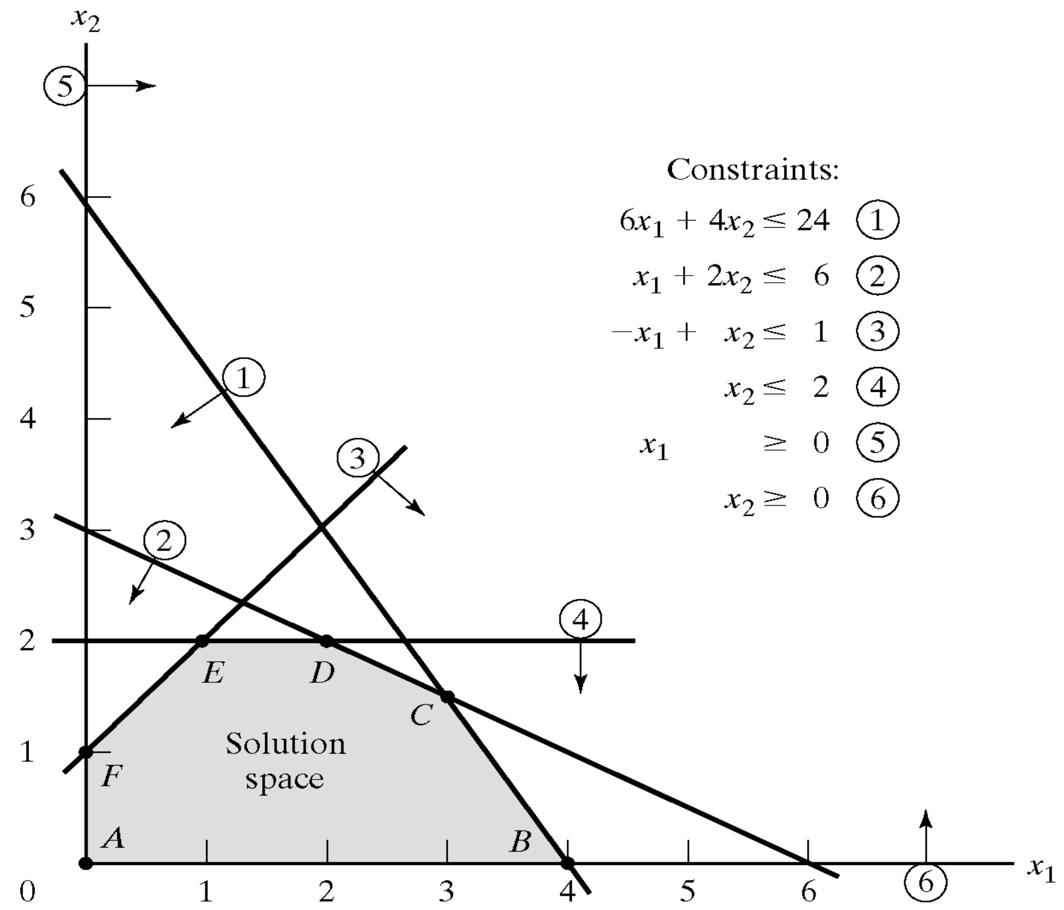
## Constraints:

1. $6X_1 + 4X_2 \leq 24$ (raw material M1)
2. $6X_1 + 2X_2 \leq 6$ (raw material M2)
3. $X_2 - X_1 \leq 1$
4. $X_2 \leq 2$
5. $X_1 \geq 0;\ X_2 \geq 0$

# Corner Point Method

## EXAMPLE 2 – Solution (cont'd)

### Graphical Representation - Feasible Solution



# Corner Point Method

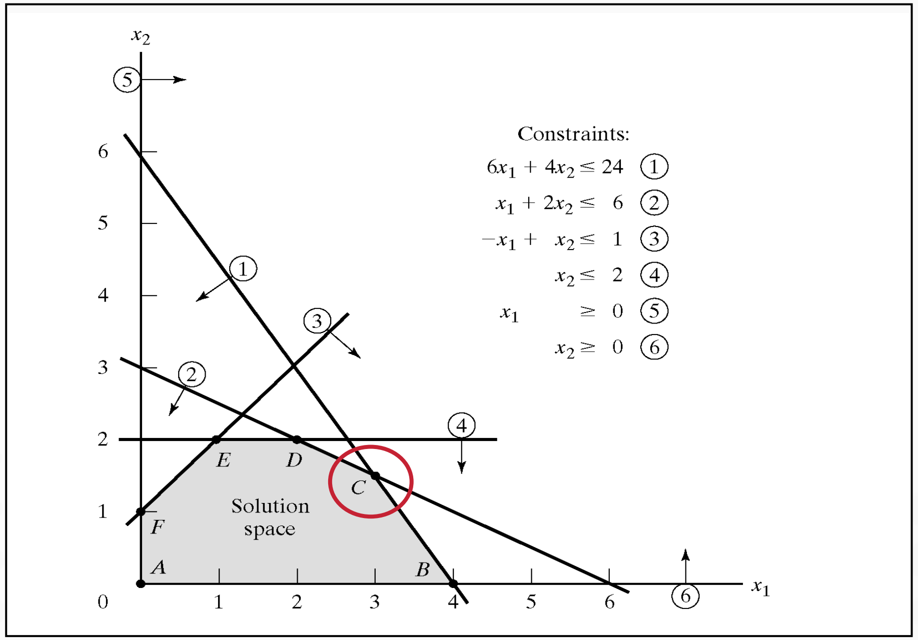
## EXAMPLE 2 – Solution (cont'd)

- To find **optimum solution**, identify the direction in which the maximum profit increases (Profit, Z $= 5X_1 + 4X_2$)
- Assign random increasing values to Z. Example: Z = 10 and Z = 15

$$5X_1 + 4X_2 = 10$$

$$5X_1 + 4X_2 = 15$$

- Plot graphs of above two equations.
- Thus in this way the optimum solution occurs at corner point C which is the point in the solution space.





UTM UNIVERSITI TEKNOLOGI MALAYSIA

# Corner Point Method

## EXAMPLE 2 – Solution (cont'd)

- Any further increase in Z that is beyond corner point C will put points outside the boundaries of ABCDEF feasible space.
- Values of $X_1$ and $X_2$ associated with optimum corner point C are determined by solving the equations:

$$6X_1 + 4X_2 = 24 \quad \text{...(eq. 1)}$$

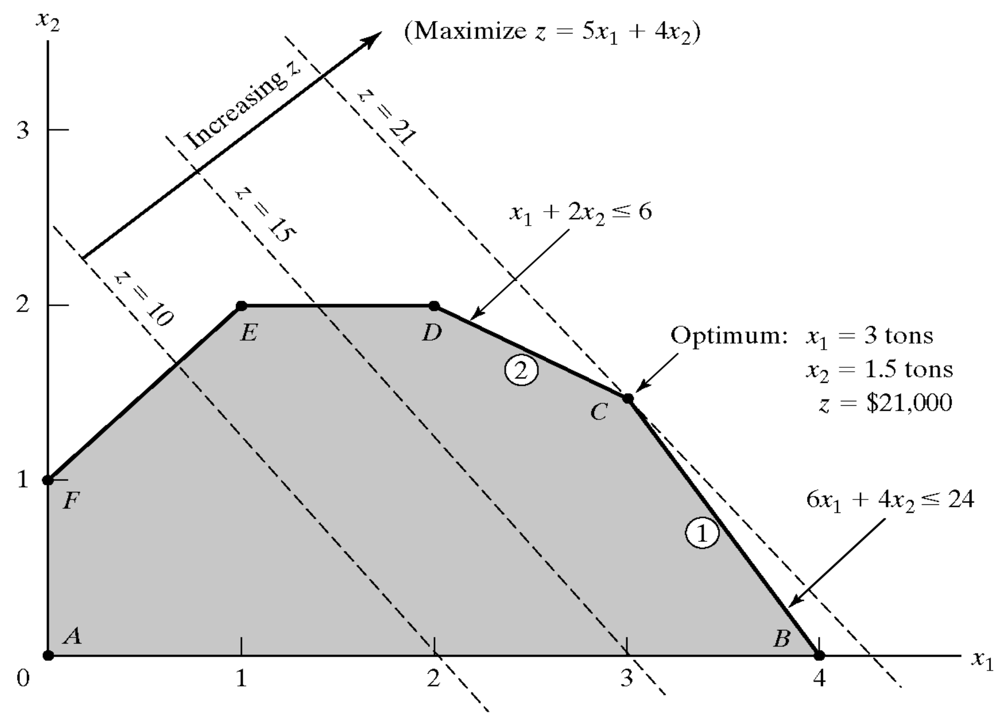
$$X_1 + 2X_2 = 6 \quad \text{...(eq. 2)}$$

- Solving the above equations give $X_1 = 3$ and $X_2 = 1.5$ with $Z = 5(3) + 4(1.5) = 21$
- So daily production mix of 3 tons of exterior paint and 1.5 tons of interior paint produces the daily profit of $21,000 .

# Corner Point Method

## EXAMPLE 2 – Solution (cont'd)

### Optimal Solution



# Corner Point Method

## EXAMPLE 2 – Solution (cont'd)

Since optimum LP solution is always associated with a corner point of the solution space, so optimum solution can be found by enumerating all the corner points as below:-

| Corner Points | $(X_1, X_2)$ | Z ($1000) |
|---|---|---|
| A | (0, 0) | 0 |
| B | (4, 0) | 20 |
| **C** | **(3, 1.5)** | **21** |
| D | (2, 2) | 18 |
| E | (1, 0) | 13 |
| F | (0, 1) | 4 |

As number of constraints and variables increases , the number of corner points also increases.

# LP - Minimization Model

## EXAMPLE 1:

A firm has two bottling plant. One plant located at Coimbatore and other plant located at Chennai. Each plant produces three types of drinks; Coca-Cola , Fanta and Thumps-up. The following table show the data.

| | Number of bottles produced per day by plant at | |
|---|---|---|
| Product | Coimbatore | Chennai |
| Coca-Cola | 15000 | 15000 |
| Fanta | 30000 | 10000 |
| Thumps-Up | 20000 | 50000 |
| | | |
| Cost per day (per unit) | 600 | 400 |

Market survey indicates that during the month of April there will be a demand of 200,000 bottles of Coca-cola , 400,000 bottles of Fanta and 440,000 bottles of Thumps-up. For how many days each plant be run in April to minimize the production cost, while still meeting the market demand?

# LP - Minimization Model

## EXAMPLE 1 - Solution

Let, $X_1$ = number of days to produce all the three types of bottles by plant at Coimbatore.

$X_2$ = number of days to produce all the three types of bottles by plant at Chennai.

**Objective:**

$$\text{Minimize } Z = 600X_1 + 400X_2$$

**Constraint:**

$$15{,}000X_1 + 15{,}000X_2 \geq 200{,}000 \quad \ldots(1)$$

$$30{,}000X_1 + 10{,}000X_2 \geq 400{,}000 \quad \ldots(2)$$

$$20{,}000X_1 + 50{,}000X_2 \geq 440{,}000 \quad \ldots(3)$$

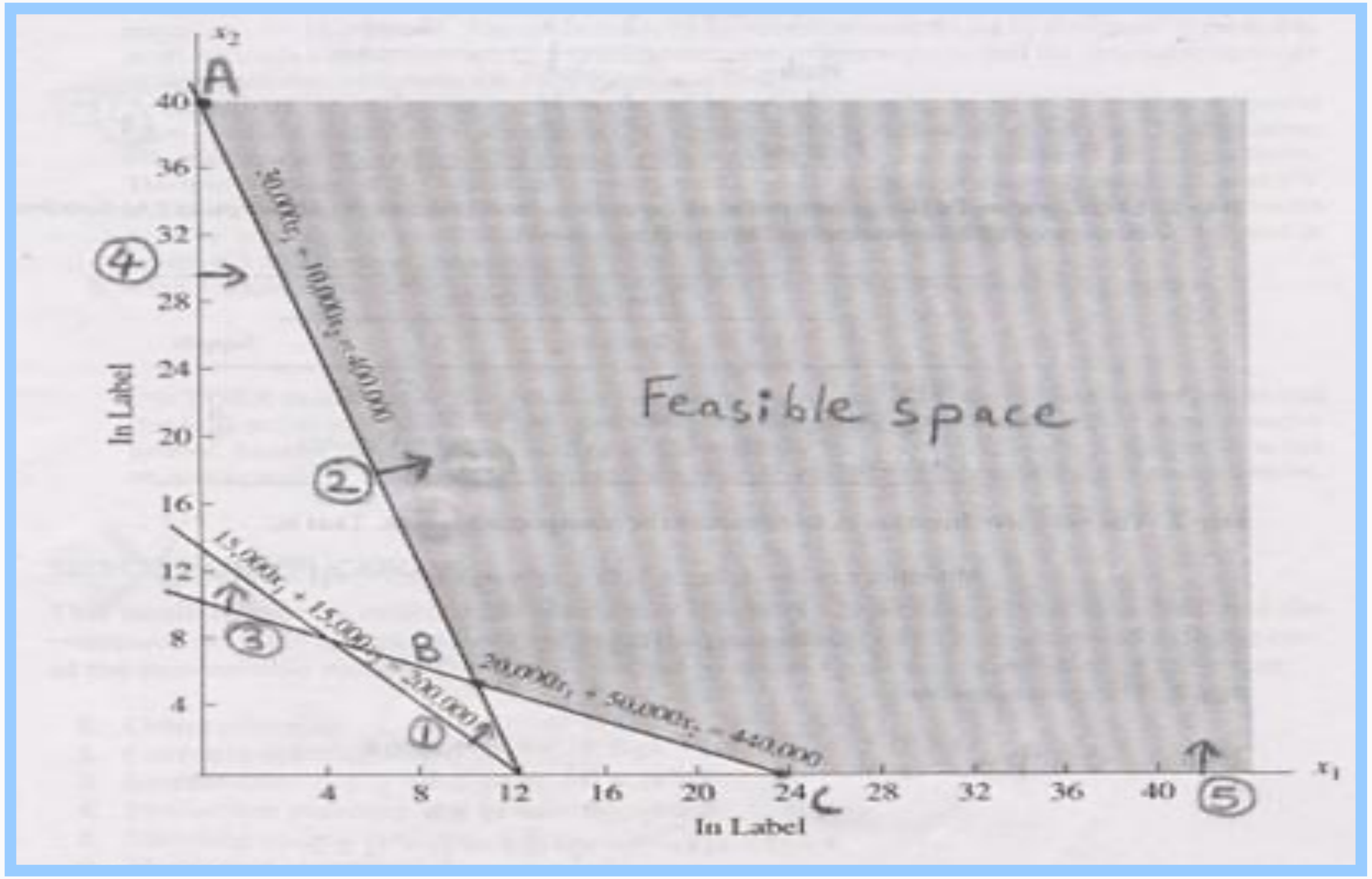
$$X_1 \geq 0 \quad \ldots(4)$$

$$X_2 \geq 0 \quad \ldots(5)$$

# LP - Minimization Model

## EXAMPLE 1 – Solution (cont'd)

Graphical Representation - Feasible Solution



# LP - Minimization Model

## EXAMPLE 2 – Solution (cont'd)

Use a corner point method to find the optimum solution.

$$\text{Minimize } Z = 600X_1 + 400X_2$$

| Corner Points | $(X_1, X_2)$ | Z |
|---|---|---|
| A | (10, 40) | 16000 |
| **B** | **(12, 4)** | **8800** |
| C | (22, 0) | 13200 |

- In 12 days, all the three types of bottles (Coca-Cola, Fanta, Thumps-up) are produced by plant at Coimbatore.
- In 4 days all the three types of bottles (Coca-Cola, Fanta, Thumps-up) are produced by plant at Chennai.
- So minimum production cost is 8800 units to meet the market demand of all the three types of bottles (Coca-Cola, Fanta, Thumps-up) to be produced in April.



UTM UNIVERSITI TEKNOLOGI MALAYSIA

# Exercise #2

(Continue from Exercise #1)

The Alex Garment Company manufactures men's shirts and women's blouses. The production process includes cutting, sewing, and packaging. The company employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hr shift, 5 days a week. The following table gives the time requirements and profits per unit to produce the two garments. **Determine the optimal** weekly production schedule for Alex Garment Company.

| | Minutes per unit | | | |
|---|---|---|---|---|
| Garment | *Cutting* | *Sewing* | *Packaging* | Unit profit ($) |
| Shirts | 20 | 70 | 12 | 8 |
| Blouses | 60 | 60 | 4 | 12 |

# Exercise #2

In Hamid grocery store, shelf space is limited and must be used effectively to increase profit. Two cereal items, Grano and Wheatie, compete for a total shelf space of 60 $ft^2$ . A box of Grano occupies 0.2 $ft^2$ and a box of Wheatie needs 0.4 $ft^2$ . The maximum daily demands of Grano and Wheatie are 200 and 120 boxes, respectively. A box of Grano nets $1.00 in profit and a box of Wheatie $1.35. Hamid thinks that because the unit profit of Wheatie is 35% higher than that of Grano, Wheatie should be allocated 35% more space than Grano, which amounts to allocating about 57% to Wheatie and 43% to Grano. What do you think?



UTM UNIVERSITI TEKNOLOGI MALAYSIA