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CHAPTER 4: LP – SENSIVITY ANALYSIS

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Innovating Solutions

The Role of Sensitivity Analysis of the Optimal Solution

- Is the optimal solution sensitive to changes in input parameters? The effective of this change is known as "**sensitivity**"
- Sensitivity of the optimal solution to the changes in the available resources i.e., the right-hand sight (RHS) of the constraints.
- Sensitivity of the optimal solution to the changes in unit profit or unit cost , i.e., coefficients of the objective function.

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Sensitivity Analysis To The Changes In The Available Resources.

- A **shadow price** for a constraint is the increase in the objective function value resulting from a one unit increase in its right-hand side value.
- The **range of feasibility** for a right-hand side coefficient is the range of that coefficient for which the shadow price remains unchanged.
- The range of feasibility is also the range for which the current set of basic variables remains the optimal set of basic variables (although their values change.)

Example 1

- JOBCO manufactures two products on two machines. All information as shown below.

	Product 1	Product 2	Daily availability
Machine 1	2 hours	1 hours	8 hours
Machine 2	1 hours	3 hours	8 hours
Unit Profit	30	20	

- LP formulation:

Maximize $Z = 30X_1 + 20X_2$

Subject to:

$$2X_1 + X_2 \leq 8$$

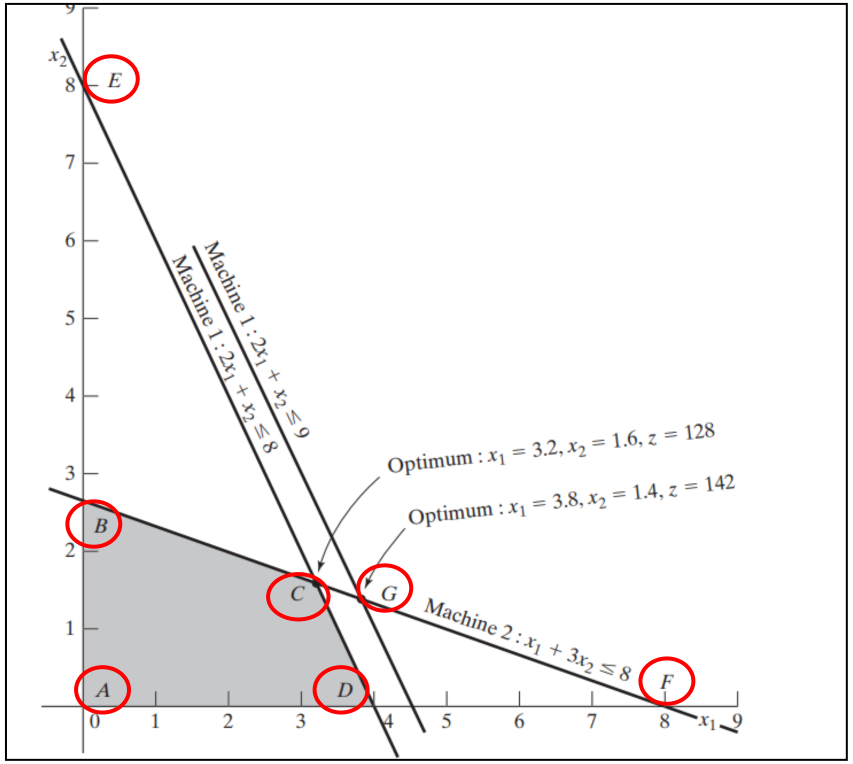
$$X_1 + 3X_2 \leq 8$$

$$X_1 \geq 0, X_2 \geq 0$$

Example 1 – Shadow Price/Dual Price (cont'd)

- The change in the optimum solution when changes are made in the capacity of machine 1.
- If the daily capacity is increased from **8 to 9 hrs**, the new optimum will move to point G. The rate of change in optimum z resulting from changing machine 1 capacity from 8 to 9 hrs can be computed as:

$$= \frac{z_G - z_C}{(\text{Capacity change})} = \frac{142 - 128}{9 - 8} = \$14/\text{hr}$$



- A unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by $14.

Example 1 – Shadow Price/Dual Price (cont'd)

- The dual price of \$14/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment BF.
- Machine 1 capacities at points B and F as follows:

$$\text{Minimum machine 1 capacity [at } B = (0, 2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67 \text{ hr}$$

$$\text{Minimum machine 1 capacity [at } F = (8, 0)] = 2 \times 8 + 1 \times 0 = 16 \text{ hr}$$

The conclusion is that the dual price of \$14.00/hr remains valid only in the range

$$2.67 \text{ hr} \leq \text{Machine 1 capacity} \leq 16 \text{ hr}$$

- Changes outside this range produce a different dual price (worth per unit).

Example 1 – Shadow Price/Dual Price (cont'd)

- Using similar computations, you can verify that the dual price for machine 2 capacity is \$2/hr, and it remains valid for changes in machine 2 capacity within the line segment DE.
- A unit increase (decrease) in machine 2 capacity will increase (decrease) revenue by \$2.

$$\text{Minimum machine 2 capacity [at } D = (4,0)] = 1 \times 4 + 3 \times 0 = 4 \text{ hr}$$

$$\text{Minimum machine 2 capacity [at } E = (0,8)] = 1 \times 0 + 3 \times 8 = 24 \text{ hr}$$

Thus, the dual price of \$2/hr for machine 2 remains applicable for the range

$$4 \text{ hr} \leq \text{Machine 2 capacity} \leq 24 \text{ hr}$$

Example 1 (cont'd)

Question 1.

If JOBCO can increase the capacity of both machines, which machine should receive priority?

Question 2.

A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of $10/hr for each machine. Is this advisable?

Example 1 (cont'd)

Question 3.

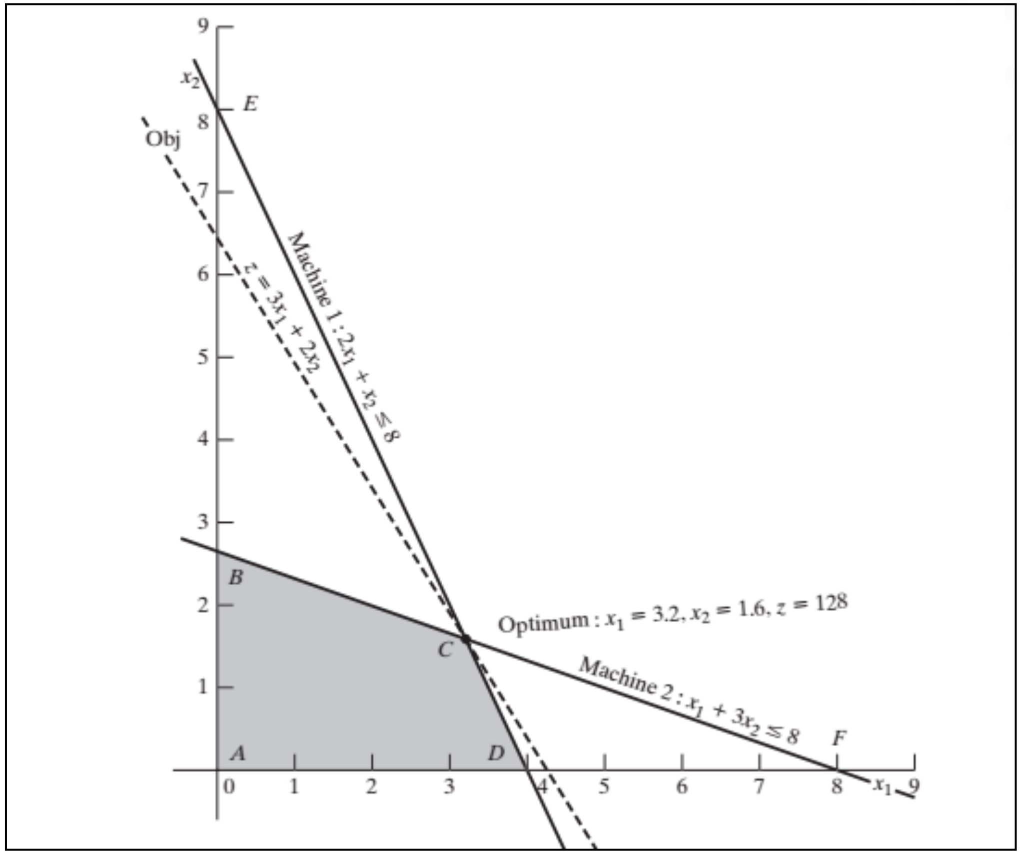
If the capacity of machine 1 is increased from 8 to 13 hrs, how will this increase impact the optimum revenue?

Sensitivity Analysis of Objective Function Coefficients.

- Range of Optimality:
  - The optimal solution will remain unchanged as long as
    - ✓ An objective function coefficient lies within its ***range of optimality***
    - ✓ There are no changes in any other input parameters(keeping all other coefficients constant)

Sensitivity Analysis of Objective Function Coefficients

Example: How can we determine the optimality range that will keep the optimal solution unchanged at C?



The slope of Z = $c_1x_1 + c_2x_2$ must lies between the slope of the two constrains so the optimal solution will be unchanged at point C.

$$\frac{1}{3} \le \frac{c_1}{c_2} \le \frac{2}{1} \;= 0.333 \le \frac{c_1}{c_2} \le 2$$

Example 1 (cont'd)

Question 4

Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same?

Question 5

Suppose that the unit revenue of product 2 is fixed at its current value c_2 = \$20. What is the associated optimality range for the unit revenue for product 1, c_1, that will keep the optimum unchanged?

Sensitivity Analysis with Solver

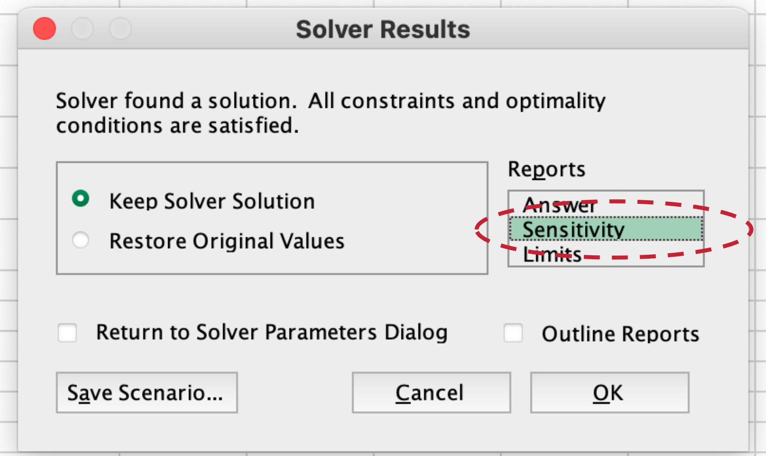
Example - JOBCO

Input Data

	X1	X2			
	Product 1	Product 2	Total		Limit
Objective	30	20	128		
M1	2	1	8	<=	8
M2	1	3	8	<=	8

Output results:

	X1	X2	Z
Solution	3.2	1.6	128



Sensitivity Analysis Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B12	Solution X1	3.2	0	30	10	23.33333333
C12	Solution X2	1.6	0	20	70	5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D7	M1 Total	8	14	8	8	5.333333333
D8	M2 Total	8	2	8	16	4

$(RHS - 5.33 = 2.67)$

2.67 hr $\leq$ Machine 1 capacity $\leq$ 16 hr

4 hr $\leq$ Machine 2 capacity $\leq$ 24 hr

$6.67 \leq c_1 \leq 40$

$15 \leq c_2 \leq 90$

$(RHS + 8) = 16)$

Example:

TOYCO uses three operations to assemble three types of toys: trains, trucks, and cars. The daily available times for the three operations are 430, 460, and 420 mins, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 mins, respectively. The corresponding times per truck and per car are (2, 0, 4) and (1, 2, 0) mins (a zero time indicates that the operation is not used).

Example - Solution:

Letting x_1, x_2, and x_3 represent the daily number of units assembled of trains, trucks, and cars, respectively, the associated LP model is given as:

Maximize $Z= 3x_1+2x_2+5x_3$

Subject to:

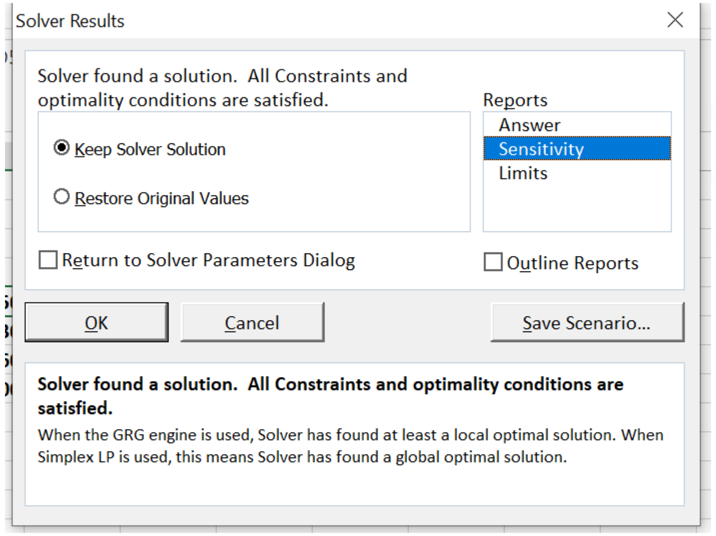
$$x_1+2x_2+x_3 <=430$$

$$3x_1+2x_3 <=460$$

$$x_1+4x_2<=420$$

Example – Solution (Solver):

	A	B	C	D	E	F	G	H
1		Example - TOYCO						
2								
3								
4								
5		Input Data						
6			X1	X2	X3			
7			Trains	Trucks	Cars	Total		Limit
8		Objective	3	2	5			
9		OP1	1	2	1	430	<=	430
10		OP2	3	0	2	460	<=	460
11		OP3	1	4	0	400	<=	420
12								
13		Output results:						
14			X1	X2	X3	Z		
15		Solution	0	100	230	1350		
16								



F9 =(C9*C15) + (D9*D15) +(E9*E15)
F10 =(C10*C15) + (D10*D15) +(E10*E15)
F11 =(C11*C15) + (D11*D15) +(E11*E15)

Example – Solution (Solver):

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C15	Solution X1	0	-4	3	4	1E+30
D15	Solution X2	100	0	2	8	2
E15	Solution X3	230	0	5	1E+30	2.666666667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F9	OP1 Total	430	1	430	10	200
F10	OP2 Total	460	2	460	400	20
F11	OP3 Total	400	0	420	1E+30	20

- Shadow Price/Dual price:

$x_1=1$; $x_2=2$; $x_3=0$

1 unit change in operation 1 capacity changes Z by $1

1 unit change in operation 2 capacity changes Z by $2

1 unit change in operation 3 capacity changes Z by $0

Example – Solution (Solver):

- Feasible ranges associated with the changing the resources

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F9	OP1 Total	430	1	430	10	200
F10	OP2 Total	460	2	460	400	20
F11	OP3 Total	400	0	420	1E+30	20

- The shadow price for operation 1 is valid in the feasible range changes between -200 and 10
- The shadow price for operation 2 is valid in the feasible range changes between -20 and 400
- The shadow price for operation 3 is valid in the feasible range changes is between -20 and infinity.

Example – Solution (Solver):

- Determine the condition that keep a solution optimal (current optimum remain the same)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C15	Solution X1	0	-4	3	4	1E+30
D15	Solution X2	100	0	2	8	2
E15	Solution X3	230	0	5	1E+30	2.666666667

- Optimality range for coefficient value of x_1 changes is less than 4
- Optimality range for coefficient value of x_2 changes is between −2 and 8
- Optimality range for coefficient value of x_3 changes is greater than −2.667

Exercise

Consider the Reddy Mikks problem. Use Solver to obtain the sensitivity report, then answer the following:

a) Determine the range for the ratio of the unit revenue of exterior paint to the unit revenue of interior paint.

b) If the revenue per ton of exterior paint remains constant at $5000 per ton, determine the maximum unit revenue of interior paint that will keep the present optimum solution unchanged.

c) If for marketing reasons the unit revenue of interior paint must be reduced to $2500, will the current optimum production mix change?

d) It is proposed that the availability of raw material 1 is increase to 30 tons. Indicate whether the changes will keep the current shadow price?

e) Determine the value of profit if the availability or raw material 2 is decrease by 2 tons.