MCSD1133 Operations Research & Optimization

UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 9:

NON-LINEAR PROGRAMMING

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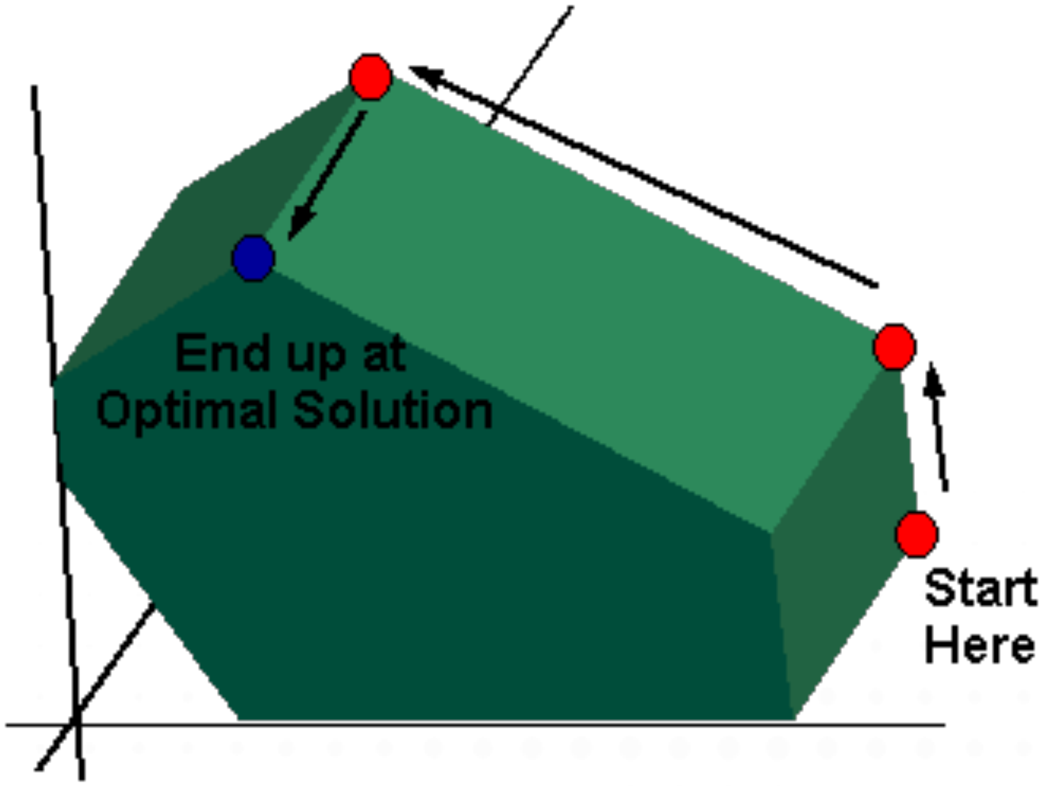
Innovating Solutions

Introduction

- In LP, the goal is to maximize or minimize a linear function subject to linear constraints.
- But in many real-world problems, either:
  - objective function may not be a linear function, or
  - some of the constraints may be nonlinear.
- Functions having exponents, logarithms, square roots, products of variables, and so on are nonlinear.

Introduction



We may not have those nice corner points as in the case of linear programming.

Non-Linear Programming (NLP)

- Optimization problems that involve non-linear functions are called nonlinear programming (NLP) optimization.
- Solution methods are more complex than linear programming methods.
- Solution techniques generally involve searching a solution surface for high or low points requiring the use of advanced mathematics.
- NLPs that do not have any constraints are called **unconstrained NLPs**

Non-Linear Programming (NLP)

- In general form, the nonlinear programming problem is to find $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to:

$$
\begin{aligned}
&\text{Maximize } f(\mathbf{x}),\\
&\text{subject to:}\\
&\qquad g_i(\mathbf{x}) \le b_i, \text{ for } i = 1, 2, \ldots, m,\\
&\text{and}\\
&\qquad \mathbf{x} \ge 0,
\end{aligned}
$$

where $f(\mathbf{x})$ and the $g_i(\mathbf{x})$ are given functions of the n decision variables.

- There are many different types of nonlinear programming problems, depending on the characteristics of the $f(\mathbf{x})$ and $g_i(\mathbf{x})$ functions.

Graphical Illustration of NLP

- LP

Maximize $Z = 3x_1 + 5x_2$,
subject to

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

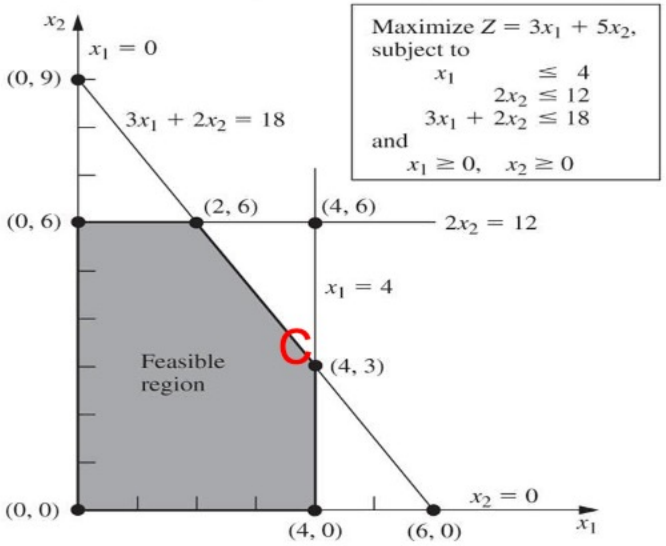


Figure 1

- Single nonlinear constraint

Maximize $Z = 3x_1 + 5x_2$,
subject to

$$x_1 \leq 4$$
$$9x_1^2 + 5x_2^2 \leq 216$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

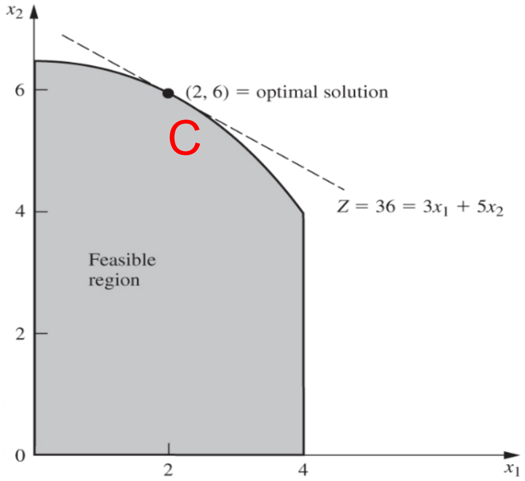


Figure 2

Graphical Illustration of NLP

- Nonlinear objective function

$$
\begin{array}{llcr}
\text{Maximize} & Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2, & & \\
\text{subject to} & x_1 & \le & 4 \\
 & 2x_2 & \le & 12 \\
 & 3x_1 + 2x_2 & \le & 18 \\
\text{and} & x_1 \ge 0, \quad x_2 \ge 0 & &
\end{array}
$$

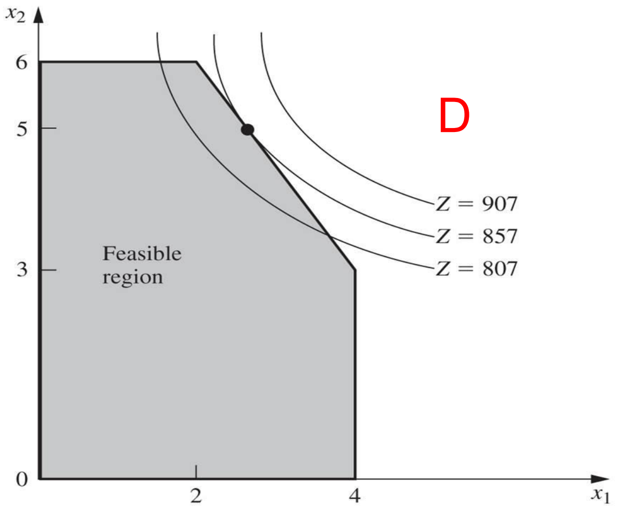


Figure 3

$$
\begin{array}{llcr}
\text{Maximize} & Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2, & & \\
\text{subject to} & x_1 & \le & 4 \\
 & 2x_2 & \le & 12 \\
 & 3x_1 + 2x_2 & \le & 18 \\
\text{and} & x_1 \ge 0, \quad x_2 \ge 0 & &
\end{array}
$$

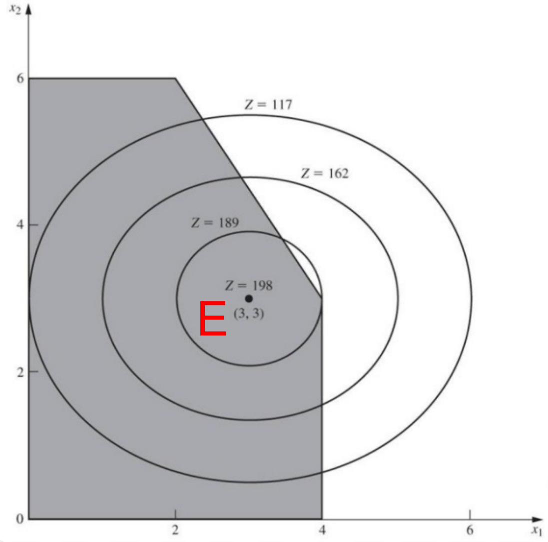
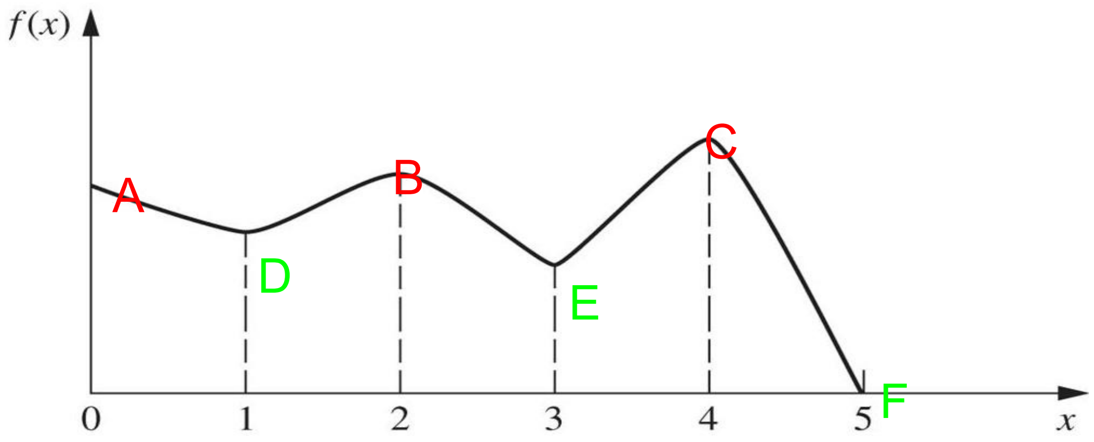


Figure 4

Local vs Global Optimum

- A local maximum need not be a global maximum (overall optimum solution).



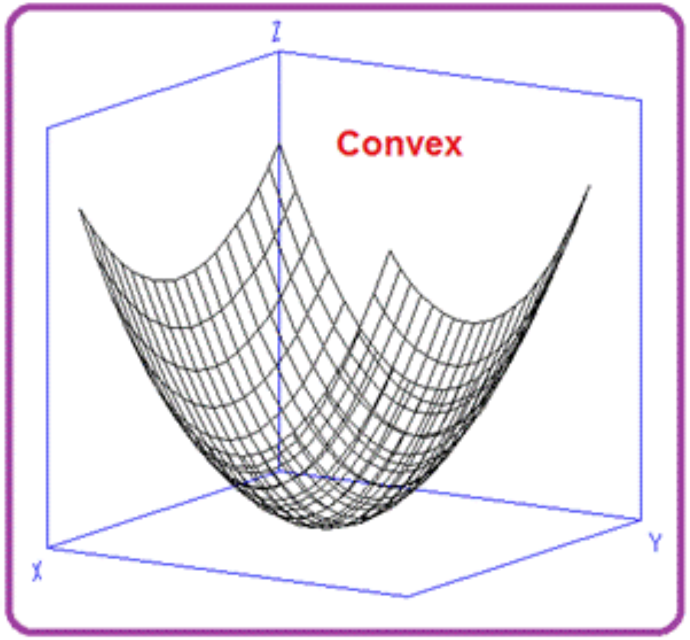
- Over the interval $0 \leq x \leq 5$, this function has three local maxima: $x = 0$(A), $x = 2$(B), and $x = 4$(C) But only one of these ($x = 4$) is a global maximum. (Similarly, there are local minima at $x = 1$(D), $x = 3$(E), and $x = 5$(F), but only $x = 5$ is a global minimum).

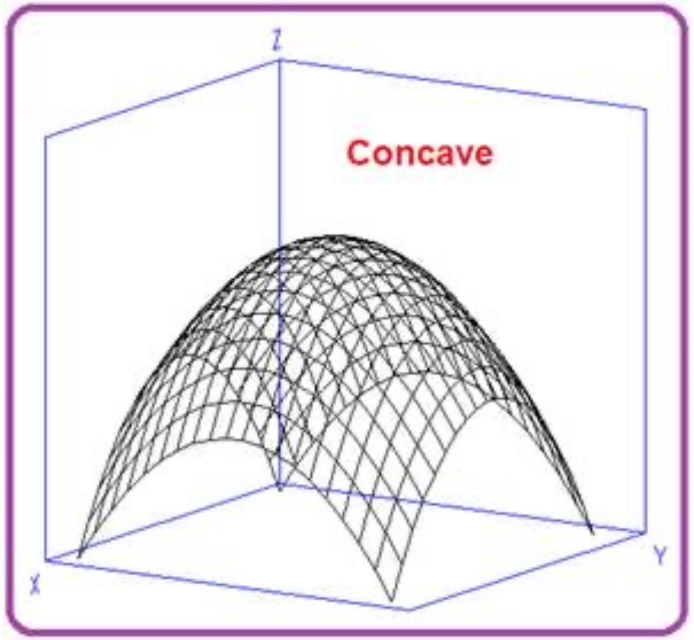
- Nonlinear Programming algorithms generally can not distinguish between a local optimal solution and a global optimal solution.

- Need to know the conditions under which any local maximum is *guaranteed* to be a global maximum over the feasible region.

Convex vs Concave

- Two important class of nonlinear functions:
  - **Convex**
    - o Increasing marginal returns.
  - **Concave**
    - o Decreasing marginal returns.





Spreadsheet Nonlinear Optimization

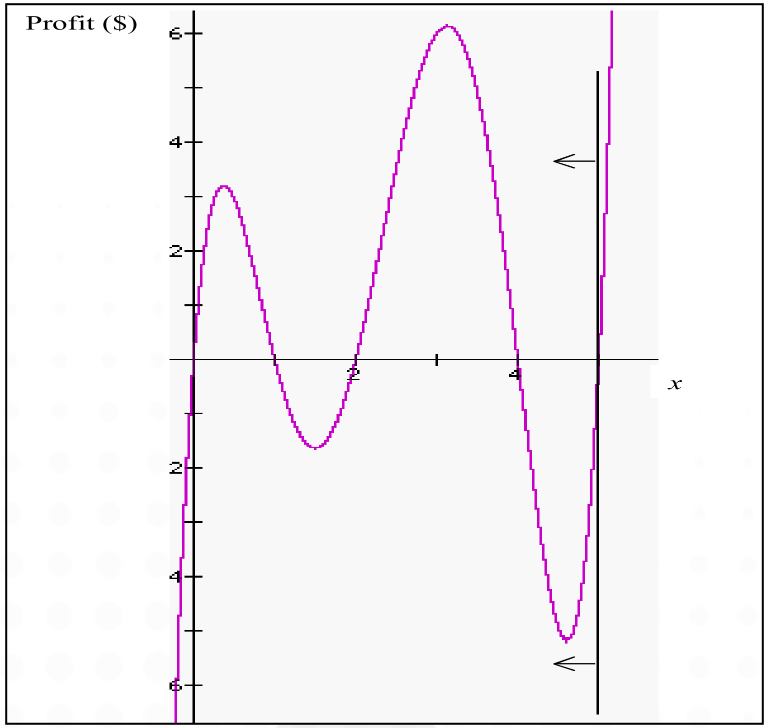
- Consider the following model in algebraic form:

$$\text{Maximize Profit} = \boxed{0.5x^5 - 6x^4 + 24.5x^3 - 39x^2 + 20x} \Rightarrow \text{Nonlinear function}$$

subject to: $x \leq 5$

$x \geq 0$

- Profit function of the model:



Spreadsheet Nonlinear Optimization

Starting with $x = 0$

	A	B	C	D	E
1	**A Simple NLP**				
2					
3					Maximum
4		x =	0.371	<=	5
5					
6		Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
7		=	$3.19		

Starting with $x = 3$

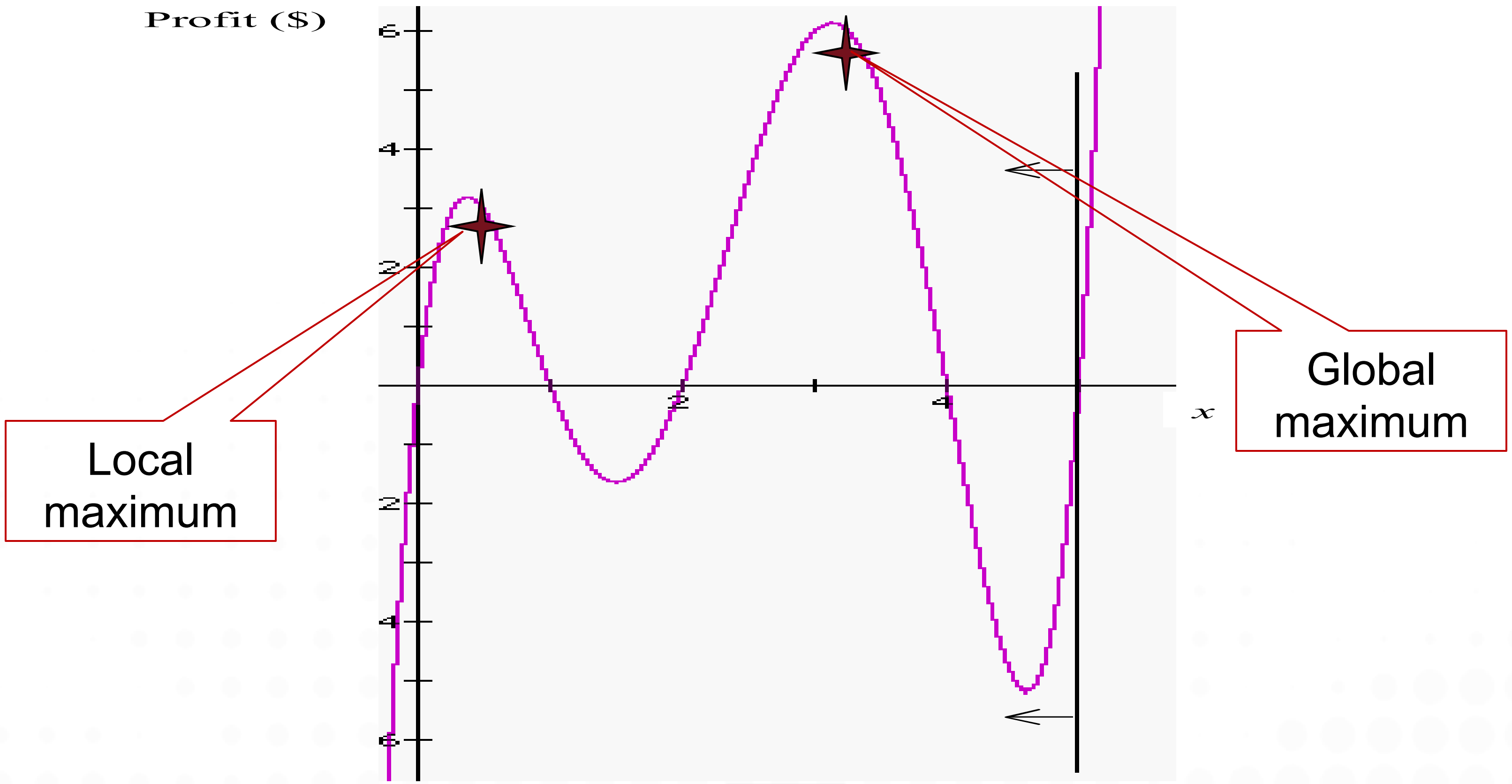
	A	B	C	D	E
1	**A Simple NLP**				
2					
3					Maximum
4		x =	3.126	<=	5
5					
6		Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
7		=	$6.13		

Starting with $x = 5$

	A	B	C	D	E
1	**A Simple NLP**				
2					
3					Maximum
4		x =	5.000	<=	5
5					
6		Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
7		=	$0.00		

Spreadsheet Nonlinear Optimization

- Local maximum vs. global maximum



Unconstrained Optimization

- **The objective function being concave** guarantees that a local maximum is a global maximum.

- **The objective function being convex** ensures that a local minimum is a global minimum.

Example: Profit Analysis - Linear

- Profit function, Z, with volume independent of price:

$$Z = vp - c_f - vc_v$$

where v = sales volume
p = price
c_f = unit fixed cost
c_v = unit variable cost

- One important but somewhat unrealistic assumption of this **break-even model** is that volume, or demand, is independent of price (i.e., volume remains constant, regardless of the price of the product).
- It would be more realistic for the demand to vary as price increased or decreased (nonlinear equation for profit that relates profit to price).

Profit function, Z, with volume independent of price:

$$Z = vp - c_f - vc_v$$

Add volume/price relationship:

$$v = 1{,}500 - 24.6p$$

With fixed cost (c_f = \$10,000) and variable cost (c_v = \$8).

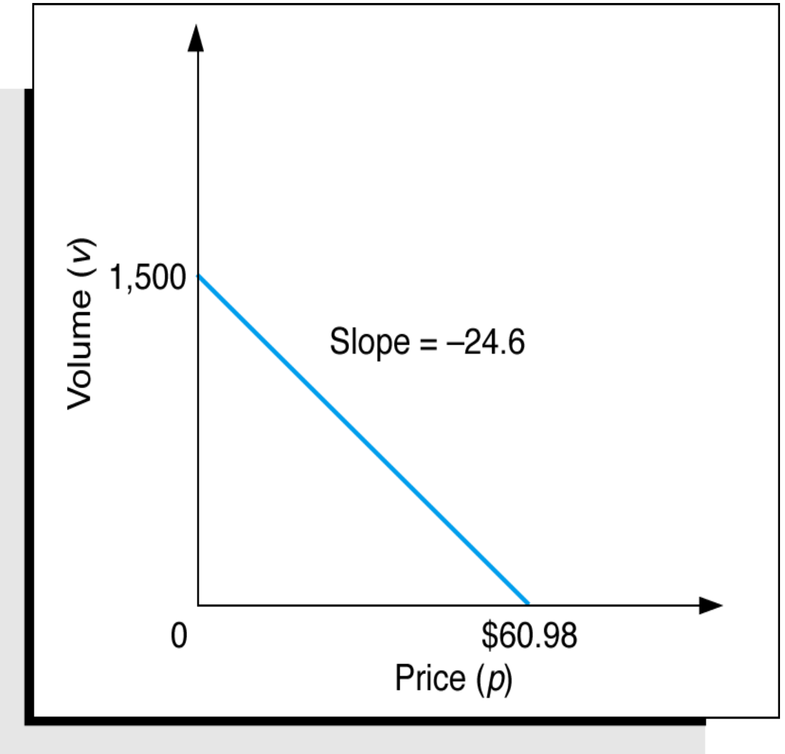


Figure 1: Linear relationship of volume to price

Example: Profit Analysis - Nonlinear

Profit, $Z = 1{,}696.8p - 24.6p^2 - 22{,}000$

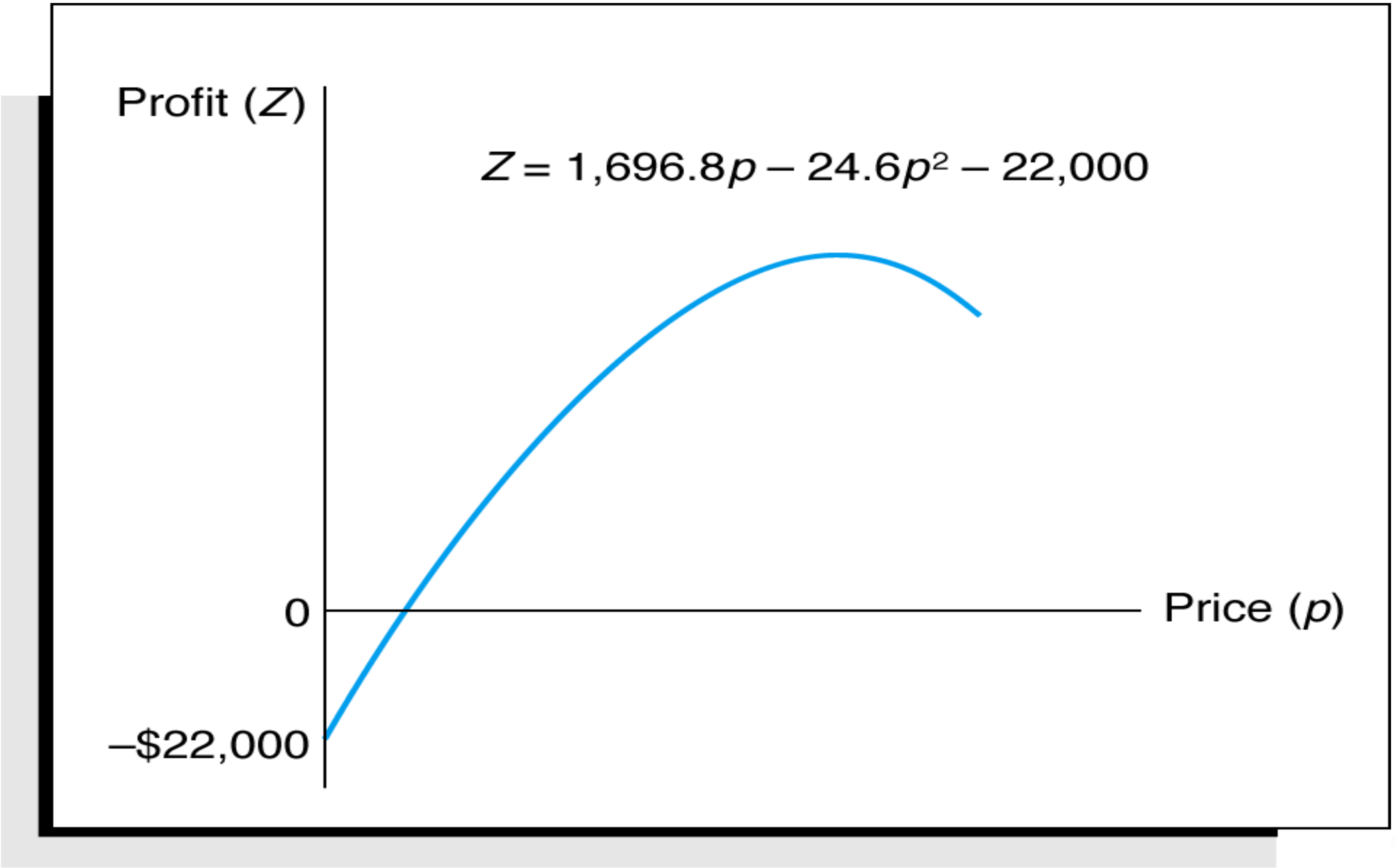


Figure 2: The nonlinear profit function

Optimal Value

- The slope of a curve at any point is equal to the derivative of the curve's function.
- The slope of a curve at its highest point equals zero.

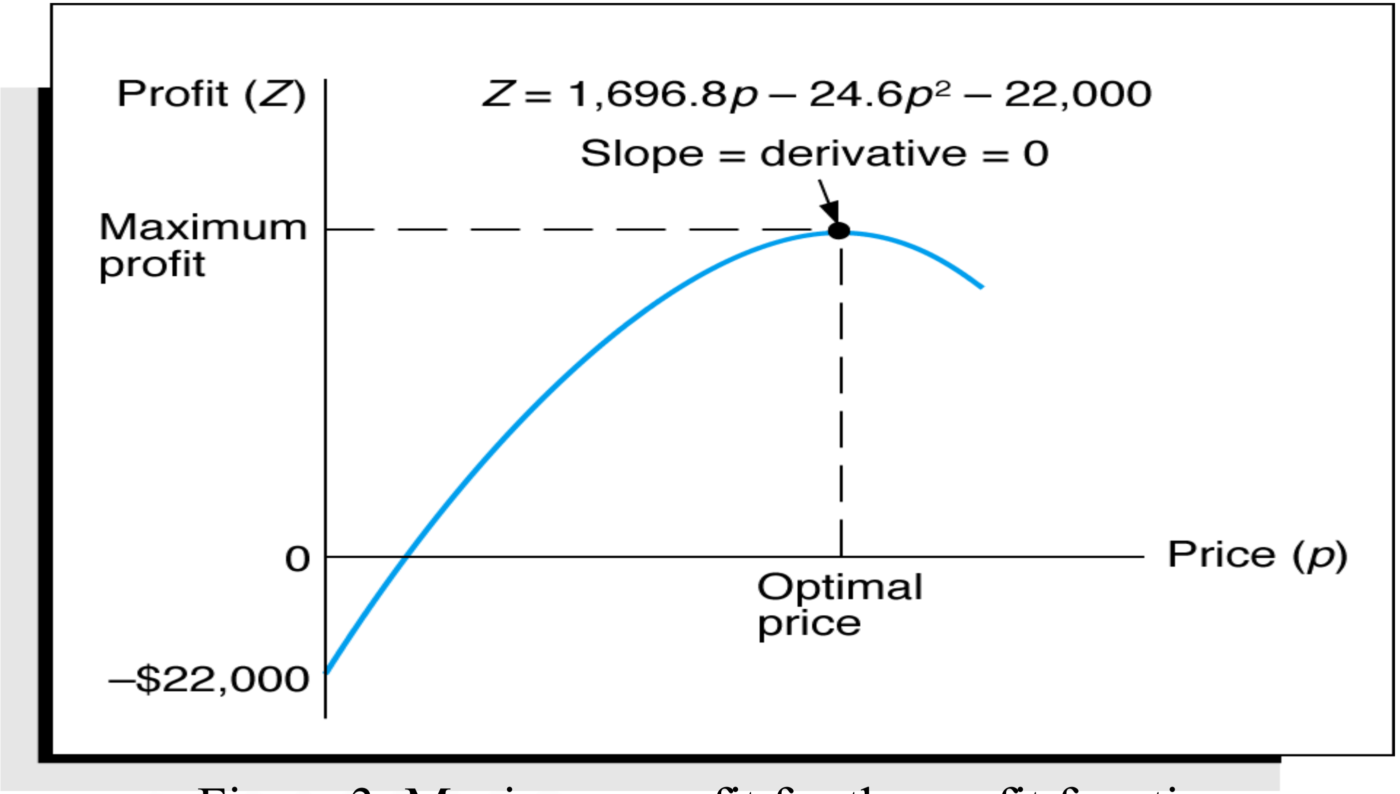


Figure 3: Maximum profit for the profit function

Optimal Value Solution Using Calculus

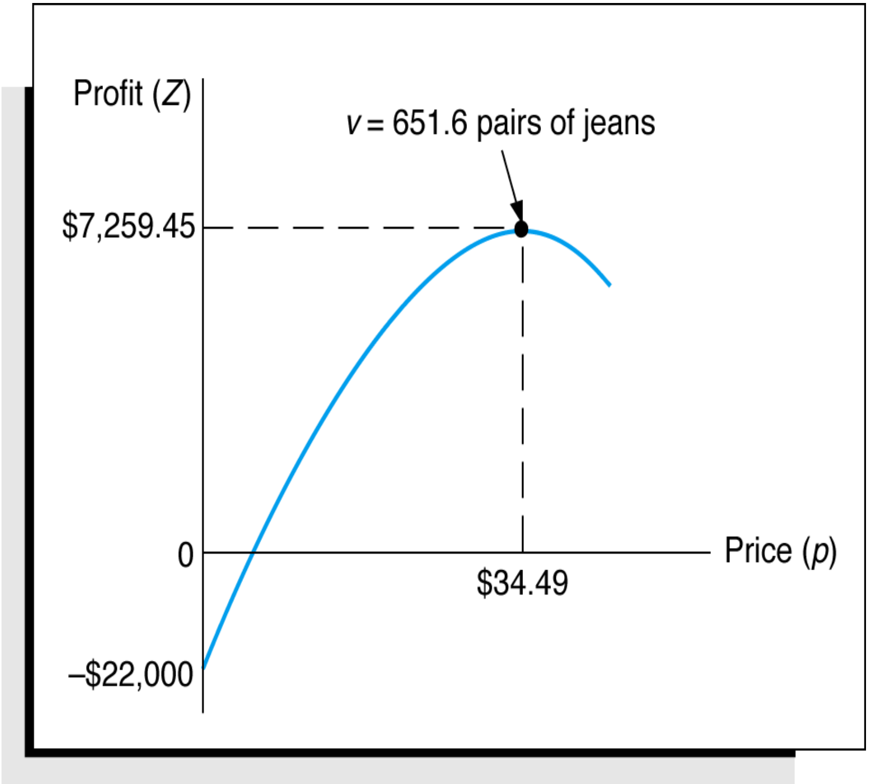


Figure 4: Maximum profit, optimal price and optimal volume

$$Z = 1{,}696.8p - 24.6p^2 - 2{,}000$$

$$\frac{dZ}{dp} = 1{,}696.8 - 49.2p = 0$$

$$p = \frac{1696.8}{49.2} = \$34.49$$

$$v = 1{,}500 - 24.6p$$

$$v = 651.6\, pairs\ of\ jeans$$

$$.:\ Z = \$7{,}259.45$$

Spreadsheet Nonlinear Optimization

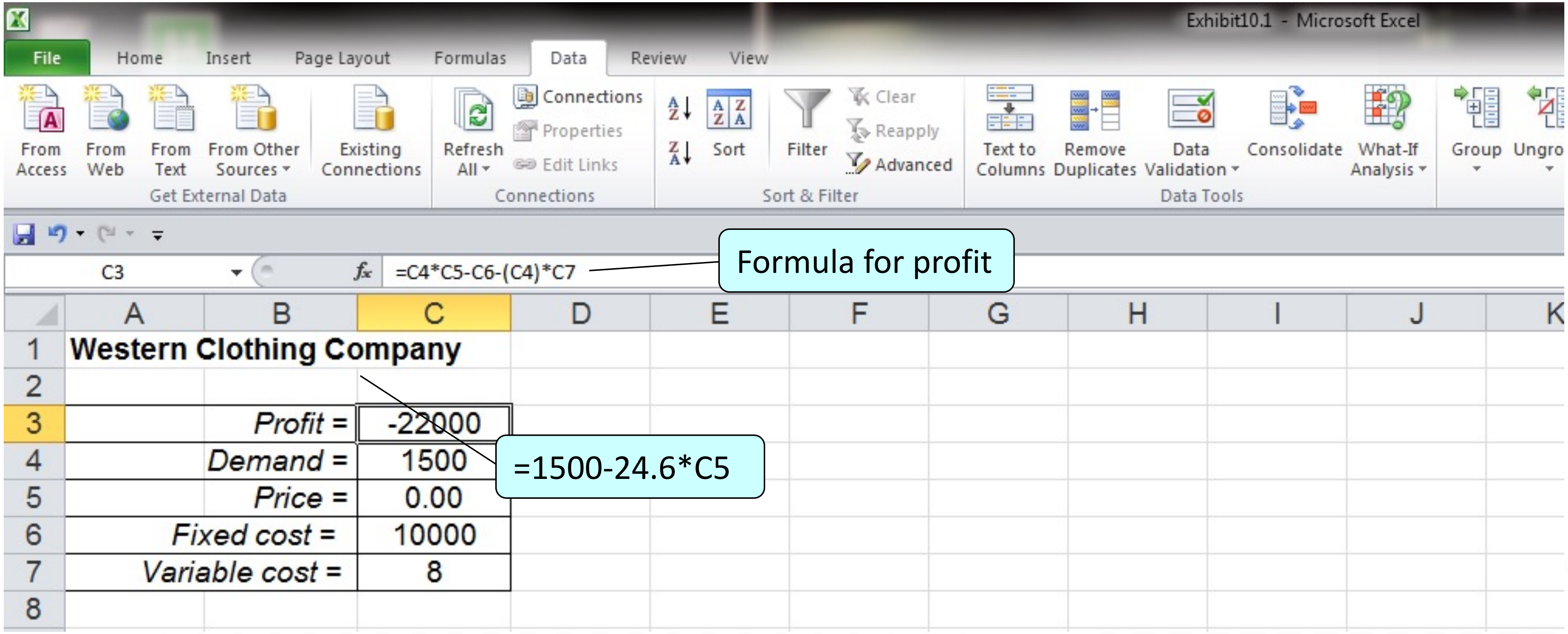
Excel Solver's nonlinear method:

- Climbing the mountain for maximization until it reaches the first peak or constraint.
- Going down for minimization until it reaches the first bottom or constraint.
- No guarantee that we will find the global optimum, unless:
  - We are maximizing a concave function, or
  - We are minimizing a convex function.

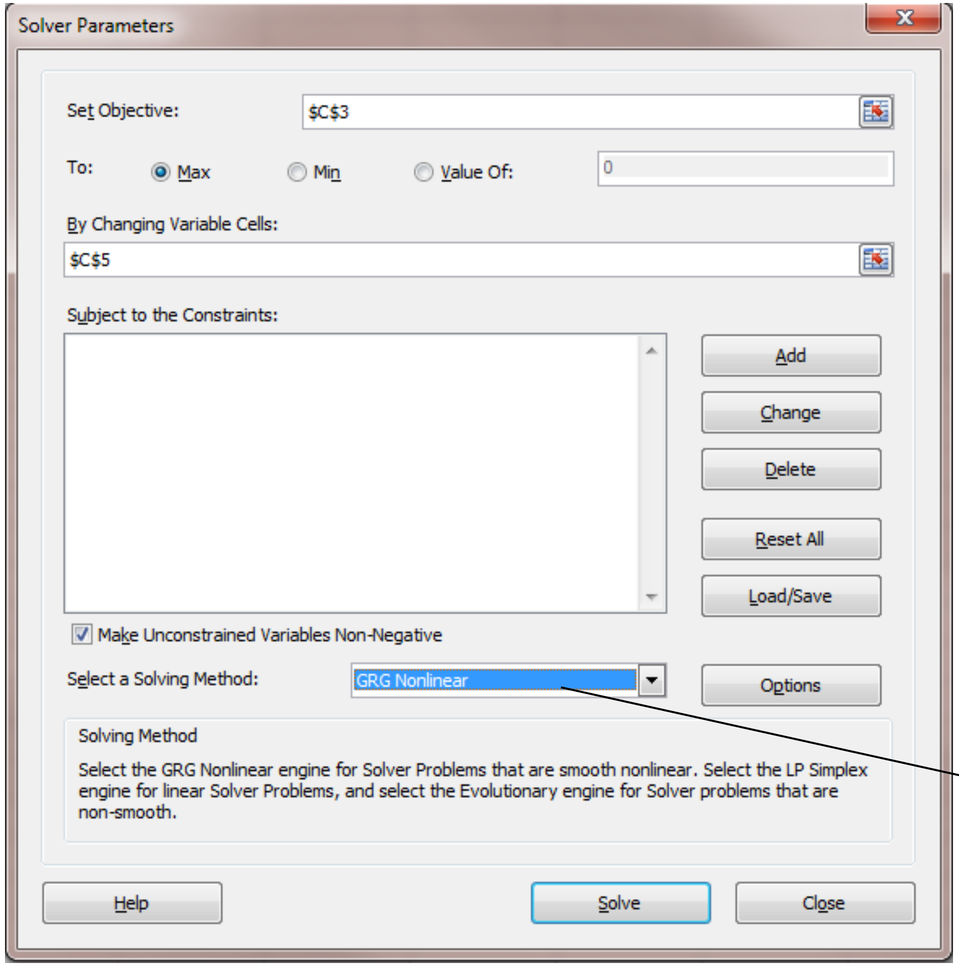
These functions have single peak and single bottom

Solve using Excel (1 of 3)

$$Z = vp - c_f - vc_v \qquad\qquad v = 1{,}500 - 24.6p$$



Solve using Excel (2 of 3)



Click on “GRG Nonlinear”

Solve using Excel (3 of 3)

	A	B	C	D	E	F
1	**Western Clothing Company**					
2						
3	Profit	7259.45				
4	Demand	651.6				
5	Price	34.49				
6	Fixed Cost	10000				
7	variable Cost	8				
8						
9						
10						

Constrained Optimization in NLP

- A nonlinear problem containing one or more constraints becomes a ***constrained optimization*** model.
- A ***nonlinear*** programming model has ***the same general form*** as the ***linear*** programming model except that the objective function and/or the constraint(s) are nonlinear.
- **The objective function being concave and the feasible region being a convex set** guarantees that a local maximum is a global maximum.

Constrained Optimization in NLP

Effect of adding constraints to nonlinear problem:

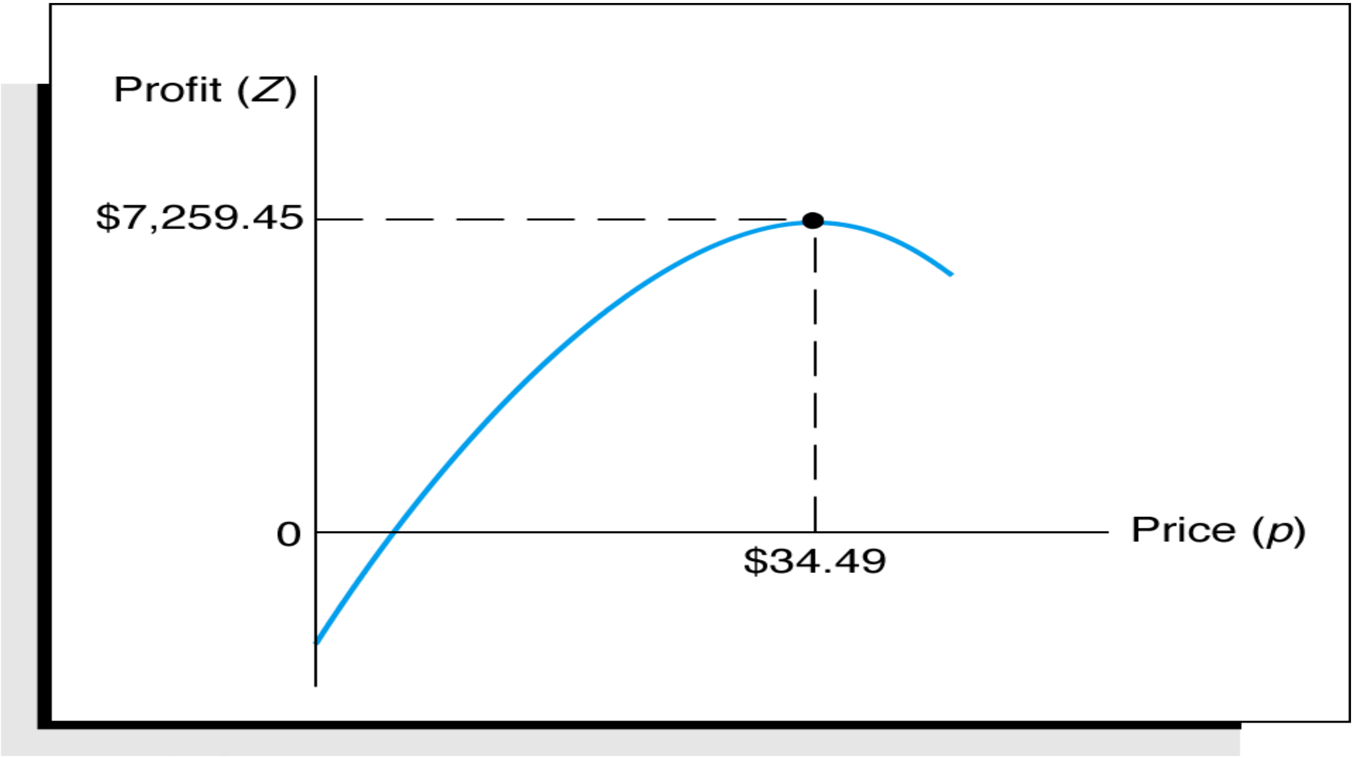


Figure 5: Nonlinear profit curve for the profit analysis model

Constrained Optimization in NLP

Adding a constrain $p \leq 20$:

- It corresponds to the maximum value of the portion of the objective
- The solution is on the boundary of the feasible solution space formed by the constraint.

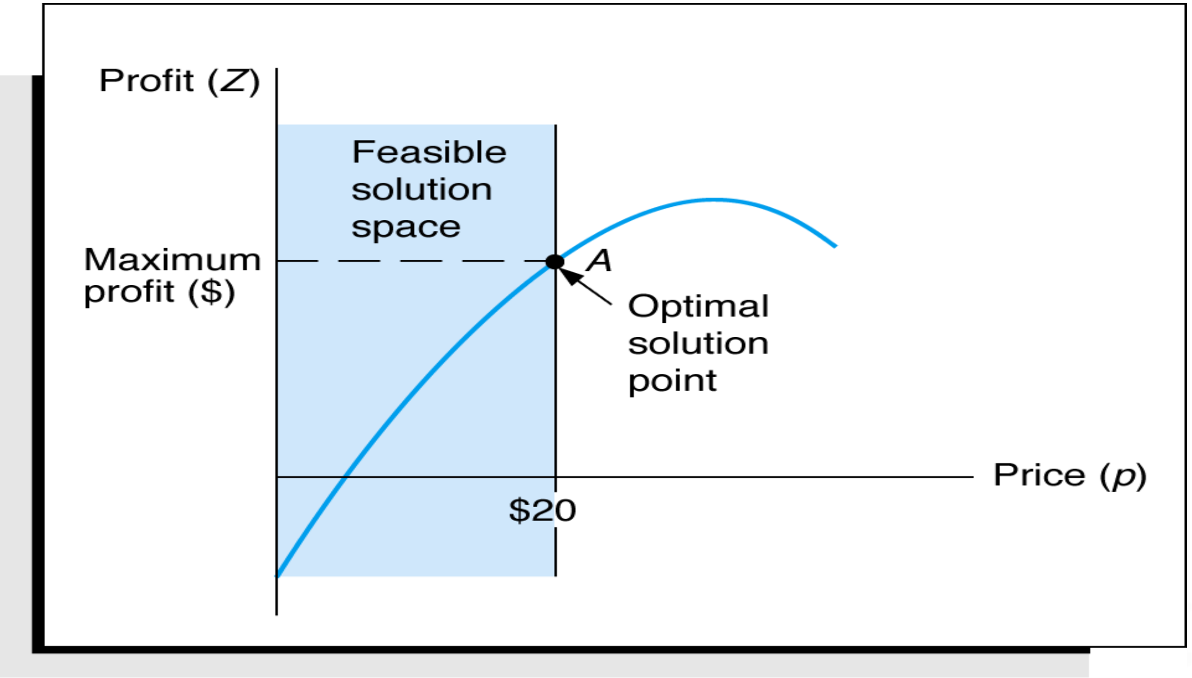


Figure 6: Constrained optimization model

Constrained Optimization in NLP

Adding a constrain $p \leq 40$:

- It corresponds to the maximum value of the portion of the objective function.

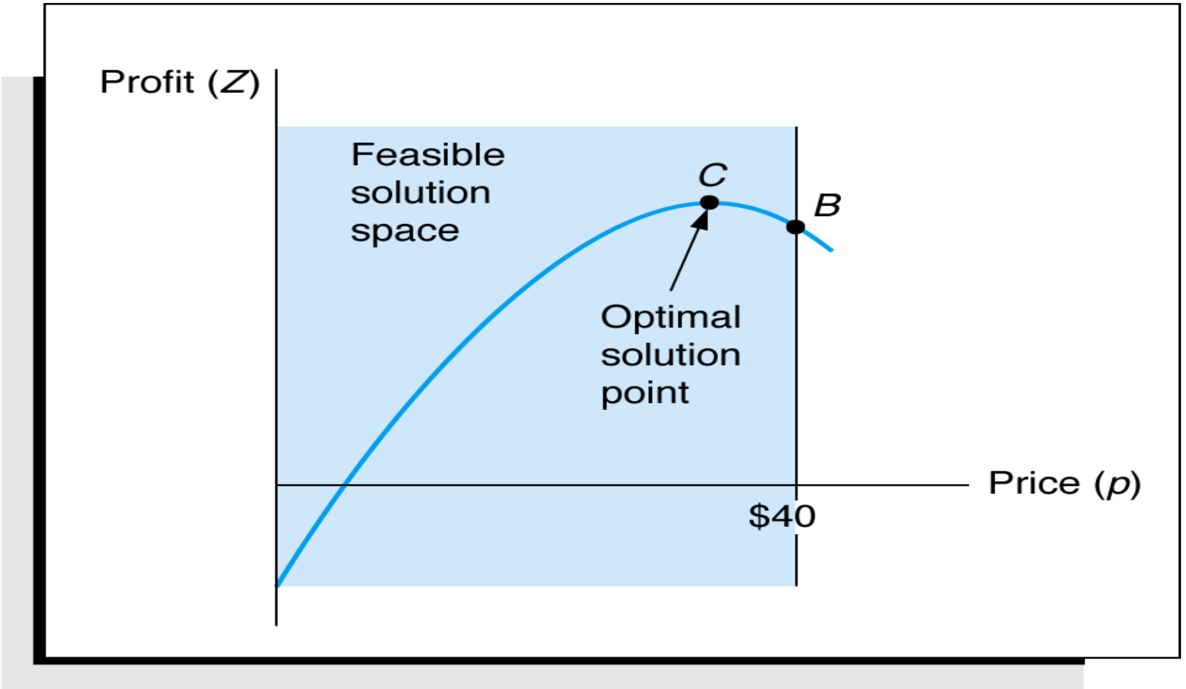


Figure 7: A constrained optimization model with a solution point not on the constraint boundary

Example

- Wyndor Glass Co. Case:
  - D: number of doors
  - W: number of windows
  - Constraints:
    - $D \leq 4$ (Plant 1 availability)
    - $2W \leq 12$ (Plant 2 availability)
    - $3D + 2W \leq 18$ (Plant 3 availability)

- Net profit for D doors

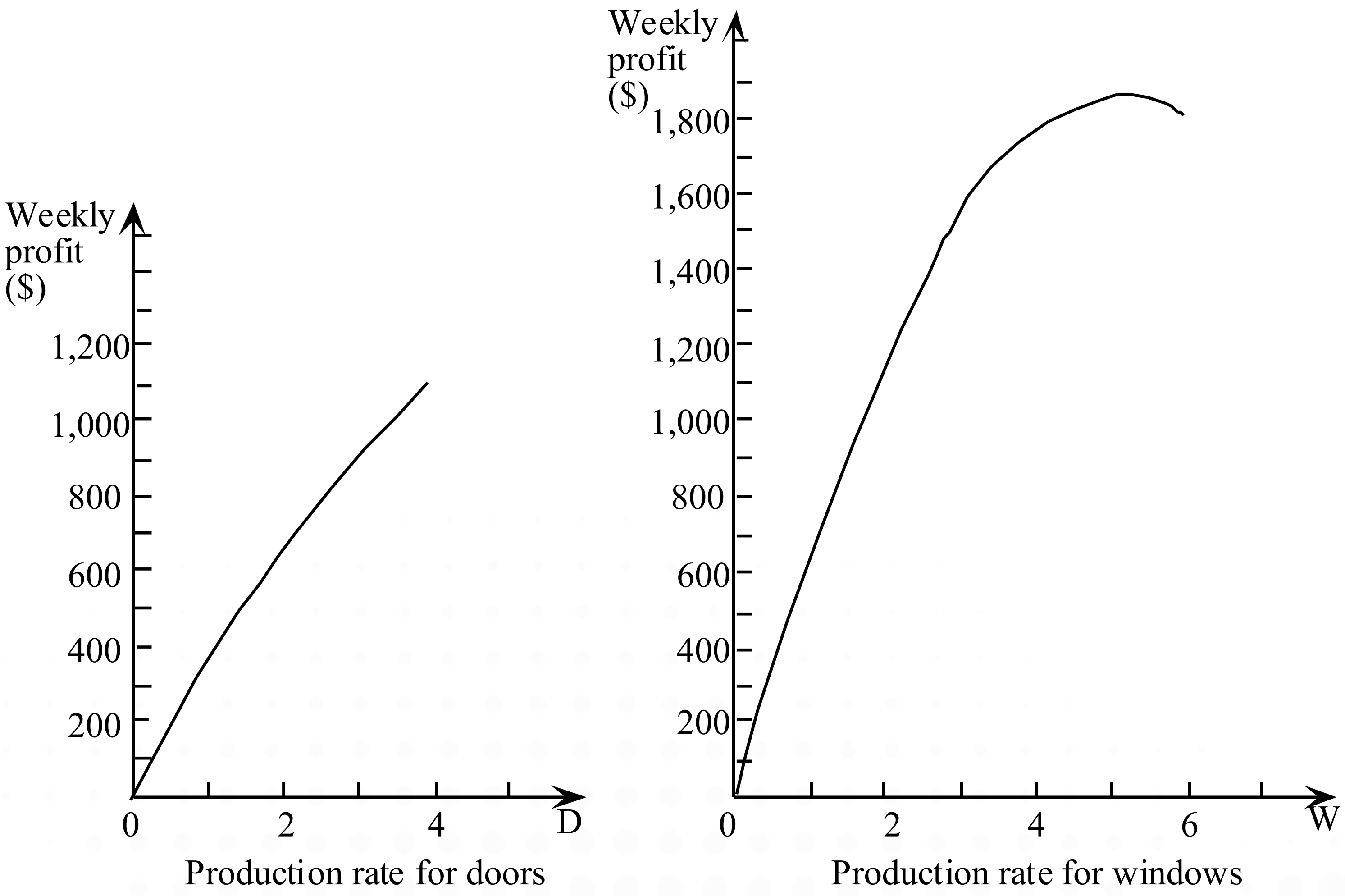
$$\underbrace{\$375D}_{\text{\$375 is the gross profit for one door}} - \underbrace{\$25D^2}_{\text{\$25}D^2\text{ is the marketing cost for D doors}}$$

- Net profit for W windows

$$\$700W - \$66\frac{2}{3}W^2$$

Example:

Decreasing Marginal Returns



Example:

Mathematical Formulation

Maximize	$P = \$375D - \$25D^2 + \$700W - \$66\frac{2}{3}W^2$	
subject to	$D \le 4$	(Plant 1 availability)
	$2W \le 12$	(Plant 2 availability)
	$3D + 2W \le 18$	(Plant 3 availability)
	$D \ge 0,\ W \ge 0$	

Example:

Solve using Excel Solver

Maximize $P = \$375D - \$25D^2 + \$700W - \$66\frac{2}{3}W^2$

subject to

$D \leq 4$ (Plant 1 availability)

$2W \leq 12$ (Plant 2 availability)

$3D + 2W \leq 18$ (Plant 3 availability)

$D \geq 0,\ W \geq 0$

H11 =C3*C11+D3*D11

	A	B	C	D	E	F	G	H	I
1									
2			Doors	Windowa					
3		Unit profit	375	700					
4					Hours		Hours		
5			Hours Used per Unit Produces		Used		Available		
6		Plant1	1	0	0	<=	4		
7		Plant2	0	2	0	<=	12		
8		Plant 3	3	2	0	<=	18		
9									
10			Doors	Windows					
11		Unit produced	0.000	0.000		Groff profit		0.0	
12									
13		marketing Cost	0.0	0.0		Total marketing cost		0.0	
14						Total profit		0.0	
15									

=25*C11^2

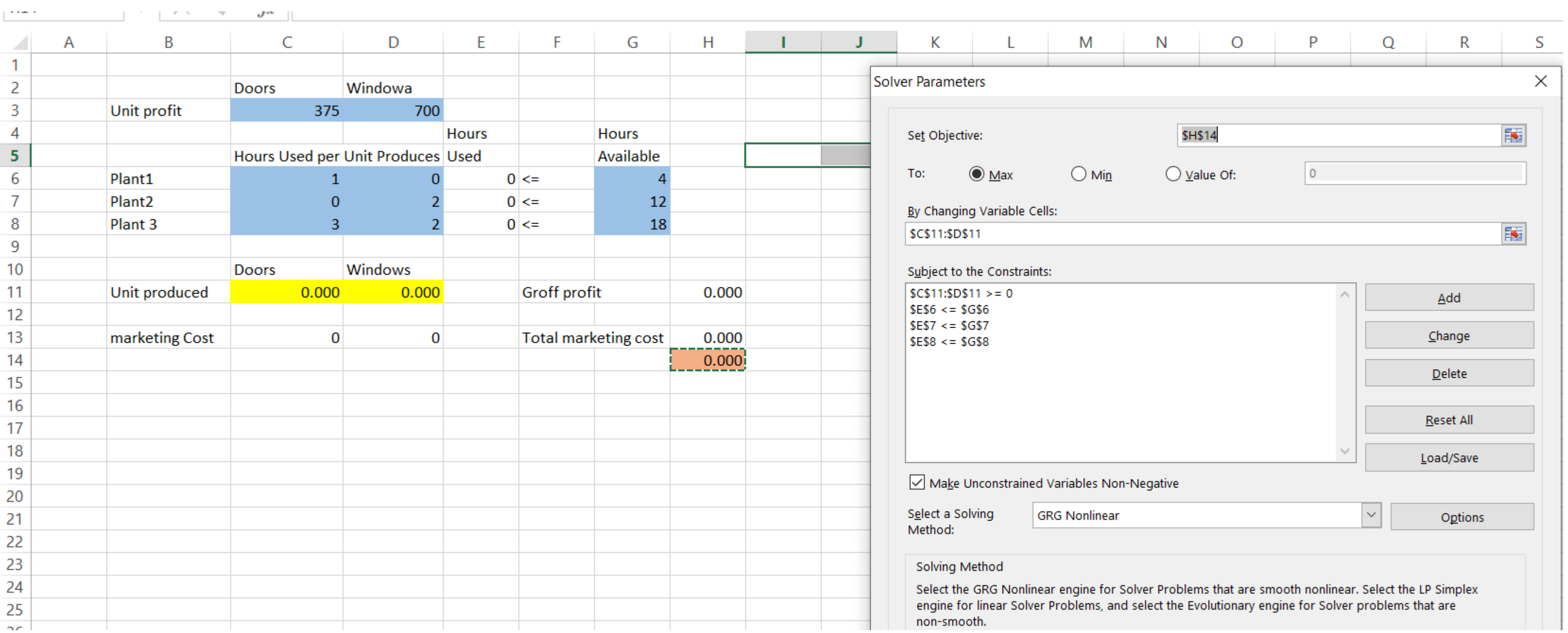
=66.67*D11^2

=C13+D13

=H11-H13

Example:

Solve using Excel Solver



Example:

Solve using Excel Solver

Wyndor Problem With Nonlinear Marketing Costs

	Doors	Windows				
Unit Profit (Gross)	$375	$700				
			Hours		Hours	
	Hours Used Per Unit Produced		Used		Available	
Plant 1	1	0	3.214	<=	4	
Plant 2	0	2	8.357	<=	12	
Plant 3	3	2	18	<=	18	
	Doors	Windows				
Units Produced	3.214	4.179	Gross Profit from Sales			$4,130
Marketing Cost	$258	$1,164	Total Marketing Cost			$1,422
					Total Profit	$2,708

Example:

Decreasing Marginal Returns