MCSD1133 Operations Research & Optimization

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CHAPTER 3 (Part 1): LINEAR PROGRAMMING – THE SIMPLEX METHOD

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Innovating Solutions

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Introduction

- The graphical method of solving linear programming problems is useful only for problems involving two decision variables and relatively few problem constraints.
- What happens when we need more decision variables and more problem constraints?
- We use an **algebraic method** called the **simplex method**, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

The Simplex Method

What is the Simplex Method?	A popular algorithm used to solve linear programming problems. It iteratively improves the solution until an optimal solution is found.
How does it Work?	The simplex method starts with an initial feasible solution and moves along the edges of the feasible region, improving the objective function value at each iteration.
Advantages of the Simplex Method	It is efficient for solving large-scale linear programming problems and provides insights into the sensitivity of the solution to changes in the problem's parameters.

Standard Maximization Problems in Standard Form

A linear programming problem is said to be a **standard maximization problem** in standard form if its mathematical model is of the following form:

Maximize $Z = 4X_1 + 6X_2 + 2X_3$

Subject to:

$2X_1 + 5x2 \leq 6$

$X_1 + 9X_2 - 7X_3 \leq 8$

$4X_1 - 10X_2 + 2X_3 \geq 3$

$X_1 \geq 0, X_2 \geq 0$

Slack Variable

- A mathematical representation of surplus resources." In real life problems, it's unlikely that all resources will be used completely, so there usually are unused resources.
- **Slack variables** represent the unused resources between the left-hand side and right-hand side of each inequality.

Slack Variable

- The difference between the Right-Hand Side (R.H.S) and Left-Hand Side (L.H.S) of the (inequality ≤) constraint thus yields the unused or slack amount of the resource.

  R.H.S - L.H.S = **slack** or **unused amount**

- To convert the (inequality ≤) to an equation (=), a non-negative slack variable is added to the left-hand side of the constraint.
- Example of Reddy Mikks model:

  Constraint of raw material M1,

  (L.H.S) $6X_1 + 4X_2 \leq 24$ (R.H.S)

  Let, S_1 be the slack or unused amount of raw material M1,

  (L.H.S) $6X_1 + 4X_2 + S_1 = 24$ (R.H.S)

  $\therefore S_1 \geq 0$ (non-negative slack variable)

  or $24 - (6X_1 + 4X_2) = S_1$

Surplus Variable

- The difference between the Left-Hand Side (L.H.S) and Right-Hand Side (R.H.S) of the (inequality $\geq$) constraint thus yields the surplus or extra amount of the resource.

  L.H.S - R.H.S = **surplus** or **extra amount**

- To convert the (inequality $\geq$) to an equation (=), a non-negative surplus variable is subtracted from the left-hand side of the constraint.
- Example:

  (L.H.S) $5X_1 + 7X_2 \geq 30$ (R.H.S)

  Let, S_2 be the surplus or extra amount,

  (L.H.S) $5X_1 + 7X_2 - \mathbf{S_2} = 30$ (R.H.S)

  $.: S_2 \geq 0$ (non-negative surplus variable)

  or $(5X_1 + 7X_2) - 30 = S_2$

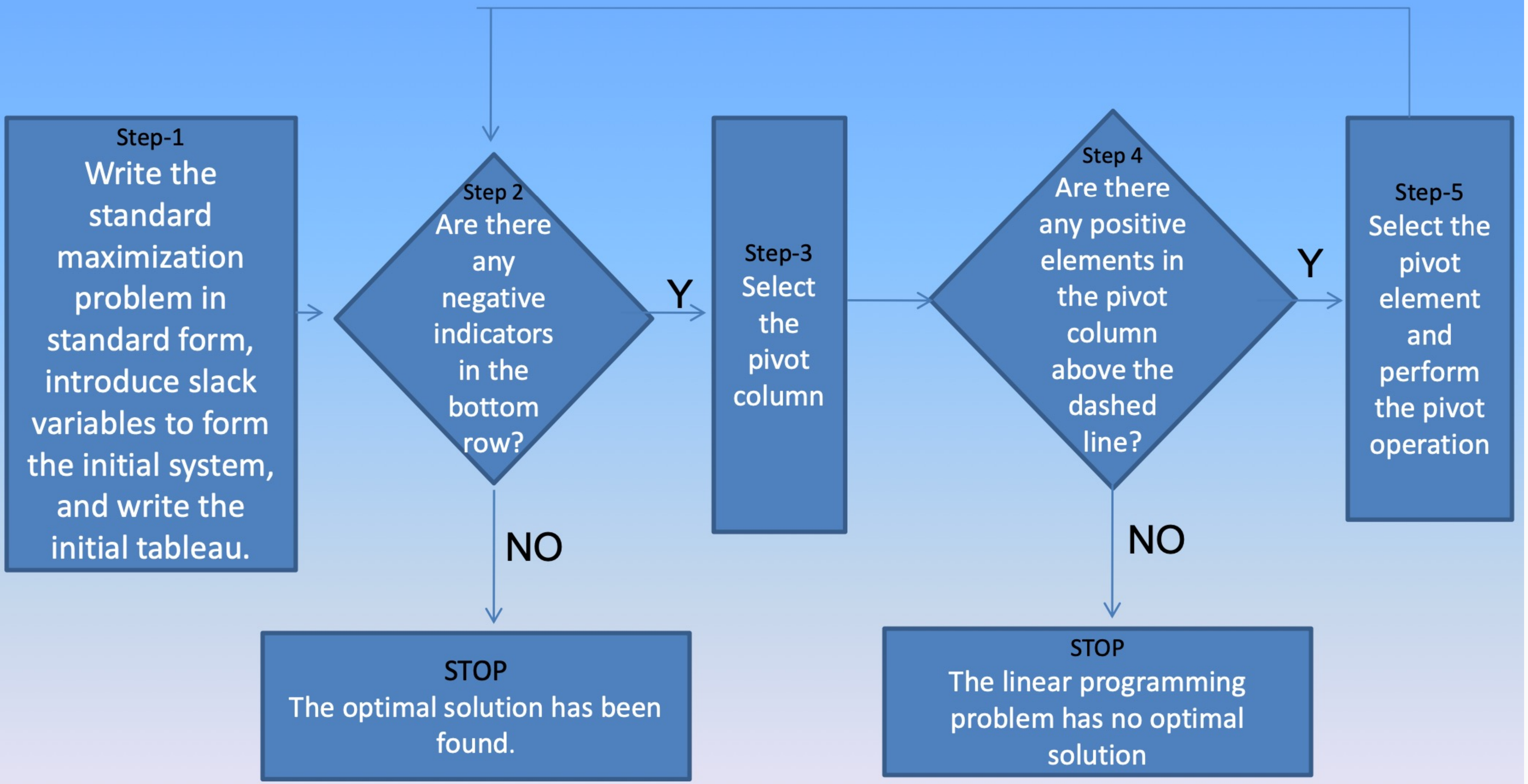
Basic and Non-basic Variables

- **Basic variables** are selected arbitrarily with the restriction that there be as many basic variables as there are equations. The remaining variables are **non-basic variables**.

$$x_1 + 2x_2 + s_1 = 32$$

$$3x_1 + 4x_2 + s_2 = 84$$

- This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables.
- A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a basic solution. If a basic solution has no negative values, it is a basic feasible solution.



Simplex algorithm for standard maximization problems

Steps to Solve LP problem using Simplex Method

To solve a linear programming problem in standard form, use the following steps.

1) Convert each inequality in the set of constraints to an equation by adding slack variables.
2) Create the initial simplex tableau.
3) Select the pivot column. (The column with the "most negative value" element in the last row.)
4) Select the pivot row. (The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column.)
5) Use elementary row operations to calculate new values for the pivot row so that the pivot is 1 (Divide every number in the row by the pivot number.)
6) Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot number. If all entries in the bottom row are zero or positive, this the final tableau. If not, go back to step 3.
7) If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Pivot

Pivot Column: The column of the tableau representing the variable to be entered into the solution mix.

Pivot Row: The row of the tableau representing the variable to be replaced in the solution mix.

Pivot Number: The element in both the pivot column and the pivot row.

Simplex Tableau

- Most real-world problems are too complex to solve graphically. They have too many corners to evaluate, and the algebraic solutions are lengthy.

- A simplex tableau is a way to systematically evaluate variable mixes in order to find the best one.

Simplex Tableau

Initial Simplex Tableau

	All Variables	Solution
Basic Variables	Coefficients	
		0

Example

The Cannon Hill furniture Company produces tables and chairs. Each table takes four hours of labour from the carpentry department and two hours of labour from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of $70 and each chair a profit of $50. How many chairs and tables should be made?

Example – Solution

Informations			
Resource	Tables (X_1)	Chairs (X_2)	Constraints
Carpentry (hr)	4	3	240
Finishing (hr)	2	1	100
Unit Profit	\$70	\$50	

Maximize $Z = 70X_1 + 50X_2$

Subject to:

$4X_1 + 3X_2 \leq 240$

$2X_1 + X_2 \leq 100$

$X_1 \geq 0, X_2 \geq 0$

Example - Solution

- The **1st step** of the simplex method requires that each inequality be converted into an equation.
- The "less than or equal to" inequalities are converted to equations by including slack variables.
- Suppose carpentry hours and finishing hours remain unused in a week.
- The constraints become;

$$4X_1 + 3X_2 + S_1 = 240 \text{ or } 4X_1 + 3X_2 + S_1 + 0S_2 = 240$$

$$2X_1 + X_2 + S_2 = 100 \text{ or } 2X_1 + X_2 + 0S_1 + S_2 = 100$$

- As unused hours result in no profit, the slack variables can be included in the objective function with zero coefficients:

$$Z = 70X_1 + 50X_2 + 0S_1 + 0S_2$$

$$Z - 70X_1 - 50X_2 - 0S_1 - 0S_2 = 0$$

Example - Solution

The problem can now be considered as solving a system of 3 linear equations involving the 5 variables; X_1, X_2, S_1, S_2, Z in such a way that Z has the maximum value;

$$4X_1 + 3X_2 + S_1 + 0S_2 = 240$$

$$2X_1 + X_2 + 0S_1 + S_2 = 100$$

$$Z - 70X_1 - 50X_2 - 0S_1 - 0S_2 = 0$$

Now, the system of linear equations can be written in matrix form or as a 3x6 augmented matrix.

Example – Solution

- The **2nd step:**

Basic Variables	X_1	X_2	S_1	S_2	Z	Right-Hand Side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
Z	-70	-50	0	0	1	0

The tableau represents the initial solution:

$$X_1 = 0;\ \ X_2 = 0;\ S_1 = 240;\ S_2 = 100; Z = 0$$

The slack variables S_1 and S_2 form the initial solution mix. The initial solution assumes that all available hours are unused. i.e. The slack variables take the largest possible values.

Example - Solution

- Variables in the solution mix are called basic variables.
- Each basic variables has a column consisting of all 0’s except for a single 1.
- All variables not in the solution mix take the value 0.
- The simplex process, a basic variable in the solution mix is replaced by another variable previously not in the solution mix.
- The value of the replaced variable is set to 0.

Example – Solution

- The **3rd step:**

Select the pivot column (determine which variable to enter the solution mix). Choose the column with the "most negative" element in the objective function row.

Basic Variables	X_1	X_2	S_1	S_2	**Z**	**Right-Hand Side**
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
Z	-70	-50	0	0	1	0

Pivot Column

- X_1 should enter the solution mix because each unit of X_1 (a table) contributes a profit of \$70 compared with only \$50 for each unit of X_1 (a chair).

Example – Solution

- The **4th step:**

Select the pivot row (determine which variable to replace in the solution mix). The pivot row is the row with the smallest non-negative result. Divide the last element in each row by the corresponding element in the pivot column.

Enter

Basic Variables	X_1	X_2	S_1	S_2	**Z**	Right-Hand Side
S_1	4	3	1	0	0	(240/4) = 60
Exit S_2	2	1	0	1	0	(100/2) = 50
Z	-70	-50	0	0	1	0

Pivot Number

Pivot Column

Pivot Row

Example - Solution

S_2 should be replaced by X_1 in the solution mix. 60 tables can be made with 240 unused carpentry hours but only 50 tables can be made with 100 finishing hours. Therefore, we decide to make 50 tables. Now calculate new values for the pivot row. Divide every number in the row by the pivot number.

Basic Variables	X_1	X_2	S_1	S_2	Z	Right-Hand Side
S_1	4	3	1	0	0	60
X_1	1	1/2	0	1/2	0	50
Z	-70	-50	0	0	1	0

Example - Solution

Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number which remains as 1.

Basic Variables	X_1	X_2	S_1	S_2	Z	Right-Hand Side	
S_1	0	1	1	-2	0	40	$-4\times R_2 + R_1$
X_1	1	1/2	0	1/2	0	50	
Z	0	-15	0	35	1	3500	$70\times R_2 + R_3$

If 50 tables are made, then the unused carpentry hours are reduced by 200 hours (4 h/table multiplied by 50 tables); the value changes from 240 hours to 40 hours. Making 50 tables results in the profit being increased by $3500; the value changes from $0 to $3500.

Example - Solution

Now repeat the steps until there are no negative numbers in the last row.
Select the new pivot column. X_2 should enter the solution mix.
Select the new pivot row. S_1 should be replaced by X_2 in the solution mix.

Enter

Basic Variables	X_1	X_2	S_1	S_2	Z	Right-Hand Side
Exit S_1	0	1	1	-2	0	40
X_1	1	1/2	0	1/2	0	50
Z	0	-15	0	35	1	3500

Pivot Row

New Pivot Column

Pivot Number

Example - Solution

Calculate new values for the pivot row. As the pivot number is already 1, there is no need to calculate new values for the pivot row.
Use row operations to make all numbers in the pivot column equal to except for the pivot number.

Basic Variables	X_1	X_2	S_1	S_2	Z	Right-Hand Side	
X_2	0	1	1	-2	0	40	
X_1	1	0	-1/2	3/2	0	30	$-\frac{1}{2}\times R_1 + R_2$
Z	0	0	15	5	1	4100	$15\times R_1 + R_3$

If 40 chairs are made, then the number of tables are reduced by 20 tables (1/2 table/chair multiplied by 40 chairs); the value changes from 50 tables to 30 tables. The replacement of 20 tables by 40 chairs results in the profit being increased by \$600; the value changes from \$3500 to \$4100.

Example - Solution

Result:

As the last row contains no negative numbers, this solution gives the maximum value of Z. This simplex tableau represents the optimal solution to the LP problem and is interpreted as:

$$X_1 = 30\ (Tables);\ X_2 = 40(Chairs);\ S_1 = 0;\ S_2 = 0; Z = \$4100\ (Profit)$$

Exercise

A farmer owns a 100-acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs \$40, \$20, and \$30 per acre, respectively. A maximum of \$3200 can be spent on seed. Crops A, B, and C require 1, 2, and 1 workdays per acre, respectively, and there are maximum of 160 workdays available. If the farmer can make a profit of \$100 per acre on crop A, \$300 per acre on crop B, and \$200 per acre on crop C, how many acres of each crop should be planted to maximize profit?