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CHAPTER 6: INTEGER PROGRAMMING (PART 2) – BRANCH & BOUND

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Innovating Solutions

Method for Solving Integer Programming

The two commonly used methods are:

1. Branch and Bound (B & B) Method

2. Cutting Plane Method

Neither method is consistently effective; but B&B is far more successful.

Branch-and-Bound (B&B)

- Developed in 1960 by A Land and G Doig
- Relax the integer restrictions in the problem and solve it as a regular LP. Let's call this LP_0 (to imply node-zero LP)
- Since the original "large" problem is hard to solve directly, it is divided into smaller and smaller sub-problems until these sub-problems can be conquered.
- The ***conquering*** (*fathoming*) is done partially by:

  (i) giving a ***bound*** for the best solution in the subset;

  (ii) discarding the subset if the bound indicates that it can't contain an optimal solution.

Branching

- If LP_0 (in general LP_i) fails to yield integer solution, branch on any variable that fails to meet this requirement. The process of branching is illustrated below.

If LP_i yields $x_1 = 3.5$ and x_1 is taken as the branching variable, we get two sub-problems, $LP_{i+1} = LP_i$ & $(x_1 \leq 3)$ and $LP_{i+2} = LP_i$ & $(x_1 \geq 4)$.

Bounding / Fathoming

- Select LP_1 (including new constraint in general LP_i) and solve.
- Three conditions arise:

1) Infeasible solution: Declare fathomed (no further investigation of LP_i)

2) Integer solution: If it is superior to the current best(Z*) solution update the current best.

   Declare fathomed.

3) Non-integer solution: If it is inferior to the current best(Z*), declare fathomed. Meaning it cannot yield any better Integer Linear Programming (ILP) solution and no further branching is required. Else branch again.

Best Bound

- In **maximisation**, the solution to a sub-problem is superior if it raises the current lower bound.

- In **minimisation**, the solution to a sub-problem is superior if it lowers the current upper bound.

- When all sub-problems have been fathomed, stop. The current bound is the best bound.

Example 1

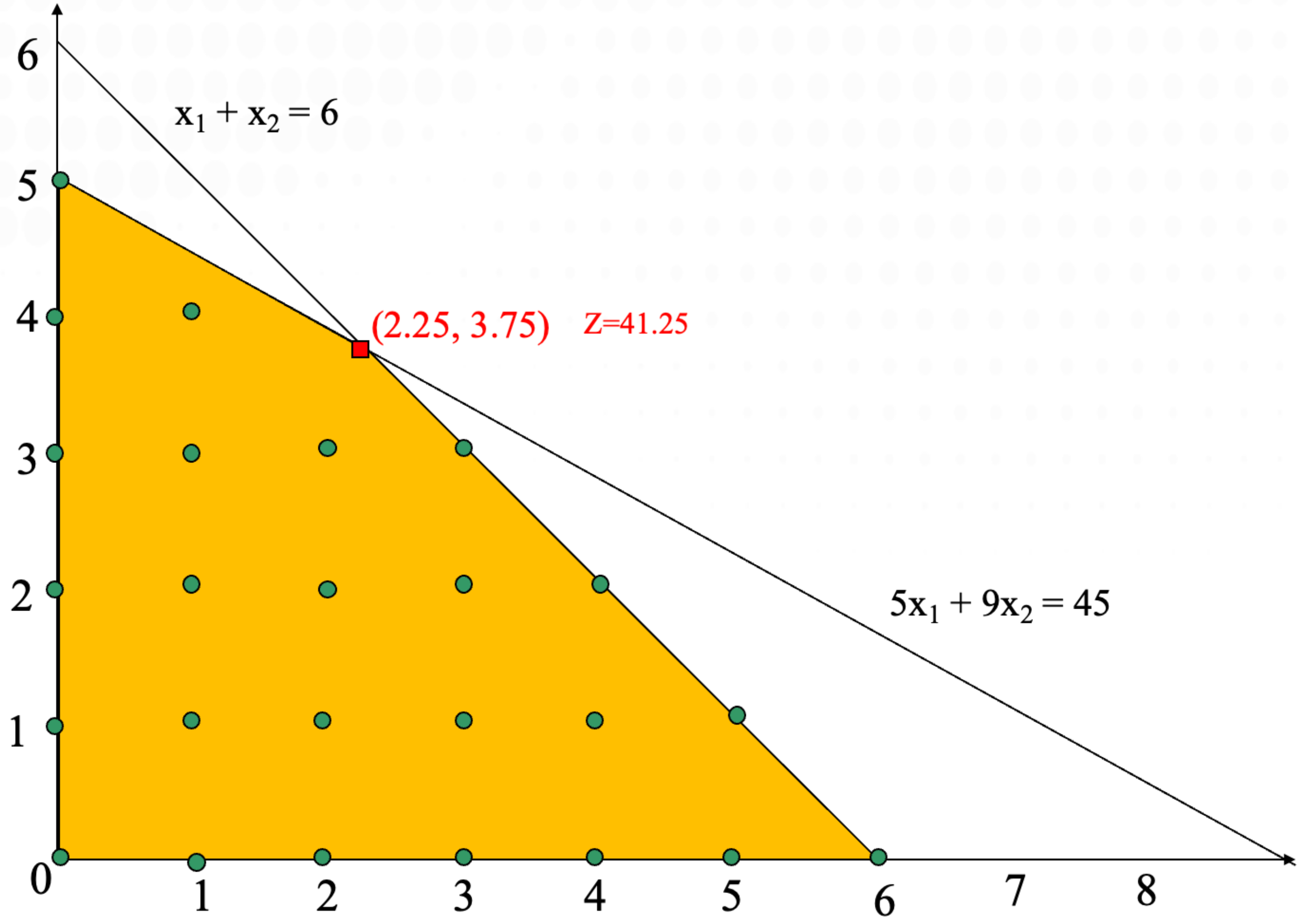
Max, $Z = 5x_1 + 8x_2$

Subject to:

$x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$ integer



LP-Relaxation

- *Fact*: If LP-relaxation has integral optimal solution x*, then x* is optimal for IP too.
  - In our case, $(x_1, x_2) = (2.25, 3.75)$ is the optimal solution of the LP-relaxation. Unfortunately, it is ***not*** integral.
  - The optimal value is 41.25

- *Fact*: OPT(LP-relaxation) ≥ OPT(IP) (for maximization problems)
  - The optimal value of the LP-relaxation is an *upper bound* for the optimal value of the IP.
  - Thus 41.25 is an upper bound value for OPT(IP). Current best value = 41.25

Branching Steps

- To find out more about the location of the IP's optimal solution, *partition* the feasible region of the LP-relaxation.
- Choose a variable that is fractional in the optimal solution to the LP-relaxation (variable has the greatest fractional part – x_2). Observe that every feasible IP point must have either $x_2 \leq 3$ or $x_2 \geq 4$.
- With this in mind, branch on the variable x_2 to create the following two subproblems:

Subproblem 1(LP1)

Max $Z = 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_2 \leq 3$

$x_1 , x_2 \geq 0$

Subproblem 2(LP2)

Max $Z = 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

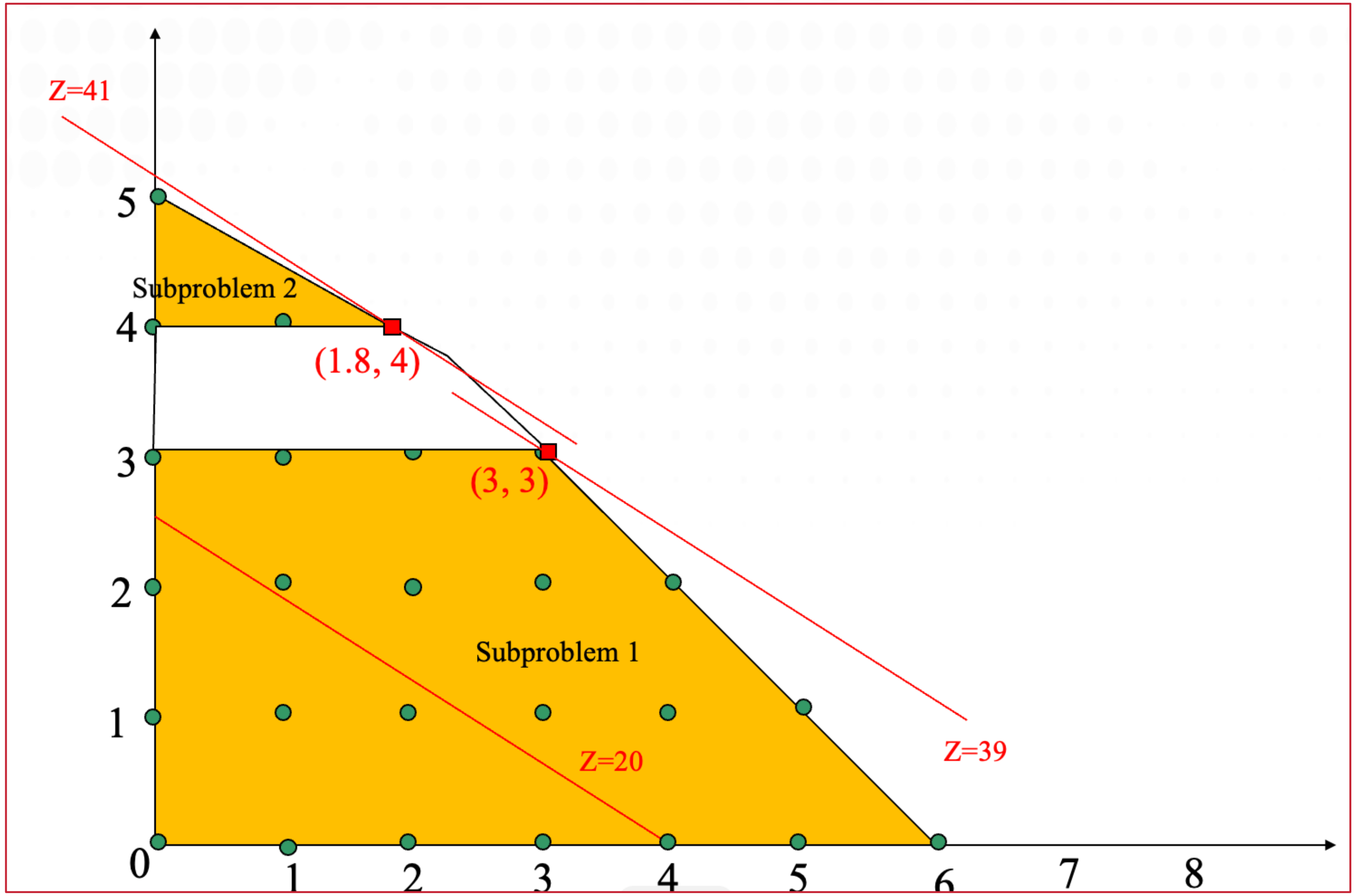
$x_2 \geq 4$

$x_1 , x_2 \geq 0$

- Solve both subproblems separately.

Branching Steps (Graphically)

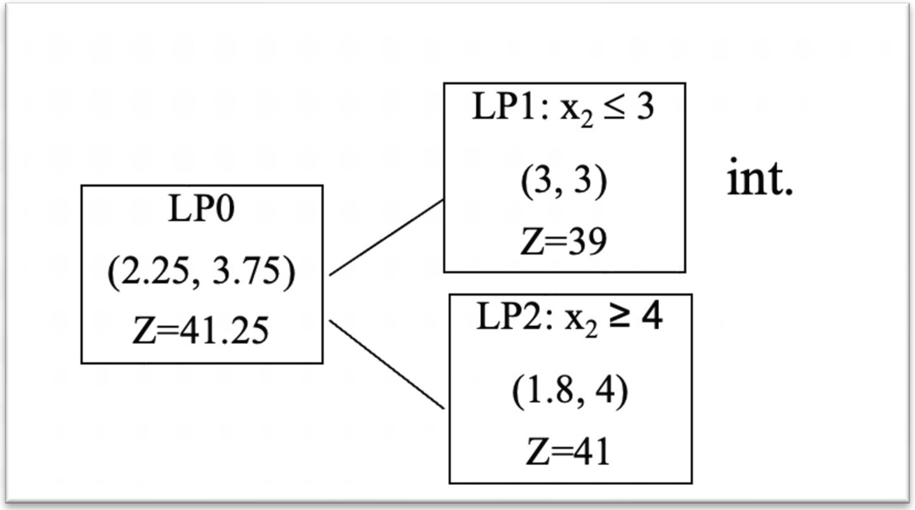
Subproblem 1: Optimal solution (3,3) with value z = 39
Subproblem 2: Optimal solution (1.8,4) with value z = 41



Solution Tree

For each subproblem, we record:

- ✓ the restriction that creates the subproblem.
- ✓ the optimal LP solution.
- ✓ the LP optimum value.



The optimal solution for Subproblem 1 is integral: (3, 3).

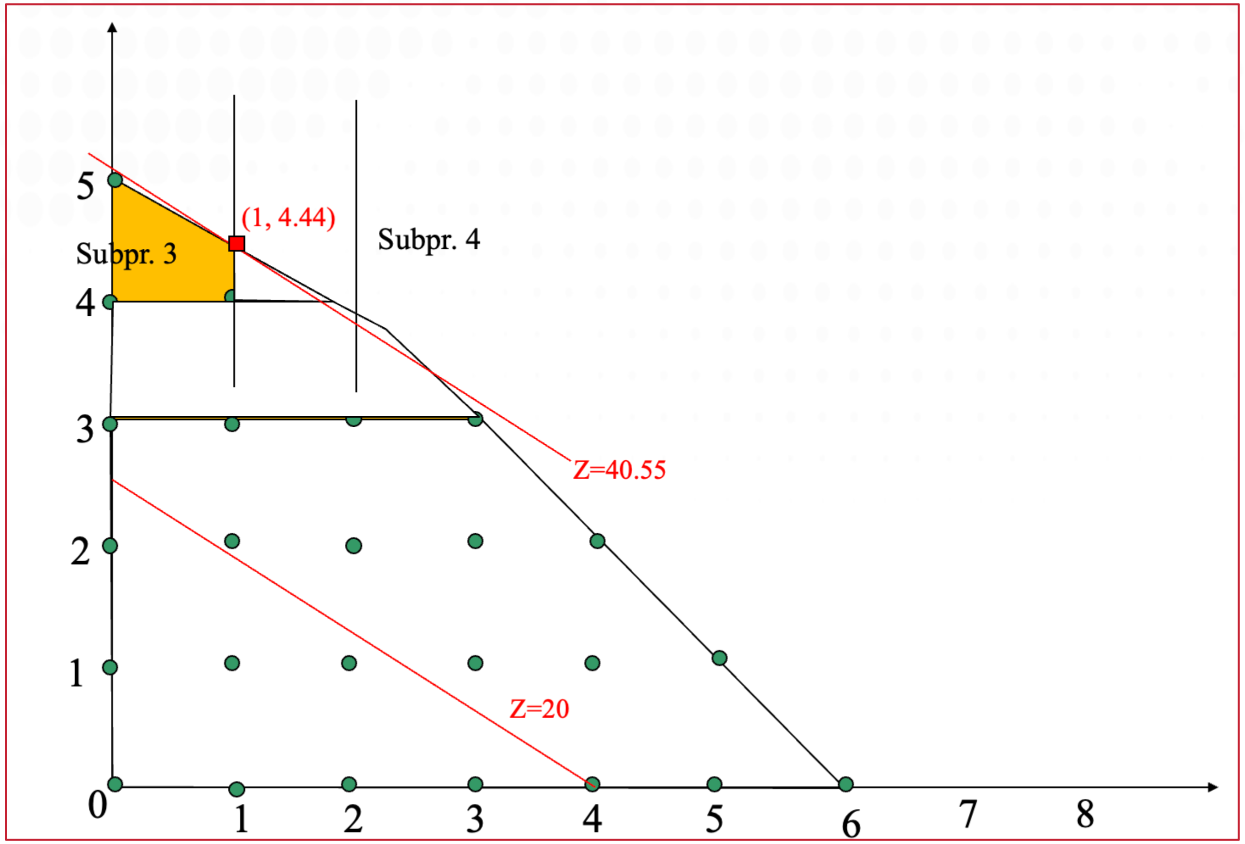
- In this case, we can fathom(dismiss) Subproblem 1 because its solution is integral.
- The best integer solution found so far is denoted by Z*. In our case, Z*=39.
- Z* is a lower bound for OPT(IP): OPT(IP) ≥ Z*. In our case, OPT(IP) ≥ 39.
- The optimal solution for Subproblem 2 is (1.8, 4) with Z=41.
- The upper bound is 41: OPT(IP) ≤ 41.

Next Branching Steps (Graphically)

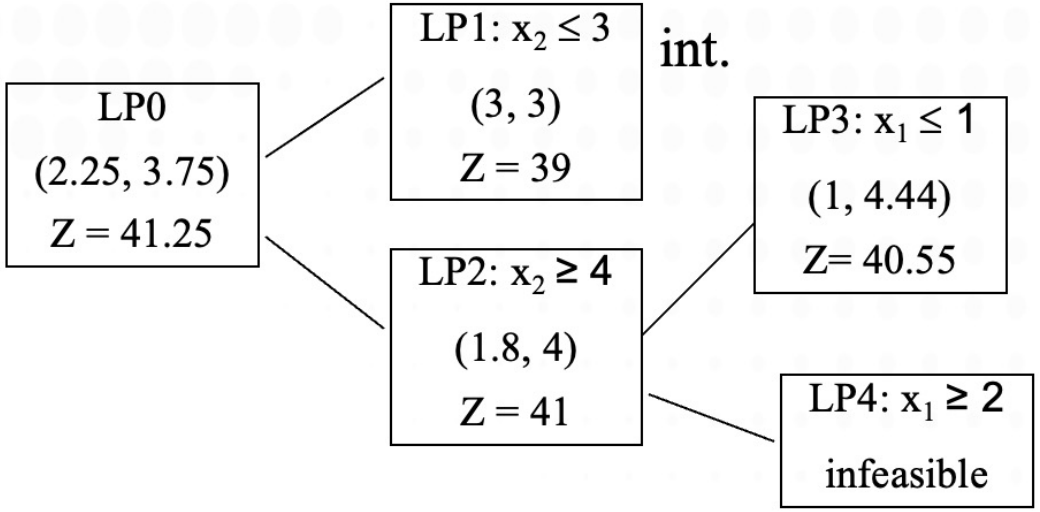
- Fathom Subproblem 1.
- Branch Subproblem 2 on x_1 :
  - Subproblem 3: New restriction is $x_1 \leq 1$.

    Opt. solution (1, 4.44) with value z = 40.55
  - Subproblem 4: New restriction is $x_1 \geq 2$.

    The subproblem is infeasible.



Solution Tree (cont'd)



Max $Z = 5x_1 + 8x_2$
s.t:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_2 \geq 4$
$x_1 \leq 1$
$x_1, x_2 \geq 0$

Max $Z = 5x_1 + 8x_2$
s.t:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_2 \geq 4$
$x_1 \geq 2$
$x_1, x_2 \geq 0$

- If a subproblem is infeasible, then it is fathomed. In our case, Subproblem 4 is infeasible; fathom it.

  The upper bound for OPT(IP) is updated: $39 \leq \text{OPT(IP)} \leq 40.55$.

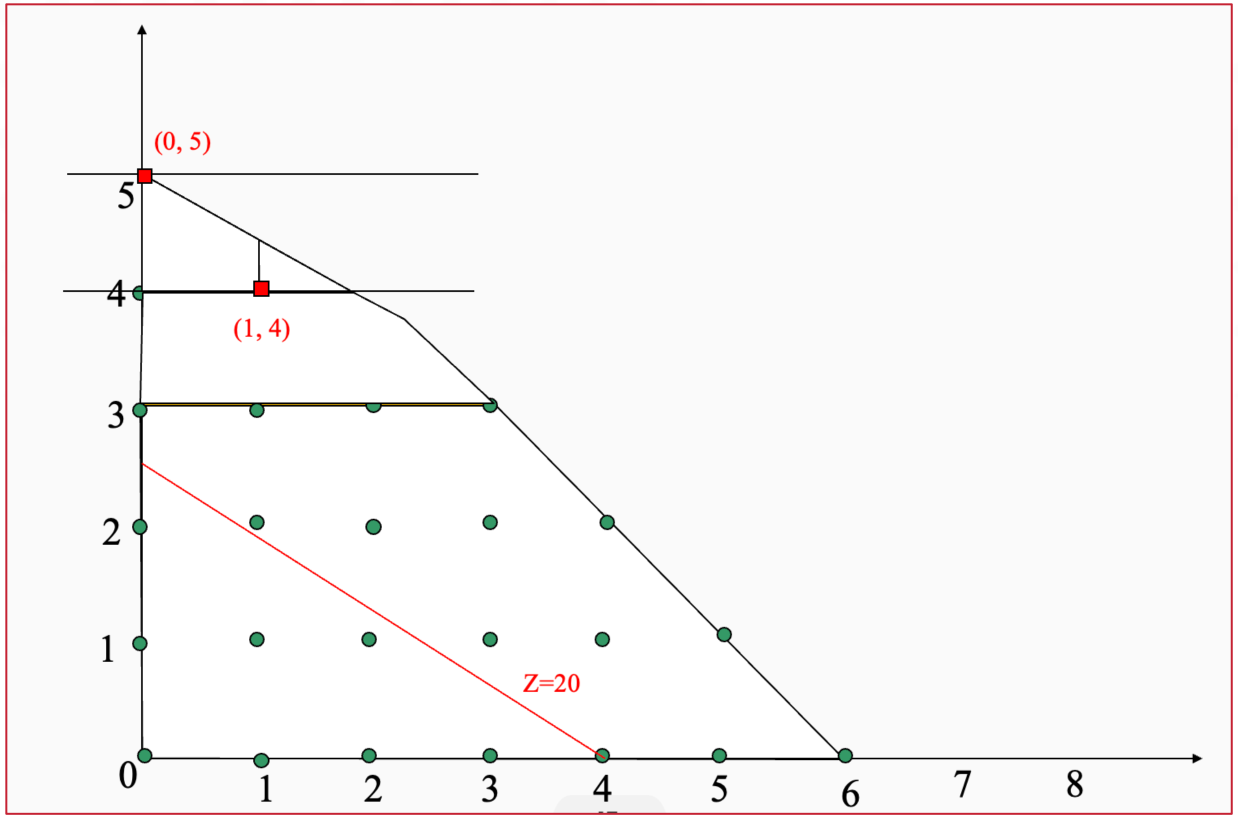
- Next branch Subproblem 3 on x_2 .

  (Note that the branching variable might recur).

Solution Tree (cont'd)

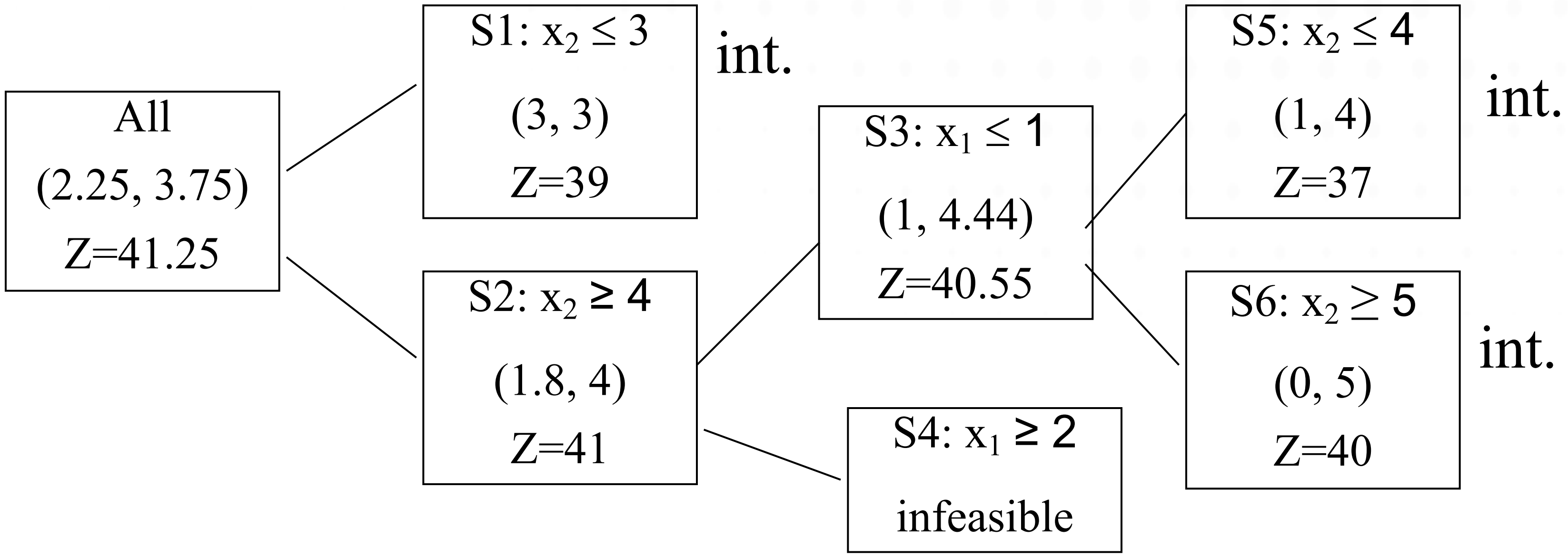
Branch Subproblem 3 on x_2 :

- Subproblem 5: New restriction is $x_2 \leq 4$.
  Feasible region:
  - The segment joining (0,4) and (1,4)
  - Opt. solution (1, 4) with value 37
- Subproblem 6: New restriction is $x_2 \geq 5$.
  - Feasible region is just one point: (0, 5)
  - Opt. solution (0, 5) with value 40



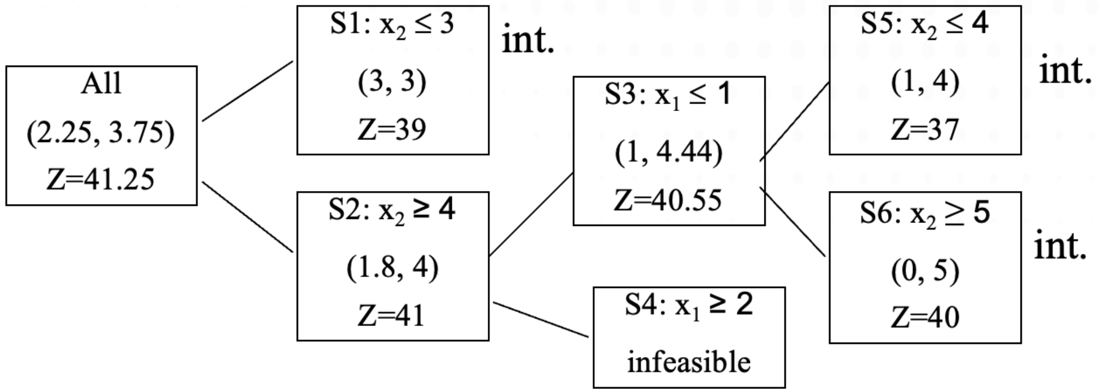
Solution Tree - Final

Max $Z = 5x_1 + 8x_2$
s.t:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_2 \geq 4$
$x_1 \leq 1$
$x_2 \leq 4$
$x_1 , x_2 \geq 0$



Max $Z = 5x_1 + 8x_2$
s.t:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_2 \geq 4$
$x_1 \leq 1$
$x_2 \geq 5$
$x_1 , x_2 \geq 0$

Solution Tree - Final



- If the optimal value of a subproblem is $\leq Z^*$, then it is fathomed.
  - In our case, Subproblem 5 is fathomed because $37 \leq Z^*$.
- If a subproblem has integral optimal solution x^*, and its value $> Z^*$, then x^* replaces the current best integer solution.
  - In our case, Subproblem 6 has integral optimal solution, and its value $40 > 39 = Z^*$. Thus, (0,5) is the new best integer solution, and new $Z^* = 40$.
- If there are no unfathomed subproblems left, then the current Z^* is an optimal solution for (IP).
  - In our case, (0, 5) is an optimal solution with optimal value $Z = 40$.

Solution – ILP Model (final LP model)

Max, $Z = 5x_1 + 8x_2$

Subject to:

$$x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_2 \geq 4$$
$$x_1 \leq 1$$
$$x_2 \geq 5$$
$$x_1 , x_2 \geq 0 \text{ and integer}$$

Solution:

$x_1 = 0; x_2 = 5; Z = 40$

Branch & Bound (for Minimization IP)

The optimal value of the LP-relaxation is a *lower bound* for the optimal value of the IP.

- The best integer solution found is denoted by Z*.

  Z* is an upper bound for OPT(IP):

  $$\mathrm{OPT(IP)} \leq Z^* .$$

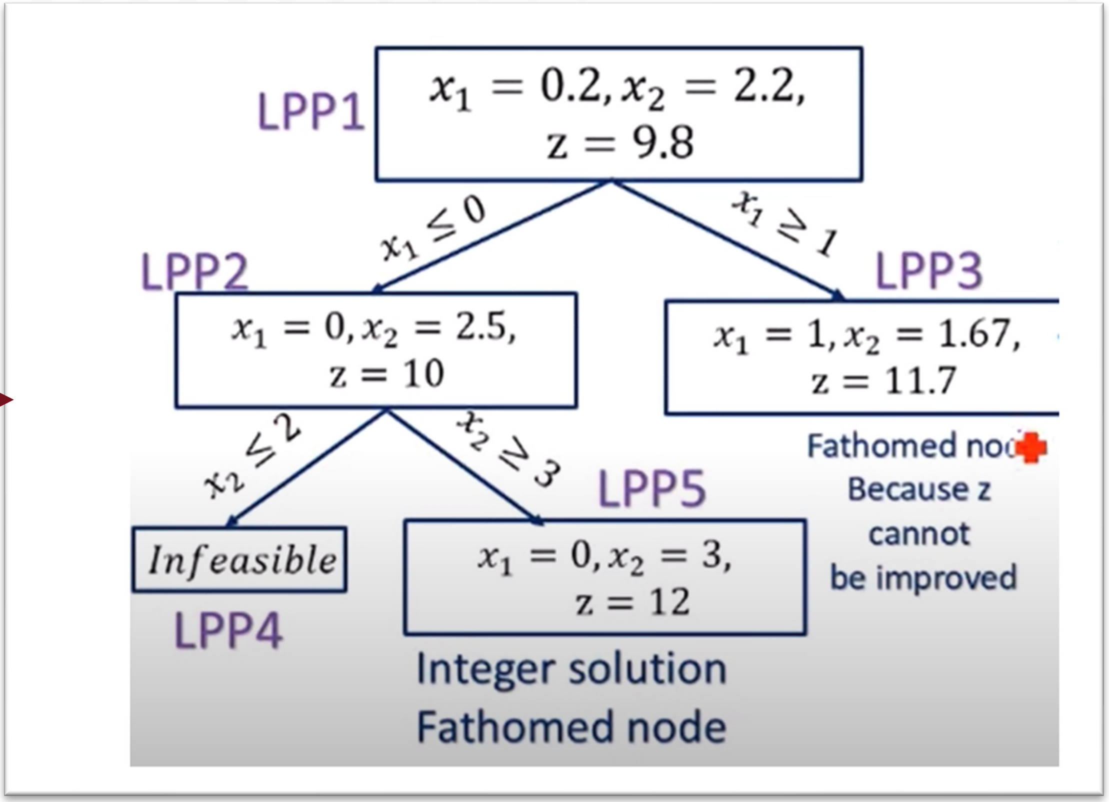
- If solution exceeds upper bound, branch is fathomed.
- If solution is integer and if it is superior to the current best(Z*) solution (< Z*) update the current best, replace the Z* (upper bound on cost)

Example 2

$$Minimize\ z = 5x_1 + 4x_2$$

subject to

$$3x_1 + 2x_2 \geq 5$$
$$2x_1 + 3x_2 \geq 7$$
$$x_1, x_2\ non-negative\ integers$$



Exercise

Solve the following ILP problem using B&B method.

Maximize $Z = 5x_1 + 4x_2$

subject to:

$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

- Draw the B&B tree.
- Provide your final solution with its ILP model (i.e., the final LP model)