UTM
UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 5: DUALITY AND POST-OPTIMAL ANALYSIS

nzah@utm.my-20232024(S2)

Innovating Solutions

Introduction

- In Chapter 2 presents the (primal) simplex algorithm that starts feasible and continues until the optimum is reached.
- Post-Optimality analysis means examining changes after the optimal solution has been reached.
- The analysis does not begin until optimal solution to the original LP problem has been obtained.
- For this reason, Sensitivity Analysis is often referred to as post-optimality analysis.
- After an LP problem has been solved, we determine the range of changes that will not affect the optimal solution in the primal.
- This is done without solving the whole problem again.

Dual Simplex Method

- In the dual simplex algorithm, the LP problem starts (better than) optimum and infeasible.
- Successive iterations are designed to move toward feasibility without violating optimality.
- At the iteration when feasibility is restored, the algorithm ends.
- The starting tableau must have an optimum objective row with at least one infeasible (< 0) Basic Variable (BV).
- To maintain optimality and simultaneously, move forward feasibility at each new iteration, the following two conditions are employed.

Dual Simplex Method

i) Dual feasibility condition:

- The leaving variable, x_r, is the BV having the most negative value with ties broken arbitrarily. If all the BV are nonnegative, the algorithm ends.

ii) Dual optimality condition:

- The entering variable is determined from among the non-BV(NBV) as the one corresponding to,

$$\min_{NBV\ x_j}\left\{\left|\frac{Z_j - c_j}{\alpha_{rj}}\right|, \alpha_{rj} < 0\right\}$$

Where α_{rj} is the constraint coefficient of the tableau associated with the row of the leaving variables x_r and the column of the entering variable x_j. Ties are broken arbitrarily.

Requirement

1) The objective function must satisfy the optimality condition of the regular simplex method:

- Minimum Z is attained when all the Z row coefficient are non positive.

2) All the constraints must be of the type (≤)

- Inequalities of the type (≥) are converted to (≤) by multiplying both sides of the inequality by -1.

- If the LP includes (=) constraints, the equation can be replaced by two inequalities. For example, $x_1 + x_2 = 1$ is equivalent to $x_1 + x_2 \leq 1$ and $x_1 + x_2 \geq 1$ $(-x_1 - x_2 \leq -1)$

Example 1

Minimize, $Z = 3x_1 + 2x_2 + x_3$

Subject to:

$$3x_1 + x_2 + x_3 \geq 3$$

$$-3x_1 + 3x_2 + x_3 \geq 6$$

$$x_1 + x_2 + x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

The first two inequalities are multiplied by −1 to convert them to (≤) constraints.

$-3x_1 - x_2 - x_3 \leq -3$

$3x_1 - 3x_2 - x_3 \leq -6$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	−3	−2	−1	0	0	0	0
x_4	−3	−1	−1	1	0	0	−3
x_5	3	−3	−1	0	1	0	−6
x_6	1	1	1	0	0	1	3

Example 1 (cont'd)

- The tableau is optimal because all the reduced costs in the Z-row are ≤ 0
- It is also infeasible because at least one of the basic variables is negative.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$\boxed{-3}$	$\boxed{-2}$	$\boxed{-1}$	0	0	0	0
x_4	−3	−1	−1	1	0	0	−3
x_5	3	−3	−1	0	1	0	−6
x_6	1	1	1	0	0	1	3

→ Leaving variable

- According to the dual feasibility condition, x_5 is the leaving variable (basic variable having the most negative value).

Example 1 (cont'd)

- Entering variable:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

- The ratios show that x_2 is the entering variable (min ratio).

	$j = 1$	$j = 2$	$j = 3$
Nonbasic variable	x_1	x_2	x_3
z-row $(\bar{c}_j)$	-3	-2	-1
x_5-row, α_{5j}	3	-3	-1
Ratio, $\left\|\frac{\bar{c}_j}{\alpha_{5j}}\right\|, \alpha_{5j} < 0$	—	$\frac{2}{3}$	1

Example 1(cont'd)

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

New x_2 row = (3, -3, -1, 0, 1,0,-6) ÷ -3 = (-1, 1, 1/3, 0, -1/3, 0,2)

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	1	1	1	0	0	1	3

Example 1(cont'd)

New Tableau-Iteration 2

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	1	1	1	0	0	1	3

- New z row = Current z row - (-2) *New x_2 row
- New x_4 row =Current x_4 row - (-1)* New x_2 row
- New x_6 row =Current x_6 row - (1)* New x_2 row

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

Example 1 (cont'd) New Tableau–Iteration 3

- Preceding tableau shows that x_4 leaves and x_3 enters.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

- New x_3 row = (-4, 0, -2/3, 1, -1/3,0,-1) ÷ -2/3 = (6, 0, 1, -3/2,1/2, 0, 0,3/2)

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

Example 1(cont'd)

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1

- New z row =Current z row –(−1/3) *New x_3 row
- New x_2 row =Current $x_{"}$ row –(1/3) *New x_3 row
- New x_6 row =Current $x_{\$}$ row –(2/3) *New x_3 row

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

The process ends with the optimal feasible solution given as $x_1 = 0, x_2 = 3/2$, $x_2 = 3/2$, and $z = 9/2$.

Post Optimal Analysis

- We dealt with the sensitivity analysis of the optimum solution by determining the ranges for the different LP parameters that would keep the optimum basic variables unchanged.
- In this section, we deal with making changes in the parameters of the model and finding the new optimum solution.
- Post-optimal analysis determines the new solution in an efficient way.

Post Optimal Analysis

The following table lists the cases that can arise in post-optimal analysis and the actions needed to obtain the new solution (assuming one exists):

Condition after parameters change	Recommended action
Current solution remains optimal and feasible.	No further action is necessary.
Current solution becomes infeasible.	Use dual simplex to recover feasibility.
Current solution becomes nonoptimal.	Use primal simplex to recover optimality.
Current solution becomes both nonoptimal and infeasible.	Use the generalized simplex method to recover optimality and feasibility.

Changes Affective Feasibility

- The feasibility of the current optimum solution is affected only if the right-hand side of the constraints is changed, or a new constraint is added to the model. In both cases, infeasibility occurs when one or more of the current basic variables become negative.

- Changes in the right-hand side: This change requires recomputing the right-hand side of the tableau using formula:

$$\begin{pmatrix}\text{New right-hand side of} \\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in} \\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand} \\ \text{side of constraints}\end{pmatrix}$$

Example 2: TOYCO Model

Letting $x_1, x_2,$ and x_3 represent the daily number of units assembled of trains, trucks, and cars, respectively. The associated LP model is given as:

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to:

$$x_1 + 2x_2 + x_3 <= 430$$

$$3x_1 + 2x_3 <= 460$$

$$x_1 + 4x_2 <= 420$$

Example 2 (cont.)

- The associated optimum tableau for the primal simplex is given as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	4	0	0	1	2	0	1,350
X_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
X_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
X_6	2	0	0	-2	1	1	20

Example 2 (cont.) Situation 1

Suppose that TOYCO is increasing the daily capacity of operations 1, 2, and 3 to 600, 640, and 590 minutes, respectively. How would this change affect the total revenue?

$$\begin{pmatrix}\text{New right-hand side of}\\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand}\\ \text{side of constraints}\end{pmatrix}$$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	4	0	0	1	2	0	1,350
X_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
X_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
X_6	2	0	0	-2	1	1	20

$$\begin{pmatrix}x_2\\ x_3\\ x_6\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & 0\\ 0 & \frac{1}{2} & 0\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}600\\ 640\\ 590\end{pmatrix} = \begin{pmatrix}140\\ 320\\ 30\end{pmatrix}$$

Example 2 (cont.)

- The current basic variables, x_2, x_3, and x_6, remain feasible (≥ 0) at the new values 140, 320, and 30 units, respectively.
- The associated optimum revenue is

$$Z = 3 * (0) + 2 * (140) + 5 * (320) = \$1870.$$

Example 2 (cont.) Situation 2

Although the new solution is appealing from the standpoint of increased revenue, TOYCO recognizes that its implementation may take time. Another proposal shifts the slack capacity of operation 3 ($x_6 = 20$ minutes) to the capacity of Operation 1. How would this change impact the optimum solution?

Solution:

$$x_1 + 2x_2 + x_3 \leq 430$$

$$3x_1 + 2x_3 \leq 460$$

$$x_1 + 4x_2 \leq 420$$

The capacity mix of the three operations changes to 450, 460, and 400 minutes, respectively.

Example 2 (cont.)

New optimal solutions:

$$\begin{pmatrix}\text{New right-hand side of}\\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand}\\ \text{side of constraints}\end{pmatrix}$$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	4	0	0	1	2	0	1,350
X_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
X_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
X_6	2	0	0	-2	1	1	20

New right –hand side = 450, 460, 400

$$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & 0\\ 0 & \frac{1}{2} & 0\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}450\\460\\400\end{pmatrix} = \begin{pmatrix}110\\230\\-40\end{pmatrix}$$

The resulting solution is infeasible because x_6 = -40, which requires applying the dual simplex method to recover feasibility.

Example 2 (cont.)

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix} = \begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix}$$

- Modify the right-hand side of the tableau as shown by the shaded column.
- Associated value of z = 3 * (0) + 2 * (110) + 5 *(230) = $1370.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1370
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	110
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	-40

- Solve using dual simplex method.

Example 2 (cont.)

- x_6 leaves and x_4 enters, which yields the following optimal feasible tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	5	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1350
x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_4	-1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	20

- The optimum solution (in terms of $x_1, x_2,$ and x_3) remains the same as in the original model. This means that the proposed shift in capacity allocation is not advantageous because it simply shifts the slack capacity from operation 3 to a slack capacity in operation 1.

Exercise #1

1) In the TOYCO model (Example 2), would it be more advantageous to assign the 20-minute excess capacity of operation 3 to operation 2 instead of operation 1?

2) Suppose that TOYCO wants to change the capacities of the three operations to 460, 500 and 400 minutes, respectively. Use post-optimal analysis to determine the optimum solution

Exercise #2

Reddy Mikks model:

Given its optimal tableau as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

If the daily availabilities of raw materials M1 and M2 are increased to 35 and 10 tons, respectively, use post-optimal analysis to determine the new optimal solution.